Lectures on Algebraic Statistics Ch. 3.1 & 3.2

Lukas Gustafsson

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Today's goal

- Define Conditional Probability explicitly

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- Define Conditional Probability explicitly
- Define Conditional Independence (CI)
- Describe discrete models with CI
- Describe Gaussian models with CI
- Graphical models, undirected

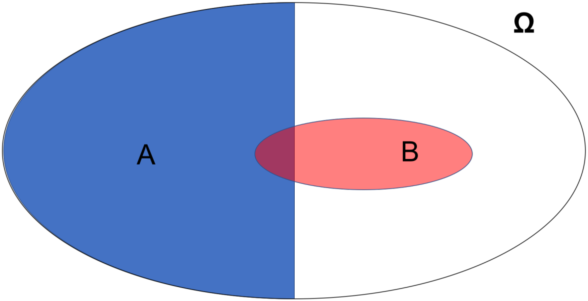
What is a conditional probability?

What is a conditional probability?

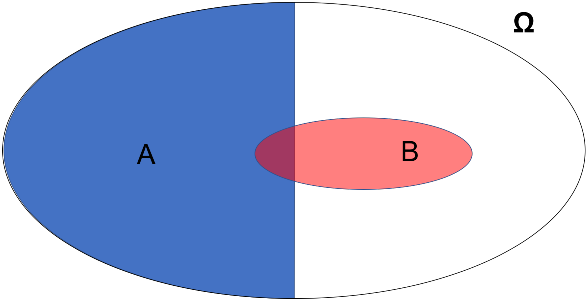
Recall that for two events A, B we define it to be

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

Probability intro



Probability intro



$\mathbb{P}(A) = 1/2$ while $\mathbb{P}(A|B)$ is much smaller.

Conditional probability

How do we define conditional probability of random variables?

Notation

Let X be a random vector then for $c \in \mathbb{R}^m$ let $X = c$ denote

$$\{\omega \in \Omega : \quad X(\omega) = c\}$$

Conditional probability

If X, Y are discrete

$$\mathbb{P}(X = i | Y = j) = \frac{\mathbb{P}(X = i, Y = j)}{\mathbb{P}(Y = j)}$$

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In other notation

$$\frac{p_{ij}}{p_{+j}}$$

Conditional probability

Let $X : \Omega \to \mathbb{R}^m$ with density f_X.

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If $A \subset [m]$, define subvector of X

$$X_A : \Omega \to \mathbb{R}^A$$

Conditional probability

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$$X_A : \Omega \to \mathbb{R}^A$$

The marginal density

$$f_A(x_A) := \int_{\mathbb{R}^{A^C}} f_X(x) dx_{A^C}$$

Conditional probability

Given a continuous $X : \Omega \to \mathbb{R}^m$ and $A, B \subset [m]$ disjoint. Then

$$f_{A|B}(x_A|x_B) := \frac{f_{A\cup B}(x_A, x_B)}{f_B(x_B)}$$

Where are we?

Defined conditional probability. Check!

Where are we?

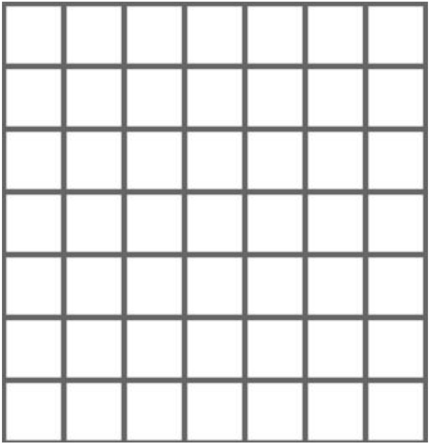
Defined conditional probability. Check! Next?

Where are we?

Defined conditional probability. Check! Next?

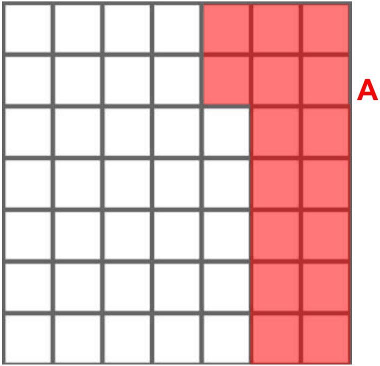
Define conditional independence

Conditional independence



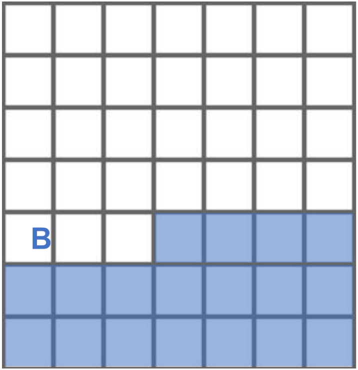
Ω

Conditional independence



P(A) = 16/49

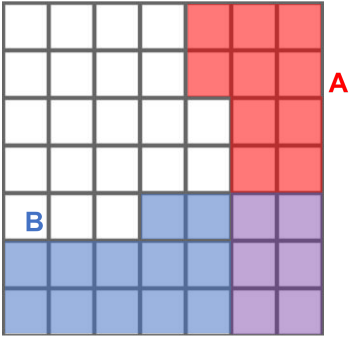
Conditional independence



Ω

P(B) = 18/49

Conditional independence



Ω

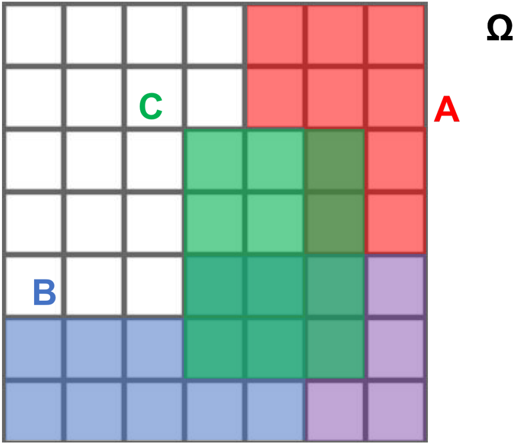
P(A) = 16/49

P(B) = 18/49

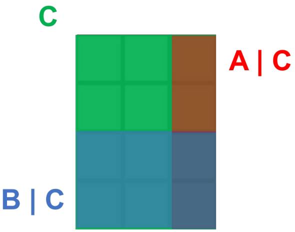
P(A ∩ B) = 6/49

Not independent!

Conditional independence



Conditional independence



P(A|C) = 1/3

P(B|C) = 1/2

$P(A \cap B | C) = 1/6$

Conditional independence

A is independent of B if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

Conditional independence

A is independent of B if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$$

A is *conditionally* independent of B given C if

$$\mathbb{P}(A \cap B | C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

Conditional independence

How do we translate this to random variables?

$$X_A \perp\!\!\!\perp X_B | X_C$$

Conditional independence

Two discrete random vectors X, Y are conditionally indpendent given Z iff

$$\forall i, j, k \quad X = i \perp\!\!\!\perp Y = j \text{ given } Z = k$$

Conditional independence

For continuous random variables conditional independence means

$$f_{A \cup B|C}(x_A, x_B|x_C) = f_{A|C}(x_A|x_C) \cdot f_{B|C}(x_B|x_C)$$

Conditional independence

Let $A, B, C, D \subset [m]$ be pairwise disjoint subsets.

Conditional Independence

Symmetry

$$X_A \perp\!\!\!\perp X_B | X_C \implies X_B \perp\!\!\!\perp X_A | X_C$$

Decomposition

$$X_A \perp\!\!\!\perp X_{B \cup D} | X_C \implies X_A \perp\!\!\!\perp X_B | X_C$$

Conditional Independence

Decomposition

$$X_A \perp\!\!\!\perp X_{B \cup D} | X_C \implies X_A \perp\!\!\!\perp X_B | X_C$$

'If two combined items of information are judged irrelevant to A, then each separate item is irrelevant as well'

Judea Pearl. *Causality: Reasoning and Inference.*

Weak union

$$X_A \perp\!\!\!\perp X_{B\cup D} | X_C \implies X_A \perp\!\!\!\perp X_B | X_{C\cup D}$$

.

Weak union

$$X_A \perp\!\!\!\perp X_{B \cup D} | X_C \implies X_A \perp\!\!\!\perp X_B | X_{C \cup D}$$

.

'Learning irrelevant information D cannot help the irrelevant information B become relevant to A'

Contraction

$$X_A \perp\!\!\!\perp X_B | X_{C \cup D} \text{ and } X_A \perp\!\!\!\perp X_D | X_C \implies X_A \perp\!\!\!\perp X_{B \cup D} | X_C$$

.

Contraction

$$X_A \perp\!\!\!\perp X_B | X_{C \cup D} \text{ and } X_A \perp\!\!\!\perp X_D | X_C \implies X_A \perp\!\!\!\perp X_{B \cup D} | X_C$$

.

'If we judge B irrelevant to A after learning some irrelevant information D , then B must have been irrelevant before we learned D'

Conditional Independence

Intersection (only for strictly positive distributions)

$$X_A \perp\!\!\!\perp X_B | X_{C \cup D} \text{ and } X_A \perp\!\!\!\perp X_C | X_{B \cup D} \implies X_A \perp\!\!\!\perp X_{B \cup C} | X_D$$

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Intersection (only for strictly positive distributions)

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.

'If B is irrelevant to A when we know C and if C is irrelevant to A when we know B, then neither C nor B (nor their combination) is relevant to A'

Conditional Independence

$$X_4 \perp\!\!\!\perp X_1 | X_{2\cup 3} \text{ and } X_4 \perp\!\!\!\perp X_2 | X_{1\cup 3} \implies X_4 \perp\!\!\!\perp X_{1\cup 2} | X_3$$

Where are we?

Define and gain intuition for conditional independence. Check!

Where are we?

Define and gain intuition for conditional independence. Check!
Next?

Where are we?

Define and gain intuition for conditional independence. Check! Next?

For a discrete random vector, can we describe conditional independence of subvectors purely in terms of the 'probability tensor' $p_{i_1 \ldots i_m}$?

Conditional independence

Notation:
Given a discrete random vector $X = (X_1, \ldots, X_m)$ we will denote

$$p_{i_1 i_2 \ldots i_m} := \mathbb{P}(X_1 = i_1, \ldots, X_m = i_m)$$

the distribution tensor.

Discrete CI models

If X is a discrete, then $X_A \perp\!\!\!\perp X_B | X_C$ iff

$$p_{(i_A, i_B, i_C, +)} \cdot p_{(k_A, k_B, i_C, +)} - p_{(i_A, k_B, i_C, +)} \cdot p_{(k_A, i_B, i_C, +)} = 0$$

for all $i_A, k_A \in \mathcal{R}_A$, $i_B, k_B \in \mathcal{R}_B$, and $i_C \in \mathcal{R}_C$.

Proof:

The first step is just to untangle the notation.

$$p_{(i_A, i_B, i_C, +)} := \mathbb{P}(X_A = i_A, X_B = i_B, X_C = i_C)$$

Discrete CI models

By definition all events $X_A = i_A$ must be conditionally independent of $X_B = i_B$ given $X_C = i_C$. This means exactly that

$$\frac{p_{(i_A,i_B,i_C,+)}}{p_{(+,+,i_C,+)}} = \frac{p_{(i_A,+,i_C,+)}}{p_{(+,+,i_C,+)}} \frac{p_{(+,i_B,i_C,+)}}{p_{(+,+,i_C,+)}}$$

This means that the matrix for a fix i_C, the conditional probability matrix $\frac{p_{(i_A,i_B,i_C,+)}}{p_{(+,+,i_C,+)}}$ is rank 1.

Thus $p_{(i_A,i_B,i_C,+)}$ must be rank 1 as well.

However, matrices of rank 1 is a famous variety, namely a Segre variety with well known equations, these are all the 2×2 minors of the matrix.

$$p_{(i_A,i_B,i_C,+)} \cdot p_{(k_A,k_B,i_C,+)} - p_{(i_A,k_B,i_C,+)} \cdot p_{(k_A,i_B,i_C,+)} = 0$$

For the other direction, we regain the conditional probabilities of A and B by taking marginals of the matrix.

$\square$

Discrete CI models

We define the conditional independence ideal

$$\mathcal{I}_{A \perp\!\!\!\perp B | C} = \langle p_{(i_A, i_B, i_C, +)} \cdot p_{(k_A, k_B, i_C, +)} - p_{(i_A, k_B, i_C, +)} \cdot p_{(k_A, i_B, i_C, +)} \rangle$$

It can be shown that this is prime (book omits it).

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It can be shown that this is prime (book omits it).

Satisfying several conditional statements amounts to adding the corresponding CI. ideals together.

Example (Marginal independence)

The (marginal) independence statement $X_1 \perp\!\!\!\perp X_2$ (or $X_1 \perp\!\!\!\perp X_2 | X_\emptyset$) checks whether

$$\operatorname{rk} \begin{bmatrix} p_{11} & \cdots & p_{1r_2} \\ \vdots & \cdots & \\ p_{r_1 1} & \cdots & p_{r_1 r_2} \end{bmatrix} \leq 1$$

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More generally,

$$\forall i, j \quad X_i \perp\!\!\!\perp X_j \iff \text{tensorrank}(p_{i_1 i_2 \ldots i_n}) \leq 1$$

Theorem 3.1.11 (Binomial primary decomposition)

Can we paramterize these models?

Theorem 3.1.11 (Binomial primary decomposition)

Every primary component and associated prime of a binomial ideal is a binomial ideal. In particular, every irreducible component of a binomial variety is a toric variety, and is unirational.

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Corollary:
If $\mathcal{C}$ consists of CI statements of the form $A \perp\!\!\!\perp B | C$ such that $A \cup B \cup C = [m]$, then every irreducible component of $I_{\mathcal{C}}$ is a unirational variety.

Where are we?

Discrete CI models. Check!

Where are we?

Discrete CI models. Check! Next?

Where are we?

Discrete CI models. Check! Next?

CI models of normal distributions.

Gaussian CI models

The statement $X_A \perp\!\!\!\perp X_B | X_C$ holds for $X \sim \mathcal{N}(\mu, \Sigma)$ if and only

$$\operatorname{rk} \Sigma_{A \cup C, B \cup C} \leq \# C$$

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These are varieties in the entries of Σ, $\sigma_{i_1 \ldots i_m}$.

Proof idea

We have a formula for the conditional densities of a normally distributed r.v.

$$\Sigma_{A\cup B|C} = (\Sigma_{A\cup B, A\cup B} - \Sigma_{A\cup B, C}\Sigma_{C,C}^{-1}\Sigma_{C, A\cup B})$$

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$$(\Sigma_{A\cup B,A\cup B} - \Sigma_{A\cup B,C}\Sigma_{C,C}^{-1}\Sigma_{C,A\cup B})_{A,B} = \Sigma_{A,B} - \Sigma_{A,C}\Sigma_{C,C}^{-1}\Sigma_{C,B}$$

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$X_A \perp\!\!\!\perp X_B | X_C$ iff

$$\Sigma_{A,B} - \Sigma_{A,C}\Sigma_{C,C}^{-1}\Sigma_{C,B} = 0$$

This is the 'Schur complement' of $\Sigma_{C,C}$ in

$$\Sigma_{A\cup C, B\cup C} = \begin{bmatrix} \Sigma_{A,B} & \Sigma_{A,C} \\ \Sigma_{C,B} & \Sigma_{C,C} \end{bmatrix}$$

Proof idea

Now using that $\Sigma_{C,C}$ has rank $\#C$ the Guttman rank additivity formula gives us the desired result

$$\operatorname{rk} \Sigma_{A \cup C, B \cup C} = \operatorname{rk} \mathrm{Schur} + \operatorname{rk} \Sigma_{C,C}$$

$\square$

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$$\operatorname{rk} \Sigma_{A \cup C, B \cup C} = \#C$$

□

Gaussian CI models

Given disjoint $A, B, C \subset [m]$, define

$$J_{A \perp\!\!\!\perp B|C} = \langle (\#C + 1) \times (\#C + 1) \text{ minors of } \Sigma_{A \cup C, B \cup C} \rangle.$$

This is an ideal of $\mathbb{R}[\sigma_{ij}, 1 \leq i \leq j \leq m]$.

Example (Gaussian conditional and marginal independence)

Let $\mathcal{C} = \{1 \perp\!\!\!\perp 3, \quad 1 \perp\!\!\!\perp 3|2\}$.

$$J_{\mathcal{C}} = J_{1 \perp\!\!\!\perp 3} + J_{1 \perp\!\!\!\perp 3|2} = \{\mathrm{rk}[\sigma_{13}] = 0\} \text{ and } \{\mathrm{rk} \begin{bmatrix} \sigma_{12} & \sigma_{13} \\ \sigma_{22} & \sigma_{23} \end{bmatrix} = 1\}$$

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$$\implies \langle \sigma_{13}, \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22} \rangle.$$

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$$\implies \langle \sigma_{13}, \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22} \rangle.$$

$$J_{\mathcal{C}} = \langle \sigma_{13}, \sigma_{12}\sigma_{23} \rangle = \langle \sigma_{12}, \sigma_{13} \rangle \cap \langle \sigma_{13}, \sigma_{23} \rangle = J_{1 \perp\!\!\!\perp \{2,3\}} \cap J_{\{1,2\} \perp\!\!\!\perp 3}$$

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$$\implies \langle \sigma_{13}, \sigma_{12}\sigma_{23} - \sigma_{13}\sigma_{22} \rangle.$$

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It follows that

$$X_1 \perp\!\!\!\perp X_3 | X_2 \text{ and } X_1 \perp\!\!\!\perp X_3 \iff X_1 \perp\!\!\!\perp (X_2, X_3) \text{ or } (X_1, X_2) \perp\!\!\!\perp X_3,$$

holds for multivariate normal random vectors.

Where are we?

CI models of normal distributions. Check!

Where are we?

CI models of normal distributions. Check! Next?

Where are we?

CI models of normal distributions. Check!   Next?

Graphical models!

A graph will describe the conditional independence relations. i.e. it describes C which can generate either I_C or J_C.

Undirected graphical CI models

Suppose all edges in the graph $G = (V, E)$ are undirected.

A random vector X satisfies the undirected pairwise Markov property associated to G iff

$$\forall (v, w) \notin E \qquad X_v \perp\!\!\!\perp X_w | X_{V \setminus \{v,w\}}$$

In this chain, every pair of non-neighbours are independent given the other two.

Undirected graphical CI models

For Gaussians these conditions correspond to

$$\det(\Sigma_{(V\setminus\{w\})\times(V\setminus\{v\})}) = 0 \iff (\Sigma^{-1})_{vw} = 0.$$

It is a linear concentration model!

Undirected graphical CI models

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It is a linear concentration model!

For discrete models this will correspond to the hierarchical model associated to the simplicial complex whose facets are maximal cliques of G.

Undirected global Markov property

The global Markov property corresponding to G is the set of constraints

$$A, B \text{ separated in } G_{V \setminus C} \qquad X_A \perp\!\!\!\perp X_B | X_C$$

A and B non-empty

Undirected global Markov property

Theorem 3.2.2 . If the random vector X has a joint distribution $\mathcal{P}^X$ that satisfies the intersection axiom.

Then $\mathcal{P}^X$ obeys the pairwise Markov property for an undirected graph G if and only if it obeys the global Markov property for G.

From this we could easily see that the conditions

$$X_i \perp\!\!\!\perp X_{\{1,2,\ldots,i-2\}} | X_{i-1}$$

Proof:

One direction is trivial. Every pairwise condition is a global condition since every pair of non-neighbouring vertices are separated by the complement.

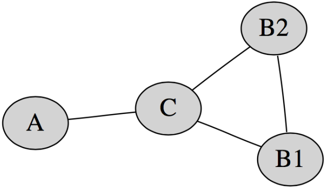
Undirected global Markov property

Induction

$$[A \perp\!\!\!\perp B | C] \leq [A' \perp\!\!\!\perp B' | C'] \quad \Longleftrightarrow \quad \#C \geq \#C'$$

Undirected global Markov property

Case 1: $A \cup B \cup C = [m]$



Induction gives us that

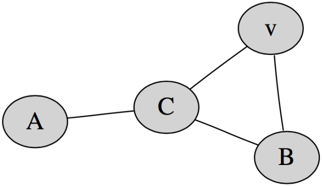
$$X_A \perp\!\!\!\perp B_i | X_{B_j \cup C}$$

Intersection axiom

$$X_A \perp\!\!\!\perp X_B | X_C$$

Undirected global Markov property

Case 2: $A \cup B \cup C \subsetneq [m]$



Induction gives us that

$$X_A \perp\!\!\!\perp X_B | X_{\{v\} \cup C}$$

$$X_A \perp\!\!\!\perp X_{\{v\}} | X_{B \cup C}$$

Intersection and decomposition gives

$$X_A \perp\!\!\!\perp X_{B \cup \{v\}} | X_C \implies X_A \perp\!\!\!\perp X_B | X_C$$

Proposition 3.2.3 (Completeness of the undirected global Markov property).

Suppose $A, B, C \subset V$ are pairwise disjoint subset with A and B non-empty. If C does not separate A and B in the undirected graph G, then there exists a joint distribution for the random vector X that obeys the undirected global Markov property for G but for which $X_A \perp\!\!\!\perp X_B | X_C$ does not hold.

Thanks for listening!