ALGEBRAIC SPACES AND QUOTIENTS BY EQUIVALENCE RELATION OF SCHEMES

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ABSTRACT. We consider quotient sheaves by equivalence relations of schemes, in subcanonical topologies. We introduce the notion of stability, and we show that algebraic spaces are stable. We provide some general criteria for stability of quotient sheaves, and we apply these results to give a global construction of the Grassmannians and some Hilbert schemes.

INTRODUCTION

By definition an algebraic space A is the étale quotient sheaf by an étale equivalence relation of schemes $R \rightrightarrows X$. The quotient map $X \longrightarrow A$ becomes representable, and étale. Properties of the quotient sheaf are related to properties of its cover, and as such, algebraic spaces are very close to being schemes.

There are however many subcanonical topologies available, and other properties than étaleness, that one could impose on the equivalence relation. Furthermore, in many moduli situations one encounters the situation when one would like to quotient out by an equivalence relation that is not étale. For these two reasons alone, it is natural and interesting to consider sheaf quotients by schemes in general.

Given an equivalence relation $R \rightrightarrows X$ of schemes, one forms the quotient functor of equivalence classes X_R. Then the natural question is in which topology one should sheafify the functor X_R within. This and related questions were considered in ample generality in the SGA-seminars ([Dem63], [Ver72]), and in particular the question wheter such a quotient were representable by a scheme was of natural interest (see e.g. [Ray67], [Gro95], [Mur95]).

The approach we undertake in this article is however different as we are not concerned wheter the quotient sheaf becomes representable by a scheme. Instead we focus on which properties a quotient sheaf should posess in order to be treated as it were a scheme. Or put more

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speficially, what exactly makes the notion of an algebraic space behave so arguably well.

Let A denote the sheaf quotient of an equivalence relation of schemes $R \rightrightarrows X$ in some fixed subcanonical topology. Under some mild hypothesis the quotient map of sheaves $X \longrightarrow A$ will be representable, and as such it will be a covering map for the topology. A sheaf with a representable cover $X \longrightarrow A$ is however not behaving as it was a scheme. In particular such a sheaf quotient might not remain a sheaf in a refinement of the topology.

In this article we introduce the notion of stability. We say that a sheafification A of a functor, in a subcanonical topology, is *stable* if A remains a sheaf in any refinement of the topology. For technical and practical purposes we limit ourselves by not going any further than the fpqc-topology.

Schemes are by definition stable in any subcanonical topology, and we show that algebraic spaces are stable (for the étale topology). In particular we have that a sheaf A which is representable in a given topology, is stable.

The particular choice of topology in which the quotient sheaf of an equivalence relation $R \rightrightarrows X$ is taken with respect to, is crucial for the stability of the quotient sheaf. It is natural to choose the coarsest, and we give some general criteria of stability. The notion of stability appears to be quite complex, and is in particular not directly related to the maps defining the equivalence relation. For instance, we give examples where the equivalence relation are Zariski open immersions, but where the quotient sheaf is not stable for the Zariski topology. And we provide examples where the equivalence relation are not Zariski open immersions, but where the quotient sheaf is stable for the Zariski topology.

As applications of our results we construct the Grassmannian and the Hilbert scheme of points on affine d-space as Zariski sheaf quotients of an equivalence relation of schemes.

Structure of the article. In the first section we recall some basic notions about contravariant functors on schemes, and their sheafification in a given pretopology. The results of Section 1 are surely well-known.

In Section 2 we consider the saturated situation, working with a topology instead of a pretopology. The results of Section 2 are found more or less explicitely in the SGA-seminars, for instance. Important for our understanding is Corollary (2.10), after which we provide some new examples that we hope will be of interest.

The main novelty of our article is in Section 3, where we give the notion of stability, and give some general criteria for stability. We apply these criteria to show that algebraic spaces are stable, and to give examples of finite group quotients that are not stable in the *finite* étale topology.

In the last section we give two examples that initated this article: Both the Grassmanninan and the Hilbert scheme of points on affine n-space, naturally occur as the quotient sheaf of an equivalence relation of schemes. The equivalence relation is given by smooth morphism, but the Zariski topology is where we form the quotient sheaf.

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1. Presheaves and associated sheaves

In this section we recall some general results about sheaves. We include these for readability and completeness, as well as to give an elementary treatment of quotient sheaves. Apart from Proposition (1.16) the statements are sheaf theoretical.

1.1. **Functors and universe.** A category $\mathfrak{c}$ will always mean an $\mathscr{U}$-category, where $\mathscr{U}$ is a fixed universe. A category $\mathfrak{c}$ means furthermore always some category of schemes over a fixed base scheme. Moreover, as our category always will be small, being a $\mathscr{U}$-category, we have that the collection of contravariant functors from $\mathfrak{c}$ to the category of sets, will again form a category. The objects are contra-variant functors and referred to as *presheaves*. Morphism are the natural transformations of functors. The category of presheaves $\mathrm{Fun}^{\mathrm{op}}(\mathfrak{c}, \mathrm{Ens})$ we denote by $\hat{\mathfrak{c}}$.

Example 1.2. This example, due to T. Ekedahl, shows that if the category $\mathfrak{c}$ is not small, then one will encounter set-theoretical problems when considering the collection of functors from $\mathfrak{c}$ to sets, or in fact any codomain.

Let $\mathrm{Fun}^{\mathrm{op}}(\mathfrak{c}, 0)$ denote the collection of functors from $\mathfrak{c}$ to the trivial category 0. Note that there is only a single functor F in this collection.

Any functor F induces a map on the class of objects $f\colon \mathrm{ob}(\mathfrak{c}) \longrightarrow 0$. A map f is a subclass Γ_f of the product $\mathrm{ob}(\mathfrak{c}) \times 0$, and we consider the image of Γ_f by the projection map to $\mathfrak{c}$. The image of a set is a set. Hence, if we assume that the category $\mathfrak{c}$ is not small, i.e. the objects form a proper class and not a set, then we have that Γ_f is a proper class. It follows that also F is a proper class.

Now, if the collection of functors were to be a category it ought to have a class of objects. In particular $F \in \mathrm{ob}(\mathrm{Fun}^{\mathrm{op}}(\mathfrak{c}, 0))$, which however is a contradiction since F is a proper class and can then, by definition, not occur as an element of a class.

Proposition 1.3 (Yoneda Lemma)**.** *Let $h(X) = \mathrm{Hom}(-, X)$ denote the associated presheaf for any scheme X. For any presheaf F on the category $\mathfrak{c}$, there is a natural bijection $\mathrm{Hom}_{\hat{\mathfrak{c}}}(h(X), F) = F(X)$. In particular the functor $h\colon \mathfrak{c} \longrightarrow \hat{\mathfrak{c}}$ is full and faithful.*

In what follows we do not distinguish a scheme X from its associated presheaf $h(X)$.

1.4. **Equivalence relations.** An equivalence relation on a presheaf X is a monomorphism of presheaves $R \longrightarrow X \times X$ such that for any scheme S the set $R(S)$ is the graph of an equivalence relation on $X(S)$. If X is a scheme itself, then an equivalence relation R on X will always be assumed to be a scheme.

Lemma 1.5. *Two morphisms* $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$ *of presheaves form an equivalence relation if and only if the following four assertions hold.*

(1) *The natural map* $R \longrightarrow R_{X \times X} R$ *is an isomorphism.*
(2) *The diagonal map* $\Delta \colon X \longrightarrow X \times X$ *has a unique factorization through* $R \longrightarrow X \times X$.
(3) *There exists a morphism* $i \colon R \longrightarrow R$ *such that the diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d, "i"] & X \times X \arrow[d, "\tau"] \\ R \arrow[r] & X \times X \end{tikzcd}$$

is commutative, where $\tau \colon X \times X \longrightarrow X \times X$ *is the involution shifting the order.*
(4) *Let* $R \times_X R$ *denote the sub-presheaf of* $R \times R$ *given by*

$$R \times_X R(T) = \{(r, s) \in R \times R(T) \mid \pi_2(r) = \pi_1(s)\}.$$

Then the image of $(\pi_1, \pi_2) \colon R \times_X R \longrightarrow X \times X$ *is included in the image of* $R \longrightarrow X \times X$.

Proof. A morphism of presheaves $R \longrightarrow U$ is a monomorphism if and only if the morphism $R \longrightarrow R \times_U R$ is an isomorphism. We therefore have that (1) is equivalent with $R \longrightarrow X \times X$ being a monomorphism.

By (1) we have that $R \longrightarrow X \times X$ is a monomorphism. Together with the reflexivity we get a factorization of Δ via $R \longrightarrow X \times X$, and the factorization is unique. Conversely, a factorization of Δ through R implies the reflexivity axiom.

The symmetry axiom together with (1) imply (3), whereas (3) implies symmetry. Itis also clear that Assertion (4) is a reformulation of the transitivity axiom.

□

1.6. **Presheaf quotients.** If $R \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} X$ is an equivalence relation then for each scheme S we can form the quotient set $X_R(S)$ of equivalence classes in $X(S)$. If $T \longrightarrow S$ is any morphism of schemes we obtain two commutative diagrams of sets

$$
\begin{tikzcd}
R(S) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d] & X(S) \arrow[r] \arrow[d] & X_R(S) \\
R(T) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X(T) \arrow[r] & X_R(T).
\end{tikzcd} \tag{1.6.1}
$$

And consequently we obtain an induced map of sets $X_R(S) \longrightarrow X_R(T)$, and we have that X_R is a presheaf.

Lemma 1.7. *Let $\pi\colon X \longrightarrow F$ be an epimorphism of presheaves. Then the presheaf $R = X \times_F X$ with its two projections to X, form an equivalence relation on X, and the quotient presheaf X_R is canonically identitfied with F.*

Proof. The map $R = X \times_F X \longrightarrow X \times X$ is a monomorphism. There is furthermore a natural involution $i\colon R \longrightarrow R$, and the diagonal map $X \longrightarrow X \times X$ factorizes through $R \longrightarrow X \times X$. So $R \longrightarrow X \times X$ satisfies (1), (2) and (3) of Lemma (1.5). One checks that the image of $R \times_X R$ in $X \times X$ is included in the image of R, hence $R \longrightarrow X \times X$ is an equivalence relation. The map π factors through the quotient of $R \rightrightarrows X$, and since π was assumed to be an epimorphism it follows that F equals the quotient of $R \rightrightarrows X$. □

1.8. **Pretopologies.** For each scheme S we assume that we have a collection $\mathrm{Cov}(S)$ of morphisms of schemes with codomain S, satisfying the axioms of a pretopology (see e.g. [Ver72] Expose II, Definition 1.3, p. 221). An element of $\mathrm{Cov}(S)$ is a collection of maps $\{S_\alpha \longrightarrow S\}_{\alpha\in\mathscr{A}}$, we occasionally replace with the single morphism $T \longrightarrow S$, where $T = \sqcup S_\alpha$ is the disjoint union of the domains S_α of the collection.

1.9. **Sheaves.** Let F be a presheaf. For any covering $\{S_\alpha \longrightarrow S\}_{\alpha\in\mathscr{A}}$ of a scheme S, we get the induced sequence

$$
F(S) \longrightarrow \prod_{\alpha\in\mathscr{A}} F(S_\alpha) \underset{q}{\overset{p}{\rightrightarrows}} \prod_{\alpha,\beta\in\mathscr{A}} F(S_\alpha \times_S S_\beta)\ . \tag{1.9.1}
$$

The presheaf F is *separated* if the leftmost map (1.9.1) is injective, and a *sheaf* if the sequence is exact, for all schemes S, and all coverings.

A pretopology is *subcanonical* if the associated presheaf X is a sheaf, for any scheme X.

1.10. **Sheafification.** For any presheaf F we can form its sheafification with respect to a given pretopology. For each scheme S, the set $\mathrm{Cov}(S)$ is filtered via the fiber product. We form the directed limit

$$
LF(S) := \varinjlim_{T\to S\in \mathrm{Cov}(S)} F(T). \tag{1.10.1}
$$

This assignment gives a map of presheaves $F \longrightarrow LF$.

Proposition 1.11. *For any presheaf F the presheaf LF is separated. Furthermore we have that*

1. *The morphism $F \longrightarrow LF$ is a monomorphism if and only if F is separated. And in that case LF is a sheaf.*
2. *The morphism $F \longrightarrow LF$ is an isomorphism if and only if F is a sheaf.*

In particular we have that L^2F is a sheaf, for any presheaf F.

Proof. See Demazure [Dem63], IV, 4, Proposition 4.3.11, p. 200 or Verdier [Ver72], II, 4, Proposition 3.2, p. 232. □

Remark 1.12. There is one point to make about the usual pretopologies and our fixed universe $\mathscr{U}$ lying in the background. For a fixed scheme X the category of Zariski coverings is small, and the category of surjective, étale morphism (as well as fppf-coverings), is essentially small. In particular we can assume that the coverings are all included in our universe $\mathscr{U}$. The category of fpqc-coverings is however not small, and consequently only some of the fpqc-coverings will fit in our universe. It is therefore plausible that the sheafification of a presheaf in the fpqc-topology will depend on the universe, see [BLR90] p. 201, and [Wat75].

Proposition 1.13. *Let* Cov *be a pretopology, and let R and X be two sheaves. Assume that we have an equivalence relation $R \rightrightarrows X$, and let X_R denote its presheaf quotient. Then the presheaf X_R is separated. In particular we have that LX_R is a sheaf, and that the equivalence relation is effective, i.e. $R = X \times_{LX_R} X$.*

Proof. Let S be a scheme and $\{S_\alpha \longrightarrow S\}_{\alpha \in \mathscr{A}}$ some covering. We will show that the induced map $X_R(S) \longrightarrow \prod X_R(S_\alpha)$ is injective. To see this we consider the commutative diagram of sets

(1.13.1)
$$\begin{tikzcd}
R(S) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d] & X(S) \arrow[r] \arrow[d] & X_R(S) \arrow[d] \\
\prod R(S_\alpha) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] & \prod X(S_\alpha) \arrow[r] \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] & \prod X_R(S_\alpha) \arrow[d, shift right, "p"'] \arrow[d, shift left, "q"] \\
\prod R(S_{\alpha,\beta}) \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & \prod X(S_{\alpha,\beta}) \arrow[r] & \prod X_R(S_{\alpha,\beta}),
\end{tikzcd}$$

where $S_{\alpha,\beta} = S_\alpha \times_S S_\beta$. The horizontal sequences are exact by definition, whereas the two leftmost vertical sequences are exact since R and X are sheaves. We want to check that the upper right vertical map is injective. Let $f, g \in X(S)$ be two elements that are mapped to the same element in $\prod X_R(S_\alpha)$, and let f_1, g_1 be the images of f, g in $\prod X(S_\alpha)$. Then there exists $z \in \prod R(S_\alpha)$ such that $\pi_1(z) = f_1$ and $\pi_2(z) = g_1$.

We have furthemore that $p(f_1) = q(f_1)$ by the exactness of the middle coloumn, hence by the commutativity we have $\pi_1(p(z)) = \pi_1(q(z))$. Similarily using g_1, we get that $\pi_2(p(z)) = \pi_2(q(z))$. Since $R \longrightarrow X \times X$ is a monomorphism it follows that $p(z) = q(z)$. Since the left vertical sequence is exact we get an element $z' \in R(S)$ such that $\pi_1(z') = f$ and $\pi_2(z') = g$, and we have that the map $X_R(S) \longrightarrow \prod X_R(S_\alpha)$ is injective. Consequently the quotient sheaf X_R is separated, and by Proposition (1.11) we have that LX_R is a sheaf.

To see that the equivalence relation is effective we need to establish that $R = X \times_{LX_R} X$. By Lemma (1.7) we have that $R = X \times_{X_R} X$ since $X \longrightarrow X_R$ clearly is an epimorphism of presheaves. Moreover, as just proven, the map $X_R \longrightarrow LX_R$ is a monomorphism, and we get that

$$X \times_{LX_R} X(S) = X \times_{X_R} X(S),$$

for any scheme S. Hence $X \times_{LX_R} X = X \times_{X_R} X = R$, and we have proven the proposition. □

Corollary 1.14. *Let $X \longrightarrow A$ be an epimorphism of sheaves. Then the sheaf $R = X \times_A X$ with its projections to X, form an equivalence relation on X, and the sheaf quotient LX_R is A.*

Proof. For each scheme S let $F(S) \subseteq A(S)$ denote the image subset of the sheaf epimorphism $\pi\colon X \longrightarrow A$. We have that the map $X \longrightarrow F$ is an epimorphism of presheaves. Furthermore, as

$$X \times_F X(S) = X \times_A X(S),$$

for any scheme S, we have $R = X \times_A X = X \times_F X$. By Lemma (1.7) we have that $R \to X \times X$ forms an equivalence relation, and that the presheaf quotient $X_R = F$. From the proposition we get that the sheaf quotient of the equivalence relation $R \rightrightarrows X$ is LF. The inclusion of presheaves $X_R = F \subseteq A$ induces an isomorphism of sheaves $LX_R \longrightarrow A$, proving the corollary. □

Lemma 1.15. *Let R and X be two sheaves in some pretopology* Cov*. Let $R \rightrightarrows X$ be an equivalence relation, and let $A = LX_R$ denote the sheaf quotient. Let $s\colon S \longrightarrow A$ be a morphism, with S a scheme. Then there exists $T \to S$ in* $\mathrm{Cov}(S)$*, making the cartesian diagram*

(1.15.1)
$$\begin{tikzcd} T \times_X R \arrow[r] \arrow[d] & S \times_A X \arrow[d] \\ T \times X \arrow[r] & S \times X \end{tikzcd}$$

Proof. Via the Yoneda Lemma (1.3) we identify $s\colon S \longrightarrow A$ with an element $s \in A(S)$. As $A = LX_R$ we get from the definition of the direct limit that there exists a cover $j\colon T \longrightarrow S$ of S, and an element $\bar{t} \in X_R(T)$ that is mapped to s via the natural map

$$X_R(T) \longrightarrow LX_R(S) = \varinjlim_{\{T \longrightarrow S\} \in \mathrm{Cov}(S)} X_R(T).$$

Let $t \in X(T)$ be a representative of the equivalence class $\bar{t} \in X_R(T)$. Then again by the Yoneda lemma we have that $t\colon T \longrightarrow X$. If $\pi\colon X \longrightarrow A$ denotes the quotient map, then the covering $j\colon T \longrightarrow S$ is such that $\pi \circ t = j \circ s$. Since the composition $\pi \circ t\colon T \longrightarrow A$ factors via X, we can write

$$T \times_A X = T \times_X X \times_A X. \tag{1.15.2}$$

By Proposition (1.13) we have that $R = X \times_A X$, and it follows that the diagram (1.15.1) is cartesian. □

Proposition 1.16. *Let* Cov *be a subcanonical pretopology, and let A be a sheaf. Then the following are equivalent.*

(1) *There exists an epimorphism of sheaves $\pi\colon X \longrightarrow A$, with X a scheme, the morphism π being representable, and such that the induced morphism of schemes $X \times_A X \longrightarrow X \times X$ satisfies effective descent (*[Knu71, p.32]*) with respect to the pretopology* Cov*.*

(2) *There exists a morphism of schemes $R \longrightarrow X \times X$ satisfying effective descent with respect to* Cov*, and where the induced maps $R \rightrightarrows X$ form an equivalence relation with sheaf quotient A.*

Proof. Assume (1). We obtain from the representability of the map $\pi\colon X \longrightarrow A$ that the sheaf $X \times_A X$ is a scheme R. From Corollary (1.14) we have that $R \rightrightarrows X$ is an equivalence relation and where the sheaf quotient is A. Thus we have that (1) implies (2).

To show the converse, assume that (2) holds. Let $S \longrightarrow A$ be a morphism with S a scheme. By Lemma (1.15) there is a cover $T \longrightarrow S$ of S, such that we obtain a cartesian diagram as (1.15.1). The left vertical morphism in (1.15.1) is a morphism of schemes that satisfies effective descent with respect to Cov. Since $T \longrightarrow S$ is a covering map the effective descent assumption implies that the right vertical map in (1.15.1) is also a morphism of schemes. In particular $S \times_A X$ is a scheme, and hence the quotient map $X \longrightarrow A$ is representable. Since A is the sheaf quotient we have that $A = LX_R$ by Corollary (1.14), and in particular the map $X \longrightarrow A$ is an epimorphism of sheaves. We have shown that (2) implies (1). □

Remark 1.17. The result above is a generalization of the defining property of algebraic spaces (cf. [Knu71], Proposition 1.3, p. 93).

Example 1.18. With this example we show that a quotient sheaf with a representable cover can behave unexpectedly. Let $K \longrightarrow L$ be a separable field extension. Then $\mathrm{Spec}(L \otimes_K L) \rightrightarrows \mathrm{Spec}(L)$ form an equivalence relation on $\mathrm{Spec}(L)$. The quotient sheaf, in the étale topology, is well-known to be $\mathrm{Spec}(K)$ (see also Example (2.13)). If one instead consider the sheaf quotient A in the Zariski topology we obtain a different sheaf.

By Proposition (1.16) the quotient map $\mathrm{Spec}(L) \longrightarrow A$ is representable, and there is a natural inclusion of sheaves

$$A \subset \mathrm{Spec}(K),$$

which however is not an isomorphism: Consider the identity morphism id_K on $\mathrm{Spec}(K)$. Assume that there exists $s \in A(\mathrm{Spec}(K))$ mapping to the identity map id_K. Since A is the Zariski sheafification of the quotient presheaf, the element $s \in A(\mathrm{Spec}(K))$ would after a Zariski refinement $S' \longrightarrow \mathrm{Spec}(K)$ lift to $\mathrm{Spec}(L)$. As there are only trivial Zariski refinements of $\mathrm{Spec}(K)$ we get that $s\colon \mathrm{Spec}(K) \longrightarrow A$ lifts to a morphism $\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(L)$. Which is impossible as long as $K \neq L$, and consequently $A \neq \mathrm{Spec}(K)$.

2. Effective quotients of schemes

The results in this section are perhaps lesser known that the result in the previous section. The upshot of this section is Corollary (2.10) that characterizes when a scheme Y is the sheaf quotient of a scheme X by a morphism $f\colon X \longrightarrow Y$.

2.1. **Sieves.** A subpresheaf $\mathtt{F} \subseteq Y$ of a scheme Y is a *sieve* on Y. A morphism $\pi\colon X \longrightarrow Y$ of schemes determines the sieve $\mathtt{F}_\pi \subseteq Y$, by $\mathtt{F}_\pi(T) = \{f\colon T \longrightarrow Y$ that factorizes through $\pi\colon X \longrightarrow Y\}$, for any scheme T.

Lemma 2.2. *Let $\pi\colon X \longrightarrow Y$ be a morphism of schemes that has a section. Then the associated sieve $\mathtt{F}_\pi = Y$.*

Proof. Let $s\colon Y \longrightarrow X$ be a section of π. If $f\colon T \longrightarrow Y$ is a morphism of schemes, then f composed with $\pi \circ s$ gives a factorization of f via $\pi\colon X \longrightarrow Y$. Hence we have equality $\mathtt{F}_\pi = Y$. □

Let $\pi\colon X \longrightarrow Y$ be a morphism of schemes, and let $\mathtt{F}$ be a sieve on Y. Then we have the pull-back $X \times_Y \mathtt{F}$, denoted $\pi^*\mathtt{F}$, which is a sieve on X.

Definition 2.3. A *topology* J is, for any scheme Y, a collection of sieves $J(Y)$, called *covering sieves*, satisfying the following three axioms.

(t1) If $\mathtt{F} \in J(Y)$ is a covering sieve, then $f^*\mathtt{F} \in J(Y')$ is a covering sieve, for a, any morphism of schemes $f\colon Y' \longrightarrow Y$.

(t2) Let $\mathtt{F} \in J(Y)$ be a covering sieve, and let $i_Y\colon \mathtt{F} \subseteq Y$ denote the inclusion. Let $\mathtt{G}$ be any sieve on Y. If, for any scheme Y', any morphism of presheaves $f\colon Y' \longrightarrow \mathtt{F}$, we have that the pullback $(i_Y \circ f)^*\mathtt{G}$ is a covering sieve on Y', then $\mathtt{G}$ is a covering sieve on Y.

(t3) For any scheme Y we have that $Y \in J(Y)$.

Remark 2.4. A pretopology determines a collection of sieves, which not necessarily satisfy Axiom (t2). By adding the needed sieves we get the topology J *generated* by the pretopology. We can assume that our universe $\mathscr{U}$ also contains the sieves needed to make a pretopology a topology ([Ver72], Proposition 3.0.4, p. 229).

Remark 2.5. A sheaf A on a pretopology will remain a sheaf in the topology generated by the pretopology ([Ver72], Corollary 2.4, p. 226). Thus, the results about sheaves given in the preceding sections remains true also for topologies.

2.6. A morphism of schemes $\pi \colon X \longrightarrow Y$ is a *covering map* for the topology J if the sieve $\mathtt{F}_\pi$ is a covering sieve, that is if $\mathtt{F}_\pi \in J(Y)$.

Proposition 2.7. *Let J be a topology, and let $\pi \colon X \longrightarrow Y$ be a morphism of schemes. Then π is a covering map for the topology J if and only if there exist a covering map $Y' \longrightarrow Y$ such that the induced morphism $X' = X \times_Y Y' \longrightarrow Y'$ has a section.*

Proof. If $\pi \colon X \longrightarrow Y$ is a covering map then the result is clear since we can take $R = X \times_Y X \longrightarrow X$, which has a section. To prove the converse we need to show that $\pi \colon X \longrightarrow Y$ is a covering map, that is $\mathtt{F}_\pi \in J(Y)$. By assumption there exists a covering map $i \colon Y' \longrightarrow Y$ such that the induced morphism of schemes

$$\pi' \colon X' = X \times_Y X' \longrightarrow Y'$$

has a section. The pull-back $i^*\mathtt{F}_\pi$ is the sieve on Y' consisting of all maps to Y' that factors through $X' \longrightarrow Y'$. In other words $i^*\mathtt{F}_\pi = \mathtt{F}_{\pi'}$. By assumption π' has a section, and we have from the Lemma (2.2) that $\mathtt{F}_{\pi'} = Y'$. By the Axiom (t3) we have $Y' \in J(Y')$, hence $i^*\mathtt{F}_\pi \in J(Y')$.

Consider now the covering sieve $\mathtt{F}_i$ on Y, where $i \colon Y' \longrightarrow Y$ is the covering map as above. Let $f \colon Z \longrightarrow \mathtt{F}_i$ be any morphism, with Z a scheme. The pull-back of the sieve $\mathtt{F}_\pi$ along the composition of f and the inclusion $\mathtt{F}_i \longrightarrow Y$ is $f^*i^*\mathtt{F}_\pi$. As just seen $i^*\mathtt{F}_\pi = \mathtt{F}_{\pi'}$ is a covering sieve, hence by Axiom (t1) so is the pull-back $f^*i^*\mathtt{F}_\pi$. It then follows from Axiom (t2) that $\mathtt{F}_\pi \in J(Y)$. Thus, $\pi \colon X \longrightarrow Y$ is a covering map. □

Corollary 2.8. *The topology generated by surjective, smooth morphisms equals the topology generated by surjective, étale morphisms.*

Proof. A smooth surjective morphism of schemes $\pi \colon X \longrightarrow Y$ has étale locally a section ([Gro67] Corollary 17.16.3). A consequence of the Proposition is then that the topology generated by smooth, surjective morphisms equals the topology generated by étale, surjective morphisms. □

Corollary 2.9. *If $\pi \colon X \longrightarrow Y$ is a morphism of schemes that has a section, then $\pi \colon X \longrightarrow Y$ is a covering map in any topology.*

Proof. By Axiom (t3) the identity morphism $\mathtt{F}_{\mathrm{id}} = Y$ is a covering map in any topology. Consequently the map $\pi\colon X \longrightarrow Y$ has a section locally in any topology, and then the corollary follows from the proposition. $\square$

Corollary 2.10. *Let J be a subcanonical topology, and $\pi\colon X \longrightarrow Y$ a morphism of schemes. Let $R = X \times_Y X$ and let A denote the sheaf quotient of the equivalence relation $R \rightrightarrows X$. Then $A = Y$ if and only if π is a covering map.*

Proof. If Y equals the sheaf quotient of $R \rightrightarrows X$ then $Y = LX_R$, and in particular we have that the morphism $\pi\colon X \longrightarrow Y$ is an epimorphism of sheaves. By considering the identity morphism id_Y we have that there exists a covering $i\colon Y' \longrightarrow Y$ such that $\pi'\colon X \times_Y Y' \longrightarrow Y'$ has a section. By Proposition (2.7) we have that $\pi\colon X \longrightarrow Y$ is a covering map.

To prove the converse, assume that π is a covering map. By Proposition (2.7) we get that the morphism $\pi\colon X \longrightarrow Y$ is an epimorphism of sheaves. Then the result follows from Corollary (1.14).

$\square$

Example 2.11. Let $X = \mathrm{Spec}(k[x,y]/(xy))$ denote the coordinate cross, and let $\pi\colon X \longrightarrow \mathrm{Spec}(k[x]) = \mathbf{A}^1$ denote the natural projection onto one of its axis. Since π is not flat, then neither is the equivalence relation $R = X \times_{\mathbf{A}^1} X$ on X. However, as the projection

$$\pi\colon X = \mathrm{Spec}(k[x,y]/(xy)) \longrightarrow \mathbf{A}^1$$

has a section, we obtain, by Corollary (2.10) that the quotient sheaf of $R \rightrightarrows X$ is the affine line $\mathbf{A}^1$, in any subcanonical site.

Example 2.12. Let $\pi\colon X = \mathrm{Spec}(k[x,y]) \longrightarrow C = \mathrm{Spec}(k[x^2,xy,y^2])$ be the natural map from the affine plane to the cone C. We have the equivalence relation

$$\begin{aligned} R = X \times_C X &= \mathrm{Spec}(k[x,y] \otimes_{k[x^2,xy,y^2]} k[x,y]) \\ &= \mathrm{Spec}(k[x_1,y_1,x_2,y_2]/(x_1^2 - x_2^2, y_1^2 - y_2^2, x_1y_1 - x_2y_2). \end{aligned}$$

This equivalence relation is the equivalence relation *generated* by the natural action of $\mathbf{Z}_2$ on the affine plane, and its topological quotient $|C|$ is discussed in ([Kol08]). However, as there is no section of $\pi\colon X \longrightarrow C$, not locally in any subcanonical topology, we get that the quotient sheaf A of $R \rightrightarrows X$ does not equal the cone C.

Topologically $|A| = |C|$, but over the vertex of the cone C, the sheaf quotient A has additional points sticking out . Any infinitesimal thickening of the origin of $\mathbf{A}^2$ will give a point of A. In particular the two tangent directions, from the origin pointing towards (x,y) and $(-x,-y)$ in the plane will give two different tangent directions in A pointing at the same point.

Example 2.13. Let $K \subseteq L$ be a field extension, not necessarily separated. Let $X = \operatorname{Spec}(L)$, and let $R = \operatorname{Spec}(L \otimes_K L)$. Depending on the topology, the sheaf quotient of $R \rightrightarrows X$ will not necessarily be $\operatorname{Spec}(K)$. However, the morphism $X = \operatorname{Spec}(L) \longrightarrow \operatorname{Spec}(K)$ has a section in the fpqc-topology, and consequently we can always find some subcanonical site where the quotient sheaf $R \rightrightarrows X$ equals $\operatorname{Spec}(K)$.

For instance, we could have $K = \mathbf{Q}$ included in its algebraic closure $L = \overline{\mathbf{Q}}$. Then $\operatorname{Spec}(\mathbf{Q})$ would not be the sheaf quotient $R \rightrightarrows X$ unless the map $\operatorname{Spec}(\overline{\mathbf{Q}}) \longrightarrow \operatorname{Spec}(\mathbf{Q})$ is a covering map for the topology.

2.14. Glueing. It is well-known that schemes can be glued along Zariski open subsets as long the identification of the glueing charts satisfy the cocycle condition. The scheme one obtains by glueing X along the open subsets R is the quotient or push-out of $R \rightrightarrows X$, in the category of schemes. With the following example we show that the quotient depends on the category.

Example 2.15. . The following example is based on ([Knu71], page 9). Let $X = \operatorname{Spec}(k[x,y]/(xy))$ again be the coordinate cross. Let R be the disjoint union of X and X_0, where $X_0 = X \setminus (0,0)$ is the complement of the origin. We consider the two morphisms $\pi_i \colon R \longrightarrow X$, where π_1 is the natural inclusion on both components of R. The morphism π_2 is the identity on the component X, but on the other component X_0 the morphism π_2 switches the axes. Note that the two maps π_1 and π_2 are Zariski local open immersions.

The glueing of X along R will give the affine line, but the quotient sheaf A, of $R \rightrightarrows X$ is not a scheme, in any topology. To see this note that the presheaf quotient has, over the origo, different tangent directions sticking out. Since the presheaf quotient is separated (Proposition (1.13)), these tangent directions will also appear in the sheaf quotient. Hence, since the sheaf quotient is different from the affine line, it follows that the sheaf quotient can not be a scheme.

Thus, the non-scheme like points, as the tangent directions in the above example, are not particular for algebraic spaces being étale quotient sheaves by étale equivalence relations. One encounters these non-scheme points when taking Zariski quotient sheaves by Zariski equivalence relations.

Proposition 2.16. *Let J be a subcanonical topology. Let $R \rightrightarrows X$ be an equivalence relation, where $R \longrightarrow X \times X$ is a morphism of schemes that satisfies effective descent with respect to J. Denote by $\pi \colon X \longrightarrow A$ the sheaf quotient. Then we have for any $Y \longrightarrow A$, with Y a scheme, that the induced morphism of schemes $\pi_Y \colon Y \times_A X \longrightarrow Y$ is a covering map for the topology J. Furthermore, we have that $R \times_A Y \rightrightarrows X \times_A Y$ is an equivalence relation of schemes, and the quotient sheaf is Y.*

Proof. For any morphism $s\colon Y \longrightarrow A$, with Y a scheme, we have by Proposition (1.16) that the induced morphism $\pi_Y\colon X_Y = Y \times_A X \longrightarrow Y$ is a morphism of schemes. We need to see that π_Y is a covering map for the topology J. By Lemma (1.15) there exist a covering $Y' \longrightarrow Y$ making the cartesian diagram

(2.16.1)
$$\begin{tikzcd} Y' \times_X R \arrow[r] \arrow[d] & Y \times_A X \arrow[d] \\ Y' \times X \arrow[r] & Y \times X \end{tikzcd}$$

By Lemma (1.5) (2) the diagonal map $X \longrightarrow X \times X$ factorizes through $R \longrightarrow X \times X$. It follows that the morphism $\pi_1\colon R \longrightarrow X$ has a section, and consequently that the left most vertical arrow in the diagram above has a section. Furthermore, as $Y' \longrightarrow Y$ is a covering of Y, we get from Axiom (t1) that $Y' \times X \longrightarrow Y \times X$ is a covering of $Y \times X$. Then, by Proposition (2.7) we have that π_Y is a covering map.

And finally, as the morphism $\pi_Y\colon X_Y \longrightarrow Y$ is a covering map, we get from Corollary (2.10) that the sheaf quotient equals Y. $\square$

Remark 2.17. Note that there is no assumption on the maps in the equivalence relation. The notion of equivalence relation makes these maps covering maps in any subcanonical topology.

3. Stability

The main novelty of this article is the notion of stability that we introduce below. After giving some general criteria for stability, we show that algebraic spaces satisfy the stability condition. Thereafter we give examples of group quotients that are unstable in some topologies coarser than the étale topology.

3.1. Notation. We use the definition of fpqc as in ([FGI+05], p. 23). In particular any of the standard subcanonical topologies (Zariski, étale, fppf) can be refined to the fpqc. Our interest is to understand, given an equivalence relation $R \rightrightarrows X$, which topology J provides a well-behaved quotient sheaf LX_R. Note that by Proposition (2.16) we have that the geometry, that is the points, of a quotient sheaf behaves well in any topology, whereas it is clear, e.g. Example (1.18), that in some topologies the quotient sheaves do not behave well.

Definition 3.2. Let F be a presheaf, and let J be a topology that can be refined to the fpqc-topology. The sheafification $F_J = L^2F$ with respect to J is *stable* if the presheaf F_J is a sheaf in the fpqc-topology as well.

Remark 3.3. Note that a scheme Y is stable, in any subcanonical topology. In particular if the sheafification L^2F of a presheaf F in some subcanonical topology J, is isomorphic to a scheme $L^2F = Y$, then the sheafifcation L^2F is stable with respect to J.

Remark 3.4. The Zariski sheaf quotient of Example (1.18) is not stable. The étale sheafification of that quotient is stable since the étale quotient is representable by a scheme.

Remark 3.5. As any sheaf in the fpqc-topology is stable, the definition is a bit unsatisfactory. This means also that one can always find a topology J, namely the fpqc, that makes the sheafification stable. Our interest is however in finding coarser topologies than the fpqc that will provide stable sheafification.

Lemma 3.6. *Let $\pi\colon X \longrightarrow Y$ be a morphism of schemes where π is a covering map for the fpqc-topology, and let $R = X \times_Y X$. Assume that π is not a covering map for a subcanonical topology J. Then the quotient sheaf of the equivalence relation $R \rightrightarrows X$ in the J-topology is not stable.*

Proof. By Corollary (2.10) we have the sheaf quotient LX_R in the J-topology is not equal to Y, but that the sheaf quotient in the fpqc-topology is Y. Consequently the sheafification of the quotient X_R is not stable with respect to the J-topology. $\square$

Proposition 3.7. *Let $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$ be an equivalence relation of schemes, and assume that the two morphism π_1 and π_2 have a property $\mathscr{P}$. Let I and J be two subcanonical topologies with J being a refinement of I. Assume that the three following conditions hold.*

1. *If π is a morphism of schemes with property $\mathscr{P}$, then π is a covering map for the I-topology.*
2. *The property $\mathscr{P}$ is local on the domain with respect to J.*
3. *The morphism $R \longrightarrow X \times X$ satisfies effective descent with respect to J.*

Then the quotient sheaf of the equivalence relation $R \rightrightarrows X$ in the I-topology, equals the sheaf quotient in the J-topology.

Proof. Let X_R denote the quotient functor, and let IX_R and JX_R denote the quotient sheaves in the I-topology and J-topology, respectively. Since J is a refinement of I, we have an induced map of presheaves

$$IX_R \longrightarrow JX_R. \tag{3.7.1}$$

One can check that the map (3.7.1) is a monomorphism as follows. Assume $f, g \in IX_R(S)$ are two elements that are mapped to the same element in $JX_R(S)$, where S is some scheme. The two elements f and g in $IX_R(S)$ we represent with two elements in $X_R(U)$, for some I-covering $i\colon U \to S \in I(S)$. And we have that $f = g$ in $X_R(T)$ for some

J-covering $j\colon T \longrightarrow S \in J(S)$. We get that $i^*f = i^*g$ in $X_R(T \times_S U)$. As $T \times_S U \longrightarrow U$ is an element of $J(U)$, we get that the map

$$X_R(U) \longrightarrow JX_R(U) = \varinjlim_{T\to U\in J(U)} X_R(T)$$

sends f and g to the same element in $JX_R(U)$. By Proposition (1.13) we have that X_R is separated, and it follows that (3.7.1) is a monomorphism.

Next we show that (3.7.1) is an epimorphism of presheaves. Let $\pi\colon X \longrightarrow JX_R$ denote the quotient map of sheaves. By Assumption (3) we have that $R \longrightarrow X \times X$ satisfies effective descent with respect to J, and consequently by Proposition (1.16) we have that the quotient map $\pi\colon X \longrightarrow JX_R$ is representable. Let $s \in JX_R(S)$ be an element, with S a scheme. Consider the cartesian diagram of sheaves in the J-topology

$$\begin{tikzcd} X \arrow[r, "\pi"] & JX_R . \\ X \times_{JX_R} S \arrow[u] \arrow[r, "\pi_S"] & S \arrow[u, "s"'] \end{tikzcd}$$

We have that π_S is a morphism of schemes. From Lemma (1.15) we have that there exist a J-cover $T \longrightarrow S$ such that the pull-back of the morphism π_S is

$$\mathrm{id} \times \pi_1 \colon T \times_X R \longrightarrow T \times X.$$

Since $\pi_1\colon R \longrightarrow X$ has property $\mathscr{P}$, so has $\mathrm{id} \times \pi_1$. By assumption the property $\mathscr{P}$ is local on the domain for the J-topology, which guarantees that π_S has property $\mathscr{P}$. By the first assumption we get that π_S is a covering map for the I-topology, hence by Proposition (2.7) the morphism π_S has I-locally a section. In other words the element s in $JX_R(S)$ is also in $IX_R(S)$, and we have proven that (3.7.1) is an epimorphism of presheaves. Thus $IX_R = JX_R$ as claimed. $\square$

Corollary 3.8. [LMB00, Théorème A.4] *Let J denote the topology generated by surjective étale morphisms, let $R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$ be an equivalence relation of schemes, where π_1 and π_2 are étale morphisms. Assume furthermore that the diagonal morphism $R \longrightarrow X \times X$ is quasi-affine. Then we have that the quotient sheaf is stable.*

Proof. Quasi-coherent sheaves with descent data are effective for the fpqc-topology ([FGI+05], Theorem 4.23, p.82), from where one obtains that quasi-affine morphisms satisfy effective descent for the fpqc-topology ([Gro63], Expose VIII, Corollary 7.9, p.31). As étale morphisms are local on the domain for the fpqc-topology ([FGI+05], Proposition 2.36, p.29) the corollary follows from the proposition. $\square$

Remark 3.9. In (Théorème A.4[LMB00]) one only assumes that the diagonal map $R \longrightarrow X \times X$ is quasi-compact.

Remark 3.10. An algebraic space is defined as the étale quotient sheaf of an étale equivalence relation $R \rightrightarrows X$ of schemes. The diagonal map $R \longrightarrow X \times_S X$ of an étale equivalence relation will satisfy effective descent for the fppf-topology [RG71]. It is unclear if algebraic spaces, not being quasi-separated, are stable.

3.11. Examples: Parametrizing finite closed subschemes. Let X be a separated scheme (over the base), and let $X^n = X \times \cdots \times X$ denote the n-fold product. Let $U = X^n \setminus \Delta$ denote the open complement of the closed set Δ given as the union of all diagonals. The group $\mathfrak{S}_n$ acts on X^n by permuting the factors, and the induced action on U is free.

Proposition 3.12. *Let J be a subcanonical topology that can be refined to the fpqc-topology. Let X be a separated scheme, and let E^n_X denote the sheaf quotient of the equivalence relation $\mathfrak{S}_n \times U \rightrightarrows U$, where U is the open complement of the diagonals $U = X^n \setminus \Delta$. Then we have for any scheme S that*

$$E^n_X(S) = \left\{ \begin{array}{c} \textit{closed subschemes } Z \subseteq X \times S, \textit{ such that there exists} \\ \textit{a } J\textit{-covering } T \longrightarrow S \textit{ such that } Z \times_S T = \sqcup_{i=1}^n T \end{array} \right\}$$

Proof. Elements in $X^n(S)$ correspond to n ordered sections $s_i \colon S \longrightarrow X \times S$, $i = 1, \dots, n$. The separatedness assumption implies that each section $s_i \colon S \longrightarrow X \times S$ gives a closed subscheme $Z_i \subseteq X \times S$ isomorphic to S. An element in $U(S) \subseteq X^n(S)$ corresponds to n *disjoint* ordered sections $s_i \colon S \longrightarrow X \times S$, which together give a closed subscheme $Z \subseteq X \times S$, with Z being the disjoint union of n-copies of S.

Let F denote the presheaf quotient of $\mathfrak{S}_n \times U \rightrightarrows U$. By definition $F(S)$ is the quotient set $U(S)/\mathfrak{S}_n$, and we get that elements in $F(S)$ corresponds to closed subschemes $Z \subseteq X \times S$ that are split $Z = \sqcup Z_i$, where $Z_i = S$, for each $i = 1, \dots, n$.

Since $E^n_X = LF$ is the sheafification of F we have that an element $s \in E^n_X(S)$ is J-locally an element of $F(S)$. Thus, J-locally the element s corresponds to a closed subscheme in X. Closed immersions are local on the domain for the fpqc-topology ([Gro63], Corollary 4.4), and then in particular for the J-topology. Hence we have that an element in $s \in E^n_X(S)$ will give a closed subscheme $Z \subseteq X \times S$, such that Z splits J-locally. Since the presheaf F is separated (Proposition (1.13)) it follows that elements in $E^n_X(S)$ correspond precisely to closed subschemes $Z \subseteq X \times S$ that J-locally splits. $\square$

Corollary 3.13. *Consider the topology generated by étale surjective morphisms, and let E^n_X denote the sheaf quotient of the equivalence relation given in the proposition. We then have the modular description*

$$E^n_X(S) = \{\textit{closed subschemes } Z \subseteq X \times S, \textit{ étale of rank } n \textit{ over } S\}.$$

In particular E^n_X is stable.

Proof. We have by the proposition that S-valued points of E^n_X are closed subschemes $Z \subseteq X \times S$ that étale locally splits. Conversely, if $Z \subseteq X \times S$ is finite and étale, then there exists an étale morphism $T \longrightarrow S$ such that $Z \times_S T$ splits. Stability follows from Corollary (3.8). □

Remark 3.14. The modular description is also given in [LMB00] and [RS07]).

Corollary 3.15. *Let $f\colon Z \longrightarrow S$ be a finite étale morphism of rank n. Let J be a subcanonical topology where $\mathtt{F}_f \notin J(S)$ is not a covering map for the topology J. Then for any separated scheme X containing $Z \subseteq X$ as a closed subscheme, the sheafification E^n_X in the J-topology is not stable.*

Proof. Let E_{et} denote the étale sheaf quotient of $\mathfrak{S}_n \times U \rightrightarrows U$, where $U = X^n \setminus \Delta$ is the complement of the diagonals. By Corollary (3.13) we have that $f\colon Z \longrightarrow S$ is an element of the étale sheaf $E_{et}(S)$. By assumption $\mathtt{F}_f \notin J(S)$, hence by Theorem (2.7) we have that $f\colon Z \longrightarrow S$ has no section J-locally. Consequently $f\colon Z \longrightarrow S$ does not split J-locally, and we get that Z is not an element of $E^n_X(S)$. Thus E^n_X is not equal to E_{et}. As the étale sheafification of E^n_X is the sheaf E_{et} we have proven that E^n_X is not stable. □

Remark 3.16. If G is a finite, discrete, group acting on a scheme X, separated over the base, then there exist a group quotient X/G in the category of algebraic spaces. That particular sheaf X/G is stable, being an algebraic space, and the question is wheter group quotients are stable in a topology coarser than the étale topology. The results above show that the topology must contain Zariski coverings, and finite étale coverings.

4. Applications

4.1. Notation. Let M be a quasi-coherent sheaf on the fixed base scheme S. For any scheme $T \longrightarrow S$, let $E_T = \oplus_{i=1}^n \mathscr{O}_T$ denote the free $\mathscr{O}_T$-module of rank n. We have that the functor of linear maps from M to E is represented by $L(M,E) = \mathbf{S}\mathrm{pec}(\mathscr{S}(M_S \otimes_{\mathscr{O}_S} E^*_S))$, where E^*_S is the dual of E, and where $\mathscr{S}(-)$ is the sheaf of symmetric tensor algebra [Die62].

Lemma 4.2. *There is an open subset $X^M_E \subseteq L(M,E)$ parameterizing surjective linear maps $M \longrightarrow E$.*

Proof. We may assume that the base schemes $S = \mathrm{Spec}(A)$ is affine, and that the quasi-coherent sheaf M is given by an A-module M. On the scheme $L = L(M,E)$ we have the universal map $M_L \longrightarrow E_L$. The open set $X^M_E \subseteq X(M,E)$ is the union of the open schemes obtained by inverting all $(n \times n)$-minors of the universal map. □

4.3. Grassmannians. We give here a global construction of the Grassmannian $\mathrm{Grass}^n(M)$ of quasi-coherent $\mathscr{O}_S$-modules that are locally free rank n-quotients of a quasi-coherent sheaf M ([GD71], [Kle69]).

Let $\mathrm{GL}_n \subset \mathbf{A}_S^{n^2}$ denote the open subscheme obtained by inverting the determinant. The group scheme GL_n acts naturally by composition on the space $L(M, E)$ of linear maps from $M \longrightarrow E$, and we get an induced action on the set of surjective maps $X_E^M \subseteq L(M, E)$. Indeed let $\varphi\colon M_T \longrightarrow E_T$ be a surjective $\mathscr{O}_T$-linear map corresponding to $t\colon T \longrightarrow X_E^M$, with T a scheme. For any group element $g \in \mathrm{GL}_n(T)$ we consider the composition

$$g \circ \varphi_t\colon M_T \longrightarrow E_T. \tag{4.3.1}$$

Since the composition also is surjective it corresponds to an element $gt \in X_E^M(T)$.

Lemma 4.4. *Let $R = \mathrm{GL}_n \times_S X_E^M$. The projection on the second factor, together with the action (4.3.1), form an equivalence relation on X_E^M. The presheaf quotient X_R parameterizes quotient modules of M that are free of rank n over the base scheme S.*

Proof. One checks that the action is free, that is $g \cdot t = t$ implies $g = \mathrm{id}$. Furthermore, as a group scheme acting freely on a scheme form an equivalence relation, we have the first statement of the lemma.

The value of X_R at a scheme T is the set of surjective linear maps $M_T \longrightarrow E_T$ modulo the equivalence given by GL_n. If $t \in X_E^M(T)$, then the kernel of the corresponding surjective linear map $\varphi_t\colon M_T \longrightarrow E_T$ equals the kernel of $g \cdot t$, for any $g \in \mathrm{GL}_n(T)$. And conversely, if $t_1, t_2 \in X_E^M(T)$ are two elements where the corresponding surjective linear maps φ_i have the same kernel, then there exists $g_{12} \in \mathrm{GL}_n(T)$ such that $g_{12} \cdot t_1 = t_2$. Thus $X_R(T)$ is the set of surjective linear maps where two maps are identified if they have the same kernel. $\square$

Proposition 4.5. *Let M be a quasi-coherent sheaf on a scheme S, and let LX_R denote Zariski sheaf quotient of the equivalence relation given in Lemma (4.4). Then LX_R is the Grassmannian $\mathrm{Grass}_S^n(M)$.*

Proof. Let $t \in LX_R(T)$, for some scheme T. Then there exist a Zariski covering $T' \longrightarrow T$ such that t lifts to $t' \in X_E^M(T')$. That is a surjective quotient map of $\mathscr{O}_{T'}$-modules $M_{T'} \longrightarrow E_{T'}$, with descent datum. Effective descent for quasi-coherent surjective module homomorphism ([Gro63], Corollary 1.8, p.6) gives a Zariski locally free $\mathscr{O}_T$-module L being the quotient of M_T.

Conversely, if $M_T \longrightarrow L$ is a surjective quotient with L a Zariski-locally free $\mathscr{O}_T$-module, then by definition there exist a Zariski trivialization $\{T_\alpha \longrightarrow T\}$ such that L_{T_α} trivializes. For each α we choose a trivialization, and let

$$\varphi_\alpha\colon M_{T_\alpha} \longrightarrow L_{T_\alpha} \longrightarrow E_{T_\alpha}$$

denote the composition. Let $p_i \colon T_\alpha \times_T T_\beta \longrightarrow T_\alpha$ denote the projection on the $i'th$ factor. We have that there exist $g_{\alpha,\beta} \in \mathrm{GL}_n(T_\alpha \times_T T_\beta)$ such that $g_{\alpha,\beta} p_1^*(\varphi_\alpha) = p_2^*(\varphi_\beta)$. We get a map $\sqcup T_\alpha \times T_\beta \longrightarrow \mathrm{GL} \times X_E^m$. The commutative diagrams

$$\begin{tikzcd} \mathrm{GL}_n \times X_E^M \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X_N^m \\ \sqcup T_\alpha \times T_\beta \arrow[u, "{g_{\alpha,\beta}, p_1^*\varphi_\alpha}"] \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & \sqcup T_\alpha \arrow[u, "\varphi_\alpha"'] \end{tikzcd}$$

gives a map of sheaves $T \longrightarrow LX_R$, which corresponds to the quotient $M_T \longrightarrow L$. Consequently the sheaf LX_R is the Grassmannian $\mathrm{Grass}^n(M)$.

□

Corollary 4.6. *Let $\mathbf{A}_S^{n+1} \setminus 0$ denote the open complement of the intersection of the coordinate axis of the affine $n+1$-space. The group $\mathbf{G}_m$ of multiplicative units acts naturally on $\mathbf{A}_S^{n+1} \setminus 0$, and the Zariski quotient is the projective n space $\mathbf{P}_S^n$ over S.*

Proof. Set M as the free $\mathscr{O}_S$-module of rank $n+1$. Then $L(M, \mathscr{O}_S)$ equals the affine space $\mathbf{A}_S^{n+1}$, given by $\mathscr{O}_S[x_1, \ldots, x_n]$. Each variable determines an open set $D(x_i)$ where the variabel x_i is non-vanishing. The open set $X_{\mathscr{O}}^M \subseteq L(M, \mathscr{O}_S)$ is identified with the open union $\cup_{i=1}^{n+1} D(x_i)$, which is $\mathbf{A}_S^{n+1} \setminus 0$. The result now follows from the proposition as the Grassmannian of locally rank one quotients of $M = \oplus_{i=1}^{n+1} \mathscr{O}_S$ is $\mathbf{P}_S^n$. □

4.7. Hilbert scheme of points. We give here a global construction of the Hilbert scheme of n-points on the affine d-space $\mathbf{A}_S^d$, over an arbitrary base scheme S. In the special situation with the base being a field, the construction is known ([Nor78], [**?**], [**?**], and the non-commutative version [Ber88]).

4.8. Notation. As before we let E_T denote the free $\mathscr{O}_T$-module of rank n, and we let $L(E, E)$ the space of endomorphisms of E_S, where S is the base scheme. Let $L_S^{d,n}$ denote the space parameterizing d-ordered sequence of linear maps, and vectors in E. Thus

$$L_S^{d,n} = L(E, E) \times_S \cdots \times_S L(E, E) \times_S L(\mathscr{O}_S, E).$$

For any S-scheme T we have that a point in $t \in L_S^{d,n}(T)$ is given by

$$t = (Z_1, \ldots, Z_d, v)$$

where $Z_i \colon E_T \longrightarrow E_T$ are endomorphisms ($i = 1, \ldots, d$), and $v \in E_T$ a vector. Such a point induces by evaluation a map

$$\varphi_t \colon \mathscr{O}_T < z_1, \ldots, z_n > \longrightarrow E_T \tag{4.8.1}$$

by sending the variable $z_i \mapsto Z_i(v)$. A point $t \in X_S^{d,n}(T)$ is called *cyclic* if the corresponding map (4.8.1) is surjective. Moreover, there is a natural action of GL_n on $L_S^{d,n}$ given by

$$(g,(Z_1,\dots,Z_d,v)) \mapsto (gZ_1g^{-1},\dots,gZ_dg^{-1},gv)$$

for all $g \in \mathrm{GL}_n(T), (Z_1,\dots,Z_d,v) \in L_S^{d,n}(T)$, all schemes T.

Lemma 4.9. *The set of cyclic vectors form an open subset of $L_S^{d,n}$, and the induced action of GL_n on the set of cyclic vectors is free.*

Proof. That the cyclic vectors form an open subset is clear. Assume that $t = gt$ for some cyclic vector $t \in L^{d,n}(T)$, and some $g \in \mathrm{GL}_n(T)$. Let $t = (Z_1,\dots,Z_d,v)$, and let $w \in E_T$ be any vector. Since t is cyclic, there exists an element $f(z_1,\dots,z_d)$ in the free algebra, such that

$$f(Z_1,\dots,Z_d)v = w.$$

By assumption we have that $gZ_ig^{-1} = Z_i$ for all i, and that $gv = v$. We then get that

$$w = f(gZ_1g^{-1},\dots,gZ_dg^{-1}) \cdot gv = g \cdot f(Z_1,\dots,Z_d)g^{-1}gv = gw.$$

As $gw = w$ for all vectors $w \in E_T$ we have that $g = \mathrm{id}$, and the action is free. $\square$

4.10. Commuting matrices. Let $X_E^d \subseteq L_S^{d,n}$ be the locally closed subset consisting of cyclic vectors having commuting endomorphisms. That is $t = (Z_1,\dots,Z_d,v) \in X_E^d(T)$, with T a scheme, if t is cyclic and

$$[Z_i,Z_j] = 0 \quad \text{for all } i,j = 1,\dots,d.$$

Note that for $t \in X_E^d(T)$ the evaluation map (4.8.1) now factorizes to give a surjective map

$$\varphi_t \colon \mathscr{O}_T[z_1,\dots,z_d] \longrightarrow E_T.$$

And as such we get that E_T corresponds to a closed subscheme $Z \subseteq \mathbf{A}_T^d$, which is finite and given by a free $\mathscr{O}_T$-module of rank n .

Moreover, it is clear that the action of GL_n on the cyclic vectors, induces an action on X_E^d. By Lemma (4.9) the action of GL_n on X_E^d is free, and consequently the induced map

$$R = \mathrm{GL}_n \times_S X_E^d \longrightarrow X_E^d \times_S X_E^d$$

form an equivalence relation on X_E^d.

Proposition 4.11. *Let $LX_R = L(X_E^d)_R$ denote the Zariski sheaf quotient of the equivalence relation on X_E^d given by the action of GL_n. The evaluation map induces an morphism of sheaves*

$$LX_R \longrightarrow \mathrm{Hilb}^n_{\mathbf{A}_S^d},$$

which is an isomorphism.

Proof. It is clear that any point $t \in X^d_E(T)$, and gt for any $g \in \mathrm{GL}_n(T)$ the associated evaluation maps φ_t and φ_{gt} have equal kernels. Hence we get an induced map of presheaves

$$(X^d_E)_R \longrightarrow \mathrm{Hilb}^n_{\mathbf{A}^d_S}. \tag{4.11.1}$$

This map is a monomorphism: Assume that $t = (Z_1, \ldots, Z_d, v)$ and $u = (Z'_1, \ldots, Z'_d, v')$ in $X^d_E(T)$ give the same point in the Hilbert functor. We can find a $g \in \mathrm{GL}_n(T)$ such that $gv' = v$. By letting $Y_i = gZ'_i g^{-1}$ we may assume that $u = (Y_1, \ldots, Y_d, v)$. By assumption we would then have $Z_i v = Y_i v$, for all $i = 1, \ldots, d$. As the vectors are cyclic, it follows that $Z_i w = Y_i w$, for all vectors $w \in E_T$, as well. Hence $t = gu$, and the map (4.11.1) is monomorphism. As the quotient sheaf is separated (Proposition 1.13) the induced morphism of sheaves

$$L(X^d_E)_R \longrightarrow \mathrm{Hilb}^n_{\mathbf{A}^d_S} \tag{4.11.2}$$

is a monomorphism.

To verify that (4.11.2) is also an epimorphism we take a T-valued point of the Hilbert funtor. That is a closed subscheme $Z \subseteq \mathbf{A}^d_T$ which is flat, finite of rank n over T. One can then find a Zariski trivialization $T' \longrightarrow T$, on where $Z \times_T T'$ is given by a finite $\mathscr{O}_{T'}$-algebra, which is free as a module. This amounts to giving an element $t' \in X^d_E(T')$. Arguing as in the case with Grassmanian, one gets by descent a morphism $T \longrightarrow L(X^d_E)_R$ and thereby proving that (4.11.2) is an isomorphism. □

Remark 4.12. In both of these two examples of this section, the two functors $\mathrm{Grass}^n(M)$ and $\mathrm{Hilb}^n_{\mathbf{A^d}}$ are representable by schemes.In particular the quotient sheaves of Proposition (4.5) and of Proposition (4.11) are stable for the Zariski-topology. That these two quotient sheaves are stable for the Zariski-topology can be proven without knowledge of representability in the following way. Since the group GL_n is *special* we have that a torsor for fpqc-topology, already trivializes in the Zariski topology ([Ser95]). This proves stability for the quotient sheaf construction of the Grassmannian. That result and the fact that closed immersions are local on the domain for the fpqc-topology shows that the quotient sheaf construciton for the Hilbert scheme also is stable.

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