

Empirical Coordination for Controlled Markov Processes with Strictly Causal Encoder

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Abstract—We study joint source–channel coding for a Controlled Markov Process (CMP) through the lens of empirical coordination. Our objective is to characterize the joint empirical distributions of all the involved states that can be induced by a coding scheme. The encoder is strictly causal: it selects each channel input based on past source symbols and channel feedback, while the Markov channel evolves according to a transition law determined by the current input and previous output state. For stationary input kernels, we establish the optimal characterization of the achievable joint distributions with matching inner and outer bounds. The achievability proof introduces a notion of controlled Markov typicality and uses a random coding construction with a tree structure to establish the resulting information constraint.

Index Terms—Controlled Markov Process, Joint Source-channel Coding, Empirical Coordination Coding.

I. INTRODUCTION

Controlled Markov Process (CMP) [1]–[8] describes the fundamental interaction between a controller and an environment: at each time, the environment is in some state, and the controller selects an action. Upon executing this action, the environment stochastically transitions to a new state. It provides a canonical framework for sequential decision-making [9] in control and reinforcement learning [10], [11].

We study this interaction through a joint source–channel coding (JSCC) formulation [12]–[14] over a Markov channel with feedback, where channel inputs represent control actions and channel outputs represent environmental states. Markov channels, also known as finite-state channels (FSCs), have been extensively studied from the perspectives of communication capacity and control strategies [15]–[21]. Feedback allows the encoder to adapt its inputs to past channel outputs, giving rise to a substantial literature on FSCs with feedback [22]–[27]. A similar setup for studying communication in Markov Decision Process is in [28]. In our setting, an i.i.d. source provides a sequence of symbols, and a strictly causal encoder selects each channel input based on past source symbols and channel feedback sequence. The current input and previous output state jointly determine the distribution of the next channel output. A noncausal decoder then observes the entire channel output sequence and produces a reconstruction

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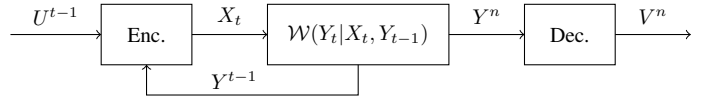


Fig. 1: Joint source-channel coding with strictly causal encoder and noncausal decoder over Markov channel

sequence. Feedback thus establishes a closed-loop interaction between the encoder and the channel.

The empirical coordination framework [29]–[31] characterizes the joint behavior attainable by distributed agents by requiring the empirical distribution of their symbol sequences to approach a prescribed target distribution. Within this framework, communication serves to coordinate the actions of encoders and decoders under their respective observation constraints. Empirical coordination results for JSCC with causal or strictly causal decision-makers over discrete memoryless channels were established in [32], [33]. Extensions incorporating channel state or feedback were developed in [34], [35], with further applications to distributed decision-making in [36]–[39]. More recently, [40] considered empirical coordination over a Markov channel with a strictly causal encoder and no channel feedback.

In this work, we extend the setting of [40] by introducing channel feedback, thereby enabling closed-loop control of the Markov channel. For stationary input kernels, we establish the optimal characterization of the achievable joint distributions through matching inner and outer bounds. To prove achievability, we introduce controlled Markov typicality and develop its fundamental properties. We then develop a random coding argument based on a tree structure, together with an error analysis that accounts for dependencies across coding blocks without imposing an additional rate penalty.

II. SYSTEM MODEL

We study the problem depicted in Fig. 1. Capital letters, like U , denote random variables, calligraphic fonts, like $\mathcal{U}$, denote alphabets, and lowercase letters, like $u \in \mathcal{U}$, denote the realizations of random variables. We assume all the alphabets $\mathcal{U}, \mathcal{X}, \mathcal{Y}, \mathcal{V}$ considered here are finite. Sequences of length $t \in \mathbb{N}$ are usually denoted by $X^t = (X_1, \dots, X_t)$. Moreover, we write $Y_0^{n-1} = (Y_0, \dots, Y_{n-1})$ and $Y_1^n = (Y_1, \dots, Y_n)$ for the length- n subsequences obtained by removing the last and first symbols, respectively, from the same sequence $Y_0^n = (Y_0, Y_1, \dots, Y_n)$ of length $(n+1)$. The ℓ_1 distance

between two probability distributions $\mathcal{Q}$ and $\mathcal{P}$ is $\|\mathcal{Q} - \mathcal{P}\|_1 = \sum_u |\mathcal{Q}(u) - \mathcal{P}(u)|$. The probability is denoted by $\mathbb{P}(\cdot)$.

We assume the source is drawn i.i.d. $\sim \mathcal{P}_U$, and the channel is stationary Markov with one step memory of the latent state, with conditional distribution $\mathcal{W}_{Y|XY'}$. Moreover, the statistics of the source and channel are known by both agents. We assume $U_0 = \emptyset$, and Y_0 is given.

Definition 1. A code with strictly causal encoders with channel feedback and noncausal decoder is a tuple of (deterministic) functions $c = (\{f_{X_t|U^{t-1}, Y^{t-1}}^{(t)}\}_{t=1}^n, g_{V^n|Y^n})$ defined by

$$f_{X_t|U^{t-1}, Y^{t-1}}^{(t)} : \mathcal{U}^{t-1} \times \mathcal{Y}^{t-1} \rightarrow \mathcal{X}, \quad g_{V^n|Y^n} : \mathcal{Y}^n \rightarrow \mathcal{V}^n,$$

We denote by $\mathcal{C}(n)$ the set of control designs with strictly causal encoders and non-causal decoder.

In this way, a code $c \in \mathcal{C}(n)$ defined above induces a joint distribution on random sequences (U^n, X^n, Y^n, V^n) given by

$$\mathcal{P}_U^{\otimes n} \prod_{t=1}^n \left[f_{X_t|U^{t-1}, Y^{t-1}}^{(t)} \mathcal{W}_{Y|XY'}(Y_t|X_t, Y_{t-1}) \right] g_{V^n|Y^n}, \quad (1)$$

which induces the following Markov chains

$$\begin{aligned} U_t &\perp\!\!\!\perp (Y_{t-1}, X_t, Y_t), \\ X_t &\ominus (U^{t-1}, Y^{t-1}) \ominus U_t^n, \\ Y_t &\ominus (X_t, Y_{t-1}) \ominus (U^{t-1}, X^{t-1}, Y^{t-2}). \end{aligned}$$

We denote the empirical distribution $Q^n \in \Delta(\mathcal{U} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V})$ of sequences $(u^n, y_0^{n-1}, x^n, y_1^n, v^n)$ with starting state y_0 by

$$\begin{aligned} Q^n(u, y', x, y, v) & \\ &= \frac{1}{n} \sum_{t=1}^n \mathbf{1}\{u_t = u, y_{t-1} = y', x_t = x, y_t = y, v_t = v\}, \\ &\forall (u, y', x, y, v) \in \mathcal{U} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}. \end{aligned} \quad (2)$$

Since the empirical distribution Q^n is induced by (1), it is a random variable. Note here, we take into account the adjacent pairs (y_{t-1}, y_t) in the Markov sequence y^n .

Definition 2. We define a joint distribution $\mathcal{P}_{UY'XYV} \in \Delta(\mathcal{U} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V})$ of symbols (U, Y', X, Y, V) to be achievable, if for all $\varepsilon \geq 0$, there exists $N_0 \in \mathbb{N}$, such that for all $n \geq N_0$, we can find a sequence of codes $c \in \mathcal{C}(n)$ with joint empirical distribution given by (2) satisfying that

$$\mathbb{P}(\|Q^n(U, Y', X, Y, V) - \mathcal{P}_{UY'XYV}\|_1 \geq \varepsilon) \leq \varepsilon. \quad (3)$$

We impose the following assumption on the Markov channel. This is the same as the assumption in [28].

Assumption A (Unichain Markov channel). The Markov channel $\mathcal{W}_{Y|XY'}$ is unichain: for every stationary input kernel $q_{X|Y'} \in \mathcal{P}(\mathcal{X}|\mathcal{Y})$, the induced transition matrix

$$T_q(y|y') \triangleq \sum_{x \in \mathcal{X}} q_{X|Y'}(x|y') \mathcal{W}_{Y|XY'}(y|x, y') \quad (4)$$

has a single recurrent communicating class $\mathcal{R}_Y$, possibly together with transient states.

This guarantees the following standard result [41]

Lemma 1. The Markov chain Y^n admits a unique equilibrium distribution $\pi_{Y'}^q = \pi_Y^q \in \Delta(\mathcal{R}_Y)$, which is uniquely determined by the stationary input distribution $q_{X|Y'}$ and the channel transition kernel $\mathcal{W}_{Y|XY'}$ by

$$\pi_Y^q(y) = \sum_{x, y'} \pi_{Y'}^q(y') q_{X|Y'}(x|y') \mathcal{W}_{Y|XY'}(y|x, y'). \quad (5)$$

By taking $\pi_Y^q(y) = 0$ for all $y \in \mathcal{Y} \setminus \mathcal{R}_Y$, from now on, we omit explicit specification of the support $\mathcal{R}_Y$ when referring to π_Y^q , and use a unifying set $\mathcal{Y}$.

III. MAIN RESULT

Our main result is the optimal characterization of the achievable joint distribution region with a stationary input kernel of form $q_{X|Y'}$:

Theorem 1. A joint distribution $\mathcal{P}_{UY'XYV}$ is achievable if and only if it factorizes as

$$\mathcal{P}_U \pi_{Y'}^q q_{X|Y'} \mathcal{W}_{Y|XY'} \mathcal{P}_{V|UXY'Y}, \quad (6)$$

and satisfies the information constraint

$$I(X; Y|Y') - I(U; V|X, Y, Y') \geq 0. \quad (7)$$

Remark 1. The information constraint in (7) has a form analogous to that of the empirical coordination theorem for memoryless channels in [35], with both mutual information terms conditioned on Y' , which is available to both the encoder and the decoder. Although (X, Y') can be viewed as an augmented channel input in (6), its distribution is constrained by the channel dynamics: Y' is the previous channel output, and its stationary distribution $\pi_{Y'}^q$ is induced by $q_{X|Y'}$ and $\mathcal{W}_{Y|XY'}$. Furthermore, our result reduces to the form with memoryless channel in [35] if Y' is a constant.

Remark 2. When the random variables (U, V) are independent of the channel (Y', X, Y) , our results boil down to the solution to a joint source-channel separation scenario with Markov channel and feedback which is uncovered in [23, Theorem 20] with the channel capacity in the single-letter form of [27, Theorem 1].

The rest of the paper focuses on the proof of the main theorem. To establish the inner bound, we introduce a new notion of typicality – the controlled Markov typicality – for sequences (X^n, Y^n) in Sec. IV. The proof of the inner bound is a random-coding achievability argument, given in the Sec. V. Finally, the outer bound is proved in Sec. VI.

IV. CONTROLLED MARKOV TYPICALITY

This section develops a notion of typicality for a stationary finite-alphabet Markov channel whose input is selected according to the previous channel output. It extends the input-driven Markov typicality for open-loop setups introduced in [40].

For a given initial state $Y_0 = y_0$, the controlled sequence (X^n, Y^n) is generated according to

$$\prod_{t=1}^n q_{X|Y'}(X_t|Y_{t-1}) \mathcal{W}_{Y|XY'}(Y_t|X_t, Y_{t-1}). \quad (8)$$

The corresponding stationary state-action-transition target law is denoted by

$$\mathcal{P}_{Y'XY}(y', x, y) \triangleq \pi_{Y'}^q(y') q_{X|Y'}(x|y') \mathcal{W}_{Y|XY'}(y|x, y'). \quad (9)$$

Define the admissible support

$$\mathcal{A}_q \triangleq \{(i, x, j) : q_{X|Y'}(x|i) \mathcal{W}_{Y|XY'}(j|x, i) > 0\}.$$

Definition 3 (Controlled Markov Typicality). *For sequences (y_0^{n-1}, x^n, y_1^n) with a fixed initial state y_0 , define the empirical distribution of the triple (y', x, y) by*

$$Q_{Y'XY}^n(y', x, y) \triangleq \frac{1}{n} \sum_{t=1}^n \mathbf{1}\{(y_{t-1}, x_t, y_t) = (y', x, y)\}.$$

Define the controlled Markov typical set by

$$\mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY}) \triangleq \{(y_0^{n-1}, x^n, y_1^n) : \|Q_{Y'XY}^n - \mathcal{P}_{Y'XY}\|_1 \leq \varepsilon, \\ Q_{Y'XY}^n(y', x, y) = 0, \forall (y', x, y) \notin \mathcal{A}_q\}.$$

and the conditioning typical set given y_0^n by

$$\mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) \triangleq \{x^n : (y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY})\}.$$

Theorem 2 (Controlled Markov ergodicity). *Suppose Assumption A holds for $q_{X|Y'}$. For every initial state y_0 ,*

$$\|Q_{Y'XY}^n - \mathcal{P}_{Y'XY}\|_1 \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0. \quad (10)$$

Consequently, for every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{P}\left((Y_0^{n-1}, X^n, Y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY})\right) = 1. \quad (11)$$

Since $\mathcal{Y}$ is finite, the convergence in (11) is uniform over y_0 .

Theorem 3 (Controlled AEP and type-size bound). *For every $\delta > 0$, there exist $\varepsilon_0 > 0$ and $N_0 \in \mathbb{N}$ such that, for $0 < \varepsilon < \varepsilon_0$, $n \geq N_0$, every initial state y_0 , and every*

$$(y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY}),$$

the following statements hold:

1) *If (X^n, Y^n) is generated according to (8), then*

$$2^{-n(H(X, Y|Y') + \delta)} \leq \mathbb{P}((X^n, Y^n) = (x^n, y^n)) \\ \leq 2^{-n(H(X, Y|Y') - \delta)}.$$

$$2) \left| \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY}) \right| \leq 2^{n(H(X, Y|Y') + \delta)}.$$

$$3) 2^{n(H(X|Y', Y) - \delta)} \leq \left| \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) \right| \leq 2^{n(H(X|Y', Y) + \delta)}.$$

More properties of the controlled Markov typicality and the proofs for the above theorems are given in the appendices of the supplementary material [42].

V. ACHIEVABILITY PROOF

We fix a joint probability distribution over symbols (U, Y', X, Y, V) that factorizes as in (6) such that there exists a $\delta > 0$ and a rate $R > 0$ satisfying

$$R \geq I(U, Y', Y; V|X) + \delta, \quad (12)$$

$$R \leq I(V; Y, Y'|X) + I(X; Y|Y') - \delta. \quad (13)$$

We consider a block-Markov random code $c \in \mathcal{C}(nB)$ over $B \in \mathbb{N}$ blocks of length $n \in \mathbb{N}$:

Random Codebook Construction: Define

$$K_n \triangleq \left\lfloor \frac{2^{nR}}{B} \right\rfloor, \quad \mathcal{M}_b \triangleq \{(b-1)K_n + 1, \dots, bK_n\}, b \in [1 : B].$$

The sets $\mathcal{M}_1, \dots, \mathcal{M}_B$ are pairwise disjoint. Let $\mathcal{M} \triangleq \bigcup_{b=1}^B \mathcal{M}_b$. Then we have $|\mathcal{M}| = BK_n \leq 2^{nR}$.

For every block $b \in [1 : B-1]$, $m \in \mathcal{M}_b$, $t \in [1 : n]$, and $y \in \mathcal{Y}$, independently generate an input symbol $X_t(m, y) \sim q_{X|Y'}(\cdot | y)$. In this way, for each fixed state sequence $y^n \in \mathcal{Y}^n$, we denote the codeword associated with y^n with index $m \in \mathcal{M}_b$ by $X^n(m, y^n) \triangleq (X_1(m, y_1), \dots, X_n(m, y_n))$ drawn from the distribution $\prod_{t=1}^n q_{X|Y'}(x_t|y_t)$. Next, conditioning on the sequence $X^n(m, y^n)$ where $m \in \mathcal{M}_b$, generate the same number K_n of reconstruction codewords $V^n(m, \hat{m}, y^n)$ with index $\hat{m} \in \mathcal{M}_{b+1}$ drawn from $\prod_{t=1}^n \mathcal{P}_{V|X}(v_t|X_t(m, y_t))$.

Encoding process: Let m_b denote the message uncovered during block $b \in [1 : B]$. During the first block, WLOG, the encoder takes $m_1 = 1$. Given $Y_{1,0}$,

$$X_{1,t} = X(m_1, Y_{1,t-1})$$

$$Y_{1,t} \sim \mathcal{W}_{Y|XY'}(\cdot | X_{1,t}, Y_{1,t-1}), \forall t = 1, \dots, n.$$

If we denote by $X_1^n = X^n(m_1, Y_{1,0}^{n-1})$, the sequences $(Y_{1,0}^{n-1}, X_1^n, Y_{1,1}^n)$ are available at the encoder for block $b = 2$. At the beginning of the block $b \in [2 : B]$, because the encoder is strictly causal, it receives the source sequence U_{b-1}^n of the last block. It recalls the last message $m_{b-1} \in \mathcal{M}_{b-1}$ and sequences $(Y_{b-1,0}^{n-1}, X_{b-1}^n = X^n(m_{b-1}, Y_{b-1,0}^{n-1}), Y_{b-1,1}^n)$ of the last block. Then, it searches for an index $m_b \in \mathcal{M}_b$ such that

$$(U_{b-1}^n, Y_{b-1,0}^{n-1}, X_{b-1}^n, Y_{b-1,1}^n, V^n(m_{b-1}, m_b, Y_{b-1,0}^{n-1})) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}).$$

Denote by $V_{b-1}^n = V^n(m_{b-1}, m_b, Y_{b-1,0}^{n-1})$. Upon it finds m_b , during block b , it proceeds

$$X_{b,t} = X(m_b, Y_{b,t-1}),$$

$$Y_{b,t} \sim \mathcal{W}_{Y|XY'}(\cdot | X_{b,t}, Y_{b,t-1}), \forall t = 1, \dots, n,$$

where the first feedback symbol $Y_{b,0} = Y_{b-1,n}$ is the last symbol of the last block. Denote $X_b^n = X^n(m_b, Y_{b,0}^{n-1})$.

Decoding Process: The decoder returns $\tilde{m}_1 = 1$. During block $b \in [2 : B]$, the decoder recalls the past sequence $(Y_{b-1,0}^{n-1}, Y_{b-1,1}^n)$ and the index $\tilde{m}_{b-1} \in \mathcal{M}_{b-1}$, that corresponds to the sequence $\tilde{X}_{b-1}^n = X^n(\tilde{m}_{b-1}, Y_{b-1,0}^{n-1})$. It

observes the current channel output sequences $(Y_{b,0}^{n-1}, Y_{b,1}^n)$ and tries to find the unique message $\tilde{m}_b \in \mathcal{M}_b$, s.t.,

$$\begin{aligned} (Y_{b,0}^{n-1}, X^n(\tilde{m}_b, Y_{b,0}^{n-1}), Y_{b,1}^n) &\in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \\ (Y_{b-1,0}^{n-1}, \tilde{X}_{b-1}^n, Y_{b-1,1}^n, V^n(\tilde{m}_{b-1}, \tilde{m}_b, Y_{b-1,0}^{n-1})) &\in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}). \end{aligned}$$

Then, it returns $\tilde{V}_{b-1}^n = V^n(\tilde{m}_{b-1}, \tilde{m}_b, Y_{b-1,0}^{n-1})$ as the decoded output sequence for block $b-1$. As for the last block, the decoder simply outputs an all zero sequence, i.e., $\tilde{V}_B^n = 0$. Usually, the sequences are not jointly typical in the last block.

Error Analysis: We show now that given the above coding scheme, the generated sequences $(U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n, V_b^n)$ for each block $b \in \{1, \dots, B-1\}$ are jointly (controlled Markov) typical with respect to the joint distribution (6) with high probability: For $\forall \varepsilon > 0$, there exists $N_0 \in \mathbb{N}$, such that the probability of error events for all $n \geq N_0$:

$$\begin{aligned} \text{(a)} \quad &\mathbb{E}_c \left[\mathbb{P} \left((U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n) \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \right) \right] \leq \frac{2\varepsilon}{9} \\ \text{(b)} \quad &\mathbb{E}_c \left[\mathbb{P} \left(\forall m \in \mathcal{M}_{b+1}, (U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n, \right. \right. \\ &\quad \left. \left. V^n(m_b, m, Y_{b,0}^{n-1})) \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \right) \right] \leq \frac{4\varepsilon}{9} \\ \text{(c)} \quad &\mathbb{E}_c [\mathbb{P} (\exists \tilde{m} \in \mathcal{M}_{b+1} \setminus \{m_{b+1}\}, \text{ s.t.} \\ &\quad \left\{ (Y_{b+1,0}^{n-1}, X^n(\tilde{m}, Y_{b+1,0}^{n-1}), Y_{b+1,1}^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \right\} \cap \\ &\quad \left\{ (Y_{b,0}^{n-1}, \tilde{X}_b^n, Y_{b,1}^n, V^n(\tilde{m}_b, \tilde{m}, Y_{b,0}^{n-1})) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \right\} \right] \leq \frac{\varepsilon}{3} \end{aligned}$$

Item (a) is guaranteed by the independent generation of the sequence U_b^n , and the sequential selection of the sequence $(Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n)$ according to (8), where X_b^n is the codebook generated randomly, such that Theorem 2 holds. Therefore, $(U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n)$ is jointly typical with respect to the distribution $\mathcal{P}_{UY'XY} = \mathcal{P}_U \otimes \mathcal{P}_{Y'XY}$ when n large. Item (b) is guaranteed by the covering lemma [43] extended from classical i.i.d. sequences to our setup, as well as the condition (12) for the rate. We provide the detailed proof in Appendix D in our supplementary file [42]. Item (c) results from the packing lemma that applies to our Markov channel, which is shown below in Sec. V-A, Lemma 2.

We denote by $\tilde{Q}^n \in \Delta(\mathcal{U} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V})$ the empirical distribution of symbols (U, Y', X, Y, V) over blocks $b = 1, \dots, B-1$. We now show that $\tilde{Q}^n$ is close to the empirical distribution Q^n over all B blocks for B sufficiently large. We denote by Q_B the empirical distribution of all symbols over the last block. Now,

$$\begin{aligned} \|Q^n - \tilde{Q}^n\|_1 &= \left\| \frac{1}{B} \left((B-1)\tilde{Q}^n + Q_B \right) - \tilde{Q}^n \right\|_1 \\ &= \frac{1}{B} \|Q_B - \tilde{Q}^n\|_1 \\ &\leq \frac{2}{B} \cdot |\mathcal{U} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}| \leq \varepsilon \end{aligned}$$

when $B \geq \frac{2}{\varepsilon} |\mathcal{U} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{V}|$. Then, the expected error probability

$$\mathbb{E}_c[\mathbb{P}_e(c)] = \mathbb{E}_c[\mathbb{P}(\|Q^n - \mathcal{P}\|_1 \geq 2\varepsilon)]$$

$$\begin{aligned} &\leq \mathbb{E}_c \left[\mathbb{P} \left(\|Q^n - \tilde{Q}^n\|_1 + \|\tilde{Q}^n - \mathcal{P}\|_1 \geq 2\varepsilon \right) \right] \\ &\leq \mathbb{E}_c \left[\mathbb{P} \left(\|\tilde{Q}^n - \mathcal{P}\|_1 \geq \varepsilon \right) \right] \leq \varepsilon. \end{aligned}$$

This implies the existence of a code $c^* \in \mathcal{C}(nB)$ with an error probability below ε for all $n \geq N_0B$.

For a distribution P^* satisfying the information constraint (6) with equality and $I_{P^*}(X; Y|Y') > 0$, replace its reconstruction kernel by $r^\theta = (1-\theta)P_{V|U,X,Y',Y}^* + \theta/|\mathcal{V}|$. Keeping all other factors fixed, the resulting distributions $P^\theta = \mathcal{P}_U \pi_{Y',Q_X|Y'}^q \mathcal{W}_{Y|X,Y'} r^\theta$ satisfy the strict information inequality for $0 < \theta < 1$ and converge to P^* as $\theta \rightarrow 0$. $\square$

A. Joint Packing Lemma

We denote the following events for block b by

$$\begin{aligned} A_b(\tilde{m}) &\triangleq \{(Y_{b,0}^{n-1}, X^n(\tilde{m}, Y_{b,0}^{n-1}), Y_{b,1}^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P})\}, \\ B_b(\tilde{m}) &\triangleq \{((Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n, V^n(m_b, \tilde{m}, Y_{b,0}^{n-1}))) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P})\}. \end{aligned}$$

Lemma 2 (Joint Packing Lemma). *For each block $b \in [2 : B]$ and for each previous message m_{b-1} , if $R \leq I(V; Y, Y'|X) + I(X; Y|Y') - \delta$, then $\forall \varepsilon > 0$, $\exists N_0 \in \mathbb{N}$, s.t. $\forall n \geq N_0$,*

$$\mathbb{E}_c [\mathbb{P} (\exists \tilde{m} \in \mathcal{M}_b \setminus \{m_b\}, \text{ s.t.}, A_b(\tilde{m}) \cap B_{b-1}(\tilde{m}))] \leq \varepsilon$$

Proof. Consider $\varepsilon > 0$ that satisfies $7\varepsilon < \delta$. We have

$$R - I(V; Y, Y'|X) - I(X; Y|Y') + 6\varepsilon \leq -\delta + 6\varepsilon < -\varepsilon.$$

Then,

$$\begin{aligned} &\mathbb{E}_c [\mathbb{P} (\exists \tilde{m} \in \mathcal{M}_b \setminus \{m_b\}, \text{ s.t.}, A_b(\tilde{m}) \cap B_{b-1}(\tilde{m}))] \\ &\leq \sum_{\tilde{m} \in \mathcal{M}_b \setminus \{m_b\}} \mathbb{E}_c [\mathbb{P} (A_b(\tilde{m}) \cap B_{b-1}(\tilde{m}))] \\ &= \sum_{\tilde{m} \in \mathcal{M}_b \setminus \{m_b\}} \mathbb{E}_c \left[\sum_y \mathbb{P} (Y_{b-1,n} = Y_{b,0} = y) \cdot \right. \\ &\quad \left. \mathbb{P} (B_{b-1}(\tilde{m}) | Y_{b-1,n} = y) \mathbb{P} (A_b(\tilde{m}) | B_{b-1}(\tilde{m}), Y_{b,0} = y) \right] \\ &\stackrel{\text{(a)}}{\leq} 2^{-n(I(X; Y|Y') - 3\varepsilon)} \\ &\quad \sum_{\tilde{m} \in \mathcal{M}_b \setminus \{m_b\}} \mathbb{E}_c \left[\sum_y \mathbb{P} (y) \cdot \mathbb{P} \left(B_{b-1}(\tilde{m}) | Y_{b-1,n} = y \right) \right] \\ &\stackrel{\text{(b)}}{=} 2^{-n(I(X; Y|Y') - 3\varepsilon)} \sum_{\tilde{m} \in \mathcal{M}_b \setminus \{m_b\}} \mathbb{E}_c \left[\sum_{(y_0^{n-1}, x^n, y_1^n, v^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P})} \right. \\ &\quad \left. \mathbb{P} (Y_{b-1,0}^n = y_0^{n-1}, X_{b-1}^n = x^n, Y_{b-1,1}^n = y_1^n) \right. \\ &\quad \left. \cdot \prod_{t=1}^n \mathcal{P}_{V|X} (v_t | x_t) \right] \\ &\stackrel{\text{(c)}}{\leq} 2^{-n(I(X; Y|Y') - 3\varepsilon)} \sum_{\tilde{m} \in \mathcal{M}_b \setminus \{m_b\}} 2^{n(H(X, Y, V|Y') + \varepsilon)} \\ &\quad \cdot 2^{-n(H(X, Y|Y') - \varepsilon)} \cdot 2^{-n(H(V|X) - \varepsilon)} \\ &\leq 2^{n(R - I(V; Y, Y'|X) - I(X; Y|Y') + 6\varepsilon)} \leq 2^{-n\varepsilon} \rightarrow 0 \end{aligned}$$

where (a): we bound the term $\mathbb{P}(A_b(\tilde{m}) | B_{b-1}(\tilde{m}), Y_{b,0} = y)$ shown at the end of this session; (b): Independence of the

generation of the false codeword $V^n(m_{b-1}, \tilde{m}, y_0^{n-1})$ given x^n , where $\tilde{m} \neq m_b$; (c): come from the standard AEP and controlled Markov typical AEP in Theorem 3, and the cardinality bound $\{(y_0^{n-1}, x^n, y_1^n, v^n) \in \mathcal{T}_\varepsilon^n(\mathcal{P})\}$ extends from the argument of Theorem 3 without changing the form of the result. Lastly,

$$\begin{aligned}
& \mathbb{P}(A_b(\tilde{m}) | B_{b-1}(\tilde{m}), Y_{b,0} = y) \\
& \stackrel{(a1)}{=} \sum_{(y_0^{n-1}, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(T_q): y_0=y} \mathbb{P}(Y_0^{n-1} = y_0^{n-1}, Y_1^n = y_1^n | B_{b-1}(\tilde{m})) \\
& \quad \sum_{x^n \in \mathcal{T}_\eta(\mathcal{P}_X | y_0^n)} \mathbb{P}(X^n(\tilde{m} | y_0^{n-1}) = x^n | B_{b-1}(\tilde{m}), Y_{b,0} = y_0^n) \\
& \stackrel{(a2)}{=} \sum_{(y_0^{n-1}, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(T_q): y_0=y} \mathbb{P}(Y_0^{n-1} = y_0^{n-1}, Y_1^n = y_1^n | B_{b-1}(\tilde{m})) \\
& \quad \sum_{x^n \in \mathcal{T}_\eta(\mathcal{P}_X | y_0^n)} \prod_{t=1}^n q_{X|Y'}(x_t | y_{t-1}) \\
& \stackrel{(a3)}{\leq} \sup_{(y_0^{n-1}, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(T_q): y_0=y} \sum_{x^n \in \mathcal{T}_\eta(\mathcal{P}_X | y_0^n)} \prod_{t=1}^n q_{X|Y'}(x_t | y_{t-1}) \\
& \leq \sup_{y_0^n \in \mathcal{Y}^{n+1}} \sum_{x^n \in \mathcal{T}_\eta(\mathcal{P}_X | y_0^n)} \prod_{t=1}^n q_{X|Y'}(x_t | y_{t-1}) \\
& \stackrel{(a4)}{\leq} \sup_{y_0^n} 2^{(n(H(X|Y', Y) + \varepsilon))} \prod_{t=1}^n q_{X|Y'}(x_t | y_{t-1}) \\
& \stackrel{(a5)}{\leq} 2^{-n(I(X; Y|Y') - 2\varepsilon)},
\end{aligned}$$

where,

(a1): the conditional typical set $\mathcal{T}_\eta^{(n)}(\mathcal{P}_X | y_0^n)$ is defined according to Def. 3 for some parameter $\eta > 0$ that is chosen later in step (a4). $\mathcal{T}_\varepsilon^{(n)}(T_q)$ is the standard Markov typicality [44]–[46] with the transition matrix T_q defined as (4)
(a2): Given $\tilde{m} \in \mathcal{M}_b / \{m_b\}$, $V^n(m_{b-1}, \tilde{m}, Y_{b-1,0}^{n-1})$ in the event $B_{b-1}(\tilde{m})$ is generated according to $X^n(m_{b-1}, Y_{b-1,0}^{n-1})$ with $m_{b-1} \in \mathcal{M}_{b-1}$, where $\mathcal{M}_b$ and $\mathcal{M}_{b+1}$ are disjoint. Therefore, we have the following Markov chain, hence the conditional independence

$$X^n(\tilde{m}, Y_{b,0}^{n-1}) \ominus Y_{b,0}^n \ominus V^n(m_{b-1}, \tilde{m}, Y_{b-1,0}^{n-1}).$$

(a3): comes from the weighted sum inequality

$$\begin{aligned}
\sum_s w(s) \cdot F(s) & \leq \left(\sum_s w(s) \right) \sup_s F(s) = \sup_s F(s) \\
\text{where } \sum_s w(s) & = 1.
\end{aligned}$$

(a4): follows from cardinality bound in Theorem 3, item 3). The parameter $\eta > 0$ is chosen such that $|\mathcal{T}_\eta^{(n)}(\mathcal{P}_X | y_0^n)| \leq 2^{n(H(X|Y', Y) + \varepsilon)}$.

(a5): independent generation of a typical sequence x^n given y_0^{n-1} , which follows the standard AEP arguments [43]. $\square$

VI. CONVERSE PROOF

This converse proof is extended from the converse proof for the setup with discrete memoryless channel studied in [35]. For $n \in \mathbb{N}$, we consider a code $c \in \mathcal{C}(n)$ achieving a joint distribution $\mathcal{P}_{UY'XYV}$. Because of noncausal decoding, we have the Markov chain $V^n \ominus Y_0^n \ominus U^n$. Therefore,

$$\begin{aligned}
0 & = I(U^n; Y_0^n) - I(U^n; Y_0^n, V^n) \\
& = \sum_{t=1}^n I(U^n; Y_t | Y^{t-1}) - \sum_{t=1}^n I(U_t; Y_0^n, V^n | U^{t-1}) \\
& \stackrel{(a)}{\leq} \sum_{t=1}^n [H(Y_t | Y^{t-1}) - H(Y_t | Y^{t-1}, X_t, U^n)] \\
& \quad H(U_t) - H(U_t | Y^n, V^n, U^{t-1}, X_t) \\
& \stackrel{(b)}{=} \sum_{t=1}^n H(Y_t | Y^{t-1}) - nH(Y | X, Y') \\
& \quad - nH(U) - \sum_{t=1}^n H(U_t | Y^n, V^n, U^{t-1}, X_t) \\
& \leq n[H(Y_T | Y_{T-1}, T) - H(Y | X, Y') - H(U) \\
& \quad - H(U_T | Y_T, Y_{T-1}, V_T, X_T, T)] \\
& \stackrel{(c)}{\leq} n[I(X; Y | Y') - I(U; V | Y', X, Y)]
\end{aligned}$$

where (a): conditioning reduces entropy and strictly causal encoding; (b): the Markov channel is stationary and the source is i.i.d., here (U, Y', X, Y) are introduced as time-average random variables, and later we define V similarly; (c): concavity of the entropy.

It is easy to see that for random variables $(U, Y'X, Y, V)$, we have $\mathcal{P}_{UY'XYV} = \mathcal{P}_U \mathcal{P}_{Y'} q_{X|Y'} \mathcal{W}_{Y|XY'} \mathcal{P}_{V|UXY'Y}$. Now we show $\mathcal{P}_{Y'} = \mathcal{P}_Y = \pi_{Y'}^q$.

Since the code achieves $\mathcal{P}_{UY'XYV}$, then, for every $i \in \mathcal{Y}$, we have the marginal empirical distribution

$$\begin{aligned}
Q_{Y'}^n(i) - Q_Y^n(i) & = \frac{\mathbf{1}\{Y_0 = i\} - \mathbf{1}\{Y_n = i\}}{n} \rightarrow 0, \\
\implies \|Q_{Y'}^n - Q_Y^n\|_1 & \rightarrow 0.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\mathcal{P}_{Y'} - \mathcal{P}_Y\|_1 & \leq \mathbb{E}\|\mathcal{P}_{Y'} - Q_{Y'}^n\|_1 \\
& \quad + \mathbb{E}\|Q_{Y'}^n - Q_Y^n\|_1 + \mathbb{E}\|Q_Y^n - \mathcal{P}_Y\|_1 \rightarrow 0.
\end{aligned}$$

Therefore, $\mathcal{P}_{Y'} = \mathcal{P}_Y$. Moreover, we have

$$\begin{aligned}
\mathcal{P}_{Y'}(i) & = \sum_{j,x} \mathcal{P}_{Y'}(j) q_{X|Y'}(x|j) \mathcal{W}_{Y|XY'}(i|x, j) \\
& = \sum_{j,x} \mathcal{P}_{Y'}(j) T_q(i|j),
\end{aligned}$$

i.e., $\mathcal{P}_{Y'}$ is an invariant distribution of the Markov chain with transition matrix T_q . Since the unichain assumption guarantees the uniqueness of the invariant distribution, $\mathcal{P}_{Y'} = \pi_{Y'}^q$. $\square$

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APPENDIX A
PROPERTY OF THE CONTROLLED MARKOV TYPICALITY

Two conditional notions are useful. Given a state history y_0^{n-1} , define the policy-conditional typical set

$$\mathcal{T}_\varepsilon^{(n)}(q_{X|Y'} | y_0^{n-1}) \triangleq \{x^n : \|Q_{Y',X}^n - Q_{Y',X}^n q_{X|Y'}\|_1 \leq \varepsilon, Q_{Y',X}^n(y', x) = 0, \text{ if } q_{X|Y'}(x|y') = 0\}, \quad (14)$$

where

$$Q_{Y',X}^n(i, x) = \sum_j Q_{Y',XY}^n(i, x, j), \quad Q_{Y'}^n(i) = \sum_{x,j} Q_{Y',XY}^n(i, x, j).$$

Given x^n and y_0 , define the channel-conditional Markov typical set

$$\mathcal{T}_\varepsilon^{(n)}(\mathcal{W}_{Y|XY'} | x^n, y_0) \triangleq \{y^n : \|Q_{Y',XY}^n - Q_{Y',XY}^n \mathcal{W}_{Y|XY'}\|_1 \leq \varepsilon, Q_{Y',XY}^n(y', x, y) = 0, \text{ if } \mathcal{W}_{Y|XY'}(y|x, y') = 0\}. \quad (15)$$

Proposition 1 (Marginal and conditional consistency). *For every n and $\varepsilon > 0$:*

1) *If $(y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y',XY})$, then*

$$\|Q_{Y'}^n - \pi_{Y'}^q\|_1 \leq \varepsilon, \quad (16)$$

$$\|Q_{Y',X}^n - \pi_{Y'}^q q_{X|Y'}\|_1 \leq \varepsilon, \quad (17)$$

$$\|Q_{Y',Y}^n - \pi_{Y'}^q T_q\|_1 \leq \varepsilon, \quad (18)$$

$$\|Q_{Y',X}^n - Q_{Y',X}^n q_{X|Y'}\|_1 \leq 2\varepsilon, \quad (19)$$

$$\|Q_{Y',XY}^n - Q_{Y',X}^n \mathcal{W}_{Y|XY'}\|_1 \leq 2\varepsilon. \quad (20)$$

2) *Conversely, if*

$$\|Q_{Y'}^n - \pi_{Y'}^q\|_1 \leq \varepsilon,$$

$$\|Q_{Y',X}^n - Q_{Y',X}^n q_{X|Y'}\|_1 \leq \varepsilon,$$

$$\|Q_{Y',XY}^n - \hat{P}_{Y',X}^n \mathcal{W}_{Y|XY'}\|_1 \leq \varepsilon,$$

then

$$\|Q_{Y',XY}^n - \mathcal{P}_{Y',XY}\|_1 \leq 3\varepsilon.$$

Proof. The first three inequalities follow from marginalization, which is a contraction in ℓ_1 . Now we prove each inequality individually:

Since $(y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y',XY})$, we have

$$\begin{aligned} \varepsilon &\geq \sum_{x \in \mathcal{X}, (i,j) \in \mathcal{Y}^2} |Q_{Y',XY}^n(i, x, j) - \pi_{Y'}^q(i) q_{X|Y'}(x|i) \mathcal{W}_{Y|XY'}(j|x, i)| \\ &\geq \sum_{i \in \mathcal{Y}} \left| \sum_{x \in \mathcal{X}, j \in \mathcal{Y}} (Q_{Y',XY}^n(i, x, j) - \pi_{Y'}^q(i) q_{X|Y'}(x|i) \mathcal{W}_{Y|XY'}(j|x, i)) \right| \\ &= \sum_{i \in \mathcal{Y}} \left| Q_{Y'}^n(i) - \pi_{Y'}^q(i) \left(\sum_{x \in \mathcal{X}, j \in \mathcal{Y}} q_{X|Y'}(x|i) \mathcal{W}_{Y|XY'}(j|x, i) \right) \right| \\ &= \|Q_{Y'}^n - \pi_{Y'}^q\|_1 \end{aligned}$$

We can similarly prove eq. (17) - (18). Now

$$\|Q_{Y',X}^n - Q_{Y',X}^n q_{X|Y'}\|_1 \leq \|Q_{Y',X}^n - \pi_{Y'}^q q_{X|Y'}\|_1 + \|(\pi_{Y'}^q - Q_{Y'}^n) q_{X|Y'}\|_1 \leq 2\varepsilon$$

$$\|Q_{Y',XY}^n - Q_{Y',X}^n \mathcal{W}_{Y|XY'}\|_1 \leq \|Q_{Y',XY}^n - \pi_{Y'}^q q_{X|Y'} \mathcal{W}_{Y|XY'}\|_1 + \|(\pi_{Y'}^q - Q_{Y'}^n) q_{X|Y'} \mathcal{W}_{Y|XY'}\|_1 \leq 2\varepsilon$$

which proves (19), (20). For the converse, insert the intermediate distributions $Q_{Y',X}^n \mathcal{W}_{Y|XY'}$ and $Q_{Y',X}^n q_{X|Y'} \mathcal{W}_{Y|XY'}$, and apply the triangle inequality:

$$\begin{aligned} \|Q_{Y',XY}^n - \pi_{Y'}^q q \mathcal{W}\|_1 &\leq \|Q_{Y',XY}^n - Q_{Y',X}^n \mathcal{W}\|_1 \\ &\quad + \|Q_{Y',X}^n \mathcal{W} - Q_{Y',X}^n q \mathcal{W}\|_1 + \|Q_{Y',X}^n q \mathcal{W} - \pi_{Y'}^q q \mathcal{W}\|_1 \\ &\leq 3\varepsilon. \end{aligned}$$

□

APPENDIX B
PROOF FOR THEOREM 2

Define

$$S_t \triangleq (Y_{t-1}, X_t, Y_t).$$

Under (8), $\{S_t\}$ is a time-homogeneous Markov chain with transition kernel

$$\mathbb{P}\left(S_{t+1} = (j, x', k) \middle| S_t = (i, x, j)\right) = q_{X|Y'}(x'|j) \mathcal{W}_{Y|XY'}(k|x', j). \quad (21)$$

which does not depend on time (hence time-homogeneous). The distribution in (9) is invariant, because

$$\begin{aligned} & \sum_{i,x} \mathcal{P}_{Y'XY}(i, x, j) \mathbb{P}\left(S_{t+1} = (j, x', k) \middle| S_t = (i, x, j)\right) \\ &= \left[\sum_{i,x} \pi^q(i) q_{X|Y'}(x|i) \mathcal{W}_{Y|XY'}(j|x, i) \right] q_{X|Y'}(x'|j) \mathcal{W}_{Y|XY'}(k|x', j) \\ &= \pi^q(j) q_{X|Y'}(x'|j) \mathcal{W}_{Y|XY'}(k|x', j) \\ &= \mathcal{P}_{Y'XY}(j, x', k). \end{aligned}$$

That is to say, $\mathcal{P}_{Y'XY}(j, x', k)$ is the unique equilibrium distribution of Markov chain $\{S_t\}$. Hence, by the strong law of large numbers for finite-state unichain Markov chains, for every initial state and every (i, x, j)

$$\frac{1}{n} \sum_{t=1}^n \mathbf{1}\{S_t = (i, x, j)\} \xrightarrow{\text{a.s.}} \mathcal{P}_{Y'XY}^q(i, x, j)$$

This is exactly (10). □

APPENDIX C
PROOF FOR THEOREM 3

1) For every initial state y_0 , and every fixed pair $(y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY})$ we have that

$$\begin{aligned} & \mathbb{P}((Y_0^{n-1}, X^n, Y_1^n) = (y_0^{n-1}, x^n, y_1^n)) \\ &= \prod_{t=1}^n q_{X|Y'}(x_t|y_{t-1}) \mathcal{W}_{Y|XY'}(y_t|x_t, y_{t-1}) \\ &= \prod_{x \in \mathcal{X}, i \in \mathcal{Y}} q_{X|Y'}(x|i)^{n \cdot Q_{X,Y'}^n(x,i)} \prod_{x \in \mathcal{X}, i, j \in \mathcal{Y}} \mathcal{W}_{Y|XY'}(j|x, i)^{n \cdot Q_{Y',X,Y}^n(i,x,j)} \end{aligned}$$

Therefore

$$\begin{aligned} & -\frac{1}{n} \log \mathbb{P}((Y_0^{n-1}, X^n, Y_1^n) = (y_0^{n-1}, x^n, y_1^n)) \\ &= - \sum_{x \in \mathcal{X}, i \in \mathcal{Y}} Q_{X,Y'}^n(x, i) \log q_{X|Y'}(x|i) - \sum_{x \in \mathcal{X}, i, j \in \mathcal{Y}} Q_{Y',X,Y}^n(i, x, j) \log \mathcal{W}_{Y|XY'}(j|x, i) \\ &= - \sum_{x \in \mathcal{X}, i \in \mathcal{Y}} (Q_{X,Y'}^n(x, i) - \pi^q(i) q_{X|Y'}(x|i)) \log q_{X|Y'}(x|i) - \sum_{x \in \mathcal{X}, i \in \mathcal{Y}} \pi^q(i) q_{X|Y'}(x|i) \log q_{X|Y'}(x|i) \\ &\quad - \sum_{x \in \mathcal{X}, i, j \in \mathcal{Y}} (Q_{Y',X,Y}^n(i, x, j) - \mathcal{P}_{Y'XY}(i, x, j)) \log \mathcal{W}_{Y|XY'}(j|x, i) \\ &\quad - \sum_{x \in \mathcal{X}, i, j \in \mathcal{Y}} \mathcal{P}_{Y'XY}(i, x, j) \log \mathcal{W}_{Y|XY'}(j|x, i) \end{aligned}$$

Since the alphabets are finite, the following quantities

$$\begin{aligned} L_q &\triangleq \max_{(i,x): q(x|i) > 0} |\log q(x|i)|, \\ L_{\mathcal{W}} &\triangleq \max_{(i,x,j): \mathcal{W}(j|x,i) > 0} |\log \mathcal{W}(j|x, i)|. \end{aligned}$$

are both finite numbers. Now, choose N_0 big enough such that $(y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY})$, $\forall n < N_0$, based on Proposition 1,

$$\begin{aligned} -\varepsilon L_q &\leq \sum_{(i,x): q(x|i)>0} (Q_{X,Y'}^n(x,i) - \pi^q(i)q_{X|Y'}(x|i)) \log q_{X|Y'}(x|i) \leq \varepsilon L_q \\ -\varepsilon L_W &\leq \sum_{(i,x,j): \mathcal{W}(j|x,i)>0} (Q_{Y',XY}^n(i,x,j) - \mathcal{P}_{Y'XY}(i,x,j)) \log \mathcal{W}_{Y|XY'}(j|x,i) \leq \varepsilon L_W, \\ \forall n &> N_0. \end{aligned}$$

For each fixed δ , choose ε_0 small enough such that

$$\varepsilon_0(L_q + L_W) \leq \delta$$

Therefore, we conclude that

$$-\delta + H(X, Y|Y') \leq -\frac{1}{n} \log \mathbb{P}((X^n, Y^n) = (x^n, y^n)) \leq \delta + H(X, Y|Y').$$

Hence,

$$2^{-n(H(X,Y|Y')+\delta)} \leq \mathbb{P}((X^n, Y^n) = (x^n, y^n)) \leq 2^{-n(H(X,Y|Y')-\delta)} \quad (22)$$

2) Sum the lower bound in the above inequality over all typical pairs with the same initial state y_0 ; their total probability is at most one. This gives the cardinality upperbound in Theorem 3.

3) The proof extends the standard type counting argument of [47, Lemma 2.3] to controlled Markov typicality:

First, we upperbound the number of elements in $\mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) = \{x^n : (y_0^{n-1}, x^n, y_1^n) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_{Y'XY})\}$: Given a fixed path y_0^n of length $n+1$, for each possible neighboring state pair $(i, j) \in \mathcal{Y}^2$, denote by

$$N_{i,j} \triangleq |\{t : (y_{t-1}, y_t) = (i, j)\}|$$

To reach the desired joint empirical distribution $Q_{Y',XY}^n(i, x, j)$, we need to assign input symbols $x \in \mathcal{X}$ to the positions $\{t : (y_{t-1}, y_t) = (i, j)\}$ such that

$$N_{i,x,j} = |\{t : (y_{t-1}, x_t, y_t) = (i, x, j)\}| = n \cdot Q_{Y',XY}^n(i, x, j)$$

number of them receive symbol x . Moreover, we need

$$\sum_{x \in \mathcal{X}} N_{i,x,j} = N_{i,j}$$

Therefore, for a fixed state pair (i, j) , the number of assignment is

$$\binom{N_{i,j}}{(N_{i,x,j})_{x \in \mathcal{X}}} = \frac{N_{i,j}!}{\prod_x N_{i,x,j}!}$$

Using the multinomial bound, we have

$$\frac{N!}{\prod_x N_x!} \leq 2^{NH(p)}, \text{ where } H(p) \text{ is the entropy for } p(x) = \frac{N_x}{N}$$

Now, for a fixed sequence y_0^n , denote the conditional typical set such that the actual induced joint empirical distribution *equals* to a target distribution Q is

$$\mathcal{A}_Q^{(n)}(\mathcal{P}_X|y_0^n) = \{x^n \in \mathcal{X}^n : Q_{Y',XY}^n = Q\}.$$

Therefore, the number of compatible input sequences that induce the target type Q is

$$\begin{aligned} \left| \mathcal{A}_Q^{(n)}(\mathcal{P}_X|y_0^n) \right| &= \prod_{i,j} \frac{N_{i,j}!}{\prod_x N_{i,x,j}!} \\ &\leq \prod_{i,j} 2^{N_{i,j} H_Q(X|Y'=i, Y=j)} \\ &= 2^{\sum_{i,j} N_{i,j} H_Q(X|Y'=i, Y=j)} \end{aligned}$$

$$= 2^{nH_Q(X|Y'Y)}$$

In this way, compared to the definition of the conditional ε -type set, we have

$$\begin{aligned} \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) &= \{x^n \in \mathcal{X}^n : \|Q - \mathcal{P}_{Y'XY}\|_1 \leq \varepsilon\} \\ &= \bigcup_{Q: \|Q - \mathcal{P}_{Y'XY}\|_1 \leq \varepsilon} \mathcal{A}_Q^{(n)}(\mathcal{P}_X|y_0^n) \end{aligned}$$

Therefore, by choosing ε small enough such that

$$H_Q(X|Y', Y) \leq H(X|Y', Y) + \frac{\delta}{2}$$

where the second entropy is evaluated with respect to the true target joint distribution $\mathcal{P}_{Y'XY}$, we obtain

$$\begin{aligned} \left| \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) \right| &\leq \sum_{Q: \|Q - \mathcal{P}_{Y'XY}\|_1 \leq \varepsilon} 2^{nH_Q(X|Y'Y)} \\ &\leq (n+1)^{|\mathcal{Y}|^2 \times |\mathcal{X}|} 2^{n(H(X|Y', Y) + \frac{\delta}{2})} \\ &\leq 2^{n(H(X|Y', Y) + \delta)} \end{aligned}$$

where the last equality holds by choosing sufficiently large N_0 such that $\forall n \geq N_0$, $(n+1)^{|\mathcal{Y}|^2 \times |\mathcal{X}|} \leq 2^{n\delta/2}$.

Now suppose $\mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) \neq \emptyset$, we lowerbound the cardinality of the conditional typical set by considering only one fixed candidate distribution $Q : \|Q - \mathcal{P}_{Y'XY}\|_1 \leq \varepsilon$. Hence we have

$$\begin{aligned} \left| \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) \right| &\geq \left| \mathcal{A}_Q^{(n)}(\mathcal{P}_X|y_0^n) \right| \\ &= \prod_{i,j} \frac{N_{i,j}!}{\prod_x N_{i,x,j}!} \\ &\geq \prod_{i,j} (N_{i,j} + 1)^{-|\mathcal{X}|} 2^{N_{i,j} H_Q(X|Y'=i, Y=j)} \\ &\geq (n+1)^{-|\mathcal{Y}|^2 |\mathcal{X}|} 2^{nH_Q(X|Y', Y)}, \end{aligned}$$

where in the second inequality we are using the multinomial lower bound

$$\frac{N!}{\prod_x N_x!} \geq (N+1)^{-|\mathcal{X}|} 2^{NH(p)}, \quad p(x) = \frac{N_x}{N},$$

and pairs with $N_{i,j} = 0$ contribute a factor of one.

For any $\delta > 0$, continuity of conditional entropy allows us to choose $\varepsilon > 0$ sufficiently small that

$$\|Q - \mathcal{P}_{Y'XY}\|_1 \leq \varepsilon \implies H_Q(X|Y', Y) \geq H(X|Y', Y) - \frac{\delta}{2},$$

where $H(X|Y', Y)$ is evaluated under $\mathcal{P}_{Y'XY}$. Consequently,

$$\begin{aligned} \left| \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_X|y_0^n) \right| &\geq |\mathcal{A}_Q^{(n)}(y_0^n)| \\ &\geq (n+1)^{-|\mathcal{Y}|^2 |\mathcal{X}|} 2^{n(H(X|Y', Y) - \delta/2)} \\ &\geq 2^{n(H(X|Y', Y) - \delta)}, \end{aligned}$$

where the last inequality holds for all $n \geq N_0$, with N_0 chosen sufficiently large that $(n+1)^{|\mathcal{Y}|^2 |\mathcal{X}|} \leq 2^{n\delta/2}$. □

APPENDIX D

PROOF OF THE COVERING LEMMA FOR THE ERROR ANALYSIS

Now, given the term $\mathbb{E}_c \left[\mathbb{P} \left(\left(U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n \right) \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \right) \right]$ happens with low probability, we have

$$\begin{aligned} &\mathbb{E}_c \left[\mathbb{P} \left(\forall m \in \mathcal{M}_{b+1}, \left(U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n, V^n(m_b, m, Y_{b,0}^{n-1}) \right) \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \right) \right] \\ &= \mathbb{E}_c \left[\sum_{u^n, y_0^{n-1}, x^n, y_1^n} \mathbb{P} \left(U_b^n = u^n, Y_{b,0}^{n-1} = y_0^{n-1}, X_b^n = x^n, Y_{b,1}^n = y_1^n \right) \right] \end{aligned}$$

$$\begin{aligned}
& \mathbb{P} \left(\forall m \in \mathcal{M}_{b+1}, (u^n, y_0^{n-1}, x^n, y_1^n, V^n(m_b, m, y_0^{n-1})) \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \middle| U_b^n = u^n, Y_{b,0}^{n-1} = y_0^{n-1}, X_b^n = x^n, Y_{b,1}^n = y_1^n \right) \\
& \stackrel{(a)}{\leq} \mathbb{E}_c \left[\sum_{z^n \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)} \mathbb{P}(Z^n = z^n) \prod_{m=1}^{K_n} \mathbb{P} \left((z^n, V^n(m_b, m, y_0^{n-1})) \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \middle| Z^n = z^n \right) + \mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \\
& = \mathbb{E}_c \left[\sum_{z^n \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)} \mathbb{P}(Z^n = z^n) \prod_{m=1}^{K_n} \left(1 - \mathbb{P} \left((z^n, V^n(m_b, m, y_0^{n-1})) \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}) \middle| Z^n = z^n \right) \right) + \mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \\
& \stackrel{(b)}{\leq} \mathbb{E}_c \left[\sum_{z^n \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)} \mathbb{P}(Z^n = z^n) \prod_{m=1}^{K_n} \left(1 - 2^{-n(I(V;U,Y',Y|X)+\delta)} \right) + \mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \\
& \leq \left(1 - 2^{-n(I(V;U,Y',Y|X)+\delta)} \right)^{2^{nR}} + \mathbb{E}_c \left[\mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \\
& \stackrel{(c)}{\leq} \exp \left(-2^{nR} 2^{-n(I(V;U,Y',Y|X)+\delta)} \right) + \mathbb{E}_c \left[\mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \\
& = \exp \left(-2^{n(R-I(V;U,Y',Y|X)-\delta)} \right) + \mathbb{E}_c \left[\mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \\
& \leq \frac{4\varepsilon}{9}.
\end{aligned}$$

Here in (a), we define $Z^n = (U_b^n, Y_{b,0}^{n-1}, X_b^n, Y_{b,1}^n)$ for simplicity, and $\mathcal{P}_Z = \mathcal{P}_{U\pi_{Y'}^q, q_{X|Y'}, \mathcal{W}_{Y|XY'}}$; and in (b), given $z^n \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)$, extending the cardinality lowerbound argument of Theorem 3, item 3), the inequality comes from the Joint Typicality Lemma in [43, Chapter 2.5.1] given that $z^n \in \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)$, and $V^n \sim \prod_{t=1}^n \mathcal{P}_{V|X}(v_t|x_t)$; (c) comes from $(1-x)^y \leq \exp(-xy)$, see in [48, Lemma 10.5.3]. Therefore, for $R \geq I(V;U,Y',Y|X) + \delta$, and for sufficiently large n such that $\mathbb{E}_c \left[\mathbb{P}(Z^n \notin \mathcal{T}_\varepsilon^{(n)}(\mathcal{P}_Z)) \right] \leq \frac{2\varepsilon}{9}$, the whole quantity above could be made smaller than $\frac{4\varepsilon}{9}$. $\square$