THE DYNAMICAL MORDELL-LANG CONJECTURE

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ABSTRACT. We prove a special case of a dynamical analogue of the classical Mordell-Lang conjecture. In particular, let φ be a rational function with no superattracting periodic points other than exceptional points. If the coefficients of φ are algebraic, we show that the orbit of a point outside the union of proper preperiodic subvarieties of $(\mathbb{P}^1)^g$ has only finite intersection with any curve contained in $(\mathbb{P}^1)^g$. We also show that our result holds for indecomposable polynomials φ with coefficients in $\mathbb{C}$. Our proof uses results from p-adic dynamics together with an integrality argument. The extension to polynomials defined over $\mathbb{C}$ uses the method of specializations coupled with some new results of Medvedev and Scanlon for describing the periodic plane curves under the action of (φ, φ) on $\mathbb{A}^2$.

1. INTRODUCTION

Let X be a variety over the complex numbers $\mathbb{C}$, let $\Phi : X \longrightarrow X$ be a morphism, and let V be a subvariety of X. For any integer $m \geq 0$, denote by Φ^m the m^{th} iterate $\Phi \circ \cdots \circ \Phi$. If $\alpha \in X(\mathbb{C})$ has the property that there is some integer $\ell \geq 0$ such that $\Phi^\ell(\alpha) \in W(\mathbb{C})$, where W is a periodic subvariety of V, then there are infinitely many integers $n \geq 0$ such that $\Phi^n(\alpha) \in V$. More precisely, if $k \geq 1$ is the period of W (the smallest positive integer m for which $\Phi^m(W) = W$), then $\Phi^{nk+\ell}(\alpha) \in W(\mathbb{C}) \subseteq V(\mathbb{C})$ for all integers $n \geq 0$. It is natural then to pose the following question: given $\alpha \in X(\mathbb{C})$, if there are infinitely many integers $m \geq 0$ such that $\Phi^m(\alpha) \in V(\mathbb{C})$, are there necessarily integers $k \geq 1$ and $\ell \geq 0$ such that $\Phi^{nk+\ell}(\alpha) \in V(\mathbb{C})$ for all integers $n \geq 0$?

This question has a positive answer in many special cases. When X is a semiabelian variety and Φ is a multiplication-by-m map, this follows from Faltings' [Fal94] and Vojta's proof [Voj96] of the Mordell-Lang conjecture in characteristic 0. More generally, the question has a positive answer when Φ is any endomorphism of a semiabelian variety (see [GT]). Denis [Den94] treated the general question under the additional hypothesis that the integers n for which $\Phi^n(\alpha) \in V(\mathbb{C})$ are sufficiently dense in the set of all positive integers; he also obtained results for automorphisms of projective

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space without using this additional hypothesis. Bell [Bel06] later solved the problem completely in the case of automorphisms of affine space. In [GT], a general framework for attacking the problem was developed and the following conjecture was made.

Conjecture 1.1. *Let $f_1, \dots, f_g \in \mathbb{C}[t]$ be polynomials, let Φ be their action coordinatewise on $\mathbb{A}^g$, let $\mathcal{O}_\Phi((x_1, \dots, x_g))$ denote the Φ-orbit of the point $(x_1, \dots, x_g) \in \mathbb{A}^g(\mathbb{C})$, and let V be a subvariety of $\mathbb{A}^g$. Then V intersects $\mathcal{O}_\Phi((x_1, \dots, x_g))$ in at most a finite union of orbits of the form $\mathcal{O}_{\Phi^k}(\Phi^\ell(x_1, \dots, x_g))$, for some nonnegative integers k and ℓ.*

See Section 2 for the definition of the orbit $\mathcal{O}_\Phi(\alpha)$. Note that the orbits for which $k = 0$ are singletons, so that the conjecture allows not only infinite forward orbits but also finitely many extra points.

Note also that if Conjecture 1.1 holds for a given map Φ, variety V, and non-preperiodic point $\alpha = (x_1, \dots, x_g)$, and if V intersects the Φ-orbit of α in infinitely many points, then V must contain a positive-dimensional subvariety V_0 that is periodic under Φ. Indeed, the conjecture says that there are integers $k \geq 1$ and $\ell \geq 0$ such that $\Phi^{nk+\ell}(\alpha)$ lies on V for all $n \geq 0$. Since α is not preperiodic, the set $S = \{\Phi^{nk+\ell}(\alpha)\}_{n\geq 0}$ is infinite, and therefore its Zariski closure V_0' contains positive-dimensional components. Thus, if we let V_0 be the union of the positive-dimensional irreducible subvarieties of V_0', then V_0 is positive-dimensional and fixed by Φ^k, as claimed.

Conjecture 1.1 fits into Zhang's far-reaching system of dynamical conjectures [Zha06]. Zhang's conjectures include dynamical analogues of the Manin-Mumford and Bogomolov conjectures for abelian varieties (now theorems of Raynaud [Ray83a, Ray83b], Ullmo [Ull98], and Zhang [Zha98]), as well as a conjecture about the Zariski density of orbits of points under fairly general maps from a projective variety to itself. This latter conjecture of Zhang takes the following form in the case of polynomial actions on $\mathbb{A}^g$.

Conjecture 1.2. *Let $f_1, \dots, f_g \in \overline{\mathbb{Q}}[t]$ be polynomials of the same degree $d \geq 2$, and let Φ be their action coordinatewise on $\mathbb{A}^g$. Then there is a point $(x_1, \dots, x_g) \in \mathbb{A}^g(\overline{\mathbb{Q}})$ such that $\mathcal{O}_\Phi((x_1, \dots, x_g))$ is Zariski dense in $\mathbb{A}^g$.*

Conjectures 1.2 and 1.1 may be thought of as complementary. Conjecture 1.2 posits that there is a point in $\mathbb{A}^g$ outside the union of the preperiodic proper subvarieties of $\mathbb{A}^g$ under the action of Φ, while Conjecture 1.1 asserts if a point α lies outside this union of preperiodic subvarieties, then the orbit of α under Φ intersects any subvariety V of $\mathbb{A}^g$ in at most finitely many points. We view our Conjecture 1.1 as an analogue of the classical Mordell-Lang conjecture for arithmetic dynamics where groups of rank one are replaced by single orbits. We also note that a stronger form of Conjecture 1.2 was proved in [MS, Theorem 5.11].

In this paper, we prove Conjecture 1.1 over number fields for curves embedded in $\mathbb{A}^g$ under the diagonal action of any polynomial which has no periodic superattracting points. (Roughly speaking, a superattracting periodic point is a periodic point at which the derivative vanishes; for a formal definition of superattracting points, see Section 2.) In fact, we prove the following more general statement.

Theorem 1.3. *Let $C \subset (\mathbb{P}^1)^g$ be a curve defined over $\overline{\mathbb{Q}}$, and let $\Phi := (\varphi, \dots, \varphi)$ act on $(\mathbb{P}^1)^g$ coordinatewise, where $\varphi \in \overline{\mathbb{Q}}(t)$ is a rational function with no periodic superattracting points other than exceptional points. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \dots, x_g) \in (\mathbb{P}^1)^g(\overline{\mathbb{Q}})$. Then $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \dots, x_g)\}_{n\geq 0}$ for nonnegative integers k and ℓ.*

See Section 2 for a definition of exceptional points.

Using recent results of Medvedev and Scanlon [MS] from model theory and polynomial decomposition, we will extend Theorem 1.3 to the complex numbers, at least under the action of indecomposable polynomials. (See Definition 7.1.) Our method from Section 7 also extends to the case of any polynomials with complex coefficients, as long as they do not have periodic superattracting points other than exceptional points (see Remark 7.10).

Theorem 1.4. *Let $\varphi \in \mathbb{C}[t]$ be an indecomposable polynomial with no periodic superattracting points other than exceptional points, and let Φ be its diagonal action on $\mathbb{A}^g$ (for some $g \geq 1$). Let $\mathcal{O}$ be the Φ-orbit of a point P in $\mathbb{A}^g(\mathbb{C})$, and let C be a curve defined over $\mathbb{C}$. Then $C(\mathbb{C}) \cap \mathcal{O}$ is at most a finite union of orbits of the form $\{\Phi^{nk+\ell}(P)\}_{n\geq 0}$, for some nonnegative integers k and ℓ.*

When the function φ is a quadratic polynomial, we can prove a similar result for subvarieties of any dimension.

Theorem 1.5. *Let $V \subset (\mathbb{P}^1)^g$ be a subvariety defined over $\overline{\mathbb{Q}}$, and let $\Phi := (f, \dots, f)$ act on $(\mathbb{P}^1)^g$ coordinatewise, where $f \in \overline{\mathbb{Q}}[t]$ is a quadratic polynomial with no periodic superattracting points in $\overline{\mathbb{Q}}$. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \dots, x_g) \in (\mathbb{P}^1)^g(\overline{\mathbb{Q}})$. Then $V(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \dots, x_g)\}_{n\geq 0}$ for nonnegative integers k and ℓ.*

For quadratic polynomials over the rational numbers, we can remove the hypothesis on superattracting points and obtain a stronger result.

Theorem 1.6. *Let $V \subset (\mathbb{P}^1)^g$ be a subvariety defined over $\mathbb{Q}$, and let $\Phi := (f, \dots, f)$ act on $(\mathbb{P}^1)^g$ coordinatewise, where $f \in \mathbb{Q}[t]$ is a quadratic polynomial. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \dots, x_g) \in (\mathbb{P}^1)^g(\mathbb{Q})$. Then $V(\mathbb{Q}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \dots, x_g)\}_{n\geq 0}$ for nonnegative integers k and ℓ.*

Using results of Jones [Jon08], we can prove the corresponding result for maps of the form $\Phi = (f_1, \dots, f_g)$, without the restriction that $f_i = f_j$, if each f_j is of the form $f_j(t) = t^2 + c_j$ with $c_j \in \mathbb{Z}$.

Theorem 1.7. *Let $V \subset \left(\mathbb{P}^1\right)^g$ be a subvariety defined over $\mathbb{Q}$, and let $\Phi := (f_1, \dots, f_g)$ act on $\mathbb{A}^g$ coordinatewise, where $f_i(t) = t^2 + c_i$ with $c_i \in \mathbb{Z}$ for each i. Let $\mathcal{O}$ be the Φ-orbit of a point $(x_1, \dots, x_g) \in (\mathbb{Z})^g$. Then $V(\mathbb{Q}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_1, \dots, x_g)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

The strategy used for proving the theorems above involves an interplay between arithmetic geometry and p-adic dynamics, and it is based in part on a non-linear analog of the technique used by Skolem [Sko34] (and later extended by Mahler [Mah35] and Lech [Lec53]) to treat linear recurrence sequences. However, unlike in the linear recurrence case (where all but finitely many p-adic absolute values will work), finding a suitable prime p is far more difficult and involves using intersection theory on $\mathbb{P}^1 \times \mathbb{P}^1$ together with an application of the classical Siegel's theorem (see Section 4.) A further complication is the problem of finding fixed points around which the dynamics can be linearized; instead, we invoke the work of Rivera-Letelier [RL03] from p-adic dynamics, as described in Section 3. More precisely, we find arithmetic progression $\mathcal{S}$ of integers such that there are infinitely many $m \in \mathcal{S}$ with $\Phi^m(\alpha)$ lying on V, and then we construct a p-adic analytic map θ sending $\mathcal{S}$ into $\mathbb{A}^g(\mathbb{C}_p)$ such that $\theta(m) = \Phi^m(x_1, \dots, x_g)$ for each integer m in the sequence. Then, for any polynomial F that vanishes on V, we have $F(\theta(k)) = 0$ for infinitely many k. Since the zeros of a nonzero p-adic analytic function are isolated, $F \circ \theta$ must vanish at *all* k in the sequence. Rivera-Letelier's results, which are used in the construction of θ, apply whenever there is a positive integer ℓ such that $\varphi^\ell(x_i)$ is in a p-adic quasiperiodicity disk for each i. (A quasiperiodicity disk is a periodic residue class on which the derivative has absolute value equal to one; see Section 3 for a formal definition.) However, one cannot expect every place to admit such an integer ℓ, but in our case the above mentioned diophantine techniques can be used to show that at least one such place exists.

We note that the Skolem-Mahler-Lech technique has played a role in other work done on this subject. Bell's [Bel06] and Denis's [Den94] work on automorphisms may be viewed as algebro-geometric realizations of the Skolem-Mahler-Lech theorem. Evertse, Schlickewei, and Schmidt [ESS02] have given a strong quantitative version of the Skolem-Mahler-Lech theorem. It may be possible to use their result to give more precise versions of the theorems of this paper.

We remark that Conjecture 1.1 has been proved (cf. [GTZ08]) by conceptually quite different methods in the special case that $g = 2$ and V is a line in $\mathbb{A}^2$. However, the methods used there (involving Ritt's classification for functional decomposition of complex polynomials) do not appear to work for more general subvarieties of affine space.

We exclude the case that the rational function φ has superattracting points because we have been unable, thus far, to extend the method of

Skolem-Mahler-Lech to this situation. Although there is a logarithm associated to a φ (see [GT]) in a neighborhood of a superattracting point, it does not have the required properties that Rivera-Letelier's logarithms for quasiperiodicity disks [RL03] provide. It should not be surprising that superattracting points pose difficulties; while they are relatively simple dynamically, they cause ramification issues in a diophantine context. In the cases of endomorphisms of semiabelian varieties (see [Voj96, Fal94, GT]) and of automorphisms of affine space (see [Den94, Bel06]), the underlying maps are étale and hence have no ramification. In fact, it is possible to prove a very general dynamical Mordell-Lang theorem for unramified maps (see [BGT08]) without using the techniques from diophantine approximation that appear in Sections 4 and 6. Thus, at this point, it seems the main obstacle to proving Conjecture 1.1 is overcoming the difficulties that ramification presents. When φ has no superattracting points, the ramification indices of φ^n remain bounded for all n; this fact plays an important role in Section 4. However, when φ has a superattracting point, these indices may become arbitrarily large. Hence, the ramification of the iterates of φ is more complicated when φ has a superattracting point.

In general, we believe that there should be a broader *Mordell-Lang principle* which holds for any sufficiently *rigid* space X (i.e. the space does not have a large set of endomorphisms). This principle would say that any definable subset of X (in the sense of model theory; for algebraic geometry, the definable sets are algebraic varieties) intersects the orbit of a point $P \in X$ under an endomorphism Φ of X in at most finitely many orbits of the form $\{\Phi^{nk+\ell}(P)\}_{n\geq 0}$, for some nonnegative integers k and ℓ. If X is a semiabelian variety, the above principle can be found at the heart of the classical Mordell-Lang conjecture (see [GT]). If X is $\mathbb{A}^g$ under the action of polynomial maps f_i on each coordinate, then we recover our Conjecture 1.1. Note that in either case, X has few endomorphisms. If X is semiabelian, then $\mathrm{End}(X)$ is a finitely generated integral extension of $\mathbb{Z}$. Similarly, if X is $\mathbb{A}^g$ under a coordinatewise polynomial action, then $(H_1, \ldots, H_g)$ is an endomorphism if and only if $H_i \circ f_i = f_i \circ H_i$ for each i, which typically implies that H_i and f_i have a common iterate. (See the extensive work on this subject by Fatou [Fat21, Fat23], Julia [Jul22], Eremenko [Ere90], among many others).

The outline of our paper is as follows. In Section 2 we introduce our notation. Sections 3 and 4 provide necessary lemmas from p-adic dynamics and from intersection theory on arithmetic surfaces. In Section 5, we prove Theorem 1.3. In Section 6, we use the results of Section 5 to prove Theorems 1.5, 1.6. and 1.7. Finally, in Section 7, we describe the results of [MS] and use them to deduce Theorem 1.4.

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2. Notation

We write $\mathbb{N}$ for the set of nonnegative integers. If K is a field, we write $\overline{K}$ for an algebraic closure of K. Given a prime number p, the field $\mathbb{C}_p$ will denote the completion of an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, the field of p-adic rationals. We denote by $|\cdot|:=|\cdot|_p$ the usual absolute value on $\mathbb{C}_p$. Given $a \in \mathbb{C}_p$ and $r>0$, we write $D(a,r)$ and $\overline{D}(a,r)$ for the open disk and closed disk (respectively) of radius r centered at a.

If K is a number field, we let $\mathfrak{o}_K$ be its ring of algebraic integers, and we fix an isomorphism π between $\mathbb{P}^1_K$ and the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$. For each nonarchimedean place v of K, we let k_v be the residue field of K at v, and for each $x \in \mathbb{P}^1(K)$, we let $x_v := r_v(x)$ be the intersection of the Zariski closure of $\pi(x)$ with the fibre above v of $\mathbb{P}^1_{\mathfrak{o}_K}$. (Intuitively, x_v is x modulo v.) This map $r_v : \mathbb{P}^1(K) \longrightarrow \mathbb{P}^1(k_v)$ is the *reduction map* at v.

If $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ is a morphism defined over the field K, then (fixing a choice of homogeneous coordinates) there are relatively prime homogeneous polynomials $F, G \in K[X,Y]$ of the same degree $d = \deg \varphi$ such that $\varphi([X,Y]) = [F(X,Y) : G(X,Y)]$. (In affine coordinates, $\varphi(t) = F(t,1)/G(t,1) \in K(t)$ is a rational function in one variable.) Note that by our choice of coordinates, F and G are uniquely defined up to a nonzero constant multiple. We will need the notion of good reduction of φ, first introduced by Morton and Silverman in [MS94].

Definition 2.1. *Let K be a field, let v be a nonarchimedean valuation on K, let $\mathfrak{o}_v$ be the ring of v-adic integers of K, and let k_v be the residue field at v. Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism over K, given by $\varphi([X,Y]) = [F(X,Y) : G(X,Y)]$, where $F, G \in \mathfrak{o}_v[X,Y]$ are relatively prime homogeneous polynomials of the same degree such that at least one coefficient of F or G is a unit in $\mathfrak{o}_v$. Let $\varphi_v := [F_v, G_v]$, where $F_v, G_v \in k_v[X,Y]$ are the reductions of F and G modulo v. We say that φ has* good reduction *at v if $\varphi_v : \mathbb{P}^1(k_v) \longrightarrow \mathbb{P}^1(k_v)$ is a morphism of the same degree as φ.*

If $\varphi \in K[t]$ is a polynomial, we can give the following elementary criterion for good reduction: φ has good reduction at v if and only if all coefficients of φ are v-adic integers, and its leading coefficient is a v-adic unit.

Definition 2.2. *Two rational functions φ and ψ are* conjugate *if there is a linear fractional transformation μ such that $\varphi = \mu^{-1} \circ \psi \circ \mu$.*

In the above definition, if φ and ψ are polynomials, then we may assume that μ is a polynomial of degree one.

Definition 2.3. *If K is a field, and $\varphi \in K(t)$ is a rational function, then $z \in \mathbb{P}^1(\overline{K})$ is a* periodic point *for φ if there exists an integer $n \geq 1$ such that*

$\varphi^n(z) = z$. *The smallest such integer n is the* period *of z, and* $\lambda = (\varphi^n)'(z)$ *is the* multiplier *of z. If* $\lambda = 0$*, then z is called* superattracting. *If* $|\cdot|_v$ *is an absolute value on K, and if* $|\lambda|_v < 1$*, then z is called* attracting.

If z is a periodic point of $\varphi = \mu^{-1} \circ \psi \circ \mu$, then $\mu(z)$ is a periodic point of ψ with the same multiplier. In particular, we can define the multiplier of a periodic point at $z = \infty$ by changing coordinates.

Whether or not z is periodic, we say z is a *ramification point* or *critical point* of φ if $\varphi'(z) = 0$. If $\varphi = \mu^{-1} \circ \psi \circ \mu$, then z is a critical point of φ if and only if $\mu(z)$ is a critical point of ψ; in particular, coordinate change can again be used to determine whether $z = \infty$ is a critical point. Note that a periodic point z is superattracting if and only if at least one of $z, \varphi(z), \varphi^2(z), \ldots, \varphi^{n-1}(z)$ is critical, where n is the period of z.

Let $\varphi : V \longrightarrow V$ be a map from a variety to itself, and let $z \in V(\overline{K})$. The (*forward*) *orbit* $\mathcal{O}_\varphi(z)$ of z under φ is the set $\{\varphi^k(z) : k \in \mathbb{N}\}$. We say z is *preperiodic* if $\mathcal{O}_\varphi(z)$ is finite. If μ is an automorphism of V, and if $\varphi = \mu^{-1} \circ \psi \circ \mu$, note that $\mathcal{O}_\varphi(z) = \mu^{-1}(\mathcal{O}_\psi(\mu(z)))$.

We say z is *exceptional* (or *totally invariant*) if there are only finitely many points w such that $z \in \mathcal{O}_\varphi(w)$ (i.e. the backward orbit of z contains only finitely many points). It is a classical result in dynamics (e.g., see [Bea91], Theorem 4.1.2) that a morphism $\varphi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree larger than one has at most two exceptional points. Moreover, it has exactly two if and only if φ is conjugate to the map $t \mapsto t^n$, for some integer $n \in \mathbb{Z}$; and it has exactly one if and only if φ is conjugate to a polynomial but not to any map $t \mapsto t^n$. In particular, φ has at least one exceptional point if and only if φ^2 is conjugate to a polynomial.

3. Quasiperiodicity disks in p-adic dynamics

As in [GT], we will need a result on non-preperiodic points over local fields. By an *open disk in* $\mathbb{P}^1(\mathbb{C}_p)$, we will mean either an open disk in $\mathbb{C}_p$ or the complement (in $\mathbb{P}^1(\mathbb{C}_p)$) of a closed disk in $\mathbb{C}_p$. Equivalently, an open disk in $\mathbb{P}^1(\mathbb{C}_p)$ is the image of an open disk $D(0,r) \subseteq \mathbb{C}_p$ under a linear fractional transformation $\gamma \in \mathrm{PGL}(2, \mathbb{C}_p)$. Closed disks are defined similarly.

The following definition is borrowed from [RL03, Section 3.2], although we have used a simpler version that suffices for our purposes.

Definition 3.1. *Let p be a prime, let* $r > 0$*, let* $\gamma \in \mathrm{PGL}(2, \mathbb{C}_p)$*, and let* $U = \gamma(D(0,r))$*. Let* $f : U \to U$ *be a function such that*

$$\gamma^{-1} \circ f \circ \gamma(t) = \sum_{i \geq 0} c_i t^i \in \mathbb{C}_p[[t]],$$

with $|c_0| < r$*,* $|c_1| = 1$*, and* $|c_i| r^i \leq r$ *for all* $i \geq 1$*. Then we say U is a* quasiperiodicity disk *for f.*

The conditions on f in Definition 3.1 mean precisely that f is rigid analytic and maps U bijectively onto U. In particular, the preperiodic points of f in U are in fact periodic. By [RL03, Corollaire 3.12], our definition implies that U is indeed a quasiperiodicity domain of f in the sense of [RL03, Définition 3.7].

The main result of this section is the following.

Theorem 3.2. *Let p be a prime and $g \geq 1$. For each $i = 1, \ldots, g$, let U_i be an open disk in $\mathbb{P}^1(\mathbb{C}_p)$, and let $f_i : U_i \to U_i$ be a map for which U_i is a quasiperiodicity disk. Let Φ denote the action of $f_1 \times \cdots \times f_g$ on $U_1 \times \cdots \times U_g$, let $\alpha = (x_1, \ldots, x_g) \in U_1 \times \cdots \times U_g$ be a point, and let $\mathcal{O}$ be the Φ-orbit of α. Let V be a subvariety of $(\mathbb{P}^1)^g$ defined over $\mathbb{C}_p$. Then $V(\mathbb{C}_p) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(\alpha)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

The proof of Theorem 3.2 relies on the following lemma from p-adic dynamics, which in turn follows from the theory of quasiperiodicity domains in [RL03, Section 3.2].

Lemma 3.3. *Let $U \subseteq \mathbb{C}_p$ be an open disk, let $f : U \to U$ be a map for which U is a quasiperiodicity disk, and let $x \in U$ be a non-periodic point. Then there exist an integer $k \geq 1$, radii $r > 0$ and $s \geq |k|_p$, and, for every integer $\ell \geq 0$, a bijective rigid analytic function $h_\ell : \overline{D}(0, s) \to \overline{D}(f^\ell(x), r)$, with the following properties:*

(i) *$h_\ell(0) = f^\ell(x)$, and*

(ii) *for all $z \in \overline{D}(f^\ell(x), r)$ and $n \geq 0$, we have*

$$f^{nk}(z) = h_\ell(nk + h_\ell^{-1}(z)).$$

Proof. Write $U = D(a, R)$. By [RL03, Proposition 3.16(2)], there is an integer $k \geq 1$ and a neighborhood $U_x \subseteq U$ of x on which f^k is (analytically and bijectively) conjugate to $t \mapsto t+k$. That is, there are radii $r, s > 0$ (with $r < R$ and $s \geq |k|_p$) and a bijective analytic function $h_0 : \overline{D}(0, s) \to \overline{D}(x, r)$ such that $f^{nk}(z) = h_0(nk + h_0^{-1}(z))$ for all $z \in \overline{D}(x, r)$ and $n \geq 0$.

For each nonnegative integer ℓ, note that f^ℓ is a bijective analytic function from $\overline{D}(x, r)$ onto $\overline{D}(f^\ell(x), r)$. Thus, if we let $h_\ell := f^\ell \circ h_0$, then h_ℓ is a bijective analytic function from $\overline{D}(0, s)$ onto $\overline{D}(f^\ell(x), r)$. Moreover, for all $z \in \overline{D}(f^\ell(x), r)$, if we let $\zeta = f^{-\ell}(z) \in \overline{D}(x, r)$, then for every $n \geq 0$,

$$f^{nk}(z) = f^\ell(f^{nk}(\zeta)) = f^\ell(h_0(nk + h_0^{-1}(\zeta))) = h_\ell(nk + h_\ell^{-1}(z)).$$

Finally, replacing $h_\ell(z)$ by $h_\ell(z + h_\ell^{-1}(f^\ell(x)))$, we can also ensure that $h_\ell(0) = f^\ell(x)$. $\square$

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. By applying linear fractional transformations γ_i to each U_i, we may assume without loss of generality that each U_i is an open disk in $\mathbb{C}_p$.

For each $i = 1, \dots, g$, consider the f_i-orbit of x_i. If x_i is periodic, let $k_i \geq 1$ denote its period, and for every $\ell \geq 0$, define the power series $h_{i,\ell}$ to be the constant $f_i^{\ell}(x_i)$. Otherwise, choose $k_i \geq 1$ and radii $r_i, s_i > 0$ according to Lemma 3.3, along with the associated conjugating maps $h_{i,\ell}$ for each $\ell \geq 0$.

Let $k = \operatorname{lcm}(k_1, \dots, k_g) \geq 1$. For each $\ell \in \{0, \dots, k-1\}$ such that $V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^k}(\Phi^{\ell}(\alpha))$ is finite, we can cover $V(\mathbb{C}_p) \cap \mathcal{O}_{\Phi^k}(\Phi^{\ell}(\alpha))$ by finitely many singleton orbits.

It remains to consider those $\ell \in \{0, \dots, k-1\}$ for which there is an infinite set $\mathcal{N}$ of nonnegative integers n such that $\Phi^{nk+\ell}(\alpha) \in V(\mathbb{C}_p)$. We will show that in fact, $\Phi^{nk+\ell}(\alpha) \in V(\mathbb{C}_p)$ for *all* $n \in \mathbb{N}$.

For any $|z| \leq 1$, note that $kz \in \overline{D}(0, s_i)$ for all $i = 1, \dots, g$. Thus, it makes sense to define $\theta : \overline{D}(0,1) \to U_1 \times \cdots \times U_g$ by

$$\theta(z) = (h_{1,\ell}(kz), \dots, h_{g,\ell}(kz));$$

Then for all $n \geq 0$, we have

$$\theta(n) = \Phi^{nk+\ell}(\alpha),$$

because for each $i = 1, \dots, g$, we have $k_i | k$, and therefore

$$h_{i,\ell}(nk) = h_{i,\ell}(nk + h_{i,\ell}^{-1}(f_i^{\ell}(x_i))) = f_i^{nk}(f_i^{\ell}(x_i)) = f_i^{nk+\ell}(x_i).$$

Given any polynomial F vanishing on V, the composition $F \circ \theta$ is a convergent power series on $\overline{D}(0,1)$ that vanishes at all integers in $\mathcal{N}$. However, a nonzero convergent power series can have only finitely many zeros in $\overline{D}(0,1)$; see, for example, [Rob00, Section 6.2.1]. Thus, $F \circ \theta$ is identically zero. Therefore,

$$F(\Phi^{nk+\ell}(\alpha)) = F(\theta(n)) = 0$$

for all $n \geq 0$, not just $n \in \mathcal{N}$. This is true for all such F, and therefore $\mathcal{O}_{\Phi^k}(\Phi^{\ell}(\alpha)) \subseteq V(\mathbb{C}_p)$.

The conclusion of Theorem 3.2 now follows, because $\mathcal{O}$ is the finite union of the orbits $\mathcal{O}_{\Phi^k}(\Phi^{\ell}(\alpha))$ for $0 \leq \ell \leq k-1$. $\square$

As an immediate corollary, we have the following result, which proves Conjecture 1.1 in the case that Φ is defined over $\overline{\mathbb{Q}}$ and there is a nonarchimedean place v with the following property: for each i, the rational function f_i has good reduction at v, and $\mathcal{O}_{f_i}(x_i)$ avoids all v-adic attracting periodic points.

Theorem 3.4. *Let V be a subvariety of $\left(\mathbb{P}^1\right)^g$ defined over $\mathbb{C}_p$, let $f_1, \dots, f_g \in \mathbb{C}_p(t)$ be rational functions of good reduction on $\mathbb{P}^1$, and let Φ denote the coordinatewise action of $(f_1, \dots, f_g)$ on $\left(\mathbb{P}^1\right)^g$. Let $\mathcal{O}$ be the Φ-orbit of a point $\alpha = (x_1, \dots, x_g) \in (\mathbb{P}^1(\mathbb{C}_p))^g$, and suppose that for each i, the orbit $\mathcal{O}_{f_i}(x_i)$ does not intersect the residue class of any attracting f_i-periodic point. Then $V(\mathbb{C}_p) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(\alpha)\}_{n \geq 0}$ for nonnegative integers k and ℓ.*

Proof. For each i, the reduction $r_p(x_i) \in \mathbb{P}^1(\overline{\mathbb{F}_p})$ is preperiodic under the reduced map $(f_i)_p$. Replacing α by $\Phi^m(\alpha)$ for some $m \geq 0$, and replacing Φ by Φ^j for some $j \geq 1$, then, we may assume that for each i, the residue class U_i of x_i is mapped to itself by f_i. By hypothesis, there are no attracting periodic points in those residue classes; thus, by [RL03, Proposition 4.32] (for example), U_i is a quasiperiodicity disk for f_i. Theorem 3.2 now yields the desired conclusion. ☐

4. Preliminary results on intersection theory

In this section we prove the following result on intersection theory for arithmetic surfaces. It will be used in the proofs of our main results in Section 5.

Theorem 4.1. *Let K be a number field, and let $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ be a morphism defined over K of degree at least 2 that is not conjugate to a map of the form $t \mapsto t^n$ for any integer n. Suppose φ does not have any superattracting periodic points other than exceptional points.*

Let $\alpha, \beta \in \mathbb{P}^1(K)$ be points that are not preperiodic for φ. Suppose that there is a curve $C \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ such that there are infinitely many integers $k \geq 0$ for which $(\varphi^k(\alpha), \varphi^k(\beta)) \in C(K)$. Then there are infinitely many finite places v of K such that φ has good reduction at v and such that for some integer $n \geq 1$, the points $\varphi^n(\alpha)$ and $\varphi^n(\beta)$ are in the same residue class at the place v; i.e., $r_v(\varphi^n(\alpha)) = r_v(\varphi^n(\beta))$.

The condition on superattracting points is equivalent to stipulating that the nonexceptional critical points of φ are not periodic. Note that φ has at most one exceptional point, since it is not conjugate to $t \mapsto t^n$.

By [MS95, Proposition 4.2], φ has good reduction at all but finitely many places v of K. (See Section 2 for a discussion of good reduction and the reduction map r_v.) Thus, the content of Theorem 4.1 is the common reduction of $\varphi^n(\alpha)$ and $\varphi^n(\beta)$.

Before proving the Theorem, we set some notation. Let V be a variety over a number field K, and let $\mathcal{V}$ be a model for V over the ring of integers $\mathfrak{o}_K$ of K. Let S be a finite set of places of K that contains all of the archimedean places of K, and let Z be an effective divisor on V. We say that a point γ on V is *S-integral for Z* if the Zariski closure of γ does not meet the Zariski closure of $\operatorname{Supp} Z$ in $\mathcal{V}$ at any fibres of $\mathcal{V}$ outside of S.

More specifically, let $\mathcal{V}$ be the model $\mathbb{P}^1_{\mathfrak{o}_K} \times \mathbb{P}^1_{\mathfrak{o}_K}$ for $\mathbb{P}^1_K \times \mathbb{P}^1_K$ that comes from the isomorphism between $\mathbb{P}^1_K$ and the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$ we chose in Section 2. We will say that a point Q on $\mathbb{P}^1_K \times \mathbb{P}^1_K$ is *S-integral* for a divisor Z if it is S-integral for Z with respect to $\mathcal{V}$.

Let

$$\Phi : \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

be the map $\Phi = \varphi \times \varphi$, and let Δ denote the diagonal divisor on $\mathbb{P}^1 \times \mathbb{P}^1$. We will need the following proposition.

Proposition 4.2. *Let $\varphi : \mathbb{P}^1_K \longrightarrow \mathbb{P}^1_K$ be a rational map of degree $d > 1$ that has no periodic critical points. Let α and β be points in $\mathbb{P}^1(K)$ that are not preperiodic for φ, and let S be a finite set of places of K that contains all of the archimedean places of K. Let C be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$. Then there are at most finitely many integers $k \geq 0$ that satisfy both of the following conditions:*

(i) *$\Phi^k(\alpha, \beta) \in C$; and*
(ii) *$\Phi^k(\alpha, \beta)$ is S-integral for Δ.*

For any nonconstant morphism $h : \mathbb{P}^1 \to \mathbb{P}^1$ and any point x on $\mathbb{P}^1$, we will denote the ramification index of x over $h(x)$ by $e(x/h(x))$.

We will need the following result about ramification.

Lemma 4.3. *Let $h : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a nonconstant morphism defined over a field K of characteristic 0, and let $H := (h, h)$ its action coordinatewise on $\mathbb{P}^1 \times \mathbb{P}^1$. Then:*

(i) *For any point $(P, Q) \in \mathbb{P}^1(K) \times \mathbb{P}^1(K)$, the multiplicity of $\Delta_H := H^*(\Delta)$ at (P, Q) is at most $\max_{x \in \mathbb{P}^1} e(x/h(x))$.*
(ii) *Each irreducible component of Δ_H has multiplicity one.*

Proof. By performing the same change of coordinates on both copies of $\mathbb{P}^1$, we may assume that the point at infinity is not among the points $P, h(P), Q, h(Q)$. Hence, let $t_0, u_0 \in K$ such that $P = [t_0 : 1]$, and $Q = [u_0 : 1]$. Then h has a local power series expansion (see [Sha77, II.2]) in a neighborhood of P as $h(t) = a_0 + \sum_{i=e_1}^{\infty} a_i(t - t_0)^i$ and in a neighborhood of Q as $h(u) = b_0 + \sum_{i=e_2}^{\infty} b_i(u - u_0)^i$, where $a_i, b_i \in K$, and $e_1 \geq 1$ and $e_2 \geq 1$ are the ramification indices of h at P and Q, respectively. Clearly, $(P, Q) \in \Delta_H$ if and only if $a_0 = b_0$. Thus, we may assume $a_0 = b_0$, and so, near (P, Q), the subvariety Δ_H is defined by the equation

$$\sum_{i=e_1}^{\infty} a_i(t - t_0)^i - \sum_{i=e_2}^{\infty} b_i(u - u_0)^i = 0.$$

The multiplicity of (P, Q) as a point on Δ_H is therefore given by $\min(e_1, e_2)$ (see [Sha77, IV.1]); since $e_1, e_2 \leq \max_{x \in \mathbb{P}^1} e(x/h(x))$, statement (i) follows.

Moreover, Δ_H has multiplicity more than one at (P, Q) only if h is ramified at both P and Q. Because h is ramified at only finitely many points of $\mathbb{P}^1$ (note that $\mathrm{char}(K) = 0$), there are at most finitely many points $(P, Q) \in \mathbb{P}^1 \times \mathbb{P}^1$ at which Δ_H has multiplicity larger than one, proving statement (ii). □

We set more notation, as follows. For each $n \geq 0$, let X_n be the divisor $(\Phi^n)^*(\Delta)$. Note that $\Delta \subseteq X_n$ for each n. Therefore, more generally, for each $0 \leq m < n$, we have $X_m \subseteq X_n$. Let $Y_0 := X_0 = \Delta$, and for $n \geq 1$, let

$$Y_n = (\Phi^n)^*(\Delta) - (\Phi^{n-1})^*(\Delta).$$

Then we have

$$X_n = \bigcup_{i=0}^{n} Y_i.$$

Note that Y_n is nonempty because $\deg(\Phi) > 1$. Furthermore, by Lemma 4.3, each irreducible component of X_n, and hence of Y_n, has multiplicity one.

We also have the following important result, giving a uniform bound for the ramification of φ^n.

Lemma 4.4. *Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a map which has no periodic critical points. Let $Q_1, \dots, Q_m$ be the ramification points of φ. Then for any n and any point P on $\mathbb{P}^1$,*

$$e(P/\varphi^n(P)) \leq \prod_{i=1}^{m} e(Q_i/\varphi(Q_i)). \tag{4.4.1}$$

Proof. For each integer $i = 1, \dots, m$, there is at most one $j \geq 0$ such that $\varphi^j(P) = Q_i$, since none of the Q_i are periodic. Meanwhile, $e(\varphi^j(P)/\varphi^{j+1}(P))$ equals 1 for all $j \geq 0$ such that $\varphi^j(P)$ is not a ramification point. Thus,

$$e(P/\varphi^n(P)) = \prod_{j=0}^{n-1} e(\varphi^j(P)/\varphi^{j+1}(P)) \leq \prod_{i=1}^{m} e(Q_i/\varphi(Q_i)). \qquad \square$$

Combining Lemmas 4.3 and 4.4 gives the following result.

Lemma 4.5. *Under the hypothesis of Lemma 4.4 and with the above notation for X_n and Y_n, there is a constant $M \geq 0$ such that for any point $Q \in \mathbb{P}^1(\overline{K}) \times \mathbb{P}^1(\overline{K})$, at most M of the Y_n contain Q.*

Proof. Let M be the quantity on the right hand side of (4.4.1). If a point Q is contained in $M+1$ different Y_i, then the multiplicity of Q on X_n is at least $M+1$ for some large enough n. Lemma 4.4 and part (i) of Lemma 4.3 now give a contradiction. $\square$

We are now ready to prove Proposition 4.2.

Proof of Proposition 4.2. Enlarge S if necessary to contain not only all archimedean places of K but also all places of bad reduction for Φ. Fix an integer $n \geq 2M$, where M is as in Lemma 4.5.

Given an irreducible curve E in $\mathbb{P}^1 \times \mathbb{P}^1$ that does not map to a point under either of the projection maps on $\mathbb{P}^1 \times \mathbb{P}^1$, we claim that E intersects X_n in at least three distinct points. Indeed, E must meet Y_m in at least one point for all $m \geq 0$. However, for each point Q, at most M of the Y_m (for $0 \leq m \leq n$) contain Q. Hence, E must intersect X_n in at least three distinct points, as desired.

Now suppose that there are infinitely many k (and hence infinitely many $k > n$) such that $\Phi^k(\alpha, \beta) \in C$ and $\Phi^k(\alpha, \beta)$ is S-integral for Δ. For each such $k > n$, then, $\Phi^{k-n}(\alpha, \beta)$ is S-integral for X_n. Then there is a K-irreducible curve Z in $(\Phi^n)^{-1}(C)$ such that there are infinitely many m for which $\Phi^m(\alpha,\beta) \in Z$ and $\Phi^m(\alpha,\beta)$ is S-integral for X_n. Because α and β are not preperiodic, Z does not project to a single point on either of the two coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. In addition, Z contains infinitely many K-rational points $\Phi^m(\alpha,\beta)$; because it is also irreducible over K, it is in fact geometrically irreducible (note that if a component of Z defined over a finite extension of K has infinitely many K-rational points, then it is in fact defined over K.)

Thus, by our claim, X_n meets Z in at least three points. However, Z contains infinitely many points $\Phi^m(\alpha,\beta)$ that are S-integral for X_n; this is impossible, by Siegel's theorem on integral points. □

We treat the case of polynomials (which *do* have a periodic critical point) slightly differently. For a polynomial $f(t) = \sum_{i=0}^d a_i t^i$ with $a_d \neq 0$, we define its homogenization $F(t,u)$ by $F(t,u) = \sum_{i=0}^d a_i t^i u^{d-i}$. We then define $\Phi_f : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ by

$$\Phi_f([x:y:z]) = [F(x,z) : F(y,z) : z^d].$$

Let D be the divisor on $\mathbb{P}^2$ consisting of all points $[x:y:z]$ such that $x = y$. The divisor D will play the same role here that the diagonal Δ played on $\mathbb{P}^1 \times \mathbb{P}^1$. We let

$$A_n = (\Phi_f^n)^*(D), \quad \text{and} \quad B_n = (\Phi_f^n)^*(D) - (\Phi_f^{n-1})^*(D).$$

Then

$$A_n = \bigcup_{i=0}^n B_i.$$

Let $\mathcal{W}$ be the model $\mathbb{P}^2_{\mathfrak{o}_K}$ for $\mathbb{P}^2_K$. We will say that a point Q on $\mathbb{P}^2_K$ is S-integral for a divisor Z if it is S-integral for Z with respect to $\mathcal{W}$.

Proposition 4.6. *Let $f \in K[t]$ be a polynomial with no periodic critical points other than the point at infinity. Let α and β be points in $\mathbb{A}^1(K)$ that are not preperiodic for f, and let S be a finite set of places of K that contains all of the archimedean places. Let C be a curve in $\mathbb{P}^2$. Then there are at most finitely many k that satisfy both of the following conditions:*

(i) $\Phi_f^k([\alpha:\beta:1]) \in C$*; and*
(ii) $\Phi_f^k([\alpha:\beta:1])$ *is S-integral for D.*

Proof. The proof is almost identical to the proof of Proposition 4.2. Note that if $[x:y:0]$ lies on A_n, then $[x:y:0]$ must be in the inverse image of $[1:1:0]$ under Φ_f, which is equivalent to saying that $x^{d^n} = y^{d^n}$. There are exactly d^n such points, and d^n is also the degree of A_n; so each point of the form $[x:y:0]$ must have multiplicity one on A_n (and hence on B_n) for any n. Then, as in Proposition 4.2, we can bound the multiplicity of any point $[x:y:1]$ on A_n by

$$\Big(\prod_{i=1}^m e(Q_i/f(Q_i))\Big)$$

where the Q_i are the ramification points of f other than infinity. Thus, again, if there are infinitely many points on C that are S-integral for D, then for any n, there are infinitely many points on some irreducible curve E in $(\Phi_f^n)^{-1}(C)$ that are S-integral for A_n. When n is at least $2 \cdot \prod_{i=1}^m e(Q_i/f(Q_i))$, such a curve E must meet A_n in at least three distinct points, which gives us a contradiction by Siegel's theorem. $\square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. If $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ has no periodic critical points, Proposition 4.2 implies that for any finite set S of places of K, there are only finitely many n such that $\varphi^n(\alpha)$ does not meet $\varphi^n(\beta)$ at any v outside of S. Thus, there must be infinitely many places v such that $r_v(\varphi^n(\alpha)) = r_v(\varphi^n(\beta))$ for some $n \in \mathbb{N}$.

On the other hand, if φ has an exceptional point, then after changing coordinates, we have $\varphi = f$ for some polynomial f (note that φ does not have two exceptional points, as it is not conjugate to a map of the form $t \mapsto t^n$). Furthermore, since φ has no non-exceptional periodic critical points, it follows that f has no periodic critical points save the point at infinity. By Proposition 4.6, for any finite set S of places of K, there are at most finitely many n such that $f^n(\alpha) - f^n(\beta)$ is an S-unit. Thus, there are infinitely many places v such that $r_v(f^n(\alpha)) = r_v(f^n(\beta))$ for some $n \in \mathbb{N}$. $\square$

5. Dynamical Mordell-Lang for curves

Using Theorem 4.1 we can prove a dynamical Mordell-Lang statement for curves embedded in $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 5.1. *Let $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve defined over $\overline{\mathbb{Q}}$, and let $\Phi := (\varphi, \varphi)$ act on $\mathbb{P}^1 \times \mathbb{P}^1$, where $\varphi \in \overline{\mathbb{Q}}(t)$ is a rational function with no superattracting periodic points other than exceptional points. Let $\mathcal{O}$ be the Φ-orbit of a point $(x, y) \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)(\overline{\mathbb{Q}})$. Then $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x, y)\}_{n \geq 0}$ for $k, \ell \in \mathbb{N}$.*

Proof. When $\deg(\varphi) = 1$, the result follows immediately from work of Denis [Den94] and Bell [Bel06], since in this case Φ induces an automorphism of $\mathbb{A}^2$. Hence, we may assume $\deg(\varphi) \geq 2$.

If φ has two exceptional points, then φ is conjugate to the map $t \mapsto t^n$, for some $n \in \mathbb{Z}$. Then our result follows from [GT, Theorem 1.8], as Φ induces an endomorphism of $\mathbb{G}_m^2$. Thus, we may assume that φ has at most one exceptional point, and no other periodic critical points.

We may assume that C is irreducible, and that $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is infinite. We may also assume that neither x nor y is φ-preperiodic, because in that case the projection of C to one of the two coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ consists of a single point (which would be a φ-periodic point), and the conclusion of Theorem 5.1 would be immediate.

Let K be a number field over which φ, C, and (x, y) are defined. By the previous paragraph, the hypotheses of Theorem 4.1 hold for $(\alpha, \beta) = (x, y)$.

Thus, there are infinitely many nonarchimedean places v of K at which φ has good reduction and such that $r_v(\varphi^n(x)) = r_v(\varphi^n(y))$ for some integer $n \geq 1$. Fix such a place v.

Let $p \in \mathbb{N}$ be the prime number lying in the maximal ideal of the nonarchimedean place v, fix an embedding of K into $\mathbb{C}_p$ respecting v, and let U denote the residue class of $\mathbb{P}^1(\mathbb{C}_p)$ containing $\varphi^n(x)$ and $\varphi^n(y)$. Since φ has good reduction, every iterate $\varphi^{n+k}(U)$ is a residue class, and it contains both $\varphi^{n+k}(x)$ and $\varphi^{n+k}(y)$. If no such residue class contains an attracting periodic point, then our desired conclusion is immediate from Theorem 3.4.

The remaining case is that some residue class $\varphi^{n+k}(U)$ contains an attracting periodic point, which must therefore attract the orbits of both x and y. The Theorem now follows from [GT, Theorem 1.3]. □

We can now prove Theorem 1.3 as a consequence of Theorem 5.1.

Proof of Theorem 1.3. We may assume that C is irreducible, and that $C(\overline{\mathbb{Q}}) \cap \mathcal{O}$ is infinite. It suffices to prove that C is Φ-periodic. Indeed, if $\Phi^k(C) = C$, then for each $\ell \in \{0, \ldots, k-1\}$, the intersection of C with $\mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ either is empty or else consists of all $\Phi^{kn+\ell}(\alpha)$, for some n sufficiently large. Either way, the conclusion of Theorem 1.3 holds.

We argue by induction on g. The case $g = 1$ is obvious, while the case $g = 2$ is proved in Theorem 5.1. Assuming Theorem 1.3 for some $g \geq 2$, we will now prove it for $g+1$. We may assume that C projects dominantly onto each of the coordinates of $\left(\mathbb{P}^1\right)^{g+1}$; otherwise, we may view C as a curve in $\left(\mathbb{P}^1\right)^g$, and apply the inductive hypothesis. We may also assume that no x_i is preperiodic, lest C should fail to project dominantly on the i^{th} coordinate.

Let $\pi_1 : \left(\mathbb{P}^1\right)^{g+1} \to \left(\mathbb{P}^1\right)^g$ be the projection onto the first g coordinates, let $C_1 := \pi_1(C)$, and let $\mathcal{O}_1 := \pi_1(\mathcal{O})$. By our assumptions, C_1 is an irreducible curve that has an infinite intersection with $\mathcal{O}_1$. By the inductive hypothesis, C_1 is periodic under the coordinatewise action of φ on the first g coordinates of $\left(\mathbb{P}^1\right)^{g+1}$.

Similarly, let C_2 be the projection of C on the last g coordinates of $\left(\mathbb{P}^1\right)^{g+1}$. By the same argument, C_2 is periodic under the coordinatewise action of φ on the last g coordinates of $\left(\mathbb{P}^1\right)^{g+1}$.

Thus, C is Φ-preperiodic, because it is an irreducible component of the one-dimensional variety $\left(C_1 \times \mathbb{P}^1\right) \cap \left(\mathbb{P}^1 \times C_2\right)$, and because both $C_1 \times \mathbb{P}^1$ and $\mathbb{P}^1 \times C_2$ are Φ-periodic.

Claim 5.2. *Let X be a variety, let $\alpha \in X(\overline{K})$, let $\Phi : X \longrightarrow X$ be a morphism, and let $C \subset X$ be an irreducible curve that has infinite intersection with the orbit $\mathcal{O}_\Phi(\alpha)$. If C is Φ-preperiodic, then C is Φ-periodic.*

Proof of Claim 5.2. Assume C is not periodic. Because C is preperiodic, there exist $k_0, n_0 \geq 1$ such that $\Phi^{n_0}(C)$ is periodic of period k_0. Let $k := n_0 k_0$, and let $C' := \Phi^k(C)$, which is fixed by Φ^k. Then $C \neq C'$, since C is not periodic. Because C and C' are irreducible curves, it follows that

$$C \cap C' \text{ is finite.} \tag{5.2.1}$$

On the other hand, there exists $\ell \in \{0, \dots, k-1\}$ such that $C \cap \mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha))$ is infinite, because $C \cap \mathcal{O}_\Phi(\alpha)$ is infinite. Let $n_1 \in \mathbb{N}$ be the smallest non-negative integer n such that $\Phi^{nk+\ell}(\alpha) \in C$. Since $C' = \Phi^k(C)$ is fixed by Φ^k, we conclude that $\Phi^{nk+\ell}(\alpha) \in C'$ for each $n \geq n_1 + 1$. Therefore

$$C \cap \mathcal{O}_{\Phi^k}(\Phi^\ell(\alpha)) \cap C' \text{ is infinite.} \tag{5.2.2}$$

Statements (5.2.1) and (5.2.2) are contradictory, proving the claim. $\square$

An application of Claim 5.2 with $X = (\mathbb{P}^1)^{g+1}$ now completes the proof of Theorem 1.3. $\square$

6. Quadratic polynomials

In this Section, we will prove Theorems 1.5 and 1.6. We will continue to work with the same reduction maps $r_v : \mathbb{P}^1(K) \longrightarrow \mathbb{P}^1(k_v)$ as in Section 2, where v is a finite place of K. We begin with a lemma derived from work of Silverman [Sil93].

Lemma 6.1. *Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism of degree greater than one, let $\alpha \in \mathbb{P}^1(K)$ be a point that is not preperiodic for φ, and let $\beta \in \mathbb{P}^1(K)$ be a nonexceptional point for φ. Then there are infinitely many v such that there is some positive integer n for which $r_v(\varphi^n(\alpha)) = r_v(\beta)$.*

Proof. Suppose there were only finitely many such v; let S be the set of all such v, together with all the archimedean places. We may choose coordinates $[x : y]$ for $\mathbb{P}^1_K$ such that β is the point $[1 : 0]$. Since $[1 : 0]$ is not exceptional for φ, we see that φ^2 is not a polynomial with respect to this coordinate system. Therefore, by [Sil93, Theorem 2.2], there are at most finitely many n such that $\varphi^n(\alpha) = [t : 1]$ for $t \in \mathfrak{o}_S$, where $\mathfrak{o}_S$ is the ring of S-integers in K. Hence, for all but finitely many integers $n \geq 0$, there is some $v \notin S$ such that $r_v(\varphi^n(\alpha)) = r_v(\beta)$; but this contradicts our original supposition. $\square$

Recall that if f has good reduction at a finite place v of K, we write f_v for the reduction of f at v.

Lemma 6.2. *Let $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be a morphism of degree greater than one, and let $\alpha \in K$ be a point that is not periodic for φ. Then there are infinitely many places v of good reduction for φ such that $r_v(\alpha)$ is not periodic for φ_v.*

Proof. If α is φ-preperiodic but not periodic, then the φ-orbit $\mathcal{O}_\varphi(\alpha)$ is finite. Hence, the reduction map r_v is injective on $\mathcal{O}_\varphi(\alpha)$ for all but finitely many places v, and Lemma 6.2 holds in this case.

Thus, we may assume that α is not preperiodic. After passing to a finite extension L of K, we may also assume that φ has a nonexceptional fixed point β. We extend our isomorphism between $\mathbb{P}^1_K$ and the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_K}$ to an isomorphism from $\mathbb{P}^1_L$ to the generic fibre of $\mathbb{P}^1_{\mathfrak{o}_L}$; and for each place $w|v$ of L, we obtain reduction maps $r_w : \mathbb{P}^1(L) \longrightarrow \mathbb{P}^1(\ell_w)$, where ℓ_w is the residue field at w. For each such $w|v$, we have $r_v(\gamma) = r_w(\gamma)$ for any $\gamma \in \mathbb{P}^1(K)$. By Lemma 6.1, there are infinitely many places w such that there is some n for which $r_w(\varphi^n(\alpha)) = r_w(\beta)$. When $w|v$ for v a place of good reduction for φ, this means that $r_v(\varphi^m(\alpha)) = r_v(\varphi^n(\alpha)) = r_w(\beta)$ for all $m \geq n$, since β is fixed by φ. At all but finitely many of these v, we have $r_v(\alpha) \neq r_w(\beta)$, which means that there is no positive integer m such that $r_v(\varphi^m(\alpha)) = r_v(\alpha)$, as desired. □

We also need the following result for quadratic polynomials.

Proposition 6.3. *Let K be a number field, and let $f \in K[t]$ be a quadratic polynomial with no periodic critical points other than the point at infinity. Then there are infinitely many finite places v of K such that $|f'(z)|_v = 1$ for each $z \in K$ such that $|z|_v \leq 1$ and $r_v(z)$ is f_v-periodic.*

Proof. Since f is a quadratic polynomial, it only has one critical point α other than the point at infinity. By Lemma 6.2 and because α is not periodic, there are infinitely many places v of good reduction for f such that $r_v(\alpha)$ is not f_v-periodic, and such that $|\alpha|_v \leq 1$ and $|2|_v = 1$. (The last two conditions may be added because each excludes only finitely many v.) In particular, $|f'(z)|_v = |z - \alpha|_v$ for any $z \in K$.

Hence, for any such v, and for any $z \in K$ as in the hypotheses, we have $r_v(z) \neq r_v(\alpha)$, since $r_v(z)$ is periodic but $r_v(\alpha)$ is not. Thus, $|f'(z)|_v = |z - \alpha|_v = 1$. □

We are now ready to prove Theorems 1.5, 1.6, and 1.7.

Proof of Theorem 1.5. Let K be a number field such that V is defined over K, the polynomial f is in $K[t]$, and $x_1, \ldots, x_g$ are all in K.

Using Proposition 6.3, we may choose a place v of K such that

(a) v is a place of good reduction for f;
(b) $|x_i|_v \leq 1$, for each $i = 1, \ldots, g$;
(c) $|f'(z)|_v = 1$ for all z such that $|z|_v \leq 1$ and $r_v(z)$ is f_v-periodic.

Indeed, conditions (a) and (b) are satisfied at all but finitely many places v, while condition (c) is satisfied at infinitely many places. Because f is a polynomial, conditions (a) and (b) together imply that $|f^n(x_i)|_v \leq 1$ for all $i = 1, \ldots, g$ and $n \geq 0$. Meanwhile, condition (c) implies that f has no attracting periodic points at v. The desired conclusion now follows from Theorem 3.4. □

Proof of Theorem 1.6. After changing coordinates, we assume that $f(t) = t^2 + c$ for some $c \in \mathbb{Q}$. Thus, 0 is the only finite critical point of f. If $c \notin \mathbb{Z}$, then there is some p such that $|c|_p > 1$. But then $|f^n(0)|_p \to \infty$, so 0 cannot be periodic. Similarly, if c is an integer other than 0, -1 or -2, then we have $|f^n(0)|_\infty \to \infty$, so 0 cannot be periodic. If $c = -2$, then 0 is only f-preperiodic, but not f-periodic. In all the above cases, the hypotheses of

Theorem 1.5 are met, and our proof is done. If $c = 0$, then $f(t) = t^2$ is an endomorphism of $\mathbb{G}_m^g$, and thus our result follows from [GT, Theorem 1.8].

We are left with the case that $f(t) = t^2 - 1$. As in the proof of Theorem 1.3, we may assume (via induction on g) that no x_i is preperiodic; in particular, all x_i and $f(x_i)$ are nonzero. If $f^2(z) = 0$, then either $z = 0$, or $z = \pm\sqrt{2}$. Bearing this fact in mind, we note that there are infinitely many primes p such that 2 is not a quadratic residue modulo p. Thus, we may choose an odd prime p such that each x_i and $f(x_i)$ is a p-adic unit, and such that 2 is not a quadratic residue modulo p. Then there is no positive integer n such that $f^n(x_i)$ is in the same residue class as 0 modulo p for any i. Therefore, $|f'(f^n(x_i))|_p = 1$ for all n, and hence $f^n(x_i)$ never lies in the same residue class as an attracting periodic point. Theorem 1.6 now follows from Theorem 3.4. □

Proof of Theorem 1.7. As before, we may assume that no x_j is preperiodic for f_j. By [Jon08, Theorem 1.2(iii)], for each f_j that is not equal to $t^2 - 1$, the set of primes p such that there is an n for which $f_j^n(x_j) \equiv 0 \pmod{p}$ has Dirichlet density zero. Meanwhile, as noted in the proof of Theorem 1.6, the density of primes p such that -2 is a square modulo p is $1/2$, and therefore the set of primes p for which there are an n and and j satisfying $f_j(t) = t^2 - 1$ and $f_j^n(x_j) \equiv 0 \pmod{p}$ must have (upper) density at most $1/2$. Hence, the set of primes p such that $f_j^n(x_j) \not\equiv 0 \pmod{p}$ for all n and all $j = 1, \dots, g$ has (lower) density at least $1/2$. Choosing such a prime p, we see that $f_j^n(x_j)$ never lies in the same residue class as an attracting periodic point for any n and any $j = 1, \dots, g$, and the result follows from Theorem 3.4. □

7. Extensions over the field of complex numbers

In this section we will use recent work of Medvedev and Scanlon [MS] to prove Theorem 1.4. We begin with the following definitions.

Definition 7.1. *Let K be a field, and let $\varphi \in K[t]$ be a nonconstant polynomial. We say that φ is* indecomposable *if there are no polynomials $\psi_1, \psi_2 \in \overline{K}[t]$ of degree greater than one such that $\varphi = \psi_1 \circ \psi_2$.*

Generic polynomials of any positive degree are indecomposable. This is obvious for (all) polynomials of prime degree or degree one and easy to prove in degree at least 6 (say by reducing to monic decompositions and counting dimensions); but it can also be shown in degree 4.

Definition 7.2. *Let K be a field, and let $f \in K[t]$ be a polynomial of degree $m \geq 1$. If f is monic with trivial t^{m-1} term, we say that f is in* normal form*; that is, f is of the form*

$$t^m + c_{m-2}t^{m-2} + \cdots + c_0.$$

In that case, we say that f is of type (a, b) *if a is the smallest nonnegative integer such that $c_a \neq 0$, and b is the largest positive integer such that $f(t) = t^a u(t^b)$ for some polynomial $u \in K[t]$.*

While we have introduced this definition of "type" to aid our exposition, the accompanying notion of normal form is not new. In fact, as noted in [Bea90, Equation (2.1)], if $\operatorname{char} K = 0$ and $f \in K[t]$ is a polynomial of degree $m \geq 2$, and if K contains an $(m-1)$-st root of the leading coefficient, then there is a linear polynomial $\mu \in K[t]$ such that $\mu^{-1} \circ f \circ \mu$ is in normal form.

Definition 7.3. *For each positive integer m, define $D_m \in \mathbb{Z}[t]$ to be the unique polynomial of degree m such that $D_m(t + 1/t) = t^m + 1/t^m$.*

The usual Chebyshev polynomial T_m (satisfying $T_m(\cos(\theta)) = \cos(m\theta)$) is conjugate to D_m, since $D_m(2t) = 2T_m(t)$. However, D_m is in normal form.

The following result is an immediate consequence of Theorem 3.149 in [MS] (see also Section 3.2 in [MS]).

Theorem 7.4 (Medvedev, Scanlon)**.** *Let K be a field of characteristic 0, and let $\varphi \in K[t]$ be a nonlinear indecomposable polynomial which is not conjugate to t^m or D_m for any positive integer m. Assume that φ is in normal form, of type (a, b).*

Let Φ denote the action of (φ, φ) on $\mathbb{A}^2$. Let C be a Φ-periodic irreducible plane curve defined over K. Then C is defined by one of the following equations in the variables (x, y) of the affine plane:

(i) *$x = x_0$, for a φ-periodic point x_0; or*
(ii) *$y = y_0$, for a φ-periodic point y_0; or*
(iii) *$x = \zeta\varphi^r(y)$, for some $r \geq 0$; or*
(iv) *$y = \zeta\varphi^r(x)$, for some $r \geq 0$,*

where ζ is a d-th root of unity, where $d \mid b$ and $\gcd(d, a) = 1$.

Remark 7.5. Note that if $b = 1$ or $a = 0$ in Theorem 7.4, then $d = 1$, and hence $\zeta = 1$.

Using Theorem 7.4, we can prove the following result.

Theorem 7.6. *Let $\varphi \in \mathbb{C}[t]$ be an indecomposable polynomial with no periodic superattracting points other than exceptional points, and let $\Phi := (\varphi, \varphi)$ be its diagonal action on $\mathbb{A}^2$. Let $\mathcal{O}$ be the Φ-orbit of a point (x_0, y_0) in $\mathbb{A}^2(\mathbb{C})$, and let C be a curve defined over $\mathbb{C}$. Then $C(\mathbb{C}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(x_0, y_0)\}_{n\geq 0}$ for nonnegative integers k and ℓ.*

We will need three more ingredients to prove Theorem 7.6.

Proposition 7.7. *Fix integers $m, g \geq 1$, let $\varphi \in \mathbb{C}[t]$ be a polynomial which is a conjugate of either t^m or D_m, and let Φ be its coordinatewise action on $\mathbb{A}^g$. Let $\mathcal{O}$ be the Φ-orbit of a point $\alpha \in \mathbb{A}^g(\mathbb{C})$, and let V be an affine subvariety of $\mathbb{A}^g$ defined over $\mathbb{C}$. Then $V(\mathbb{C}) \cap \mathcal{O}$ is a union of at most finitely many orbits of the form $\{\Phi^{nk+\ell}(P)\}_{n\geq 0}$ for nonnegative integers k and ℓ.*

Proof. By hypothesis, there is a linear polynomial $h(t) \in \mathbb{C}[t]$ such that either $\varphi(h(t)) = h(t^m)$ or $\varphi(h(t)) = h(D_m(t))$. In the first case, let $k(t) =$

$h(t)$, and in the second, let $k(t) = h(t + 1/t)$. Then $\varphi(k(t)) = k(t^m)$ for some nonconstant rational function $k(t) \in \mathbb{C}(t)$. Note that in either case, $k(\mathbb{C}) \supseteq \mathbb{C}$, and the only possible poles of k are at 0 and ∞.

Let $\alpha := (x_1, \ldots, x_g)$. For each $i \in \{1, \ldots, g\}$, pick $z_i \in \mathbb{C}$ such that $k(z_i) = x_i$. Then $\mathcal{O}_\Phi(\alpha) = \left\{\left(k(z_1^{m^n}), \ldots, k(z_g^{m^n})\right) : n \geq 0\right\}$. Let W be the affine subvariety of $\mathbb{G}_m^g$ defined by the equations $f(k(t_1), \ldots, k(t_g)) = 0$, where f ranges over a set of generators for the vanishing ideal of V. (Note that W is an algebraic subvariety of $\mathbb{G}_m^g$ because k has no poles on $\mathbb{G}_m$.)

Let Ψ be the endomorphism of $\mathbb{G}_m^g$ given by $\Psi(t_1, \ldots, t_g) = (t_1^m, \ldots, t_g^m)$. Then

$$\Phi^n(x_1, \ldots, x_g) \in V(\mathbb{C}) \text{ if and only if } \Psi^n(z_1, \ldots, z_g) \in W(\mathbb{C}).$$

Thus, Proposition 7.7 holds for Φ and V because, by [GT, Theorem 1.8], it holds for Ψ and W. $\square$

Proposition 7.8. *Let E be a field of characteristic 0, and K a function field of transcendence degree 1 over E. Let $\varphi \in K[t]$ be a polynomial of degree $m \geq 2$ in normal form. Assume that φ is not conjugate to t^m or D_m. Then for all but finitely places v of the function field K, the reduction φ_v of φ at v is not conjugate to t^m or D_m.*

Proof. After replacing K by a finite extension, we may assume that K contains all $(m-1)$-st roots of unity. All coefficients of φ are v-adic integers at all but finitely many places v. For any such place, write k_v for the residue field and φ_v for the reduction of φ. If φ_v is conjugate to the reduction f_v of either $f = D_m$ or $f(t) = t^m$, write $\varphi_v(t) = \mu_v^{-1} \circ f_v \circ \mu_v$ for some linear polynomial $\mu_v(t) = At + B \in k_v[t]$. Because φ_v and f_v are both in normal form, we must have $\mu_v(t) = \zeta_v t$, for some $(m-1)$-st root of unity $\zeta_v \in k_v$. (Indeed, because $\operatorname{char} K_v = \operatorname{char} E = 0$ and both φ_v and f_v have trivial t^{m-1} term, we must have $B = 0$; and because both are monic, A must be an $(m-1)$-st root of unity.)

Thus, at any such place v, φ is congruent modulo v to one of the m polynomials $\zeta^{-1} D_m(\zeta t)$ or $\zeta^{-1}(\zeta t)^m = t^m$, where $\zeta \in K$ is an $(m-1)$-st root of unity. Since φ is not one of those m polynomials itself, there are only finitely many such v at which that occurs. $\square$

Proposition 7.9. *Let E be a field, and K a function field of transcendence degree 1 over E. Let $f \in K[t]$ be an indecomposable polynomial of degree greater than one. Then for all but finitely many places v of K, the reduction of f modulo v is also an indecomposable polynomial over $\overline{k}_v$ of degree greater than one, where k_v is the residue field of K at v.*

Proof. First we note that for all but finitely many places v of K, the coefficients of f are integral at v, and the leading coefficient of f is a unit at v. Thus, the reduction f_v of f modulo v is a polynomial of same degree as f.

We will show that for any given positive integers m and n (with $m, n \geq 2$) such that $mn = \deg(f)$, if f is not a composition of a polynomial of degree

m with a polynomial of degree n, then for all but finitely many places v of K, the reduction of f modulo v cannot be written as a composition of two polynomials of degrees m and n, respectively, with coefficients in $\overline{k}_v$. Because there are finitely many pairs of positive integers (m, n) such that $mn = \deg(f)$, our desired conclusion follows.

Let m and n be positive integers such that $mn = \deg(f)$ (with $m, n \geq 2$). Then the nonexistence of polynomials

$$g(t) = \sum_{i=0}^{m} a_i t^i \text{ and } h(t) = \sum_{j=0}^{n} b_j t^j$$

with coefficients in $\overline{K}$, such that $f = g \circ h$, where $f(t) = \sum_{\ell=0}^{\deg(f)} c_\ell t^\ell$, translates to the statement that the variety $X \subset \mathbb{A}^{m+n+2}$ given by the equations which must be satisfied by the a_i's and the b_j's has no $\overline{K}$-points. Furthermore, X is a variety defined over a subring R of K such that all but finitely many places of K are maximal ideals of R.

Suppose there is an infinite set $\mathcal{S}$ of places v of K at which f_v is actually a composition of two polynomials of degrees m and n, with coefficients in $\overline{k}_v$. Then the special fibre of X over v is nonempty over $\overline{k}_v$ for each $v \in \mathcal{S}$. Therefore, the equations defining X determine a nonempty locus over the ultraproduct $\mathcal{K}_{\mathcal{S},\mathcal{U}}$ of all the infinitely many fields $\overline{k}_v$ with respect to a non-principal ultrafilter $\mathcal{U}$ based on $\mathcal{S}$. However, K embeds into $\mathcal{K}_{\mathcal{S},\mathcal{U}}$ (see [Hru98, p. 198–199]). Since X is defined over K and has a rational point over a field containing K, it must in fact have an algebraic point over K, giving a contradiction to the fact that $X(\overline{K})$ is empty. □

We are ready to prove Theorem 7.6.

Proof of Theorem 7.6. If φ is a linear polynomial, then the result follows from [Bel06]. If φ is conjugate to t^m or D_m, then our conclusion follows from Proposition 7.7. We may therefore assume that φ is an indecomposable, nonlinear polynomial which is neither a conjugate of t^m, nor of D_m. Furthermore, after conjugating φ by a linear polynomial μ (and replacing (x_0, y_0) by $(\mu^{-1}(x_0), \mu^{-1}(y_0))$ and C by $(\mu^{-1}, \mu^{-1})(C)$), we may assume that φ is in normal form. Let $m = \deg \varphi$.

As before, we may assume that C is an irreducible curve, and that C does not project to a single point to any of the coordinates. For example, if $C = \mathbb{A}^1 \times \{y_1\}$, then y_1 is φ-periodic, and hence C is Φ-periodic. In particular, we may assume that neither x_0 nor y_0 is φ-preperiodic.

Let K be a finitely generated field over which C, φ, x_0 and y_0 are defined. Furthermore, at the expense of replacing K by a finite extension, we may assume that C is geometrically irreducible and that K contains all critical points of φ and all $(m-1)$-st roots of unity.

We will prove Theorem 7.6 by induction on $d := \operatorname{trdeg}_{\mathbb{Q}} K$. If $d = 0$, then K is a number field, and our conclusion follows from Theorem 5.1.

Assume $d \geq 1$. Then K may be viewed as the function field of a smooth, geometrically irreducible curve Z defined over a finitely generated field E; thus, $\operatorname{trdeg}_{\mathbb{Q}} E = d-1$. Moreover, the curve C extends to a 1-dimensional scheme over Z (called $\mathfrak{C}$), all but finitely many of whose fibres $\mathfrak{C}_\gamma$ are irreducible curves.

We claim that there are infinitely many places γ of K for which all of the following statements hold. (By a place of K, we mean a valuation of the function field K/E, cf. Chapter 2 of [Ser97].)

(a) The fibre $\mathfrak{C}_\gamma$ is an irreducible curve defined over the residue field $E(\gamma)$ of γ, of the same degree as C.
(b) All nonzero coefficients of φ are units at the place γ; in particular, φ has good reduction at γ, and so we write φ_γ and $\Phi_\gamma := (\varphi_\gamma, \varphi_\gamma)$ for the reductions of φ and Φ at γ.
(c) The critical points of φ_γ are reductions at γ of the critical points of φ.
(d) For each critical point z of φ (other than infinity), the reduction z_γ is not a periodic point for φ_γ.
(e) The map $\mathcal{O} \longrightarrow \mathcal{O}_\gamma$ from the Φ-orbit of (x_0, y_0) to the Φ_γ-orbit of $(x_{0,\gamma}, y_{0,\gamma})$, induced by reduction at γ, is injective.
(f) φ_γ is not conjugate to t^m or D_m. (Recall $m = \deg \varphi$.)
(g) φ_γ is a nonlinear, indecomposable polynomial.

Conditions (a)–(c) above are satisfied at all but finitely many places γ of K. The same is true of conditions (f)–(g), by Proposition 7.8 and Proposition 7.9. Condition (d) for preperiodic (but not periodic) critical points also holds at all but finitely many places; see the first paragraph of the proof of Lemma 6.2. Meanwhile, [GTZ08, Proposition 6.2] says that the reduction of any finite set of nonpreperiodic points remains nonpreperiodic at infinitely many places γ (in fact, at all γ on Z of sufficiently large Weil height). Thus, conditions (d)–(e) hold by applying [GTZ08, Proposition 6.2] to (x_0, y_0) and the nonpreperiodic critical points, proving the claim.

Let γ be one of the infinitely many places satisfying conditions (a)–(g) above. From condition (e), we deduce that $\mathfrak{C}_\gamma(E(\gamma)) \cap \mathcal{O}_\gamma$ is infinite. Conditions (c)–(d) guarantee that φ_γ has no periodic critical points (other than the exceptional point at infinity). Because $E(\gamma)$ is a finite extension of E, we get $\operatorname{trdeg}_{\mathbb{Q}} E(\gamma) = d-1$. By the inductive hypothesis, then, $\mathfrak{C}_\gamma$ is Φ_γ-periodic. By conditions (f)–(g) and Theorem 7.4, $\mathfrak{C}_\gamma$ is the zero set of an equation from one of the four forms (i)–(iv) in Theorem 7.4. In fact, if φ has type (a, b), then the degree d in Theorem 7.4 satisfies $d|b$ and $\gcd(d, a) = 1$, because condition (b) implies that φ_γ also has type (a, b). Thus, for one of the four forms (i)–(iv), there are infinitely many places γ satisfying (a)–(g) above such that the equation for $\mathfrak{C}_\gamma$ is of that form. By symmetry, it suffices to consider only forms (i) and (iii).

Case 1. Assume there are infinitely many γ satisfying (a)–(g) such that $\mathfrak{C}_\gamma$ is given by an equation $x = x(\gamma)$, for some φ_γ-periodic point $x(\gamma) \in E(\gamma)$.

Then, since the degree of C is preserved by the reduction at γ, we see that the degree of C must be 1. Thus, C is defined by an equation of the form $ax + by + c = 0$. Since there are infinitely many γ such that the above equation reduces at γ to $x = x(\gamma)$, we must have $b = 0$; hence, the curve C must be given by an equation $x = x_1$ for some $x_1 \in K$, contradicting our assumption that C does not project to a point in any of the coordinates.

Case 2. Assume there are infinitely many γ satisfying (a)–(g) such that $\mathfrak{C}_\gamma$ is given by an equation $y = \zeta\varphi_\gamma^r(x)$, for some $r \geq 0$ and some d-th root of unity ζ, where $d \mid b$ and $\gcd(d, a) = 1$. Because there are only finitely many b-th roots of unity, we may assume ζ is the same for all of the infinitely many γ. Moreover, because $\mathfrak{C}_\gamma$ has the same degree as C, the integer r is the same for all such γ. Thus, there are infinitely many places γ for which the polynomial equation for C reduces modulo γ to $y - \zeta\varphi^r(x)$, and hence the two polynomials are the same. Thus, C is the zero set of the polynomial $y - \zeta\varphi^r(x)$. Because φ is of type (a, b), it follows that C is Φ-periodic. $\square$

Arguing precisely as in the proof of Theorem 1.3, Theorem 1.4 follows as a consequence of Theorem 7.6.

Remark 7.10. In personal communications, Medvedev and Scanlon told us that, using the methods of [MS], it is possible to prove the conclsion of Theorem 7.4 even for decomposable polynomials f that are not compositional powers of other polynomials. Using that stronger result in our proofs above, we could then extend Theorems 7.6 and 1.4 to any f that is not a compositional power of another polynomial. It would then be easy to extend those results to *all* polynomials $f \in \mathbb{C}[t]$ (with no periodic superattracting points other than exceptional points); indeed, if $f = g^k$ is a compositional power, then we may simply replace the action of f with the action of g.

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