

Distributed Prescribed-Time Observer for Nonlinear Systems in Block-Triangular Form

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Abstract—This letter proposes a design of a distributed prescribed-time observer for nonlinear systems representable in a block-triangular observable canonical form. Using a weighted average of neighbor estimates exchanged over a strongly connected digraph, each observer estimates the system state despite the limited observability of local sensor measurements. The proposed design guarantees that distributed state estimation errors converge to zero at a user-specified convergence time, irrespective of observers' initial conditions. To achieve this prescribed-time convergence, distributed observers implement time-varying local output injection gains that monotonically increase and approach infinity at the prescribed time. The theoretical convergence is rigorously proven and validated through numerical simulations, where some implementation issues due to increasing gains have also been clarified.

Index Terms—Distributed observers, sensor networks, prescribed-time state estimation, nonlinear systems.

I. INTRODUCTION

DISTRIBUTED state estimation involves estimating the state of a dynamical system through a network of spatially distributed observers, where the sensor of each observer can measure only a partial state. Due to the limited observability of local sensors, individual observers cannot independently reconstruct the system state. Therefore, observers employ information exchanged by their neighbors over a communication network to achieve full state estimation, as illustrated in Fig. 1. Such distributed approaches have emerged as a promising alternative to centralized methods for

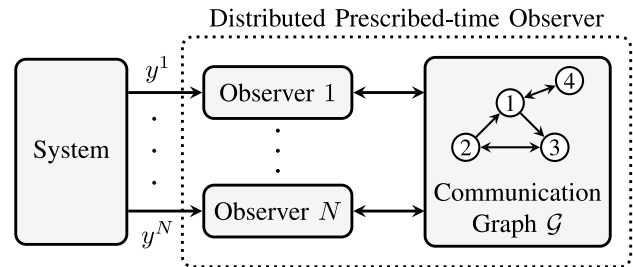


Fig. 1. Framework of distributed state estimation.

state estimation, offering significant advantages in scalability, resilience, privacy, and communication efficiency [1].

For linear systems, seminal papers [2] and [3] studied distributed Kalman filtering problem, motivating developments of distributed Luenberger observers [4], [5], [6], [7] in deterministic settings. Extensions to finite-time distributed observers, such as [8], [9], aim to estimate the state of a linear system up to a certain accuracy within a finite time, which depends on the system's initial condition. A more recent approach in [10] proposes a kernel-based method for distributed state estimation to achieve fixed-time convergence of the state estimate for a linear system, where the convergence time is independent of the system's initial condition.

Distributed observers for nonlinear systems represent a significant advancement in the field. A distributed Luenberger-like observer asymptotically estimating the state of autonomous nonlinear systems is proposed in [11]. For controlled nonlinear systems, [12] proposes a distributed observer by adapting methods from [4] and [7]. However, these methods ensure only asymptotic convergence of the estimate to the system state.

In this letter, we are interested in distributed observers that can estimate the state within finite time, which is prescribed by the user and is independent of the system's initial condition. This is because, in practical applications, especially those with uncertain initial conditions, achieving an accurate state estimate within a prescribed time is critical to meet control objectives. Existing methods for finite-time state estimation often have convergence times that depend on the system's initial conditions, which may be unavailable, inaccurate, or confidential in practical applications; whereas methods for fixed-time state estimation have convergence times that depend on observer parameters (see [10]) and cannot be chosen arbitrarily by the user. On the other hand, the prescribed-time state estimation offers a compelling alternative, as it allows the user to *a priori* specify the convergence time independently of the initial condition.

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Centralized prescribed-time observers for linear and nonlinear systems in triangular forms are proposed in [13] and [14], respectively, which employ time-varying gains that approach infinity at the prescribed time. To the best of our knowledge, the problem of distributed prescribed-time state estimation of nonlinear systems remains unresolved in current literature. While [15] attempted to address this problem, this letter contains critical errors as pointed out in [16, Appendix C].

In this letter, we address the distributed prescribed-time state estimation problem for nonlinear systems in a block-triangular observable canonical form [17], [18], [19]. Systems in this form can be obtained by transforming nonlinear systems via a specific diffeomorphic map subject to the differential observability condition [18], [20]. Our proposed distributed observer allows users to specify an arbitrary convergence time for state estimation, independent of initial conditions. Because of the limited observability of local measurements, observers communicate over a strongly connected directed graph to estimate locally observable states through output injection and unobservable states through consensus with their neighbors. Using time-varying gains that approach infinity near convergence time and synthesizing observer gains via linear matrix inequalities, we prove convergence of the distributed state estimation errors to zero within the prescribed time. We validate our results through numerical simulations and also provide guidelines on the implementation issues due to increasing time-varying gains.

II. PRELIMINARIES

Notations: Given $A \in \mathbb{R}^{n \times n}$, $\text{sym}(A) := A + A^\top$. The Kronecker product between A and B is given by $A \otimes B$. The identity matrix of dimensions $n \times n$ is I_n , a vector of ones is $\mathbf{1}_n \in \mathbb{R}^{n \times 1}$, and a zero matrix is $0_{m \times n} \in \mathbb{R}^{m \times n}$. A symmetric matrix $P = P^\top \in \mathbb{R}^{n \times n}$ is denoted as $P \geq 0$ (resp., $P \leq 0$) if it is positive (resp., negative) semi-definite. For matrices A_1, \dots, A_N , $\underline{A} = \text{diag}(A_1, \dots, A_N)$ denotes a block diagonal matrix with A_i 's as diagonal blocks. A block diagonal matrix comprising N copies of A is denoted by $\bar{A} = I_N \otimes A$. The vertical concatenation of $v_1 \in \mathbb{R}^{n_1}, \dots, v_N \in \mathbb{R}^{n_N}$ into a single column vector is denoted by $\text{col}(v_1, \dots, v_N)$.

Graph theory: A weighted directed graph is denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ with $\mathcal{N} = \{1, 2, \dots, N\}$ the set of nodes, $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ the set of edges, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ the weighted adjacency matrix. The edge $(i, j) \in \mathcal{E}$ indicates that node i receives information from node j . Each entry a_{ij} in \mathcal{A} denotes the weight associated with the edge (i, j) , with $a_{ij} > 0$ if and only if $(i, j) \in \mathcal{E}$; otherwise, $a_{ij} = 0$. A directed path from node j to node i consists of a sequence of edges $(i, i_1), \dots, (i_k, j)$, for some integer $k \geq 1$. A directed graph \mathcal{G} is called *strongly connected* if, for any $i, j \in \mathcal{N}$, there exists a directed path from j to i in \mathcal{E} .

Given \mathcal{G} is strongly connected, then (see [7], [21], [22], [23]):

Fact 1: The Laplacian matrix $\mathcal{L} \in \mathbb{R}^{N \times N}$ of \mathcal{G} has a zero eigenvalue with the corresponding eigenvector $\mathbf{1}_N$ and all other eigenvalues of \mathcal{L} lie in $\mathbb{C}_{>0}$.

Fact 2: There exists a unique $r = [r_1 \dots r_N] \in \mathbb{R}_{>0}^{1 \times N}$ such that $r\mathcal{L} = 0_{1 \times N}$ and $r\mathbf{1}_N = N$.

Fact 3: $\hat{\mathcal{L}} := r\mathcal{L} + \mathcal{L}^\top r$ is positive semi-definite, where

$$R = \text{diag}(r_1, \dots, r_N). \quad (1)$$

Moreover, $\mathbf{1}_N^\top \hat{\mathcal{L}} = 0$ and $\hat{\mathcal{L}}\mathbf{1}_N = 0$.

III. SYSTEM DEFINITION

Consider a nonlinear system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\varphi(x(t)) \\ y^i(t) &= H^i x(t), \quad i = 1, \dots, N \end{aligned} \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the state, $y^i(t) \in \mathbb{R}$ is sensor i 's measurement, and $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^N$ is a known function. Specifically, the system comprises $N \leq n$ subsystems, where the state of subsystem i is $x_i(t) \in \mathbb{R}^{n_i}$ with $\sum_{i=1}^N n_i = n$. Therefore, $x(t) = \text{col}(x_1(t), \dots, x_N(t))$ and

$$x_i(t) = \text{col}(x_{i1}(t), \dots, x_{i,n_i}(t)), \quad i = 1, \dots, N.$$

The nonlinearity φ is triangular, meaning

$$\varphi(x) = \text{col}(\varphi_1(x_1), \varphi_2(x_1, x_2), \dots, \varphi_N(x_1, \dots, x_N)) \quad (3)$$

where $\varphi_i : \mathbb{R}^{\bar{n}_i} \rightarrow \mathbb{R}$ with $\bar{n}_i = \sum_{j=1}^i n_j$.

Let $H = \text{col}(H^1, \dots, H^N)$, where $H^i \in \mathbb{R}^{1 \times n}$ given by

$$H^i = [0_{1 \times n_1} \dots 0_{1 \times n_{i-1}} H_i 0_{1 \times n_{i+1}} \dots 0_{1 \times n_N}]$$

with $H_i \in \mathbb{R}^{1 \times n_i}$ placed on the i -th block position. Notice that $y^i(t) = H^i x(t) = H_i x_i(t)$. The system matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times N}$, and $H \in \mathbb{R}^{N \times n}$ are given by

$$\begin{aligned} A &= \text{diag}(A_1, \dots, A_N), \quad B = \text{diag}(B_1, \dots, B_N), \\ H &= \text{diag}(H_1, \dots, H_N) \end{aligned}$$

with $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times 1}$, and $H_i \in \mathbb{R}^{1 \times n_i}$ in observable canonical form

$$\begin{aligned} A_i &= \begin{bmatrix} 0_{(n_i-1) \times 1} & I_{n_i-1} \\ 0 & 0_{1 \times (n_i-1)} \end{bmatrix}, \quad B_i = \begin{bmatrix} 0_{(n_i-1) \times 1} \\ 1 \end{bmatrix}, \\ H_i &= [1 \quad 0_{1 \times (n_i-1)}]. \end{aligned}$$

Assumption 1: The function $\varphi_i : \mathbb{R}^{\bar{n}_i} \rightarrow \mathbb{R}$, for $i = 1, \dots, N$, in (3) is Lipschitz continuous, i.e., there exists $\gamma_i \in \mathbb{R}_{\geq 0}$ such that for all $\bar{x}_i, \bar{\omega}_i \in \mathbb{R}^{\bar{n}_i}$, it holds that

$$|\varphi_i(\bar{x}_i + \bar{\omega}_i) - \varphi_i(\bar{x}_i)| \leq \gamma_i \|\bar{\omega}_i\| \quad (4)$$

where $\bar{x}_i := \text{col}(x_1, \dots, x_i)$ and $\bar{\omega}_i := \text{col}(\omega_1, \dots, \omega_i)$.

The communication network between the sensors is defined by a weighted directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$.

Assumption 2: \mathcal{G} is strongly connected.

We further assume that the linear part of the system is jointly observable. However, it must be noted that the class of systems defined above becomes unobservable if any of the sensors are removed. Therefore, all measurements $y^i(t)$, for $i = 1, \dots, N$, must be utilized for state estimation (cf. [12]).

Assumption 3: $\text{rank}[H^\top (HA)^\top \dots (HA^{N-1})^\top] = n$.

Subject to the assumptions above, our objective is to design a distributed estimation algorithm with N observers to cooperatively determine the full state of $x(t)$ of system (2) within a finite time $T > 0$, which is prescribed by the user arbitrarily and independent of the initial conditions.

IV. DESIGN PROBLEM

The distributed state estimation algorithm comprises N observers, where each observer i can access only the measurement of sensor i and a weighted combination of state estimates of its neighboring observers according to the communication network \mathcal{G} . Let $z^i(t) \in \mathbb{R}^n$ denote the state estimate of observer i . Each observer i estimates the locally

observable part of the state, $x_i(t)$, by using the local output injection $y^i(t) - H^i z^i(t)$. The unobservable part of the state, $\text{col}(x_1(t), \dots, x_{i-1}(t), x_{i+1}(t), \dots, x_N(t))$, is estimated by seeking consensus with the neighboring observers, i.e., $\sum_j a_{ij}[z^i(t) - z^j(t)]$. In particular, we consider the following architecture of the distributed observer

$$\begin{aligned} \dot{z}^i(t) &= A z^i(t) + B \varphi(z^i(t)) + \Gamma(t) L^i [y^i(t) - H^i z^i(t)] \\ &\quad - \mu^{1+m}(t) r_i \sum_{j=1}^N a_{ij} [z^i(t) - z^j(t)], \quad i = 1, \dots, N \end{aligned} \quad (5)$$

where $z^i(t) \in \mathbb{R}^n$ denotes the full-state estimate of observer i . The entry a_{ij} is the (i, j) -th entry of the adjacency matrix \mathcal{A} of \mathcal{G} and r_i is defined in Fact 2 in Section II. The distributed observer design problem aims to find gains $\Gamma(t) \in \mathbb{R}_{>0}^{n \times n}$, $\mu(t) \in \mathbb{R}_{>0}$, and $L^i \in \mathbb{R}^n$ such that a certain stability property (to be defined later) of the state estimation error holds.

A. Coordinate Transformation and Time-Varying Gains

The state estimation error of observer i in (5) is defined as

$$e^i(t) = x(t) - z^i(t), \quad i = 1, \dots, N.$$

Consider a time-varying coordinate transformation

$$\zeta^i(t) = \Gamma^{-1}(t) e^i(t)$$

where $\Gamma(t) \in \mathbb{R}_{>0}^{n \times n}$ is expressed as

$$\Gamma(t) := \Gamma(\mu(t)) = \text{diag}(\Gamma_1(\mu(t)), \dots, \Gamma_N(\mu(t))) \quad (6)$$

with $\Gamma_i(\mu(t)) \in \mathbb{R}^{n_i \times n_i}$, associated with subsystem i ,

$$\Gamma_i(t) := \Gamma_i(\mu(t)) = \text{diag}(\mu^{1+m}(t), \dots, \mu^{n_i(1+m)}(t)). \quad (7)$$

Here, $m \geq 1$ and $\mu: [t_0, t_0 + T) \rightarrow \mathbb{R}_{>0}$ is defined as

$$\mu(t) := \mu(t; t_0, T) = \frac{T}{T + t_0 - t} \quad (8)$$

for some initial time $t_0 \in \mathbb{R}_{\geq 0}$ and prescribed-time $T \in \mathbb{R}_{>0}$. Notice that μ is parametrized by t_0 and T . In (7), m is a design parameter that controls the convergence speed, and $\mu(t) \in [1, \infty)$ is a monotonically increasing function approaching infinity as $t \rightarrow t_0 + T$. Consequently, its multiplicative inverse $1/\mu(t) \in (0, 1]$ decreases monotonically to 0 as $t \rightarrow t_0 + T$.

B. Prescribed-Time Convergence of Estimation Error

Let the vector of the state estimates of N observers be $z(t) = \text{col}(z^1(t), \dots, z^N(t))$ and the corresponding distributed state estimation error

$$e(t) = [\mathbf{I}_N \otimes x(t)] - z(t). \quad (9)$$

By taking the derivative of (9) and combining (2) and (5), we obtain the error system as

$$\dot{e}(t) = \Lambda(t)e(t) + \bar{B}\Delta\Phi(t) - \mu^{1+m}(t)(R\mathcal{L} \otimes I_n)e(t) \quad (10)$$

where $\bar{B} = I_N \otimes B$, \mathcal{L} is the Laplacian matrix, R in (1), and

$$\begin{aligned} \Delta\Phi(t) &= \text{col}(\varphi(x(t)) - \varphi(z^1(t)), \dots, \varphi(x(t)) - \varphi(z^N(t))), \\ \Lambda(t) &= \text{diag}(A - \Gamma(t)L^1 H^1, \dots, A - \Gamma(t)L^N H^N). \end{aligned}$$

Now, consider the transformed error

$$\zeta(t) = \bar{\Gamma}^{-1}(t)e(t) \quad (11)$$

where $\bar{\Gamma}(t) := I_N \otimes \Gamma(t)$. Taking its derivative yields

$$\begin{aligned} \dot{\zeta}(t) &= \dot{\bar{\Gamma}}^{-1}(t)e(t) + \bar{\Gamma}^{-1}(t)\Lambda e(t) + \bar{\Gamma}^{-1}(t)\bar{B}\Delta\Phi(t) \\ &\quad - \mu^{1+m}(t)\bar{\Gamma}^{-1}(t)(R\mathcal{L} \otimes I_n)e(t). \end{aligned}$$

Let $D_i = \text{diag}(1, 2, \dots, n_i)$ and

$$\begin{aligned} \bar{A} &= I_N \otimes A, & \bar{D} &= I_N \otimes \text{diag}(D_1, \dots, D_N), \\ \underline{L} &= \text{diag}(L^1, \dots, L^N), & \underline{H} &= \text{diag}(H^1, \dots, H^N). \end{aligned} \quad (12)$$

Since $e(t) = \bar{\Gamma}(t)\zeta(t)$, and using the facts

$$\begin{aligned} \dot{\bar{\Gamma}}^{-1}(t) &= -\bar{\Gamma}^{-2}(t)\dot{\bar{\Gamma}}(t) = (1+m)\frac{\mu(t)}{T}\bar{D}\bar{\Gamma}^{-1}(t), \\ \bar{\Gamma}^{-1}(t)\bar{A}\bar{\Gamma}(t) &= \mu^{1+m}(t)\bar{A} \end{aligned}$$

we obtain

$$\begin{aligned} \dot{\zeta}(t) &= \mu^{1+m}(t)(\bar{A} - \underline{L}\underline{H} - (R\mathcal{L} \otimes I_n))\zeta(t) \\ &\quad - (1+m)\frac{\mu(t)}{T}\bar{D}\zeta(t) + \bar{\Gamma}^{-1}(t)\bar{B}\Delta\Phi(t). \end{aligned} \quad (13)$$

Definition 1 (FGUAS [13]): A system $\dot{e} = f(e, t)$ is *fixed-time, globally uniformly asymptotically stable* over an interval $[t_0, t_0 + T)$ with $T \in \mathbb{R}_{>0}$ if there exists a class \mathcal{KL} function¹ $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ such that

$$\|e(t)\| \leq \beta(\|e(t_0)\|, \mu(t; t_0, T) - 1), \quad \forall t \in [t_0, t_0 + T) \quad (14)$$

where $\mu(t; t_0, T)$ is defined in (8).

Similar to [13], we employ the definition of fixed-time stability, which guarantees convergence within finite time T that is independent of initial conditions. Prescribed-time observers build upon this notion but via specific time-varying gain (6) transform the fixed-time property into prescribed-time behavior—i.e., for any arbitrary T prescribed by the user, convergence is ensured at T rather than just before some constant upper bound. Therefore, when a user can arbitrarily select the fixed time T and the system continues to satisfy the FGUAS property, we call the system to be *prescribed-time, globally uniformly asymptotically stable*.

Our goal is to reconstruct the state before time $t_0 + T$. To this end, it is restrictive to satisfy (14) over the whole interval $[t_0, t_0 + T)$. Therefore, inspired by the eventual stability notion [24], we can relax Definition 1 as follows.

Definition 2 (E-FGUAS): A system $\dot{e} = f(e, t)$ is *eventually fixed-time, globally uniformly asymptotically stable* over an interval $[t_0, t_0 + T)$ if there exists $t_0 \leq t_* < t_0 + T$ such that $\|e(t)\| < \infty$, for all $t \in [t_0, t_*)$, and (14) holds for all $t \in [t_*, t_0 + T)$.

Using Definition 2, we aim to develop a design criterion for the distributed observer to ensure that the error system (10) converges to zero as $t \rightarrow t_0 + T$. Since the time T is prescribed, one can substitute “prescribed-time” for “fixed-time” in the E-FGUAS property. However, showing stability within time T for time-varying systems (10) and (13) constitutes a stability analysis problem rather than a stabilization problem. Consequently, similar to [13], we utilize fixed-time stability terminology and avoid using “prescribed-time stability” in our analysis, though it should be understood implicitly that an observer corresponding to an E-FGUAS error system achieves state reconstruction within the prescribed time T .

¹A function $\beta: \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ is of class \mathcal{KL} if (i) for each fixed $s \in \mathbb{R}_{>0}$, $\beta(\cdot, s)$ is continuous, zero at zero, and strictly increasing; and (ii) for each fixed $r \in \mathbb{R}_{\geq 0}$, $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

V. MAIN RESULT

This section provides a design criterion for the distributed prescribed-time observer (5) that achieves the E-FGUAS property of error dynamics (10). Given the time-varying gains $\Gamma(t)$ and $\mu(t)$ in (6) and (8), one can synthesize local output injection gains L^i , for $i = 1, \dots, N$, using this criterion.

Theorem 1: Given an initial time $t_0 \in \mathbb{R}_{\geq 0}$, the prescribed-time $T \in \mathbb{R}_{> 0}$, and $\mu(t; t_0, T)$ as in (8), suppose there exist symmetric positive definite matrices $P_1, \dots, P_N \in \mathbb{R}^{n \times n}$, row vectors $Q_1, \dots, Q_N \in \mathbb{R}^{1 \times n}$, scalars $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{R}_{> 0}$, and $\mu_* := \mu(t_*)$, for some $t_* \in [t_0, t_0 + T)$, such that

$$\text{sym}(P\bar{A} - Q^\top \bar{H} - P(R\mathcal{L} \otimes I_n)) + \epsilon_1 I_{nN} \leq 0 \quad (15a)$$

$$\text{sym}(P\bar{D}) - \epsilon_2 I_{nN} \geq 0 \quad (15b)$$

$$P - \epsilon_3 I_{nN} \leq 0 \quad (15c)$$

$$\epsilon_2 - \frac{2\epsilon_3 k_f T}{\mu_*(1+m)} \geq 0 \quad (15d)$$

where $P = \text{diag}(P_1, \dots, P_N)$, $Q = \text{diag}(Q_1, \dots, Q_N)$, R is given in (1), $k_f = \sqrt{N \sum_{i=1}^N (n_i \gamma_i)^2}$ with γ_i in (4), and $\bar{A}, \bar{H}, \bar{L}, \bar{D}$ in (12). Then, the error system (10) is E-FGUAS over the interval $[t_0, t_0 + T)$ with the observer gains given by

$$L^i = P_i^{-1} Q_i^\top, \quad i = 1, \dots, N. \quad (16)$$

Before proving the theorem, we provide some remarks regarding the feasibility of (15). Firstly, the feasibility of LMI (15a) is guaranteed by the detectability of the pair $(\bar{H}, \bar{A} - (R\mathcal{L} \otimes I_n))$, which holds when Assumption 2 and Assumption 3 are satisfied [25]. This implies the existence of symmetric positive definite matrices P_1, \dots, P_N satisfying (15a). However, additional requirements, i.e., LMIs (15b) and (15c), and scalar condition (15d), restrict the search space of feasible solutions P , but they are necessary for establishing the E-FGUAS property.

Due to the required prescribed-time convergence, the inequalities (15) may become infeasible at t_0 , where $\mu(t_0) = 1$. Allowing a larger threshold $\mu_* \geq \mu(t_0)$ reduces conservatism in ϵ_2 and ensures feasibility. Since $\mu(t)$ increases to $+\infty$ on $[t_0, t_0 + T)$, it necessarily reaches μ_* at some $t_* \in [t_0, t_0 + T)$, so the resulting design ensures the observer error converges to zero strictly before $t = t_0 + T$. This is the reason why it is generally impossible, as also observed in [13], to establish the FGUAS property of the error system (10) over the interval $[t_0, t_0 + T)$. However, one can show that there exists $t_* \in [t_0, t_0 + T)$ such that the system exhibits E-FGUAS property, ensuring the error $e(t) \rightarrow 0$ as $t \rightarrow t_0 + T$.

Proof of Theorem 1: We perform Lyapunov analysis to show the E-FGUAS property of the transformed error system (13) and then establish the E-FGUAS property of the original error system (10). The proof is constructive in the sense that we derive an explicit upper bound of the distributed state estimation error that converges to zero as $t \rightarrow t_0 + T$.

Let $P = \text{diag}(P_1, \dots, P_N)$ with $P_i \in \mathbb{R}^{n \times n}$ some positive definite matrices. Consider a candidate Lyapunov function

$$V(\zeta(t)) = \zeta(t)^\top P \zeta(t).$$

Define $\Theta := \bar{A} - \bar{L}\bar{H} - (R\mathcal{L} \otimes I_n)$. Then,

$$\begin{aligned} \dot{V}(\zeta(t)) &= \mu^{1+m}(t) \zeta(t)^\top \left(\Theta^\top P + P\Theta \right) \zeta(t) \\ &\quad - (1+m) \frac{\mu(t)}{T} \zeta(t)^\top \left(\bar{D}^\top P + P\bar{D} \right) \zeta(t) \\ &\quad + 2\zeta(t)^\top P \bar{\Gamma}^{-1}(t) \bar{B} \Delta \Phi(t). \end{aligned} \quad (17)$$

From [16, Lemma 1], [26, Lemma 1], and [27, eq. (23)]:

$$\begin{aligned} 2\zeta(t)^\top P \bar{\Gamma}^{-1}(t) \bar{B} \Delta \Phi(t) &\leq 2\lambda_{\max}(P) \|\zeta(t)\| \|\bar{\Gamma}^{-1}(t) \bar{B} \Delta \Phi(t)\| \\ &\leq 2k_f \lambda_{\max}(P) \|\zeta(t)\|^2 \end{aligned}$$

where $k_f = \sqrt{N \sum_{i=1}^N (n_i \gamma_i)^2}$. Using (15a), (15b), and (16), we can upper bound (17) as

$$\begin{aligned} \dot{V}(\zeta(t)) &\leq \left(-\epsilon_1 \mu^{1+m}(t) - \epsilon_2 \frac{(1+m)\mu(t)}{T} \right. \\ &\quad \left. + 2k_f \lambda_{\max}(P) \right) \|\zeta(t)\|^2. \end{aligned} \quad (18)$$

Firstly, to establish that $\|\zeta(t)\|$ remains bounded on $[t_0, t_*)$, we can further upper bound (18) as

$$\dot{V}(\zeta(t)) \leq 2k_f \lambda_{\max}(P) \|\zeta(t)\|^2.$$

Since $V(\zeta) > 0$ for all $\zeta \in \mathbb{R}^{nN} \setminus \{0\}$, by integrating over $[t_0, t)$, for $t_0 \leq t < t_*$, we obtain

$$V(\zeta(t)) \leq V(\zeta(t_0)) \exp\left(\frac{2k_f \lambda_{\max}(P)(t-t_0)}{\lambda_{\min}(P)}\right).$$

Secondly, we show $\zeta(t)$ satisfies (14) for $t \in [t_*, t_0 + T)$. From (15c), we have $\lambda_{\max}(P) \leq \epsilon_3$. Moreover, (15d) guarantees that on the interval $[t_*, t_0 + T)$,

$$\epsilon_2 \frac{(1+m)\mu(t)}{T} \geq 2k_f \epsilon_3 \geq 2k_f \lambda_{\max}(P).$$

Therefore, for $t \in [t_*, t_0 + T)$, (15) implies

$$\dot{V}(\zeta(t)) \leq -\epsilon_1 \mu^{1+m}(t) \|\zeta(t)\|^2.$$

Since $V(\zeta(t)) \leq \lambda_{\max}(P) \|\zeta(t)\|^2$, we have

$$\frac{\dot{V}(\zeta(t))}{V(\zeta(t))} \leq -\frac{\epsilon_1 \mu^{1+m}(t)}{\lambda_{\max}(P)}, \quad \text{for } \zeta(t) \neq 0.$$

Integrating over $[t_*, t)$, for $t_* \leq t \leq t_0 + T$, we have

$$\begin{aligned} \int_{t_*}^t \frac{\dot{V}(\zeta(\tau))}{V(\zeta(\tau))} d\tau &\leq \int_{t_*}^t -\frac{\epsilon_1 \mu^{1+m}(\tau)}{\lambda_{\max}(P)} d\tau \\ &= -\frac{\epsilon_1 T [\mu^m(t) - \mu_*^m]}{m \lambda_{\max}(P)} \end{aligned}$$

where we used the fact that

$$\frac{d}{d\tau} \mu^m(\tau) = \frac{m}{T} \mu^{1+m}(\tau), \quad \mu^m(t_*) = \mu_*^m.$$

Since $\int_{t_*}^t \frac{\dot{V}(\zeta(\tau))}{V(\zeta(\tau))} d\tau = \ln\left(\frac{V(\zeta(t))}{V(\zeta(t_*))}\right)$, we obtain

$$V(\zeta(t)) \leq V(\zeta(t_*)) \exp\left(-\frac{\epsilon_1 T [\mu^m(t) - \mu_*^m]}{m \lambda_{\max}(P)}\right).$$

Using $\lambda_{\min}(P) \|\zeta(t)\|^2 \leq V(\zeta(t)) \leq \lambda_{\max}(P) \|\zeta(t)\|^2$, we obtain

$$\|\zeta(t)\| \leq \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}} \|\zeta(t_*)\| \exp\left(-\frac{\epsilon_1 T [\mu^m(t) - \mu_*^m]}{2m \lambda_{\max}(P)}\right). \quad (19)$$

The right-hand side of (19) can be written as

$$\beta(r, s(t)) = \Omega r \exp(-\alpha s(t)) \quad (20)$$

where $r = \|\zeta(t_*)\|$, $s(t) = \mu^m(t) - \mu_*^m$, $\Omega = \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}$, and $\alpha = \frac{\epsilon_1 T}{2m \lambda_{\max}(P)}$. For each fixed $s(t)$, the function $\beta(r, s(t))$ is strictly increasing in r with $\beta(0, s(t)) = 0$. For each fixed

$r, \beta(r, s(t)) \rightarrow 0$ because $s(t) \rightarrow \infty$ and $\exp(-\alpha s(t)) \rightarrow 0$ as $t \rightarrow t_0 + T$. Hence, β qualifies as a class \mathcal{KL} function, which by Definition 2 implies that the system (13) is E-FGUAS.

Having established the bound for the transformed error $\zeta(t)$, we now consider the original error $e(t) = \bar{\Gamma}(t)\zeta(t)$, where we have $\|e(t)\| \leq \|\bar{\Gamma}(t)\|\|\zeta(t)\|$. From (19), we have

$$\|e(t)\| \leq \Omega \|\bar{\Gamma}(t)\| \|\bar{\Gamma}^{-1}(t_*)\| \|e(t_*)\| \exp(-\alpha[\mu^m(t) - \mu_*^m])$$

where

$$\|\bar{\Gamma}(t)\| \|\bar{\Gamma}^{-1}(t_*)\| = \frac{\mu^p(t)}{\mu_*^{1+m}}$$

with $p = (1 + m) \max(n_1, \dots, n_N)$. Therefore,

$$\|e(t)\| \leq \frac{\Omega \|e(t_*)\|}{\mu_*^{1+m}} \exp(p \ln \mu(t) - \alpha[\mu^m(t) - \mu_*^m]).$$

Since $\mu^m(t) = s(t) + \mu_*^m$, we can rewrite the right-hand side

$$\beta_e(r_e, s) = \frac{\Omega r_e}{\mu_*^{1+m}} (s(t) + \mu_*^m)^{\frac{p}{m}} \exp(-\alpha s(t))$$

where $r_e = \|e(t_*)\|$. Again, for fixed $s(t)$, β_e is strictly increasing in r_e with $\beta_e(0, s(t)) = 0$. Moreover, as $t \rightarrow t_0 + T$, we have $s(t) \rightarrow \infty$ due to $\mu(t) \rightarrow \infty$. Using L'Hôpital's rule, we have that exponential decay dominates polynomial growth:

$$\lim_{s \rightarrow \infty} (s + \mu_*^m)^{\frac{p}{m}} \exp(-\alpha s) = 0$$

implying $\beta_e(r_e, s(t)) \rightarrow 0$ as $t \rightarrow t_0 + T$. Hence, β_e qualifies as a class \mathcal{KL} function, which implies that the original error system (10) is E-FGUAS. ■

Remark 1: The centralized prescribed-time observer in [14] employs a proof strategy with similarities to our approach. However, the proof of [14, Theorem 1] contains several mathematical errors, as described in [16, Appendix B]. Similarly, a distributed prescribed-time observer in [15] relies on a wrong lemma, which is also falsified in [16, Appendix C]. These errors affect the validity of their stated results. We provide this observation to clarify the distinctions between our work and the existing literature, and to ensure mathematical rigor in this developing field.

VI. NUMERICAL SIMULATIONS

To demonstrate the performance of the distributed observer, we consider the example from [12], which is a four-agent ($N = 4$) system of the form (2) given by

$$\begin{aligned} \dot{x}_{11} &= x_{12}, \quad \dot{x}_{12} = -x_{11} + x_{12}(1 - x_{11}^2) \\ \dot{x}_{21} &= -x_{21} - \cos(x_{11}) \\ \dot{x}_{31} &= x_{32}, \quad \dot{x}_{32} = -2x_{31} + \sin(x_{12}) + x_{21} \\ \dot{x}_{41} &= -\frac{x_{41}}{1 + x_{41}^2} + x_{31} \\ y^1 &= x_{11}, \quad y^2 = x_{21}, \quad y^3 = x_{31}, \quad y^4 = x_{41} \end{aligned} \quad (21)$$

where $n_1 = n_3 = 2$ and $n_2 = n_4 = 1$. The agents communicate over a strongly connected directed graph illustrated in Fig. 2, whose adjacency and Laplacian matrices are

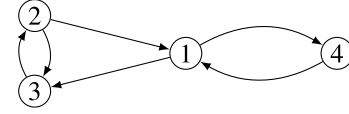


Fig. 2. Directed graph for the example.

$$\mathcal{A} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{L} = \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ -1 & -1 & 2 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

The normalized positive left eigenvector of the Laplacian matrix (see Fact 2) is computed to be $r = [0.8 \ 1.6 \ 0.8 \ 0.8]$.

The system described by (21) is not globally Lipschitz over \mathbb{R}^n . However, (21) is smooth and its trajectories are bounded inside a compact set $\mathcal{X} \subset \mathbb{R}^n$, i.e., it is forward complete within \mathcal{X} . Therefore, as also discussed in [12], the Lipschitz condition is satisfied over the system's state space \mathcal{X} and our approach is valid for this example. The Lipschitz constant of the first subsystem is computed as $\gamma_1 \approx 6$ over $t \in [0, 2]$. The remaining subsystems are globally Lipschitz, with $\gamma_2 = \gamma_4 = \sqrt{2}$, and $\gamma_3 = \sqrt{6}$. Therefore, from [16, Lemma 1], we get $k_f \approx 26.2$.

For the distributed observer, we set the prescribed convergence time $T = 2$ and convergence parameter $m = 2$. For $t_0 = 0$ and $t_* = 1.98$, we obtain $\mu_* = 100$ from (8). We find a feasible solution for the LMIs (15) using YALMIP with Sedumi solver, where we obtain $\epsilon_1 = 0.0045$, $\epsilon_2 = 0.3694$, and $\epsilon_3 = 1.0046$. The local observer gains (16) are

$$\begin{aligned} L^1 &= [5.8117 \ 7.7697 \ 0 \ 0 \ 0 \ 0]^\top, \\ L^2 &= [0 \ 0 \ 1.3578 \ 0 \ 0 \ 0]^\top, \\ L^3 &= [0 \ 0 \ 0 \ 8.7002 \ 9.3939 \ 0]^\top, \\ L^4 &= [0 \ 0 \ 0 \ 0 \ 0 \ 1.1797]^\top. \end{aligned}$$

In the simulation, the initial state of the observed system is set as $x(t_0) = [1 \ 0 \ -1 \ 2 \ 0 \ -2]^\top$ and $z(t_0) = 0_{nN}$.

All simulations are done with `ode15s` and the results are illustrated in Fig. 3. The first plot presents the trajectories of the observed system. The next four plots display the trajectories of the local observer errors. It can be seen that the errors converge to zero for the prescribed time $T = 2$. Finally, the last plot demonstrates convergence of the observer estimate $z_{12}^1(t)$ for different m values for the unmeasured state $x_{12}(t)$. As m increases, the estimate converges faster.

Remark 2 (Implementation): To mitigate numerical challenges arising from the diverging gains Γ, μ near T , practical implementation strategies proposed by [13] include: (i) saturating (e.g., when $\mu_{\max} = 10^{10}$) or disabling the gains once the estimation error reaches an acceptable threshold; (ii) selecting a slightly larger convergence time $T_\delta = T + \delta$ ($\delta > 0$) to avoid singularities at $t = T + t_0$; and (iii) implementing a reset mechanism. Specifically, if at some time t_{stop} we have $\mu(t_{\text{stop}}) = \mu_{\max}$, the observer is reinitialized by setting $t_0 \leftarrow t_{\text{stop}}$ and $z(t_0) \leftarrow z(t_{\text{stop}})$.

VII. CONCLUSION

We presented a design for a distributed prescribed-time observer for nonlinear systems in block-triangular observable canonical form. Each local observer reconstructs the state

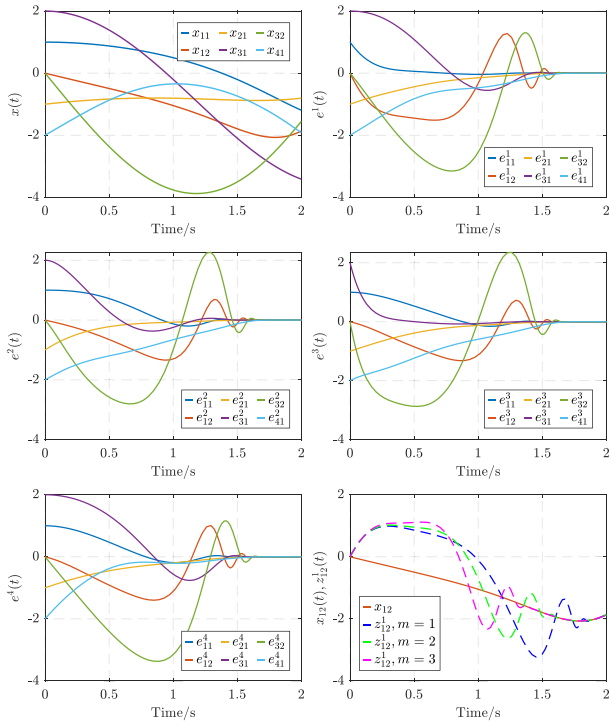


Fig. 3. From top left to bottom right: the observed system, the errors of local observers, the estimates $z_{12}^m(t)$ for various m .

within a user-specified time, regardless of initial conditions, by communicating over a strongly connected graph and utilizing time-varying, monotonically increasing gains. The significance of this letter lies in establishing the first theoretically sound framework for distributed prescribed-time observers for a class of nonlinear systems. Unlike fixed-time approaches, where convergence time depends on observer parameters, our method allows the user to arbitrarily choose the desired convergence time for the distributed observer to reconstruct the state.

Several limitations remain to be addressed in future research. The current framework does not account for measurement noise, model uncertainties, communication delays, or time-varying network topologies. Additionally, exploring extensions to more general nonlinear system classes and investigating the effect of time-varying gains under noise measurements will be addressed next. Future work will also focus on robust versions that maintain performance guarantees under noisy and uncertain conditions. Finally, while the observer operates in a distributed manner, its design procedure requires centralized computation. Developing a fully decentralized design methodology for this distributed observer remains an important direction for future research.

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