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Continuous-time damping-based mirror descent for a class of non-convex multi-player games with coupling constraints

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***Abstract*— We study the computation of the global generalized Nash equilibrium (GNE) for a class of non-convex multi-player games, where players' actions are subject to both local and coupling constraints. Due to the non-convex payoff functions, we employ canonical duality to reformulate the setting as a complementary problem. Under given conditions, we reveal the relation between the stationary point and the global GNE. According to the convex-concave properties within the complementary function, we propose a continuous-time mirror descent to compute GNE by generating functions in the Bregman divergence and the damping-based design. Then, we devise several Lyapunov functions to prove that the trajectory along the dynamics is bounded and convergent.**

I. Introduction

Game theory fully leverages its advantages in multi-player scenarios. The Nash equilibrium (NE) plays a pivotal role, transcending disciplines such as applied mathematics, computer science, engineering, and economics. This paper studies a typical class of non-convex multi-player games. Player i minimizes its own payoff function $J_i(x_i, \boldsymbol{x}_{-i}) : \mathbb{R}^{Nn} \to \mathbb{R}$ impacted by both the player's own decision $x_i \in \mathbb{R}^n$ and others' decisions $\boldsymbol{x}_{-i} \in \mathbb{R}^{(N-1)n}$. Specifically, the players' non-convex payoff structure is

$$J_i(x_i, \boldsymbol{x}_{-i}) = \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})).$$

Here, $\Lambda_i : \mathbb{R}^{Nn} \to \mathbb{R}^{q_i}$ is a vector-valued nonlinear operator, where $\Lambda_i = (\Lambda_{i,1}, \cdots, \Lambda_{i,q_i})^T$ and, for $k \in \{1, \cdots, q_i\}$, $\Lambda_{i,k} : \mathbb{R}^{Nn} \to \mathbb{R}$ is a quadratic function in x_i. Also, $\Psi_i : \mathbb{R}^{q_i} \to \mathbb{R}$ is a canonical function [1] and $\nabla\Psi_i$ is a one-to-one mapping from the primal space into the dual space.

This setting has been extensively explored in engineering applications. For example, in sensor localization tasks [2], [3], [4], x_i is a non-anchor node localization, $\Lambda_{i,k}$ represents the estimated distance between x_i and $\boldsymbol{x}_{-i}$, while Ψ_i represents a Euclidean norm to measure the error of the true distance and the estimated distance. In robust neural network training [5], [6], x_i is the model parameter, $\Lambda_{i,k}$ serves as the output of training data, and Ψ_i represents the cross-entropy function. Moreover, this setting may also serve as inspiration for addressing resource allocation problems in unmanned vehicles [7] and secure transmission [8], where x_i stands for the transmit resources and Ψ_i denotes the transmission cost together with $\Lambda_{i,k}$, which is a logarithmic-posynomial function.

There exist several efficient tools to search for NE in multi-player settings [9], [10], [11], [12]. They are, however, restricted to convex payoff functions. For non-convex settings, it is much harder to a global NE, because the aforementioned methods become trapped in local NE or approximations. Also, constrained problems represent a complex yet compelling research domain. For local constraints, the most prevalent approach involves incorporating projection operations into algorithms. Moreover, coupling constraints present another challenge in multi-agent settings, like resource allocation tasks. In such scenarios, another acceptable concept of players should be the generalized Nash equilibrium (GNE) [13], since players' strategies share common coupling constraints. The primal-dual framework, connected with decoupling techniques, has emerged as a popular methodology [14], [15], [16]. However, the aforementioned approaches often overlook the intricacies inherent within specific constraint structures. Particularly, the time complexity involved in finding optimal solutions with complex or high-dimensional constraints necessitates the development of efficient approaches tailored to special constraint structures, such as the unit simplex and the Euclidean sphere [17].

The mirror descent (MD) method is an established tool to overcome the aforementioned bottleneck. Initially introduced in [18], MD is recognized as a generalization of (sub)gradient methods. By mapping variables into a conjugate space and leveraging the Bregman divergence, MD demonstrates effectiveness in handling constraints with specific structures [19], [20], [21]. This approach yields a faster convergence rate compared to projected (sub)gradient descent algorithms concerning problem dimensions [22], rendering it suitable for addressing large-scale optimization problems. Undoubtedly, as a pivotal tool, MD has played a crucial role in multi-agent settings [23], [24].

The objective of this paper is to design a novel continuous-time MD-based dynamics to compute GNE for a class of non-convex multi-player games, where players' actions are subject to both local and coupling constraints. The main contribution is threefold.

- We reformulate the non-convex setting with constraints by a complementary problem via employing canonical

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duality theory. Under certain conditions on the domain of the dual variables, it is shown that the stationary point of the complementary function is a global GNE of the non-convex game (Theorem 1).
- We propose a continuous-time MD-based dynamics (Algorithm 1) to compute global GNE efficiently. Designing Bregman damping terms in dynamics helps to ensure that all variables are bounded.
- We provide a convergence analysis of the proposed dynamics based on Lyapunov theory and the invariance principle. Through Bregman divergence, we show that the continuous-time dynamics converges to a stationary point of the complementary function (Theorem 2).

The rest of this paper is organized in the following way. Section II formulates the non-convex multi-player game. Section III transforms the non-convex structures into a complementary problem and shows a basic assumption. Section IV proposes a novel damping-based continuous-time MD to solve the problem. Section V shows the convergence analysis of the algorithm. Section VI provides a numerical experiment. Section VII summarizes the paper.

II. Non-convex game formulation

Consider a game $\mathcal{G}$ with multiple players in $\mathcal{I} = \{1, \cdots, N\}$ defined as follows. For $i \in \mathcal{I}$, the ith player has an action variable $x_i \in \mathbb{R}^n$ Let the profile $\boldsymbol{x} = \mathrm{col}\{x_1, ..., x_N\} \in \mathbb{R}^{nN}$, while $\boldsymbol{x}_{-i} = \mathrm{col}\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N\}$ stands for all players' actions except the ith player. For constraints, player i's action x_i is constrained locally in a compact and convex set $\Omega_i \subseteq \mathbb{R}^n$, and thus denote $\boldsymbol{\Omega} = \prod_{i=1}^N \Omega_i$. Also, consider more complex coupling constraints, that is, players' actions are subject to both a convex function $g(\boldsymbol{x}) \leq 0$ and an affined function $h(\boldsymbol{x}) = 0$. The coupling constraints usually come from resource allocation conditions, which means that players' decisions should admit shared resources [16], [11].

The ith player has a payoff function $J_i(x_i, \boldsymbol{x}_{-i}) : \boldsymbol{\Omega} \to \mathbb{R}$, twice continuously differentiable in x_i. Specifically, we are interested in the following non-convex payoff

$$J_i(x_i, \boldsymbol{x}_{-i}) = \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})). \quad (1)$$

Notice that $\Lambda_i : \mathbb{R}^{Nn} \to \Theta_i \subseteq \mathbb{R}^{q_i}$ is a vector-valued operator with $\Lambda_i = (\Lambda_{i,1}, \cdots, \Lambda_{i,q_i})^T$. For $k \in \{1, \cdots, q_i\}$, each $\Lambda_{i,k} : \mathbb{R}^{Nn} \to \mathbb{R}$ is quadratic in x_i, whose second-order partial derivative in x_i is $\boldsymbol{x}$-free, e.g., $\Lambda_{i,k} = x_i^T A_{i,k} x_i + \sum_{i \neq j} x_i^T B_{i,k} x_j$. Additionally, the function $\Psi_i : \Theta_i \to \mathbb{R}$ is called a convex canonical function [1], with gradient $\nabla \Psi_i : \Theta_i \to \Theta_i^*$ being a one-to-one mapping. Such non-convex structures play an important role in many applications like robust network training [5] and sensor localization [2], [4].

Remark 1: Let $\Psi_i = \sum_{j \in \mathcal{N}_s^i} \Lambda_{i,j}^T \Lambda_{i,j}$ and $\Lambda_{i,j} = \|x_i - x_j\|^2 - d_{ij}$. Then, the payoff becomes $J_i(x_i, \boldsymbol{x}_{-i}) = \sum_{j \in \mathcal{N}_s^i} (\|x_i - x_j\|^2 - d_{i,j})^2$, which is the widely accepted payoffs in sensor localization problems [2], [3], [4], where $x_i \in \Omega_i$ is the non-anchor node's location, $\mathcal{N}_s^i$ is its neighbors, and d_{ij} is the distance.

Therefore, given $\boldsymbol{x}_{-i}$, the ith player of game $\mathcal{G}$ tries to solve the following problem:

$$\begin{aligned} &\min_{x_i} \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})) \\ &\text{s.t. } x_i \in \Omega_i, \quad g(\boldsymbol{x}) \leq 0, \quad h(\boldsymbol{x}) = 0. \end{aligned} \quad (2)$$

The following definition describes a solution to this problem.

Definition 1: A strategy profile $\boldsymbol{x}^\diamond \in \boldsymbol{\Omega}$ satisfying $g(\boldsymbol{x}^\diamond) \leq 0$ and $h(\boldsymbol{x}^\diamond) = 0$ is said to be a GNE of game $\mathcal{G}$, if for all $i \in \mathcal{I}$ and for any $x_i \in \Omega_i$ satisfying $g(x_i, \boldsymbol{x}_{-i}^\diamond) \leq 0$ and $h(x_i, \boldsymbol{x}_{-i}^\diamond) = 0$, we have

$$J_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond) \leq J_i(x_i, \boldsymbol{x}_{-i}^\diamond). \quad (3)$$

Within this non-convex game scenario, a generalized Nash equilibrium (GNE) represents a strategic profile where each player adopts their globally optimal strategy. In essence, this concept of GNE aligns with a form of *global* NE, distinct from *local* NE or stationary points [5], [25].

Clearly, in convex games, the predominant approach involves computing (global) NE or GNE by exploring stationary points [9], [11], as these two concepts are equivalent under certain conditions. However, given this rugged geometric complexity of the non-convex payoff function as highlighted by [5], it is unrealistic to rely solely on stationary points to discover a global GNE. Furthermore, due to the interdependence of constraints, seeking a GNE is even more challenging than seeking an NE, as each player must consider not only their own payoff but also the actions of others in relation to shared resources.

With these challenges in mind, our objective in this paper is to ascertain a global GNE for such a non-convex multi-player model 2, and we embark on this exploration in the subsequent discussions.

III. Complementary reformulation

In this section, we will employ the canonical duality theory to transform the non-convex game into a complementary dual problem, and investigate the relationship between the stationary points of the dual problem and the global GNE.

Let $\xi_i = \Lambda_i(x_i, \boldsymbol{x}_{-i}) \in \Theta_i$ in the payoff function of (1) as a canonical measure following the canonical functions. Considering that $\Psi_i(\xi_i)$ is convex, the one-to-one relation $\sigma_i = \nabla \Psi_i(\xi_i) : \Theta_i \to \Theta_i^*$ yields the structure of the conjugate function $\Psi_i^* : \Theta_i^* \to \mathbb{R}$. It is able to be defined uniquely by the Legendre transformation [26]: $\Psi_i^*(\sigma_i) = \xi_i^T \sigma_i - \Psi_i(\xi_i)$. Take $\boldsymbol{\sigma} = \mathrm{col}\{\sigma_1, \cdots, \sigma_N\}$ and $\boldsymbol{\Theta}^* = \prod_{i=1}^N \Theta_i^* \subseteq \mathbb{R}^q$ with $q = \sum_{i=1}^N q_i$. As well, we introduce two multipliers α, β for the coupling constraints. So far, the modified complementary function $\Gamma_i : \boldsymbol{\Omega} \times \Theta_i^* \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, according to the canonical duality theory, is

$$\Gamma_i(\boldsymbol{x}, \sigma_i, \alpha, \beta) = \sigma_i^T \Lambda_i(\boldsymbol{x}) - \Psi_i^*(\sigma_i) + \alpha g(\boldsymbol{x}) + \beta h(\boldsymbol{x}). \quad (4)$$

Briefly, we will design dynamics according to the gradient descent or ascent of Γ_i in the next section. Before designing algorithms, we need to investigate the domain of vector $\boldsymbol{\sigma}$ since the set $\boldsymbol{\Theta}^*$ is not accurate enough to ensure a good structure of Γ_i. For $i \in \mathcal{I}$, let

$$P_i(\sigma_i) = \sum\nolimits_{k=1}^{q_i} [\sigma_i]_k \nabla^2_{x_i} \Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}).$$

Because $\Lambda_i : \boldsymbol{\Omega} \to \Theta_i$ is a quadratic operator and $\nabla^2_{x_i}\Lambda_i$ is $\boldsymbol{x}$-free, $P_i(\sigma_i)$ is linearly combined of $[\sigma_i]_k$. Hence, define the set

$$\mathscr{E}_i^+ = \Theta_i^* \cap \{\sigma_i : P_i(\sigma_i) \succeq \kappa_x \boldsymbol{I}_n\}, \tag{5}$$

where the scalar parameter $\kappa_x > 0$ and $\mathscr{E}^+ = \mathscr{E}_1^+ \times \cdots \times \mathscr{E}_N^+$. Since both Ω_i and Θ_i are convex sets, Θ_i^* is compact consequently. Thus, $\mathscr{E}_i^+$ is compact for $i \in \mathcal{I}$.

When $\sigma_i \in \mathscr{E}_i^+$, the positive definiteness of $P_i(\sigma_i)$ and the convexity of function $\alpha g(\boldsymbol{x})$ imply that Γ_i is convex with respect to x_i. Besides, the convexity of $\Psi_i(\xi_i)$ derives that its Legendre conjugate $\Psi_i^*(\sigma_i)$ is also convex. Hence, the complementary function Γ_i is concave in σ_i. Also, Γ_i is linear in either α or β. This convex-concave property enables us to investigate the optimality of the stationary points of (4).

To rationally use the convex-concave property of Γ_i, we need the following basic assumption.

Assumption 1: Set $\mathscr{E}_i^+$ is nonempty for $i \in \mathcal{I}$.

Based on the above formulation, we obtain the following result to reveal the relation between its stationary point and the global GNE of the non-convex game (2)

Theorem 1: Under Assumption 1 and given a profile $\boldsymbol{x}^\diamondsuit$, if there exists the stationary point $(\boldsymbol{x}^\diamondsuit, \sigma_i^\diamondsuit, \alpha^\diamondsuit, \beta^\diamondsuit)$ of Γ_i, for $i \in \mathcal{I}$, satisfying $\sigma_i^\diamondsuit \in \mathscr{E}_i^+$ and $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\Lambda_i(\boldsymbol{x}^\diamondsuit)\right)$, then $\boldsymbol{x}^\diamondsuit$ is a GNE of game (2).

Proof. For the convenience of investigating the GNE, we need to discuss the action variable within both local constraints and coupling constraints. Given $\boldsymbol{x}_{-i}$, denote the constraint

$$\Xi_i(\boldsymbol{x}_{-i}) = \{x_i \in \Omega_i : \, g(\boldsymbol{x}) \leq 0, \, h(\boldsymbol{x}) = 0\}.$$

Since $(\boldsymbol{x}^\diamondsuit, \sigma_i^\diamondsuit, \alpha^\diamondsuit, \beta^\diamondsuit)$ is a stationary point of Γ_i, it is not hard to know

$$(\sigma_i^{\diamondsuit T} \nabla_{x_i} \Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T (x_i - x_i^\diamondsuit) \geq 0, \quad \forall x_i \in \Xi_i(\boldsymbol{x}_{-i}^\diamondsuit).$$

If $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\Lambda_i(\boldsymbol{x}^\diamondsuit)\right)$, for any $x_i \in \Xi_i(\boldsymbol{x}_{-i}^\diamondsuit)$, we can derive

$$\left(\nabla\Psi_i(\Lambda_i(\boldsymbol{x}^\diamondsuit))^T \nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)\right)^T (x_i - x_i^\diamondsuit) \geq 0,$$

which implies, by the chain rules,

$$(\nabla_{x_i} J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T (x_i - x_i^\diamondsuit) \geq 0, \quad \forall x_i \in \Xi_i(\boldsymbol{x}_{-i}^\diamondsuit). \tag{6}$$

This tells that $\boldsymbol{x}^\diamondsuit$ is a Nash stationary point of the original game (2). Moreover, when $\sigma_i \in \mathscr{E}_i^+$, the positive definiteness of $P_i(\sigma_i)$ and the convexity of function $\alpha g(\boldsymbol{x})$ imply that Γ_i is convex with respect to x_i. Besides, due to the convexity of Ψ_i, its Legendre conjugate Ψ_i^* is also convex [26], [27]. Therefore, the total complementary function Γ_i is concave in canonical dual variable σ_i. In this light, we can obtain the global optimality as below, for $i \in \mathcal{I}$ and $x_i \in \Xi_i(\boldsymbol{x}_{-i}^\diamondsuit)$,

$$\Gamma_i(x_i^\diamondsuit, \sigma_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit, \alpha^\diamondsuit, \beta^\diamondsuit) \leq \Gamma_i(x_i, \sigma_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit, \alpha^\diamondsuit, \beta^\diamondsuit). \tag{7}$$

Get together the relations (6) and (7) and it confirms that $\boldsymbol{x}^\diamondsuit$ is the global GNE of (2), which completes the proof. $\square$

IV. Damping-based mirror descent

In spite of the above complementary function handling the coupled constraints, to design efficient dynamics, we also need to notice the local constraints. Usually, sets Ω_i and $\mathscr{E}_i^+$ can be equipped with structures in specific tasks. We intend to employ conjugate properties of the generating functions within Bregman divergence to design ODE flows.

Take $\phi_i(x_i)$ and $\varphi_i(\sigma_i)$ as two generating functions. In detail, $\phi_i(x_i)$ is μ_x-strongly convex and L_x-smooth on Ω_i, while $\varphi_i(\sigma_i)$ is μ_σ-strongly convex and L_σ-smooth on $\mathscr{E}_i^+$. It follows from the Fenchel inequality [20] that the Legendre conjugates ϕ_i^* and φ_i^* are convex and differentiable. Then, for $y_i \in \mathbb{R}^n$, denote $\phi_i^*(y_i) \triangleq \min_{x_i\in\Omega_i}\{-x_i^T y_i + \phi_i(x_i)\}$. Analogously, for $\nu_i \in \mathbb{R}^{q_i}$, denote $\varphi_i^*(\nu_i) \triangleq \min_{\sigma_i \in \mathscr{E}_i^+}\{-\sigma_i^T \nu_i + \varphi_i(\sigma_i)\}$. Accordingly, their conjugate gradients yield

$$\nabla\phi_i^*(y_i) = \operatorname{argmin}_{x_i\in\Omega_i}\left\{-x_i^T y_i + \phi_i(x_i)\right\}, \tag{8}$$

$$\nabla\varphi_i^*(\nu_i) = \operatorname{argmin}_{\sigma_i\in\mathscr{E}_i^+}\{-\sigma_i^T \nu_i + \varphi_i(\sigma_i)\}. \tag{9}$$

On this basis, for each player $i \in \mathcal{I}$, the continuous-time MD for seeking a global GNE of the non-convex multi-player game (2) is proposed in Algorithm 1. We drop the time t in the dynamics for a concise expression.

Algorithm 1

Input: generating functions ϕ_i on Ω_i and φ_i on $\mathscr{E}_i^+$.
Initialize: for $i \in \mathcal{I}$, $y_i(0) = y_{i0} \in \mathbb{R}^n$, $\nu_i(0) = \nu_{i0} \in \mathbb{R}^{q_i}$, and $s(0) = s_0 \in \mathbb{R}^+$, $\beta(0) = \beta_0 \in \mathbb{R}$.

for $i \in \mathcal{I}$ **do**
  $\dot{y}_i = -\sigma_i^T \nabla_{x_i}\Lambda_i(\boldsymbol{x}) - \beta\nabla_{x_i}h(\boldsymbol{x}) - \alpha\nabla_{x_i}g(\boldsymbol{x}) + \nabla\phi_i(x_i) - y_i$
  $\dot{\nu}_i = \Lambda_i(\boldsymbol{x}) - \nabla\Psi_i^*(\sigma_i) + \nabla\varphi_i(\sigma_i) - \nu_i$
  $x_i = \nabla\phi_i^*$
  $\sigma_i = \nabla\varphi_i^*$
end for
$\dot{s} = g(\boldsymbol{x}) + \alpha - s$
$\dot{\beta} = h(\boldsymbol{x})$
$\alpha = [s]^+$

We provide an elucidation for designing the continuous-time MD Algorithm 1. Firstly, we devise the dynamics for $y_i(t)$ and $\nu_i(t)$ in dual spaces by leveraging the stationary conditions. The dynamic evolution of y_i and ν_i is influenced not only by player i's own decision, but also by the knowledge of other players' decisions $\boldsymbol{x}_{-i}$ contained in Λ_i and its partial derivative. Then $-\partial_{x_i}\Gamma_i = -\sigma_i^T\nabla_{x_i}\Lambda_i(x_i, \boldsymbol{x}_{-i}) - \beta\nabla_{x_i}h(\boldsymbol{x}) - \alpha\nabla_{x_i}g(\boldsymbol{x})$ for the dynamics of y_i and $\partial_{\sigma_i}\Gamma_i = \Lambda_i(x_i, \boldsymbol{x}_{-i}) - \nabla\Psi_i^*(\sigma_i)$ for the dynamics of ν_i represent the directions of gradient descent and ascent along Γ_i in (4), respectively.

Inspired by [24], terms $\nabla\phi_i(x_i)$ and $\nabla\varphi_i(\sigma_i)$ are introduced as *Bregman dampings*. Specifically, they not only serve as modified terms to restrict the update directions of y_i and ν_i, but also play a damping role to avoid y_i and ν_i going to infinity, respectively. Besides, we use conjugate gradients $\nabla\phi_i^*$ and $\nabla\varphi_i^*$, the mapping from dual spaces

TABLE I

CLOSED-FORM CONJUGATE GRADIENTS WITH DIFFERENT GENERATING FUNCTIONS.

	Feasible set	Generating function	Conjugate gradient
General convex set	Ω	$\frac{1}{2}\|x\|_2^2$	$\operatorname{argmin}_{x\in\Omega}\frac{1}{2}\|x-y\|^2$
Non-negative orthant	$\mathbb{R}_+^n$	$\sum_{l=1}^n x^l\log(x^l)-x^l$	$\exp(y)$
Simplex Δ^n	$\{x\in\mathbb{R}_+^n:\sum_{l=1}^n x^l=1\}$	$\sum_{l=1}^n x^l\log(x^l)$	$\mathrm{col}\{\frac{\exp(y^l)}{\sum_{j=1}^n\exp(y^j)}\}_{l=1}^n$
Euclidean sphere $\mathbf{B}_\rho^n(w)$	$\{x\in\mathbb{R}^n:\|x-w\|_2^2\le p\}$	$-\sqrt{p^2-\|x-w\|_2^2}$	$py[\sqrt{1+\|y\|_2^2}]^{-1}-w$

back to primal spaces, to update $x_i(t)$ and $\sigma_i(t)$, which play the role of output feedback in system dynamics. The mappings facilitated by conjugate gradients $\nabla\phi_i^*$ and $\nabla\varphi_i^*$ are established on valid generating functions. This design enables explicit mappings between dual and primal spaces, allowing for flexible management of various constraint conditions and yielding an efficient algorithm for practical computing. Interested readers can refer to Table I for more structures.

Remark 2: Though the proposed continuous-time MD is a centralized algorithm, it is possible to deploy it upon distributed schemes since the setting can be considered as a multi-agent system. Players would have their own counterparts α_i and β_i, and they need to communicate with neighbors to synchronize the multipliers. This modification would result in the pursuit of the concept of variational GNE (vGNE), and additional dynamics for making consensus are required.

V. DYNAMICS ANALYSIS

In this section, we will give a convergence analysis of the proposed continuous-time MD Algorithm 1. We will also point out how to utilize the convergent points to achieve a global GNE of the original non-convex game (2).

Similarly to $\boldsymbol{x}$ and $\boldsymbol{\sigma}$, compactly denote $\boldsymbol{y}\in\mathbb{R}^{nN}$ and $\boldsymbol{\nu}\in\mathbb{R}^q$. Denote the profile of all Λ_i by $\Lambda(\boldsymbol{x})=\mathrm{col}\{\Lambda_i(x_i,\boldsymbol{x}_{-i})\}_{i=1}^N$, and the augmented partial derivative profile by $Q(\boldsymbol{x},\boldsymbol{\sigma})=\mathrm{col}\{\sigma_i^T\nabla_{x_i}\Lambda_i(x_i,\boldsymbol{x}_{-i})\}_{i=1}^N$. Take $\nabla\Psi^*(\boldsymbol{\sigma})=\mathrm{col}\{\nabla\phi_i^*\}_{i=1}^N$, $\nabla g(\boldsymbol{x})=\mathrm{col}\{\nabla_{x_i}g(\boldsymbol{x})\}_{i=1}^N$, and $\nabla h(\boldsymbol{x})=\mathrm{col}\{\nabla_{x_i}h(\boldsymbol{x})\}_{i=1}^N$. Also define $\boldsymbol{z}=(\boldsymbol{x},\boldsymbol{\sigma},\alpha,\beta)$ and the pseudo-gradient

$$F(\boldsymbol{z})=\begin{bmatrix} Q(\boldsymbol{x},\boldsymbol{\sigma})+\alpha\nabla g(\boldsymbol{x})+\beta\nabla h(\boldsymbol{x})\\ -\Lambda(\boldsymbol{x})+\nabla\Psi^*(\boldsymbol{\sigma})\\ g(\boldsymbol{x})\\ h(\boldsymbol{x})\end{bmatrix}. \tag{10}$$

So far, we can get the following convergence result.

Theorem 2: Under Assumption 1, the dynamics of Algorithm 1 is bounded and converges to a stationary point of Γ_i for each $i\in\mathcal{I}$.

Proof. (i) We first prove that the trajectory is bounded along Algorithm 1. Construct a Lyapunov candidate function as

$$\begin{aligned}V_1=&\sum_{i=1}^N D_{\phi_i^*}(y_i,y_i^\diamondsuit)+D_{\varphi_i^*}(\nu_i,\nu_i^\diamondsuit)\\&+\frac{1}{2}\big((s-[s^\diamondsuit]^+)^2-(s-\alpha)^2\big)+\frac{1}{2}\|\beta-\beta^\diamondsuit\|^2,\end{aligned}$$

where Bregman divergences therein are expressed as

$$D_{\phi_i^*}(y_i,y_i^\diamondsuit)=\phi_i^*(y_i)-\phi_i^*(y_i^\diamondsuit)-\nabla\phi_i^*(y_i^\diamondsuit)^T(y_i-y_i^\diamondsuit),$$
$$D_{\varphi_i^*}(\nu_i,\nu_i^\diamondsuit)=\varphi_i^*(\nu_i)-\varphi_i^*(\nu_i^\diamondsuit)-\nabla\varphi_i^*(\nu_i^\diamondsuit)^T(\nu_i-\nu_i^\diamondsuit).$$

Consider $D_{\phi_i^*}(y_i,y_i^\diamondsuit)$ for $i\in\mathcal{I}$. Since $x_i=\nabla\phi_i^*(y_i)$ and $x_i^\diamondsuit=\nabla\phi_i^*(y_i^\diamondsuit)$, it follows from $\nabla\phi_i^*$ in (8) that

$$\phi_i^*(y_i)=x_i^Ty_i-\phi_i(x_i),\quad \phi_i^*(y_i^\diamondsuit)=x_i^{\diamondsuit T}y_i^\diamondsuit-\phi_i(x_i^\diamondsuit). \tag{11}$$

Thus, by (11), we get

$$D_{\phi_i^*}(y_i,y_i^\diamondsuit)=\phi_i(x_i^\diamondsuit)-\phi_i(x_i)-(x_i^\diamondsuit-x_i)^Ty_i.$$

Since ϕ_i is μ_x-strongly convex on Ω_i, we derive

$$D_{\phi_i^*}(y_i,y_i^\diamondsuit)\ge\frac{\mu_x}{2}\left\|x_i-x_i^\diamondsuit\right\|^2+(x_i^\diamondsuit-x_i)^T(\nabla\phi(x_i)-y_i).$$

In fact, $\nabla\phi_i^*(y_i)=\operatorname{argmin}_{x\in\Omega_i}\{-x^Ty_i+\phi_i(x)\}$. Due to the optimality of $\nabla\phi_i^*(y_i)$ and the convexity of ϕ_i,

$$0\le(x_i^\diamondsuit-x_i)^T(\nabla\phi_i(x_i)-y_i). \tag{12}$$

Thus,

$$\sum_{i=1}^N D_{\phi_i^*}(y_i,y_i^\diamondsuit)\ge\frac{\mu_x}{2}\left\|\boldsymbol{x}-\boldsymbol{x}^\diamondsuit\right\|^2.$$

Similarly, we can derive

$$\sum_{i=1}^N D_{\varphi_i^*}(\nu_i,\nu_i^\diamondsuit)\ge\frac{\mu_\sigma}{2}\left\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^\diamondsuit\right\|^2.$$

As a result,

$$V_1\ge\mu(\|\boldsymbol{x}-\boldsymbol{x}^\diamondsuit\|^2+\|\boldsymbol{\sigma}-\boldsymbol{\sigma}^\diamondsuit\|^2+\|\alpha-\alpha^\diamondsuit\|^2+\|\beta-\beta^\diamondsuit\|^2),$$

where $\mu=\min\{\mu_x/2,\mu_\sigma/2,1/2\}$. Therefore, V_1 is positive semi-definite, and $V_1=0$ if and only if $\boldsymbol{x}=\boldsymbol{x}^\diamondsuit$, $\boldsymbol{\sigma}=\boldsymbol{\sigma}^\diamondsuit$, $\alpha=\alpha^\diamondsuit$, and $\beta=\beta^\diamondsuit$. Moreover, V_1 is radially unbounded in $\boldsymbol{x}(t)$, $\boldsymbol{\sigma}(t)$, $\alpha(t)$, and $\beta(t)$.

(ii) We investigate the derivative of V_1 along Algorithm 1.

$$\begin{aligned}\frac{d}{dt}V_1(t)=&\sum_{i=1}^N(x_i-x_i^\diamondsuit)^T\dot{y}_i(t)+\sum_{i=1}^N(\sigma_i-\sigma_i^\diamondsuit))^T\dot{\nu}_i(t)\\&+(\alpha-\alpha^\diamondsuit)(g(\boldsymbol{x})+\alpha-s)+(\beta-\beta^\diamondsuit)h(\boldsymbol{x}).\end{aligned}$$

We employ the compact form for a concise statement. Hence,

$$\begin{aligned}\frac{d}{dt}V_1(t)=&-(\boldsymbol{z}-\boldsymbol{z}^\diamondsuit)^TF(\boldsymbol{z})+(\boldsymbol{x}-\boldsymbol{x}^\diamondsuit)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x})-\boldsymbol{y})\\&+(\boldsymbol{\sigma}-\boldsymbol{\sigma}^\diamondsuit)^T(\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma})-\boldsymbol{\nu}).\end{aligned}$$

Recall that (12) actually reveals

$$(\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit})^T(\nabla\boldsymbol{\phi}(\boldsymbol{x})-\boldsymbol{y}) \leq 0, \quad (\boldsymbol{\sigma}-\boldsymbol{\sigma}^{\diamondsuit})^T\left(\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma})-\boldsymbol{\nu}\right) \leq 0. \tag{13}$$

By the first-order condition of $\boldsymbol{z}^{\diamondsuit}$, we have

$$\left(\boldsymbol{z}-\boldsymbol{z}^{\diamondsuit}\right)^T F(\boldsymbol{z}^{\diamondsuit}) \geq 0. \tag{14}$$

Thus, (13) and (14) yield a further scaling

$$\dot{V}_1 \leq -(\boldsymbol{z}-\boldsymbol{z}^{\diamondsuit})^T(F(\boldsymbol{z})-F(\boldsymbol{z}^{\diamondsuit})). \tag{15}$$

With the guarantee of Assumption 1, the pseudo-gradient F is monotone in $\boldsymbol{z}$, which yields

$$\dot{V}_1 \leq -\kappa_x\|\boldsymbol{x}-\boldsymbol{x}^{\diamondsuit}\|^2 \leq 0. \tag{16}$$

Since V_1 is radially unbounded, the trajectories of $\boldsymbol{x}(t), \boldsymbol{\sigma}(t), \alpha(t)$, and $\beta(t)$ are bounded along the continuous-time MD Algorithm 1.

(iii) We show that $\boldsymbol{y}(t)$, $\boldsymbol{\nu}(t)$ and $s(t)$ are bounded. Take another Lyapunov candidate function as $V_2 = \frac{1}{2}\|\boldsymbol{y}\|^2$, which is radially unbounded in $\boldsymbol{y}$. Along the trajectory, the derivative of V_2 satisfies

$$\dot{V}_2 \leq -\|\boldsymbol{y}\|^2 + p_1\|\boldsymbol{y}\| = -2V_2 + p_1\sqrt{2V_2}.$$

Here p_1 is a positive constant because $\boldsymbol{x}$, $\boldsymbol{\sigma}$, α and β have been proved to be bounded. It can be easily verified that V_2 is bounded, so is $\boldsymbol{y}(t)$. Analogously, we can prove that $\boldsymbol{\nu}(t)$ and $s(t)$ are bounded.

(iv) Now let us investigate the set $\left\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\boldsymbol{\nu},\alpha,s,\beta) : \frac{d}{dt}V_1 = 0\right\}$, and take I_v as its largest invariant subset. It follows from the invariance principle [28, Theorem 2.41] that $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\boldsymbol{\nu},\alpha,s,\beta) \to I_v$ as $t \to \infty$, and I_v is a positive invariant set. Then it follows from the derivation in (16) that $I_v \subseteq \left\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\sigma},\boldsymbol{\nu}) : \boldsymbol{x} = \boldsymbol{x}^{\diamondsuit}\right\}$. This indicates that any trajectory along Algorithm 1 results in $\boldsymbol{x}(t) \to \boldsymbol{x}^{\diamondsuit}$ as $t \to \infty$. Thus, we complete the proof. □

Remark 3: Based on the theoretical results presented above, our work surpasses several obstacles compared to existing research. Unlike convex optimization problems addressed by continuous-time MD in [23], [24], we expand the dynamics design to non-convex settings and effectively compute global GNE under player interaction and interference. In contrast to non-convex game problems [29], [30], we tackle scenarios with coupling constraints to apply the MD approach in a more generalized formulation.

At the end of this section, we summarized the aforementioned procedure to facilitate understanding. First, we check if set $\mathscr{E}_i^+$ is nonempty once the problem is given. Second, we compute the solution via the continuous-time dynamics Algorithm 1, wherein the variable σ_i is restricted on the nonempty $\mathscr{E}_i^+$. Third, we identify that if the convergent point satisfies the duality relation $\sigma_i^{\diamondsuit} = \nabla\Psi_i\left(\Lambda_i(\boldsymbol{x}^{\diamondsuit})\right)$, then it implies a global GNE.

VI. Numerical experiment

Consider a sensor localization problem [2], [3] with $N = 10$ sensor nodes, seeing the illustration in Fig.1. The sensor i's position x_i is unknown while the anchor node k's position a_k

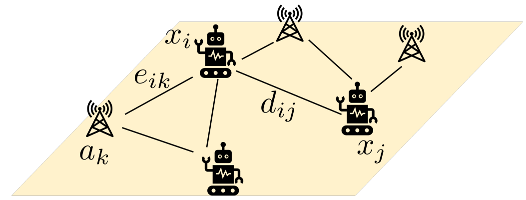


Fig. 1. Illustration of the sensor localization problem setting

is known. For $i \in \mathcal{I}$, the position strategy set Ω_i is equipped with a unit square form $\Omega_i = \{x_i \in \mathbb{R}^2 : x_{\min} \leq x_{il} \leq x_{\max}\}$ for $l = 1, 2$. Also, all position strategies need to be restricted in coupling regions $C_l^- \leq \sum_{i\in\mathcal{I}} x_{il} \leq C_l^+$. The non-convex payoff function is

$$J_i(x_i, \boldsymbol{x}_{-i}) = \sum_{j\in\mathcal{N}_s^i}(\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{k\in\mathcal{N}_a^i}(\|x_i - a_k\|^2 - e_{ik}^2)^2 + \frac{\kappa}{2}\|x_i\|^2.$$

The first term measures the localization accuracy between sensor i and its neighbor $j \in \mathcal{N}_s^i$, The second term is another localization measurement between sensor i and its neighbor anchor $k \in \mathcal{N}_a^i$. The last term is a regularization.

The problem can be regarded as a potential game and all players share the same dual variable σ. Then,

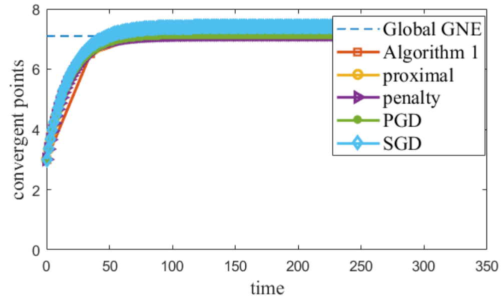
$$\mathscr{E}^+ = \Theta^* \cap \{\sigma : P(\sigma) + \kappa\boldsymbol{I}_{Nn} \succeq \kappa_x\boldsymbol{I}_{Nn}\}. \tag{17}$$

Note that the global GNE $\boldsymbol{x}^{\diamondsuit}$ represents the localization accuracy for all sensors. Deduced by the dual relation $\sigma_i^{\diamondsuit} = \nabla\Psi_i\left(\Lambda_i(\boldsymbol{x}^{\diamondsuit})\right)$, we have

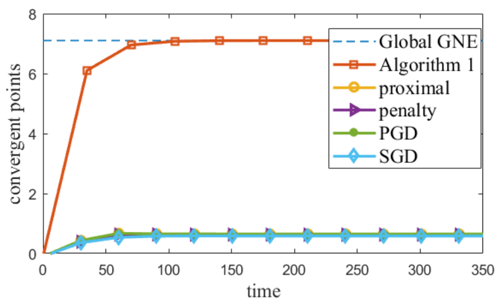
$$\sigma_{ij}^s = 2(\|x_i^{\diamondsuit} - x_j^{\diamondsuit}\|^2 - d_{ij}^2) = 0, \forall(i,j) \in \mathcal{E}_{ss},$$
$$\sigma_{ik}^a = 2(\|x_i^{\diamondsuit} - a_k\|^2 - e_{ik}^2) = 0, \forall(i,k) \in \mathcal{E}_{as}.$$

These indicate that dual variables $\sigma^{\diamondsuit}$ corresponding to the global NE $\boldsymbol{x}^{\diamondsuit}$ is subject to $\sigma^{\diamondsuit} = \mathbf{0}_q$ where $q = |\mathcal{E}_{ss}| + |\mathcal{E}_{as}|$. Since $\mathbf{0}_q \in \mathscr{E}^+$, we can replace $\mathscr{E}^+$ in (17) with a simple unit square constraint $\mathscr{E}^+ = [0, D]^q$ in the practical implementation, where D is a positive constant to reduce the computational complexity. Take generating functions $\phi_i(x_i) = \sum_{l=1}^{2}(x_{i,l} - x_{min})\log(x_{i,l} - x_{min}) + (x_{max} - x_{i,l})\log(x_{max} - x_{i,l})$ and $\varphi(\sigma) = \frac{1}{2}\|\sigma\|_2^2$. The distance parameters d_{ij} are randomly chosen from a compact region $[5, 10]$. We randomly generate three different initial points to check the convergence.

We compare Algorithm 1 with several developed methods for multi-player games, including projected gradient descent (PGD) [11], penalty-based methods [9], stochastic gradient descent (SGD) [31], and gradient-proximal methods [32]. In Fig. 2 (a) and (b), we check two cases in view of a fixed player's decision. In the first case $x_{11}^0 = 3$, all methods locate the sensors accurately. Nevertheless, with the other initial conditions $x_{11}^0 = 0$, the advantage of Algorithm 1 is outstanding that only Algorithm 1 still converges to the global GNE, while other methods can not guarantee this. This validates our theoretical results.



(a) Initialization $x_{11}(0) = 3$



(b) Initialization $x_{11}(0) = 0$

Fig. 2. Convergence results with different initial conditions

VII. Conclusion

We considered a non-convex game with players' both local and coupling constraints. We reformulated the problem with a complementary function and established the connection between stationary points and the global GNE. We proposed a novel continuous-time MD by leveraging Bregman damping. We demonstrated that trajectories along the dynamics remain bounded and convergent. Experiments evaluated our theoretical results. The idea in this paper will naturally guide a distributed design in our ongoing work.

References

[1] D. Y. Gao, V. Latorre, and N. Ruan, *Canonical Duality Theory: Unified Methodology for Multidisciplinary Study*. Springer, 2017.

[2] J. Jia, G. Zhang, X. Wang, and J. Chen, "On distributed localization for road sensor networks: A game theoretic approach," *Mathematical Problems in Engineering*, vol. 2013, 2013.

[3] M. Ke, Y. Xu, A. Anpalagan, D. Liu, and Y. Zhang, "Distributed toa-based positioning in wireless sensor networks: A potential game approach," *IEEE Communications Letters*, vol. 22, no. 2, pp. 316–319, 2017.

[4] S. Yang, Y. Guo, N. Li, and D. Fang, "DF-CSPG: A potential game approach for device-free localization exploiting joint sparsity," *IEEE Wireless Communications Letters*, vol. 8, no. 2, pp. 608–611, 2018.

[5] M. Nouiehed, M. Sanjabi, T. Huang, J. D. Lee, and M. Razaviyayn, "Solving a class of non-convex min-max games using iterative first order methods," *Advances in Neural Information Processing Systems*, vol. 32, 2019.

[6] Y. Deng and M. Mahdavi, "Local stochastic gradient descent ascent: Convergence analysis and communication efficiency," in *International Conference on Artificial Intelligence and Statistics*. PMLR, 2021, pp. 1387–1395.

[7] H. Yang and X. Xie, "Energy-efficient joint scheduling and resource management for UAV-enabled multicell networks," *IEEE Systems Journal*, vol. 14, no. 1, pp. 363–374, 2019.

[8] R. Ruby, V. C. Leung, and D. G. Michelson, "Centralized and game theoretical solutions of joint source and relay power allocation for af relay based network," *IEEE Transactions on Communications*, vol. 63, no. 8, pp. 2848–2863, 2015.

[9] F. Facchinei and C. Kanzow, "Penalty methods for the solution of generalized Nash equilibrium problems," *SIAM Journal on Optimization*, vol. 20, no. 5, pp. 2228–2253, 2010.

[10] P. Yi and L. Pavel, "An operator splitting approach for distributed generalized Nash equilibria computation," *Automatica*, vol. 102, pp. 111–121, 2019.

[11] G. Chen, Y. Ming, Y. Hong, and P. Yi, "Distributed algorithm for ε-generalized Nash equilibria with uncertain coupled constraints," *Automatica*, vol. 123, p. 109313, 2021.

[12] H. Zhang, G. Chen, and Y. Hong, "Distributed algorithm for continuous-type Bayesian Nash equilibrium in subnetwork zero-sum games," *IEEE Transactions on Control of Network Systems*, 2023.

[13] F. Facchinei and C. Kanzow, "Generalized Nash equilibrium problems," *Annals of Operations Research*, vol. 175, no. 1, pp. 177–211, 2010.

[14] X. Zeng, P. Yi, Y. Hong, and L. Xie, "Distributed continuous-time algorithms for nonsmooth extended monotropic optimization problems," *SIAM Journal on Control and Optimization*, vol. 56, no. 6, pp. 3973–3993, 2018.

[15] G. Chen, X. Zeng, and Y. Hong, "Distributed optimisation design for solving the Stein equation with constraints," *IET Control Theory & Applications*, vol. 13, no. 15, pp. 2492–2499, 2019.

[16] S. Liang, X. Zeng, G. Chen, and Y. Hong, "Distributed sub-optimal resource allocation via a projected form of singular perturbation," *Automatica*, vol. 121, p. 109180, 2020.

[17] G. Chen, P. Yi, Y. Hong, and J. Chen, "Distributed optimization with projection-free dynamics: A Frank-Wolfe perspective," *IEEE Transactions on Cybernetics*, vol. 54, no. 1, pp. 599–610, 2024.

[18] A. S. Nemirovskij and D. B. Yudin, *Problem Complexity and Method Efficiency in Optimization*. Wiley, New York, 1983.

[19] W. Krichene, A. Bayen, and P. L. Bartlett, "Accelerated mirror descent in continuous and discrete time," *Advances in Neural Information Processing Systems*, vol. 28, 2015.

[20] J. Diakonikolas and L. Orecchia, "The approximate duality gap technique: A unified theory of first-order methods," *SIAM Journal on Optimization*, vol. 29, no. 1, pp. 660–689, 2019.

[21] A. Ben-Tal, T. Margalit, and A. Nemirovski, "The ordered subsets mirror descent optimization method with applications to tomography," *SIAM Journal on Optimization*, vol. 12, no. 1, pp. 79–108, 2001.

[22] A. Beck and M. Teboulle, "Mirror descent and nonlinear projected subgradient methods for convex optimization," *Operations Research Letters*, vol. 31, no. 3, pp. 167–175, 2003.

[23] Y. Sun and S. Shahrampour, "Distributed mirror descent with integral feedback: Asymptotic convergence analysis of continuous-time dynamics," *IEEE Control Systems Letters*, vol. 5, no. 5, pp. 1507–1512, 2020.

[24] G. Chen, G. Xu, W. Li, and Y. Hong, "Distributed mirror descent algorithm with Bregman damping for nonsmooth constrained optimization," *IEEE Transactions on Automatic Control*, vol. 68, no. 11, pp. 6921–6928, 2023.

[25] M. Heusel, H. Ramsauer, T. Unterthiner, B. Nessler, and S. Hochreiter, "GANs trained by a two time-scale update rule converge to a local Nash equilibrium," *Advances in Neural Information Processing Systems*, vol. 30, 2017.

[26] R. T. Rockafellar, *Conjugate Duality and Optimization*. SIAM, 1974.

[27] D. Yang Gao, "Canonical dual transformation method and generalized triality theory in nonsmooth global optimization," *Journal of Global Optimization*, vol. 17, no. 1, pp. 127–160, 2000.

[28] W. M. Haddad and V. Chellaboina, *Nonlinear Dynamical Systems and Control: A Lyapunov-based Approach*. Princeton University Press, 2011.

[29] G. Chen, G. Xu, F. He, Y. Hong, L. Rutkowski, and D. Tao, "Global Nash equilibrium in non-convex multi-player game: Theory and algorithms," *arXiv preprint arXiv:2301.08015*, 2023.

[30] R. Li, G. Chen, D. Gan, H. Gu, and J. Lü, "Stackelberg and Nash equilibrium computation in non-convex leader-follower network aggregative games," *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 71, no. 2, pp. 898–909, 2024.

[31] P. Mertikopoulos and Z. Zhou, "Learning in games with continuous action sets and unknown payoff functions," *Mathematical Programming*, vol. 173, no. 1, pp. 465–507, 2019.

[32] K. Liu, N. Oudjane, and C. Wan, "Approximate Nash equilibria in large nonconvex aggregative games," *Mathematics of Operations Research*, vol. 48, no. 3, pp. 1791–1809, 2023.