EL3210 Multivariable Feedback Control

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Lecture 3: Introduction to MIMO Control (Ch. 3-4)

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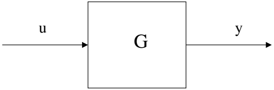
Outline

- Transfer-matrices, poles and zeros
- The closed-loop
- Performance measures and choice of norm
- The Small Gain Theorem and choice of norm
- Generalization of gain: SVD and the condition number
- Eigenvalues and the Generalized Nyquist Criterion
- Introduction to MIMO controller design
- A Generalized Control Problem

Multivariable Systems

Consider a MIMO systems with m inputs and l outputs



- all signals are vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad ; \quad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_l \end{bmatrix}$$

- the $l \times m$ transfer-matrix $G(s) = C(sI - A)^{-1}B + D$ has elements

$$G_{ij}(s) = \frac{y_i(s)}{u_j(s)}$$

- the system is said to be <u>interactive</u> is some input affects several outputs, i.e., $G(s)$ can not be made diagonal.

Poles

The pole polynomial of a system with transfer-matrix $G(s)$ is the **least common denominator** of all **minors** of all orders of $G(s)$. The **poles** of the system are the zeros of the pole polynomial

The system is **input-output stable** if and only if the poles of $G(s)$ are strictly in the complex left half plane.

Note:

- poles of $G(s)$ are also poles of some $G_{ij}(s)$
- poles = eigenvalues of A in the state-space description.
- poles can only be moved by feedback

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Zeros

Definition: z_i is a zero of $G(s)$ if the rank of $G(z_i)$ is less than the normal rank of $G(s)$

The zero polynomial of $G(s)$ is the **greatest common divisor** of all the numerators for the **maximum minors** of $G(s)$, normed so that they have the pole polynomial as the denominator. The zeros of the system are the zeros of the zero polynomial.

Note:

- need only check the determinant for square systems, but make sure denominator equals pole polynomial!
- zeros usually computed from state-space description.
  See S&P, Ch. 4.
- zeros are invariant under feedback and can only be moved by parallell interconnections

Example 1

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{s+3}{s+1} & \frac{2}{s+2} \end{pmatrix}$$

- minors are all elements and the determinant

$$\det G(s) = \frac{1-s}{(s+1)(s+2)}$$

  LCD: $(s+1)(s+2)$, thus poles are $s=-1, s=-2$
- maximum minor, with pole polynomial as denominator, is the determinant, thus zero at $s=1$

Note: there is in general no relation between the zeros of $G(s)$ and the zeros of its elements.

Example 2

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} \\ \frac{-s+3}{s+1} & \frac{2}{s+2} \end{pmatrix}$$

$$\det G(s) = \frac{s+1}{(s+1)(s+2)}$$

- LCD of det G and elements is $(s+1)(s+2)$, hence poles at $s=-1, s=-2$, zero at $s=-1$.
- No cancellation of pole and zero because they have different directions (see below)

Example 3

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{1}{s+2} & \frac{1}{s+2} \\ \frac{s+3}{s+1} & \frac{2}{s+2} & \frac{s-2}{s+1} \end{pmatrix}$$

No zeros. Why?

Zero and Pole Directions

- If z is a zero of $G(s)$ then

$$G(z)u_z = 0 \cdot y_z$$

  where u_z and y_z are the zero input and output directions, respectively

- If p is a pole of $G(s)$ then

$$G(p)u_p = \infty \cdot y_p$$

  where u_p and y_p are the pole input and output directions, respectively

  – Note that $u_p = B^H q$ and $y_p = Ct$ where q and t are the corresponding left and right eigenvectors of A

A Trivial Example

$$G(s) = \begin{pmatrix} \frac{s-1}{s+1} & 0 \\ 0 & \frac{s+1}{s-1} \end{pmatrix}$$

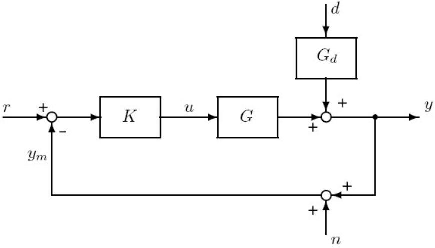
- For zero at $s = 1$

$$u_z = y_z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- For pole at $s = 1$

$$u_p = y_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The Closed Loop System



from block diagram

$$e = -Sr + SG_d d - Tn$$

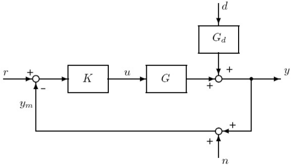
where

$$S = (I + GK)^{-1} \; ; \quad T = GK(I + GK)^{-1}$$

- similar to SISO case, e.g., want magnitude of $S(j\omega)$ "small" for reference tracking and disturbance rejection

  $\Rightarrow$ need scalar measure for size of S and T

Side remark: transfer-functions from block diagrams



To derive transfer-function from an input to an output

1. *start from output and move against the signal flow towards input*
2. *write down the blocks, from left to right, as you meet them*
3. *when you exit a loop, add the term $(I + L)^{-1}$ where L is the loop transfer-function evaluated from exit*
4. *parallell paths should be treated independently and added together*

Also useful, the "push through" rule

$$A(I + BA)^{-1} = (I + AB)^{-1}A$$

Vector (spatial) Norms

- The p-norm for a constant vector

$$\|x\|_p = \left(\Sigma_i |x_i|^p\right)^{1/p}$$

- Most common
  - $p = 1$: sum of absolute values of elements
  - $p = 2$: Euclidian vector length
  - $p = \infty$: maximum absolute value of elements
- Signal perspective: spatial norms essentially ”sum up channels”

Induced Matrix Norms

- Consider the static system $y = Ax$
- The maximum amplification from input x to output y

$$\|A\|_{ip} = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

$\|\cdot\|_{ip}$ - the induced p-norm

  - $p = 1$: $\|A\|_{i1} = \max_j \left(\Sigma_i |a_{ij}|\right)$ (maximum column sum)
  - $p = \infty$: $\|A\|_{i\infty} = \max_i \left(\Sigma_j |a_{ij}|\right)$ (maximum row sum)
  - $p = 2$: $\|A\|_{i2} = \bar{\sigma}(A) = \sqrt{\rho(A^H A)}$ (maximum singular value)

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Temporal (signal) Norms

- The temporal p-norm, or the L_p-norm, of a signal $e(t)$ is defined as

$$\|e(t)\|_p = \left(\int_{-\infty}^{\infty} \Sigma_i |e_i(\tau)|^p d\tau \right)^{1/p}$$

  - $p = 1$: $\|e(t)\|_1 = \int_{-\infty}^{\infty} \Sigma_i |e_i(\tau)| d\tau$
  - $p = 2$: $\|e(t)\|_2 = \sqrt{\int_{-\infty}^{\infty} \Sigma_i |e_i(\tau)|^2 d\tau}$
  - $p = \infty$: $\|e(t)\|_\infty = \sup_\tau \left(\max_i |e_i(\tau)| \right)$

- Signal perspective: temporal norms ”sum up in time”

(Induced) System Norms

System gains for LTI system $y = G(s)u$

	$\|u\|_2$	$\|u\|_\infty$
$\|y\|_2$	$\|G(s)\|_\infty$	∞
$\|y\|_\infty$	$\|G(s)\|_2$	$\|g(t)\|_1$

System norms on diagonal are induced norms

The L_2-gain for LTI systems equals the H_∞-norm

$$\|G(s)\|_\infty = \sup_{u \neq 0} \frac{\|y(t)\|_2}{\|u(t)\|_2} = \sup_\omega \bar{\sigma}(G(i\omega))$$

- $\sup_\omega$ picks maximum frequency, $\bar{\sigma}(\cdot)$ picks maximum direction
- $\bar{\sigma}$ generalizes the concept of frequency dependent amplification

Comparison of $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norm

Given system

$$e = N(s)w$$

- $\|N\|_2 < 1$:
  - $w(t) = \delta(t)$ implies $\|e(t)\|_2 < 1$
  - $\|w(t)\|_2 < 1$ implies $\sup_t |e(t)| < 1$
  - $w(t)$ white noise with unit variance, then variance of $e(t) < 1$
- $\|N\|_\infty < 1$:
  - $w(t) = \sin(\omega t)$ implies $\sup_t |e(t)| < 1$
  - $\|w(t)\|_2 < 1$ implies $\|e(t)\|_2 < 1$

$\mathcal{H}_2$ reasonable choice for nominal performance (cf LQG), $\mathcal{H}_\infty$ looks at worst case but can be used with Small Gain Theorem and therefore useful for robustness analysis.

The Small Gain Theorem

Small Gain Theorem. *Consider a system with a stable loop transfer-function $L(s)$. Then the closed-loop system is stable if*

$$\|L(j\omega)\| < 1 \quad \forall\omega$$

where $\|\cdot\|$ denotes any matrix norm satisfying the multiplicative property $\|AB\| \leq \|A\| \cdot \|B\|$

- The maximum singular value $\bar{\sigma}(L)$ satisifies the multiplicative property
- We will prove the Thm later when we consider robustness

MIMO Frequency Domain Analysis

frequency response (in phasor notation)

$$y(\omega) = G(j\omega)u(\omega)$$

- **Amplification for SISO system:**

$$\frac{|y(\omega)|}{|u(\omega)|} = \frac{y_0}{u_0} = |G(j\omega)|$$

  - depends on **frequency** ω **only**

- **Amplification for MIMO system:** define amplification as

$$\frac{\|y(\omega)\|_2}{\|u(\omega)\|_2}$$

  - depends on **frequency** ω and on **direction** of input $u(\omega)$

Static Example

$$G(0) = \begin{pmatrix} 1 & -0.9 \\ 2 & -2.1 \end{pmatrix}$$

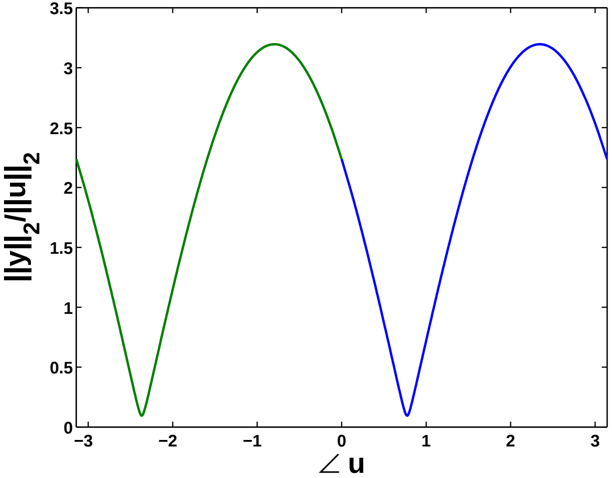
$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix} \quad : \quad \frac{\|y\|_2}{\|u\|_2} = 0.1$$

$$u = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow y = \begin{pmatrix} 1.9 \\ 4.1 \end{pmatrix} \quad : \quad \frac{\|y\|_2}{\|u\|_2} = 3.2$$

- amplification varies with at least a factor 32 with input direction

Example cont'd

- amplification as a function of input direction



Maximum and Minimum Gains (for fixed ω)

Maximum amplification:

$$\max_{u \neq 0} \frac{\|y(\omega)\|_2}{\|u(\omega)\|_2} = \bar{\sigma}\left(G(j\omega)\right)$$

$\bar{\sigma}$ – the maximum singular value

Minimum amplification:

$$\min_{u \neq 0} \frac{\|y(\omega)\|_2}{\|u(\omega)\|_2} = \underline{\sigma}\left(G(j\omega)\right)$$

$\underline{\sigma}$ – the minimum singular value

Thus,

$$\underline{\sigma}\left(G(j\omega)\right) \leq \frac{\|y(\omega)\|_2}{\|u(\omega)\|_2} \leq \bar{\sigma}\left(G(j\omega)\right)$$

Singular Value Decompositon – SVD

Let $G = G(j\omega)$ at a fixed ω. SVD of G

$$G = U\Sigma V^H$$

- $\Sigma = diag(\sigma_1, \sigma_2, \ldots \sigma_k),\ k = min(l, m)$
  - $\bar{\sigma} = \sigma_1 > \sigma_2 > \ldots > \sigma_k = \underline{\sigma}$ – singular values
- $U = (u_1, u_2, \ldots, u_l)$
  - u_i - orthonormal output singular vectors (output directions)
- $V = (v_1, v_2, \ldots, v_m)$
  - v_i - orthonormal input singular vectors (input directions)

Input-output interpretation

$$Gv_i = \sigma_i u_i$$

input in direction v_i gives output in direction u_i with amplification σ_i

SVD of Example

$$G(0) = \begin{pmatrix} 1 & -0.9 \\ 2 & -2.1 \end{pmatrix}$$

SVD yields

$$U = \begin{pmatrix} -0.42 & -0.91 \\ -0.91 & 0.42 \end{pmatrix}; \ \Sigma = \begin{pmatrix} 3.20 & 0 \\ 0 & 0.093 \end{pmatrix} \ ; \ V = \begin{pmatrix} -0.70 & -0.71 \\ 0.71 & -0.70 \end{pmatrix}$$

- thus, moving inputs in opposite directions has large effect and moves outputs in the same direction

The Condition Number

$$\gamma(G) = \frac{\bar{\sigma}(G)}{\underline{\sigma}(G)}$$

- a condition number $\gamma(G) >> 1$ implies strong directional dependence of input-output gain: *ill-conditioned system*
- to compensate for ill-conditioning, controller must also have widely differing gains in different directions; sensitive to model uncertainty
- scaling dependent ill-conditioning may not be a problem, e.g.,

$$G = \begin{pmatrix} 100 & 0 \\ 0 & 1 \end{pmatrix}$$

  has $\gamma = 100$, but can be reduced to 1 by scaling inputs/outputs
- minimized condition number often used instead

$$\gamma^*(G) = \min_{D_1, D_2} \gamma\left(D_1 G D_2\right)$$

SVD generalizes the concept of amplification, but not phase

- singular values generalize the concept of amplification (amplitude)
- but, no similar definition of phase for singular values
- however, phase can be generalized if we instead consider the eigenvalues λ_i of G

  $$Gu_{xi} = \lambda_i u_{xi}$$

  $\arg \lambda_i$ gives phase lag for eigenvector direction u_{xi}
- eigenvalues of transfer-matrices useful for analysis of closed-loop stability

Generalized Nyquist Theorem

Theorem 4.9 *Let P_{ol} denote the number of open-loop RHP poles in the loop gain $L(s)$. Then the closed-loop system $(I + L(s))^{-1}$ is stable iff the Nyquist plot of det(I+L(s))*

(i) *makes P_{ol} anti-clockwise encirclements of the origin, and*

(ii) *does not pass through the origin*

- Proof: note that $\det(I + L(s)) = c\frac{\phi_{cl}(s)}{\phi_{ol}(s)}$ (see S&P, p. 151) and apply Argument Variation Principle
- plot of det $(I + L(j\omega))$ for $\omega \in [-\infty, \infty]$ is the generalized version of the *Nyquist plot*.
- note that the critical point is 0 with this definition.

Eigenvalue loci

- the determinant can be written

$$\det(I + L) = \prod_i (1 + \lambda_i(L))$$

- change in argument (phase) as s traverses the Nyquist contour

$$\Delta \arg det\,[1 + L(j\omega)] = \sum_i \Delta \arg\,(1 + \lambda_i(j\omega))$$

- thus, can count the total number of encirclements of the origin made by all the graphs of $1 + \lambda_i(j\omega)$, or equivalently, the encirclements of -1 made by all $\lambda_i(j\omega)$
- the Nyquist plot of $\lambda_i(L)$ are called *eigenvalue loci*

Why not eigenvalues for gain?

- eigenvalues are "gains" for the special case that the inputs and outputs are completely aligned (same direction); not too useful for performance.
- also, generalization of gain should satisfy *matrix norm* properties
  - $\|G_1 + G_2\| \leq \|G_1\| + \|G_2\|$ - *triangle inequality*
  - $\|G_1 G_2\| \leq \|G_1\| \|G_2\|$ - *multiplicative property*

The **maximum eigenvalue** $\rho(G) = |\lambda_{max}(G)|$ (spectral radius) is **not a norm**

Singular values for performance

Recall that the control error for setpoints is given by

$$e = Sr$$

hence

$$\underline{\sigma}\left(S(j\omega)\right) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \bar{\sigma}\left(S(j\omega)\right)$$

- thus, to keep error "small" for all directions of setpoint r we require $\bar{\sigma}\left(S(j\omega)\right)$ small
- more generally, introduce a frequency-dependent performance weight $w_P(s)$ such that performance requirement is

$$\frac{\|e\|_2}{\|r\|_2} \leq \frac{1}{|w_P(j\omega)|}\ \forall\omega \quad \Leftarrow \quad \bar{\sigma}(S) \leq \frac{1}{|w_P|}\ \forall\omega \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1$$

Introduction to Multivariable Control Design

- diagonal (decentralized control)

$$K(s) = diag\,(k_1(s)\; k_2(s) \ldots k_m(s))$$

  - no attempt to compensate for directionality in $G(s)$
- decoupling control

$$K(s) = k(s)G^{-1}(s)$$

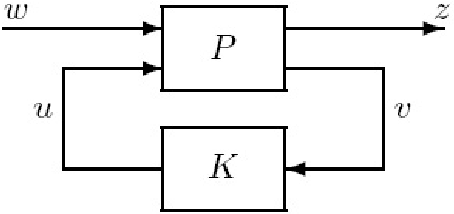
  - full compensation for directionality in $G(s)$
- "cheap" disturbance compensation, $e = SG_d d$

$$\bar{\sigma}(SG_d) = 1\ \forall\omega \quad \Rightarrow \quad SG_d = U_1 \quad \texttt{s.t.} \quad \bar{\sigma}(U_1) = \underline{\sigma}(U_1) = 1$$

yields $K(s) = G^{-1}(s)G_d(s)U_1^{-1}(s)$

  - does in general not provide decoupling

General Control Problem Formulation



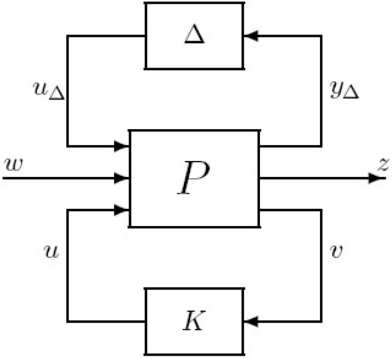
Design aim: find controller K that minimizes some norm of the transfer-function from w to z

1. *signal based approach*, e.g., $w = [r\ d\ n]^T$ and $z = [e\ u]^T$
2. *shaping the closed-loop*, e.g., minimize $\| [w_P S \;\; w_T T]^T \|$. Identify z and w so that

$$z = [w_P S \;\; w_T T]\, w$$

See S&P on how to derive P for the two cases

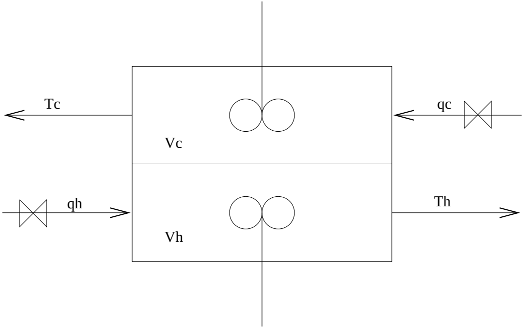
Including uncertainty in the formulation



minimize norm of transfer-function from w to z in the presence of the uncertainty $\Delta(s)$ with bound $\|\Delta\|_\infty \leq 1$

more on this later

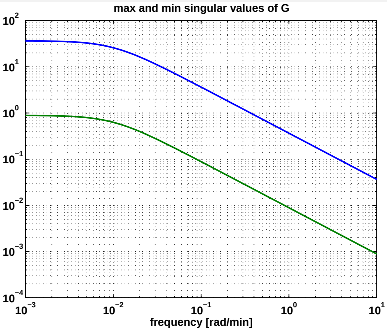
The role of uncertainty - control of heat-exchanger



- *Problem:* control temperatures T_C and T_H using flows q_C and q_H.
- *Model:*

$$\begin{pmatrix} T_c \\ T_H \end{pmatrix} = \frac{1}{100s+1} \begin{pmatrix} -18.74 & 17.85 \\ -17.85 & 18.74 \end{pmatrix} \begin{pmatrix} q_C \\ q_H \end{pmatrix}$$

Singular values of plant



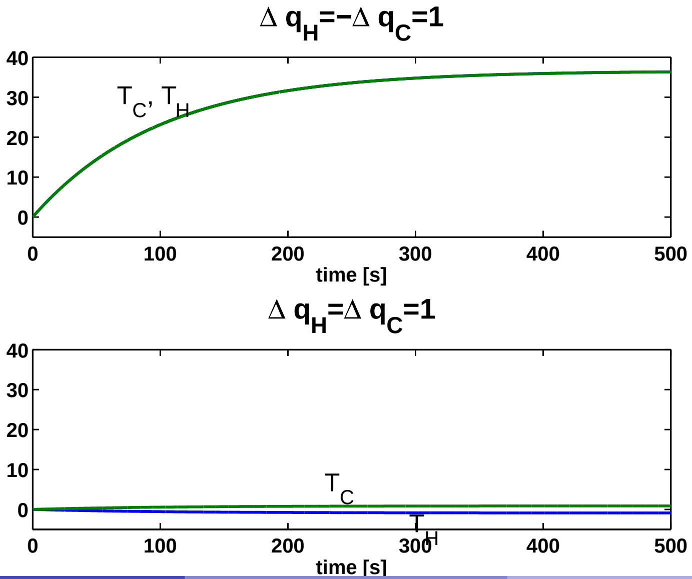
High-gain direction:

$$\bar{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \Rightarrow \quad \bar{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Low-gain direction:

$$\underline{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \underline{u} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Step Responses



Decentralized control

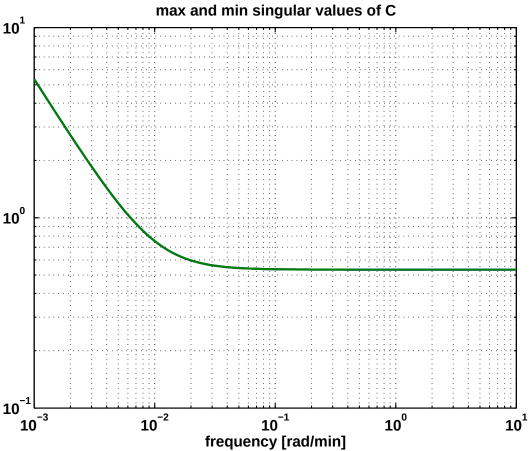
Employ controller

$$C(s) = \begin{pmatrix} c_1(s) & 0 \\ 0 & c_2(s) \end{pmatrix}$$

and use inverse based loop shaping for each loop,

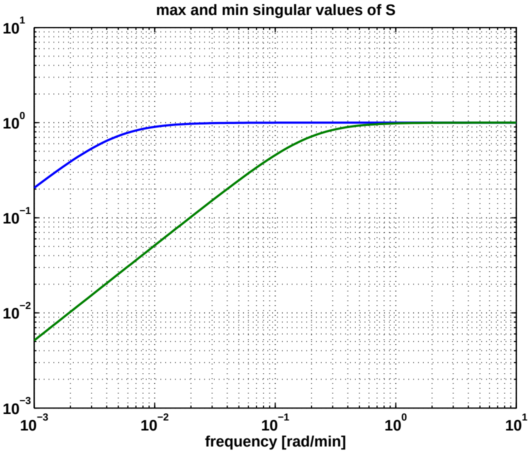
$$c_i(s) = \frac{\omega_c}{s} \frac{1}{g_{ii}(s)} \; ; \quad \omega_c = 0.1$$

Singular values of decentralized controller



same gain in all directions, no compensation for directionality in G

Singular values of sensitivity function



poor performance in some directions

Decoupling control

- Employ decoupler

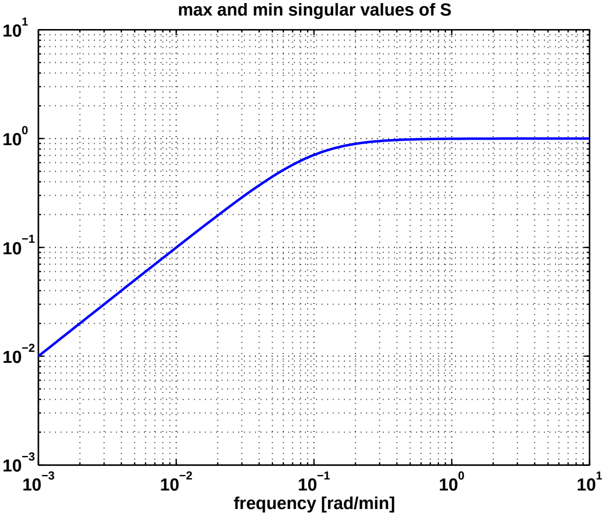
$$C(s) = \frac{\omega_c}{s} G^{-1}(s)$$

- Compensates for plant directionality by employing high (low) gain in low-gain (high-gain) direction of plant.
- Yields for sensitivity

$$S = \frac{s}{s + \omega_c} I$$

  i.e., same sensitivity in all directions.
- Excellent (nominal) performance.

Singular values of sensitivity function



Good performance in all directions

Impact of uncertainty

Assume model is uncertain such that

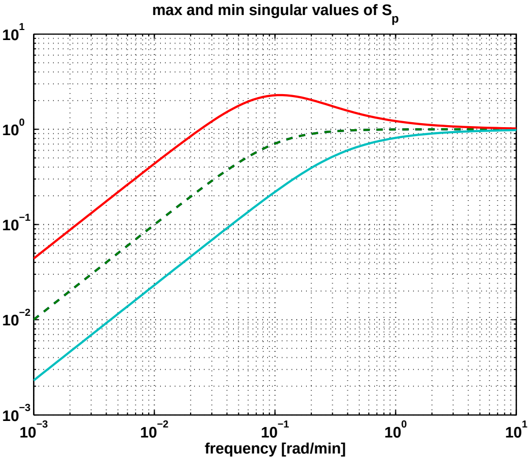
$$G_p = G(I + \Delta); \quad \Delta = \begin{pmatrix} 0.1 & 0 \\ 0 & -0.1 \end{pmatrix}$$

Corresponds to 10% input uncertainty:

$$q_H = 1.1 q_{Hc} \quad q_C = 0.9 q_{Cc}$$

- Note: all variables are deviations from nominal values, so uncertainty is on the change of the flows

Singular values of S_p



small uncertainty completely ruins performance (but no problems with stability)

Program

- Next lecture: inherent limitations in MIMO control (Ch.6)
- Lectures 5-8:
  - modeling uncertainty, analysis of robust stability (Ch. 7-8)
  - analysis of robust performance (Ch.8)
  - design/synthesis for robust stability and performance (Ch.9-10)
  - LMI formulations of robust control problems, control structure design, course summary