ASYMPTOTIC EXPANSION OF POLYANALYTIC BERGMAN KERNELS

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Abstract. It is well-known from the work of Tian, Yau, Zelditch, Catlin, that the Bergman kernel with respect to the weight $e^{-mQ}$ has an asymptotic expansion as $m \to +\infty$. In the setting of one complex variable, we extend these results to spaces of $q$-analytic functions, where a function $f$ is $q$-analytic if $\bar{\partial}^q f = 0$ for the given positive $q$. As a $q$-analytic function may be identified with a vector-valued holomorphic function, the Bergman space of $q$-analytic functions can be understood as a vector-valued holomorphic Bergman space supplied with a certain singular local metric on the vectors.

1. Introduction

1.1. Basic notation. We let $\mathbb{C}$ denote the complex plane and $\mathbb{R}$ the real line. For $z_0 \in \mathbb{C}$ and positive real $r$, let $D(z_0, r)$ be the open disk centered at $z_0$ with radius $r$; moreover, we let $T(z_0, r)$ be the boundary of $D(z_0, r)$ (which is a circle). We let

$$\Delta = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right), \quad dA(z) := dx dy,$$

denote the normalized Laplacian and the area element, respectively. Here, $z = x + iy$ is the standard decomposition into real and imaginary parts. The complex differentiation operators

$$\partial_z := \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \bar{\partial}_z := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right),$$

will be useful. It is well-known that $\Delta = \bar{\partial}_z \bar{\partial}_z$.

1.2. The weighted Bergman spaces. Let $\Omega$ be a domain in $\mathbb{C}$, and let $\omega : \Omega \to \mathbb{R}_+$ be a continuous function ($\omega$ is frequently called a weight). Here, we write $\mathbb{R}_+ := [0, +\infty[$ for the positive half-axis. The space $L^2(\Omega, \omega)$ is the weighted $L^2$-space on $\Omega$ with finite norm

$$(1.1) \quad \|f\|_{L^2(\Omega, \omega)}^2 := \int_{\Omega} |f(z)|^2 \omega(z) dA(z),$$

and associated sesquilinear inner product

$$(1.2) \quad \langle f, g \rangle_{L^2(\Omega, \omega)} := \int_{\Omega} f(z)\overline{g(z)} \omega(z) dA(z).$$

The corresponding weighted Bergman space $A^2(\Omega, \omega)$ is the linear subspace of $L^2(\Omega, \omega)$ consisting of functions holomorphic in $\Omega$, supplied with the inner product structure of $L^2(\Omega, \omega)$. Given the assumptions on the weight $\omega$, it is easy to check that point evaluations are locally uniformly bounded on $A^2(\Omega, \omega)$, and, therefore, $A^2(\Omega, \omega)$ is a norm-closed subspace of $L^2(\Omega, \omega)$. As $L^2(\Omega, \omega)$ is separable, so is $A^2(\Omega, \omega)$, and we may find a countable orthonormal basis.
\( \phi_1, \phi_2, \phi_3, \ldots \) in \( A^2(\Omega, \omega) \). We then form the function \( K = K_{\Omega, \omega} \) – called the weighted Bergman kernel – given by

\[
K(z, w) := \sum_{j=1}^{+\infty} \phi_j(z) \overline{\phi_j(w)}, \quad (z, w) \in \Omega \times \Omega,
\]

and observe that for fixed \( w \in \Omega \), the function \( K(\cdot, w) \in A^2(\Omega, \omega) \) has the reproducing property

\[
f(w) = \langle f, K(\cdot, w) \rangle_{L^2(\Omega, \omega)}, \quad w \in \Omega.
\]

Here, it is assumed that \( f \in A^2(\Omega, \omega) \). In fact, the weighted Bergman kernel \( K \) is uniquely determined by these two properties, which means that \( K \) – initially defined by (1.3) in terms of an orthonormal basis – actually is independent of the choice of basis.

1.3. Geometric considerations. Stefan Bergman [5] considered the Bergman kernel for the weight \( \omega(z) \equiv 1 \) only. He also introduced the so-called Bergman metric in two different ways. We will now discuss the ramifications of Bergman’s ideas in the presence of a non-trivial weight \( \omega \). We interpret the introduction of the weight \( \omega \) as equipping the domain \( \Omega \) with the isothermal Riemannian metric and associated two-dimensional volume form

\[
ds_\omega(z)^2 := \omega(z)|dz|^2, \quad dA_\omega(z) := \omega(z)dA(z).
\]

Bergman’s first metric on \( \Omega \) is then given by

\[
ds_{\omega, 1}^0(z)^2 := K(z, z)ds_\omega(z)^2 = K(z, z)\omega(z)|dz|^2,
\]

where \( K = K_{\Omega, \omega} \) is the weighted Bergman kernel given by (1.3). The second metric is related to the Gaussian curvature. The curvature form for the original metric (1.5) is (up to a positive constant factor) given by

\[
x := -\Delta \log \omega(z)dA(z),
\]

The curvature form for Bergman’s first metric (1.6) is similarly

\[
x^0 := -\Delta \log[K(z, z)\omega(z)]dA(z) = -[\Delta \log K(z, z) + \Delta \log \omega(z)]dA(z),
\]

and we are led to propose the difference

\[
x - x^0 = (\Delta \log K(z, z))dA(z),
\]

as the two-dimensional volume form of a metric, Bergman’s second metric:

\[
ds_{\omega, 2}^0(z)^2 := \Delta \log K(z, z)|dz|^2.
\]

We should remark that unless \( K(z, z) \equiv 0 \), the function \( z \mapsto \log K(z, z) \) is subharmonic in \( \Omega \). This is easily seen from the identity

\[
K(z, z) = \sum_{j=1}^{+\infty} |\phi_j(z)|^2,
\]

so \( \Delta \log K(z, z) \geq 0 \) holds throughout \( \Omega \). This means that we can expect (1.10) to define a Riemannian metric in \( \Omega \) except in very degenerate situations. Bergman’s second metric appears to be the more popular metric the several complex variables setting (see, e.g., Chapter 3 of [12]). In the case of the unit disk \( \Omega = \mathbb{D}(0, 1) \) and the constant weight \( \omega(z) \equiv 1/(2\pi) \), we find that

\[
K(z, w) = \frac{4}{(1 - z\overline{w})^{2}},
\]
so that
\[ ds^2(z) = \frac{2|dz|^2}{(1 - |z|^2)^2} \quad \text{and} \quad ds^2(\bar{z}) = \frac{2|d\bar{z}|^2}{(1 - |\bar{z}|^2)^2}, \]
which apparently coincide. This means that the first and second Bergman metrics are the same in this model case (the reason is that the curvature of the first Bergman metric equals \(-1\)).

1.4. Weighted polyanalytic Bergman spaces. Given an integer \( q = 1, 2, 3, \ldots \), a continuous function \( f : \Omega \to \mathbb{C} \) is said to be \( q\)-analytic (or \( q\)-holomorphic) in \( \Omega \) if it solves the partial differential equation
\[ \partial_z^q f(z) = 0, \quad z \in \Omega, \]
in the sense of distribution theory. So the 1-analytic functions are just the ordinary holomorphic functions. A function \( f \) is said to be polyanalytic if it is \( q\)-analytic for some \( q \); then the number \( q - 1 \) is said to be the polyanalytic degree of \( f \). By solving the \( \bar{\partial} \)-equation repeatedly, it is easy to see that \( f \) is \( q\)-holomorphic if and only if it can be expressed in the form
\[ f(z) = f_1(z) + z f_2(z) + \cdots + z^{q-1} f_q(z), \]
where each \( f_j \) is holomorphic in \( \Omega \), for \( j = 1, \ldots, q \). So the dependence on \( \bar{z} \) is polynomial of degree at most \( q - 1 \). We observe quickly that the vector-valued holomorphic function
\[ V[f](z) := (f_1(z), f_2(z), \ldots, f_q(z)), \]
is in a one-to-one relation with the \( q\)-analytic function \( f \). We will think of \( V[f](z) \) as a column vector. In a way, this means that we may think of a polyanalytic function as a vector-valued holomorphic function supplied with the additional structure of scalar point evaluations \( \mathbb{C}^q \to \mathbb{C} \) given by
\[ (f_1(z), f_2(z), \ldots, f_q(z)) \mapsto f_1(z) + z f_2(z) + \cdots + z^{q-1} f_q(z). \]
We associate to \( f \) not only the vector-valued holomorphic function \( V[f] \) but also the function of two complex variables
\[ E[f](z, z') = f_1(z) + z' f_2(z) + \cdots + (z')^{q-1} f_q(z), \]
which we call the extension of \( f \). The function \( E[f](z, z') \) is holomorphic in \((z, z') \in \Omega \times \mathbb{C}, \) with polynomial dependence on \( z' \). To recover the function \( f \), we just restrict to the diagonal:
\[ E[f](z, z) = f(z), \quad z \in \Omega. \]
For some general background material on polyanalytic functions, we refer to the book [6].

As before, we let the weight \( \omega : \Omega \to \mathbb{R}^+ \) be continuous, and define \( \text{PA}^q_1(\Omega, \omega) \) to be the linear subspace of \( L^2(\Omega, \omega) \) consisting of \( q\)-analytic functions in \( \Omega \). Then \( \text{PA}^q_1(\Omega, \omega) = A^2(\Omega, \omega) \), the usual weighted Bergman space we encountered in Subsection 1.2. For general \( q = 1, 2, 3, \ldots \), it is not difficult to show that point evaluations are locally uniformly bounded on \( \text{PA}^q_1(\Omega, \omega) \), and therefore, \( \text{PA}^q_1(\Omega, \omega) \) is a norm-closed subspace of \( L^2(\Omega, \omega) \). We will refer to \( \text{PA}^q_1(\Omega, \omega) \) as a weighted \( q\)-analytic Bergman space, or as a weighted polyanalytic Bergman space of degree \( q - 1 \). If we let \( \phi_1, \phi_2, \phi_3, \ldots \) be an orthonormal basis for \( \text{PA}^q_1(\Omega, \omega) \), we can form the weighted polyanalytic Bergman kernel \( K_q = K_q(\Omega, \omega) \) given by
\[ K_q(z, w) := \sum_{j=1}^{\infty} \phi_j(z) \overline{\phi_j(w)}, \quad (z, w) \in \Omega \times \Omega. \]
As was the case with the weighted Bergman kernel, \( K_q \) is independent of the choice of basis \( \phi_j \), \( j = 1, 2, 3, \ldots \), and has the reproducing property
\[ f(w) = (f, K_q(\cdot, w))_{L^2(\Omega, \omega)} \quad w \in \Omega, \]
for all $f \in \mathbb{P}A_q^2(\Omega, \omega)$. Alongside with the kernel $K_q$, we should also be interested in its lift

$$(1.17) \quad E_{\mathbb{P}A_2}[K_q](z, z'; w, w') := \sum_{j=1}^{+\infty} E[\phi_j](z, z') E[\phi_j](w, w'), \quad (z, z', w, w') \in \Omega \times \mathbb{C} \times \Omega \times \mathbb{C}. $$

The lifted kernel $E_{\mathbb{P}A_2}[K_q]$ is also independent of the choice of basis, just like the kernel $K_q$ itself.

1.5. The two polyanalytic Bergman metrics. We continue the setting of the preceding subsection. Following in the footsteps of Bergman (see Subsection 1.3), we would like to introduce polyanalytic analogues of Bergman’s first and second metrics, respectively. Let us first discuss the interpretation in terms of jet manifolds. Following in the footsteps of Bergman (see Subsection 1.3), we would like to introduce polyanalytic analogues of Bergman’s first and second metrics, respectively. Let us first discuss

$$(1.18) \quad E_{\mathbb{P}A_2}[K_q](z, z'; z, z') := \sum_{j=1}^{+\infty} |E[\phi_j](z, z')|^2, \quad (z, z') \in \Omega \times \mathbb{C}. $$

If we know just the restriction to the double diagonal $z = w$ and $z' = w'$ of $E_{\mathbb{P}A_2}[K_q]$ we are able to recover the full kernel lifted kernel $E_{\mathbb{P}A_2}[K_q]$. If we put $z' = z + \epsilon$, where $\epsilon \in \mathbb{C}$, we may expand the extension of $\phi_j$ in a finite Taylor series:

$$(1.19) \quad E[\phi_j](z, z + \epsilon) = \sum_{k=0}^{q-1} \frac{1}{k!} \frac{\partial^k \phi_j(z) \epsilon^k}{k!}. $$

As we insert (1.19) into (1.18), the result is

$$(1.20) \quad E_{\mathbb{P}A_2}[K_q](z, z + \epsilon; z, z + \epsilon) = \sum_{k\ell=0}^{q-1} \epsilon^k \epsilon^\ell \frac{\partial^k \phi_j(z) \partial^\ell \phi_j(z)}{k!\ell!} \epsilon^k \epsilon^\ell \phi_j(z) \omega(z)|dz|^2, \quad (z, \epsilon) \in \Omega \times \mathbb{C}. $$

So, to generalize the first Bergman metric we consider the family of (possibly degenerate) metrics

$$(1.21) \quad ds^\omega_{q; \epsilon}(z)^2 := E_{\mathbb{P}A_2}[K_q](z, z + \epsilon; z, z + \epsilon) \omega(z)|dz|^2 = \sum_{k\ell=0}^{q-1} \epsilon^k \epsilon^\ell \frac{\partial^k \phi_j(z) \partial^\ell \phi_j(z)}{k!\ell!} \epsilon^k \epsilon^\ell \phi_j(z) \omega(z)|dz|^2, $$

indexed by $\epsilon \in \mathbb{C}$. We observe that in (1.21) we may think of $(\epsilon^0, \ldots, \epsilon^{q-1})$ as a general vector in $\mathbb{C}^q$, by forgetting about the interpretation of the superscript as a power, and (1.21) still defines a (possibly degenerate) metric indexed by the $\mathbb{C}^q$-vector $(\epsilon^0, \ldots, \epsilon^{q-1})$. This should have an interpretation in terms of jet manifolds.

We turn to Bergman’s second metric. We calculate that

$$(1.22) \quad \Delta_2 \log E_{\mathbb{P}A_2}[K_q](z, z'; z, z') = \frac{1}{\sum_{j=1}^{+\infty} E[\phi_j](z, z')^2} \left| \sum_{j=1}^{+\infty} \partial_j E[\phi_j](z, z') \overline{E[\phi_j](z, z')}^2 \right|^2 \geq 0, $$

where the last inequality is an instance of the Cauchy-Schwarz inequality. The above expression has a convergent Taylor expansion in $\epsilon$ if we put $z' = z + \epsilon$, and keep $\epsilon$ close to 0:

$$(1.23) \quad \Delta_2 \log E_{\mathbb{P}A_2}[K_q](z, z'; z, z')|_{z' = z + \epsilon} = \sum_{k\ell=0}^{q-1} \epsilon^k \epsilon^\ell \frac{\partial^k \phi_j(z) \partial^\ell \phi_j(z)}{k!\ell!} \epsilon^k \epsilon^\ell \phi_j(z) \omega(z)|dz|^2, \quad (z, \epsilon) \in \Omega \times \mathbb{C}. $$
This suggests that we might propose as a generalization of Bergman’s second metric the expression

$$\begin{equation}
\begin{align*}
\text{d} s^2_{k, c, \epsilon}(z^2) = \sum_{k, c = 0}^{+\infty} \frac{e^k}{k!} \partial_{z^k} \overline{\partial}_{z^c} \log E_{02}(K_{\eta} z, z') \left| dz \right|^2,
\end{align*}
\end{equation}
$$

which then defines a (possibly degenerate) metric indexed by $c$, for $e \in C$ close to 0. In analogy with (1.21), we should like to cut off the summation to $0 \leq k, k' \leq q - 1$, but then it is not entirely clear that we obtain a metric:

$$\begin{equation}
\begin{align*}
\text{d} s^2_{k, c, \epsilon}(z^2) = \sum_{k, c = 0}^{q - 1} \frac{e^k}{k!} \partial_{z^k} \overline{\partial}_{z^c} \log E_{02}(K_{\eta} z, z') \left| dz \right|^2.
\end{align*}
\end{equation}
$$

Let $\Omega$ be a domain in $C$, $Q$ a real-valued function on $\Omega$ and $m$ a positive real parameter. Denote by $\text{d} A(z)$ the area measure in $C$ divided by $\pi$. The weighted Bergman space with the weight $e^{-mQ}$ is defined as

$$A^2(e^{-mQ}) := \{ f \text{ analytic in } \Omega \mid \int_{\Omega} |f(z)|^2 e^{-mQ(z)} \text{d} A(z) < \infty \}.$$

We will usually write just $A^2(e^{-mQ}) = A_m^2$, omitting the dependence on the weight $Q$. We denote the norm of a function $u$ by $\|u\|_m$. If the function $e^{-mQ(z)}$ is locally integrable, the spaces $A_m^2$ are Hilbert spaces where the point evaluation functionals are bounded. Therefore, there exists a reproducing kernel $K_m$, this is a function on $\Omega \times \Omega$ that for each fixed $z \in \Omega$ and every $f \in A_m^2$ satisfies

$$f(z) = \int_{\Omega} f(w) \overline{K_m(z, w)} e^{-mQ(w)} \text{d} A(w), \quad K_m(z, w) \in A_m^2.$$

The kernel $K_m$ is often referred to as the Bergman kernel.

This construction can very naturally be extended to line bundles over complex manifolds. It turns out that the Bergman kernel is an important object from a geometrical point of view and has therefore been well studied in these more general settings (see e.g., [21]. Given a complex manifold $X$ and a holomorphic line bundle $L$ over $X$ with a hermitian metric $h$, the weight $e^{-Q}$ corresponds to a local representation of the metric $h$. The parameter $m$ appears, when one replaces $L$ by a tensor power $L^m$. This is often done because the initial line bundle $L$ might have admit only very few holomorphic sections.

Actually, the one variable analogue of the weight studied over general complex manifolds is $e^{-mQ(z)} \Delta Q(z) \text{d} A(z)$, where $\Delta Q(z) := \partial \overline{\partial} Q(z)$ is assumed to be non-negative. We see that the spaces $A_m^2$ of the present paper are a little different, as we have replaced the volume form $\Delta Q \text{d} A$ by $\text{d} A$. An advantage is that we do not need to assume that $Q$ is globally subharmonic.

Unfortunately, the Bergman kernel cannot in general be computed exactly from the weight. However, assuming that the weight $Q$ is strictly subharmonic, one can say more. With this assumption, the work of Tian [23], Yau [25], Zelditch, [27] and Catlin [8], shows that there exists an asymptotic expansion around the diagonal as $m \to +\infty$. More precisely, when $z$ and $w$ are sufficiently near to each other, we have

$$\begin{equation}
\begin{align*}
K_m(z, w) \sim g_1(z, w) e^{mQ(z, w)} = \left( m\delta_{1,0}(z, w) + \delta_{1,1}(z, w) + \cdots + m^{-k}\delta_{1,k+1}(z, w) \right) e^{mQ(z, w)}
\end{align*}
\end{equation}
$$

for any $k \geq 0$ and coefficients $g_1$ analytic in $z$ and $w$. The function $Q(z, w)$ is the local polarization of $Q$, i.e., it a holomorphic function in $z$ and $w$ that satisfies $Q(z, z) = Q(z)$. There are several methods that can be used to derive the asymptotics, see e.g., [23], [21], [17], [7].
We should note here that Xu [26] was recently able to obtain a closed graph-theoretic formula for a general term in the expansion.

Apart from being connected to geometry, Bergman kernels have been studied by Ameur, Hedenmalm and Makarov (see [2], [3]) in relation to unitarily invariant random normal matrix ensembles. Probability density of the eigenvalues can be written in terms of the kernel of the spaces

$$\text{Pol}_{n,m} := \text{span}_{0 \leq j \leq n-1} z^j \subset L^2(e^{-mQ}dA),$$

where $n$ is the size of the matrix. Besides random matrices, the same model also describes Coulomb gas of negatively charged particles in an external field or alternatively, non-interacting electrons in a magnetic field in the ground state. The asymptotics of these kernels can be analyzed as $n, m \to \infty$ and $|n - m| = O(1)$ by similar methods as kernels $K_m$.

In this paper, we will consider the following generalizations of the spaces $A_{m,n}$:

$$A_{q,m}^2(e^{-mQ}) := \left\{ f = \sum_{r=0}^{q-1} z^r f_r(z) \mid \int_{\Omega} |f(w)|^2 e^{-mQ(w)} dA(w) < \infty, \right.$$

$$\left. f_r \text{ analytic on } \Omega \text{ for all } 0 \leq r \leq q-1 \right\}.$$

The functions of this form are called $q$-analytic; this is because an alternative characterization is through the condition $\bar{\partial} f = 0$. Functions that are $q$-analytic for some $q \geq 1$ are called polyanalytic. Identifying a polyanalytic function $\Sigma_{r=0}^{q-1} z^r f_r(z)$ with a vector $(f_0, ..., f_{q-1})$, we can alternatively think that the spaces $A_{q,m}^2$ consist of vector-valued functions.

Polyanalytic function spaces have been considered for instance in [24], [1], [20] and [13], focusing on the case $Q(z) = |z|^2$. One instance where these spaces appear is as eigenspaces of the operator

$$\Delta = -\partial_z \partial_{\bar{z}} + z \partial_{\bar{z}},$$

which is densely defined in $L^2(e^{-|z|^2})$. More precisely, the eigenspaces are of the form $A_{q,1}^2 \ominus A_{q-1,1}$, where $\ominus$ denotes the orthogonal difference within $L^2(e^{-|z|^2})$. The related operator

$$H := M_{e^{-1-\partial_{\bar{z}}}} \Delta M_{e^{-1-\partial_{\bar{z}}}},$$

where $M$ denotes the operator of multiplication by the function in the subscript, is the Hamiltonian related to a single electron in $\mathbb{C}$ with a uniform magnetic field perpendicular to the plane. The operator is called Landau Hamiltonian and the eigenspaces are commonly referred to as Landau levels.

We think of the orthogonal difference spaces

$$\Delta A_{q,m}^2(e^{-mQ}) := A_{q,m}^2(e^{-mQ}) \ominus A_{q-1,m}^2(e^{-mQ}),$$

as generalizations of these Landau levels. In [22], the authors identified approximate spectral subspaces related to a Hamiltonian describing a single electron in a magnetic field with strength $\Delta Q$. One can see that these spaces are in a certain sense dual to $\Delta A_{q,m}^2(e^{-mQ})$.

It is easily shown that the spaces $A_{q,m}^2$ possess a reproducing kernels $K_{q,m}$. Our aim is to provide an algorithm to compute a near-diagonal asymptotic expansions of $K_{q,m}$ as $m \to \infty$. By this we mean a series of the form

$$m^n \Gamma_{q,0}(z, w) + m^{n-1} \Gamma_{q,1}(z, w) + \cdots + \Gamma_{q,n}(z, w) + m^{-1} \Gamma_{q,n+1}(z, w) e^{mQ(z, w)},$$

where $n$ is the size of the matrix.
where the coefficients $\delta_{k,j}$ are $q$-analytic in $z$ and $\bar{w}$. Then it is an easy matter to obtain asymptotics for kernels of the difference spaces $\delta A_{q,m}^2$.

Our approach is based on the recent work of Berman, Berndtsson, and Sjöstrand [7], where they give an elementary algorithm to compute the coefficients $\delta_{k,j}$ of the analytic Bergman kernel. In one complex variable, we generalize this method to polyanalytic functions. We explain our results for the case $q = 2$ and compute two first terms of the asymptotic expansion. It is straightforward how our ideas generalize to general $q$, but the actual computations become quite complicated. However, we have the following theorem, the proof of which will be presented in a separate publication. The notation $L^q_k$ will stand for the generalized Laguerre polynomial of degree $q$ and parameter $k$.

**Theorem 1.1.** Let $z_0 \in \mathbb{C}$ be an arbitrary point and let $Q$ be real-analytic and strictly subharmonic in a neighborhood of $z_0$. Take a compact set $K \subset \mathbb{C}$. There exists $m_0$ such that for all $m \geq m_0$ and $z, w \in K$ we have

$$
\left| \frac{1}{m\Delta Q(z_0)} K_{q,m}(z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}}) \right| \times e^{-\frac{i\pi}{4} \left( \frac{q}{2m\Delta Q(z_0)} \right)^2 - \frac{i\pi}{2} \left( \frac{q}{2m\Delta Q(z_0)} \right)} = L_{q-1}^1((\xi - \lambda) e^{-(\xi - \lambda)^2} + O(m^{-1/2}).
$$

The constant in the error term is uniform on $K \times K$.

As the limiting kernel on the right hand side does not depend on the weight $Q$, this can be viewed as a universality result. (For more on universality, see [10]). The probabilistic interpretation is that in the large $m$ limit, the determinantal point process (for a definition, see [14]) defined by the kernel $K_{q,m}$ obeys local statistics given by the kernel $L_{q-1}^1((\xi - \lambda) e^{-(\xi - \lambda)^2}$. On the other hand, in [13] we analyzed a system of noninteracting fermions described by the Landau Hamiltonian above, so that each of the $q$ first Landau levels has $n$ particles. When $n$ goes to infinity and the magnetic field is rescaled by $n$, the particles accumulate on the unit disk and the same Laguerre polynomial kernel describes the local statistics in the limit.

This system of free fermions is a determinantal point process given by reproducing kernels of the spaces

$$
\text{Pol}_{q,m,n} := \text{span}\{|z^r \bar{z}^s| \ 0 \leq r \leq q - 1, 0 \leq j \leq n - 1\} \subset L^2(e^{-|z|^2} \text{d}A(z)).
$$

In a forthcoming paper, we replace $e^{-|z|^2}$ by a more general $e^{-Q(z)}$ and use methods of the present article to perform asymptotic analysis of the processes. On a more general complex manifold, this would correspond to studying sections on line bundles $L^q \otimes L^m$.

It would be interesting to explore how the polyanalytic Bergman kernels are connected to geometry. Here, we only want to note that the matrix

$$
B_{q,m}(z) := \left[ \partial_z^{j-1} \partial_{\bar{w}}^{k-1} K_{q,m}(z, w)|_{w=z} \right]_{j,k=1}^q
$$

defines positive semidefinite metric on $q$-dimensional vector bundles. This can be seen by writing the matrix as a sum over an orthonormal basis $e_1(z), e_2(z), \ldots$ of $A_{q,m}^2$:

$$
B_{q,m}(z) = \sum_{j=1}^q v_j(z) \otimes v_j(z), \quad v_j = \left( e_j(z), \partial e_j(z), \ldots, \partial^{q-1} e_j(z) \right).
$$

The $v_j \otimes v_j$ stands for a tensor product of the two vectors, defined as a matrix acting on $q$-dimensional vectors as $(v_j \otimes v_j)x = (x, v_j)v_j$. The individual summands are positive semidefinite.
because the tensor product of a vector with itself is always positive semidefinite. Therefore $B_{q,m}$ is positive semidefinite as well. So, we can think the kernel $K_{q,m}$ as a vector-valued analogue of the analytic Bergman kernel $K_m$. In analogy with the analytic setting, where the kernel $K_m(z,w)$ is uniquely determined by its diagonal restriction, this matrix determines the kernel $K_{q,m}(w,z)$ uniquely. Namely, we can Taylor expand $K_{q,m}(z,w)$ to get

$$K_{q,m}(z,w) = \sum_{j=0}^{q-1} (z - \bar{w})^j (w - z)^k A_{jk}(w,z)$$

for functions $A_{jk}$ that are analytic in $(z,\bar{w})$. These functions are determined by their diagonal restrictions, and therefore we see that $K_{q,m}(w,z)$ is determined by the matrix $B_{q,m}(z)$.

Finally, we note that it is probably not too difficult to extend our results to several complex variables and consequently to more general complex manifolds. Furthermore, our results raise the question whether similar asymptotic expansions hold for kernels related to more general differential operators than $\partial$.

2. The method of Berman-Berndtsson-Sjöstrand

In this section, we explain the method of [7] in one complex variable with some slight modifications. We fix a domain $\Omega$ and non-negative function $Q$ on $\Omega$. Take an arbitrary $z_0 \in \Omega$. We perform local analysis of the kernels $K_m(z,w)$, where $z, w \in D(z_0, r) \subset \Omega$. We assume that $r$ and $Q$ satisfy the following conditions:

1. $Q$ is real-analytic in $D(z_0, r)$ and $\Delta Q(z) > 0$ on $D(z_0, r)$.
2. There exists a local polarization of $Q$ in $D(z_0, r)$, i.e., a function $Q : D(z_0, r) \times D(z_0, r) \rightarrow \mathbb{C}$ which analytic in the first and anti-analytic in the second variable, and which satisfies $Q(z, z) = Q(z)$.
3. For $z, w \in D(z_0, r)$, we have $\partial_z \bar{\partial}_w Q(z, w) \neq 0$; this is possible because of condition 1.
4. The Taylor expansion of $Q(z, w)$ in the second variable and $Q(w) = Q(w, w)$ in both variables around $z$ gives

$$2\text{Re}Q(z, w) - Q(w) - Q(z) = -\Delta Q(z)|w - z|^2 + O(|z - w|^3).$$

We take $r$ such that for $z, w \in D(z_0, r)$,

$$2\text{Re}Q(z, w) - Q(w) - Q(z) = -\frac{1}{2} \Delta Q(z)|w - z|^2. \tag{2.1}$$

We take $\chi$ to be a smooth cut-off function which has its support in $D(z_0, r)$ and which equals 1 in $D(z_0, \frac{1}{2}r)$. It is possible to choose $\chi$ so that $\|\partial \chi\|_{L^2(\Omega, dA(z))}$ is independent of $z_0$ and $r$.

Define the phase function as

$$\theta(z, w) = \frac{Q(w) - Q(z, w)}{w - z}. \tag{2.2}$$

Notice that $\theta$ is analytic in the first variable and real-analytic in the second. We have $\theta(z, z) = \partial_z Q(z)$, and furthermore, as we will see later in (3.13),

$$\theta(z, w) = \frac{Q(w, w) - Q(z, w)}{w - z} = \sum_{j=0}^{\infty} \frac{1}{(j + 1)!} (w - z)^j \partial_z^{j+1} Q(z, w).$$

In [7], the starting point is an approximate reproducing identity for functions in $A^2_m$. To state the result, we define the kernel

$$R_m(z, w) := m \tilde{\theta}_{zw} \theta(z, w)e^{mQ(z, w)}.$$
**Proposition 2.1.** There exists \( \delta > 0 \) such that for all \( z \in \mathbb{D}(z_0, \frac{1}{3}r) \) and \( u \in A^2_m \),

\[
(2.3) \quad u(z) = \int_{\Omega} u(w)\chi(w)R_m(z, w)e^{-\bar{m}Q(z)}dA(w) + O(||u||_{\bar{m}e^{mQ(z)/2-\delta m}}).
\]

**Proof.** Because \( \frac{1}{z} \) is the fundamental solution of \( \partial \) w.r.t normalized area measure \( dA \), we have

\[
(2.4) \quad u(z) = \int_{\Omega} \frac{1}{z - \bar{w}} \partial_{\bar{w}}\left(u(w)\chi(w)e^{\bar{m}(z-w)\theta(z,w)}\right) dA(w)
= \int_{\Omega} u(w)\chi(w)R_m(z, w)e^{-\bar{m}Q(w)}dA(w)
+ \int_{\Omega} u(w)\frac{1}{z - \bar{w}} \partial_{\bar{w}}\chi(w)e^{\bar{m}(z-w)\theta(z,w)}dA(w).
\]

It is enough to show that the term involving \( \partial_{\bar{w}}\chi \) belongs to the error term. We will use

\[
2\text{Re}Q(z, w) - Q(w) - Q(z) \leq -\frac{1}{2}\Delta Q(z)|w - z|^2, \quad z, w \in \mathbb{D}(z_0, r),
\]

which was one of the conditions in the beginning of this section.

\[
(2.5) \quad \left| \int_{\Omega} u(w)\frac{1}{z - \bar{w}} \partial_{\bar{w}}\chi(w)e^{\bar{m}(z-w)\theta} dA(w) \right|
\leq \int_{\Omega} |u(w)| \frac{1}{z - \bar{w}} \partial_{\bar{w}}\chi(w)\left|e^{\bar{m}Q(z)}e^{\bar{m}(z-w)\theta}\right|dA(w)
\leq e^{\bar{m}Q(z)}||u||_{\bar{m}}\left(\int_{\Omega} \frac{1}{z - \bar{w}} \partial_{\bar{w}}\chi(w)^2e^{\bar{m}(z-w)\theta} dA(w)\right)^{1/2}
\leq Ce^{\bar{m}Q(z)}||u||_{\bar{m}}e^{-\frac{1}{2}\bar{m}Q(z)r^2},
\]

where \( C = \frac{3}{2}||\partial\chi||_{L^2} \). In conclusion, we managed to show that the term with \( \partial_{\bar{w}}\chi \) belongs to the error term, and the statement is proved.

A function \( L_m(z, w) \) is called a local reproducing kernel \( \text{mod}(e^{-\bar{m}m}) \) if it satisfies the statement of the proposition. There is a similar definition for a local reproducing kernel \( \text{mod}(m^{-k}) \), where the factor \( e^{-\bar{m}m} \) is replaced by \( m^{-k} \). If a local reproducing kernel is analytic in \( \bar{w} \), it will be called a local Bergman kernel. Starting from the kernel \( R_m \), the method of Berman, Berndtsson and Sjöstrand consists of an algorithm to to produce local Bergman kernels \( \text{mod}(m^{-k}) \) for arbitrary \( k \). An argument based on Hörmander’s \( L^2 \)-estimates for \( \partial \) then shows that the local Bergman kernels approximate the global Bergman kernel \( K_m \) near the diagonal. In this section, we will only present the local construction.

If the kernel \( R_m \) is analytic in \( \bar{w} \), which is the case for \( Q(z) = |z|^2 \) for instance, we have a local Bergman kernel \( \text{mod} e^{-\bar{m}m} \). Otherwise, the strategy is to add suitable correction terms to get local Bergman kernels \( \text{mod} (m^{-k}) \). These terms will be such that their contributions end up in the error term \( O(||u||_{\bar{m}e^{mQ(z)/2-\delta m}}) \).

Let us now be more specific. Define the differential operator

\[
(2.6) \quad V := \frac{1}{\partial_{\bar{w}}\theta} \partial_{\bar{w}} + M_{m(z-w)}.
\]

Here \( M \) denotes the operator of multiplication by the function in the subscript. The point in introducing this operator is that for any sufficiently smooth function \( A(z, w) \) and holomorphic
\[ u, \]

\[ \int_{\Omega} u(w)\chi(w)\bar{\partial}_{w}\theta(z, w) \cdot \nabla A(z, w) \cdot e^{m(z-w)\theta(z, w)} \, dA(w) \]

\[ = \int_{\Omega} u(w)\chi(w)\bar{\partial}_{w}\left[ A(z, w)e^{m(z-w)\theta(z, w)} \right] \, dA(w) \]

\[ = O\left(||u||_{m}e^{mQ(z)}e^{-\delta m}\right). \]

The argument is similar as in the proof of the proposition 2.1. Because of (2.7), functions of the form \( VA \) will be called negligible amplitudes. In the sequel, when we encounter error terms of this form, each appearance of \( \delta \) might refer to a different positive number.

We try look at a local Bergman kernel in the form \( \bar{\partial}_{1}(z, w)e^{mQ(z, w)} \), where

\[ \bar{\partial}_{1}(z, w) = m\bar{\partial}_{0}(z, w) + \bar{\partial}_{1,1}(z, w) + \frac{1}{m}\bar{\partial}_{1,2}(z, w) + \cdots + \frac{1}{m^{k}}\bar{\partial}_{1,k+1}(z, w) \]

is analytic in \( z \) and \( w \). By the arguments above, it is enough to find a function \( X \) such that

\[ m\bar{\partial}_{w}\theta + \bar{\partial}_{w}\theta \cdot \nabla X = \bar{\partial}_{1} + O(m^{-k-1}), \]

with the error term uniform in \( D(0, \frac{1}{r}) \times D(0, \frac{1}{r}) \).

We will now reformulate this condition. Let

\[ \bar{\partial}_{w} = \bar{\partial}_{w} - \frac{\bar{\partial}_{w}\theta}{\bar{\partial}_{w}\theta} \bar{\partial}_{w}, \quad \bar{\partial}_{\theta} = \frac{1}{\bar{\partial}_{w}\theta} \bar{\partial}_{w}. \]

The operators originate from the change of variables \((z, w, \bar{w}) \to (z, w, \theta)\). However, for us the important property is that the operators commute (this is left for the reader to check).

We write \( A(z, w) \sim mA_{1}(z, w) + A_{0}(z, w) + \frac{1}{m}A_{-1}(z, w) + \ldots \), when the function \( A \) admits an asymptotic expansion as \( m \to \infty \); in other words,

\[ \left| A - \sum_{j=1}^{k} m^{j}A_{j} \right| = O(m^{-k-1}) \]

for \( z, w \in D(0, r) \) and any \( k \geq 0 \). We consider a certain formal differential operator acting on functions with such expansion:

\[ S := e^{\frac{1}{m^{2}}\bar{\partial}_{w}\theta} = \sum_{j=0}^{\infty} \frac{1}{j!m^{j}}(\bar{\partial}_{w}\theta)^{j}. \]

We interpret \( SA \) as a formal series in \( m \) - no convergence of the series is required. For a function with an asymptotic expansion, or just a formal series in \( m \), we will use the notation

\[ A^{(k)}(z, w) = \sum_{j=1}^{k} m^{j}A_{j}(z, w), \quad k \geq 0. \]

The important property of \( S \), which is verified by a simple algebraic computation, is that

\[ SV = M_{m(z-w)}S. \]

The reason of introducing the operator \( S \) is the following proposition. The proof, which uses only the identity (2.11), can be found in [7].

**Proposition 2.2.** Fix \( k \geq 0 \). Let \( A \sim mA_{1} + A_{0} + \frac{1}{m}A_{-1} + \ldots \). The following are equivalent:
(i) \[ \left[ SA \right]_{z=w}^{(k)} = 0 \]

(ii) \[ A = \nabla C(z, w) + O(m^{-k-1}). \]

for some function \( C(z, w) = \sum_{j=1}^{k} m^j C_j(z, w). \)

Remember that our goal is to find \( X \) and \( \theta_1 \) such that (2.8) holds. Let us therefore take \( A = m - \frac{\theta_1}{\partial z \partial \theta}. \) According to the proposition, (2.8) is equivalent to

\[ (S \theta_1)_{z=w}^{(k)} = (Sm)_{z=w}^{(k)} = m. \]

By identifying the coefficients of the powers of \( m \) on both sides, it is possible to solve a desired number of terms in \( \theta_1 \) from this equation. For example, the condition for the coefficients of \( m^1 \) reads

\[ \left[ \frac{\partial \theta_1}{\partial z} \right]_{z=w} = 1. \]

Remembering the Taylor expansion (2.2), this implies that \( \theta_1(z, w) = \Delta Q(z). \) Because \( \theta_1(z, w) \) is analytic in \( w, \) we can conclude that

\[ \theta_1(z, w) = \partial \bar{z} \partial w Q(z, w). \]

For the coefficients of \( m^0 = 1, \) the condition (2.12) is

\[ \left[ \frac{\partial \theta_1}{\partial z} \frac{\partial \theta_1}{\partial w} + \frac{\partial \theta_1}{\partial z} \right]_{z=w} = 0. \]

Because we know \( \theta_1 \) already, \( \theta_1 \) can be solved. On the diagonal \( w = z, \) the result is

\[ \theta_1(z, z) = \frac{1}{2} \Delta \log Q(z), \]

and this leads us to

\[ \theta_1(z, w) = \frac{1}{2} \partial \bar{z} \partial w \log \partial \bar{z} \partial w Q(z, w). \]

3. Kernel expansion algorithm for bianalytic functions

We now turn our attention to spaces \( A^2_{\omega}, \) We start by proving an approximative representation formula similar to (2.4). The assumptions on \( Q \) and \( \chi \) as are in the section 2. We will frequently encounter functions on \( D(z_0, r) \times D(z_0, r) \) which are \( q \)-analytic in the first and real-analytic in the second variable; these classes are denoted by \( R_q. \)

Let \( u \) be bianalytic, i.e. \( \bar{\partial}^2 u = 0. \) We will use the fundamental solution \( \frac{\partial w}{w} \) of the operator \( \bar{\partial}^2. \) We have

\[ \int_{\Omega} \frac{u(w) \chi(w) \bar{\partial}^2 \left( \frac{\partial - \bar{w}}{w - z} e^{m(z-w)} \right) dA(w)}{w - z} = \int_{\Omega} \frac{\partial w}{w} \left( \frac{\partial - \bar{w}}{w - z} e^{m(z-w)} \right) dA(w) = O(||u||_m e^{\frac{1}{2}mQ(z)}). \]
The last equality follows from bianalyticity of $u$ and arguments similar to the proof of 2.1. Denoting by $\delta(w) dA(w)$ the Dirac point mass at $z$, we get

$$u(z) = \int_{\Omega} u(w) \chi(w) \left[ \delta_z - \partial^2 \left( \frac{\bar{w} - z}{w - z} e^{m|z-w|^2} \right) \right] dA(w) + O(||u||_m e^{\frac{1}{2} mQ(z) - \frac{1}{2} m})$$

$$= \int_{\Omega} u(w) \chi(w) R_{2,m}(z, w) e^{m|z-w|^2} dA(w) + O(||u||_m e^{\frac{1}{2} mQ(z) - \frac{1}{2} m}),$$

where the kernel

$$R_{2,m}(z, w) = \left[ \delta_z - \partial^2 \left( \frac{\bar{w} - z}{w - z} e^{m|z-w|^2} \right) \right] e^{-m|z-w|^2}$$

$$= 2m \partial_{w} \theta - m(z - \bar{w}) \partial_{w} \bar{\theta} - m^2 |z - \bar{w}|^2 (\partial_{w} \theta)^2$$

belongs to $\mathfrak{H}_2$.

We will look for the bianalytic Bergman kernel in the form

$$\delta_2(z, w) e^{mQ(z, w)} = \left[ m^2 \delta_{2,0}(z, w) + m \delta_{2,1}(z, w) + \delta_{2,2}(z, w) + \ldots \right] e^{mQ(z, w)},$$

where each term is bianalytic in $z$ and $\bar{w}$. First, we need to understand what the negligible amplitudes are in the present context. Recall the definition of $V$ in (2.6). Familiar arguments yield

$$\int_{\Omega} u(w) \chi(w) \left( M_{\delta, \theta} V M_{\delta, \theta} V X(z, w) \right) e^{m|z-w|^2} dA(w)$$

$$= \int_{\Omega} u(w) \chi(w) \partial^2_{w} \left( X(z, w) e^{m|z-w|^2} \right) dA(w) = O(||u||_m e^{\frac{1}{2} mQ(z) - \frac{1}{2} m})$$

for any bianalytic $u$ and smooth $X$. Motivated by this, expressions of the form $M_{\delta, \theta} V M_{\delta, \theta} V X(z, w)$ will be called negligible amplitudes. Analogously to the analytic situation, we would like to solve

$$\frac{R_{2,m}(z, w) - \delta_2}{\partial_{w} \theta} = V M_{\delta, \theta} V_m X(z, w) + O(m^{-k-1})$$

In the analytic case, the main idea was roughly speaking that if we apply the operator $S$ to a function $A$ with an asymptomatic expansion in $m$, the expression $S A$ vanishes on the diagonal $z = \bar{w}$ if and only if $A$ is a negligible amplitude. To show this, one uses the fact that for $A(z, w) \in \mathfrak{H}_1$, $A(z, z) = 0$ implies $A(z, w) = (z - \bar{w}) B(z, w)$ for some function $B \in \mathfrak{H}_1$. The function $A(z, w) = z - \bar{w}$ shows that this is no longer true if analyticity in $z$ is replaced by bianalyticity. We therefore replace the condition of vanishing on the diagonal by the requirement that $S A \in M_{\delta, \theta} \mathfrak{H}_2$.

However, this condition only ensures that $A = V X$ for some $X$, and we would like $A$ to be of the more special form $V M_{\delta, \theta} V X(z, w)$. Another condition is needed to guarantee this. This condition is written in terms of the operator $S_2$, which we will now define.

Let $f(z, w) \in \mathfrak{H}_2$. Then $f$ can be decomposed as $f(z, w) = f_1(z, w) + z f_2(z, w)$, where $f_1$ and $f_2$ are analytic in $z$. We define the operator

$$N[f](z, w) = \frac{f_1(w, w) - f_1(z, z)}{w - z} + \frac{f_2(w, w) - f_2(z, w)}{w - z}.$$ 

The important property of this operator is that $N M_{\delta, \theta}$ equals the identity. We also need the formal inverse of the operator $S$ from the section 2:

$$S^{-1} = e^{-\frac{1}{2} \partial_{w} \theta}.$$
The following proposition shows the operators that we should apply to the equation (3.6) are

\[ S_1 := S, \]

the same as in the analytic case, and

\[ S_2 := SM_{-1}^{-1}S^{-1}NS. \]

**Proposition 3.1.** Let \( A(z, w) \in \mathcal{R}_2 \) have the asymptotic expansion \( \sum_{j=0}^{+\infty} m^j A_j(z, w) \). The following are equivalent:

(i) There exist functions \( B(z, w) = \sum_{j=2}^{+\infty} m^j B_j(z, w) \in \mathcal{R}_2 \) and
\[ C(z, w) = \sum_{j=2}^{+\infty} m^j C_j(z, w) \in \mathcal{R}_2 \] such that \( (SA)^{(k)} = M_{-w}B(z, w) \) and \( (S_2A)^{(k)} = M_{-w}C(z, w) \),

(ii) There exists a function \( A(z, w) = \sum_{j=2}^{+\infty} m^j D_j(z, w) \in \mathcal{R}_2 \) such that
\[ A = VM_{-z,0}VD(z, w) + O(m^{-k-1}). \]

**Proof.** We prove the implication from (i) to (ii). We will repeatedly use that \( (SA)^{(k)} = (SA_j^{(k)})^{(k)} \) and \( (S^{-1}A)^{(k)} = (S^{-1}A)^{(j)} \).

By the first condition, we can take \( Y := (S^{-1}B)^{(k)} \) so that

\[ (SA)^{(k)} = (M_{-w}SY)^{(k)}. \]

This implies

\[ A^{(k)} = (S^{-1}M_{-w}SY)^{(k)} = \left( \frac{1}{m} VY \right)^{(k)} = \frac{1}{m} VY + O(m^{-k-1}). \]

Therefore,

\[ (S_2A)^{(k)} = (SM_{-z,0}^{-1}S^{-1}NSA)^{(k)} = (SM_{-z,0}^{-1}S^{-1}NM_{-w}SY)^{(k)} = (SM_{-z,0}^{-1}Y)^{(k)} \]

which we use with the second condition to get

\[ (SM_{-z,0}^{-1}Y)^{(k)} = (M_{-w}SZ)^{(k)}, \]

for \( Z = (S^{-1}C)^{(k)} \). This yields

\[ Y = (M_{-z,0}S^{-1}M_{-w}SZ)^{(k)} = \left( \frac{1}{m} M_{-z,0}VZ \right)^{(k)} = \frac{1}{m} M_{-z,0}VZ + O(m^{-k-1}). \]

Combine this with (3.8), and we are done. The other direction is straightforward application of similar arguments.

Before starting the computation of the terms \( c_{2,0} \) and \( c_{2,1} \), we record here some Taylor expansions that will be needed in the process. We have

\[ Q(w) = \sum_{j=0}^{+\infty} \frac{1}{j!} (w - z)^j \partial_z^j Q(z, w), \]

so that

\[ \theta(z, w) = \frac{Q(w) - Q(z, w)}{w - z} = \sum_{j=0}^{+\infty} \frac{1}{(j + 1)!} (w - z)^{j+1} \partial_z^j Q(z, w), \]

and hence

\[ \partial_w \theta = \sum_{j=0}^{+\infty} \frac{1}{(j + 1)!} (w - z)^{j+1} \partial_w \partial_z^j Q(z, w), \]
and
\[
\partial_w \theta = \sum_{j=0}^{+\infty} \frac{j + 1}{(j + 2)!} (w - z)^j \partial_z^{j+2} Q(z, w).
\]

We usually rewrite (3.13) as
\[
\partial_w \theta = \partial_z \partial_{\bar{w}} Q(z, w) \left\{ 1 + \sum_{j=1}^{+\infty} \frac{1}{(j + 1)!} (w - z)^j \partial_z^{j+1} \partial_{\bar{w}} Q(z, w) \right\}.
\]

A consequence of (3.14) and (3.15) is that
\[
\partial_w \theta \in 1 + \frac{1}{2} \partial_z \partial_{\bar{w}} Q(z, w) + M_{z-w} R_1.
\]

A more refined version is that
\[
\partial_w \theta \in 1 + \frac{3}{2} \partial_z \partial_{\bar{w}} Q(z, w) + (w - z) \left\{ \frac{1}{3} \partial_z \partial_{\bar{w}} Q(z, w) - \frac{1}{4} \frac{\partial^2 Q(z, w)[\partial^2 \partial_{\bar{w}} Q(z, w)]}{[\partial_z \partial_{\bar{w}} Q(z, w)]^2} \right\} + M_{z-w} R_1.
\]

We now turn to the computation. First, we find it convenient to rewrite the (3.6) with \( \partial_{\bar{w}} \)-derivatives instead of \( \partial_w \)-derivatives:
\[
M_{\partial_{\bar{w}}w}[6z](z, w) = 2m - m(z - \bar{w})\partial_{\bar{w}} - m^2 |z - w|^2 \partial_{\bar{w}} + V_m M_{\partial_{\bar{w}}w}^{-1} V_m [X(z, w)] + O(1).
\]

Notice that we use here the constant order error term, because we are only looking for the two first terms of the expansion. We require that
\[
\text{SM}_{\partial_{\bar{w}}w}[6z](z, w) = 2m - mS\left[(z - \bar{w})\partial_{\bar{w}} + \frac{1}{2} \partial_{\bar{w}} S \right] - m^2 S \left[ \frac{|z - w|^2}{\partial_{\bar{w}}} \right] + M_{z-w} R_2,
\]

and
\[
\text{SM}_{\partial_{\bar{w}}w}^{-1} \text{NSM}_{\partial_{\bar{w}}w}[6z](z, w)
\]
\[
\in -m \text{SM}_{\partial_{\bar{w}}w}^{-1} \text{NSM}_{\partial_{\bar{w}}w}[6z](z, w) - m^2 \text{SM}_{\partial_{\bar{w}}w}^{-1} \text{NSM}_{\partial_{\bar{w}}w}^{\frac{1}{2}} S \left[ \frac{|z - w|^2}{\partial_{\bar{w}}} \right] + M_{z-w} R_2
\]

are satisfied for the two highest powers of \( m \).

As before, we see that
\[
\text{SM}_{\partial_{\bar{w}}w}[6z] = \sum_{k=0}^{+\infty} m^{2-k} \sum_{j=0}^{k} \frac{1}{j!} (\partial_w \partial_{\bar{w}})^j M_{\partial_{\bar{w}}w} G_{2,k-j},
\]
so that in a first step
\[
\text{SNM}_{\partial_{\bar{w}}w}[6z] = \sum_{k=0}^{+\infty} m^{2-k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{j! (j - j)!} (\partial_w \partial_{\bar{w}})^{j-j} N(\partial_w \partial_{\bar{w}})^k M_{\partial_{\bar{w}}w} G_{2,k-j}.
\]
and in a second step

\[ \text{SM}_{\delta_{0},w} S^{-1} \text{NSM}_{\delta_{0},w}[\delta_{2}] \]

\[ = \sum_{k=0}^{\infty} m^{2-k} \sum_{j_{2}=0}^{k} \sum_{j_{1}=0}^{j_{2}} (-1)^{j_{2}-j_{1}} \left( \partial_{w} \partial_{\bar{w}} \right)^{j_{2} - h} N(\partial_{w} \partial_{\bar{w}})^{h} \text{M}_{\delta_{0},w} \delta_{2,k-j}. \]

We also see that

\[ 2m - m \left( \bar{w} \partial_{\bar{w}} \right) \frac{1}{\partial_{\bar{w}}} - m^{2} \left( \bar{w} \partial_{\bar{w}} \right) \]

\[ = 2m - m^{2} \left( \bar{w} \partial_{\bar{w}} \right) \frac{1}{\partial_{\bar{w}}} \quad \text{so that (3.18) asserts that} \]

\[ \sum_{k=0}^{\infty} m^{2-k} \sum_{j_{2}=0}^{k} \sum_{j_{1}=0}^{j_{2}} (-1)^{j_{2}-j_{1}} \left( \partial_{w} \partial_{\bar{w}} \right)^{j_{2} - h} N(\partial_{w} \partial_{\bar{w}})^{h} \text{M}_{\delta_{0},w} \delta_{2,k-j}. \]

Next, we use that

\[ \text{SM}_{\delta_{0},w} S^{-1} \text{NS} \]

\[ = \sum_{k=0}^{\infty} m^{2-k} \sum_{j_{2}=0}^{k} \sum_{j_{1}=0}^{j_{2}} (-1)^{j_{2}-j_{1}} \left( \partial_{w} \partial_{\bar{w}} \right)^{j_{2} - h} N(\partial_{w} \partial_{\bar{w}})^{h} \text{M}_{\delta_{0},w} \delta_{2,k-j}. \]

\[ \epsilon - \sum_{k=0}^{\infty} m^{2-k} \sum_{j_{2}=0}^{k} \sum_{j_{1}=0}^{j_{2}} (-1)^{j_{2}-j_{1}} \left( \partial_{w} \partial_{\bar{w}} \right)^{j_{2} - h} N(\partial_{w} \partial_{\bar{w}})^{h} \text{M}_{\delta_{0},w} \delta_{2,k-j}. \]
Finally, we apply (3.20) and (3.21) for each power of \( m \) separately. For (3.20), the result is, for \( k = 0 \),

(3.22) \[ M_{\delta_0,w} \zeta_{0,0} \in M_{-\omega} \mathcal{R}_2, \]

and, for \( k = 1 \),

(3.23) \[ M_{\delta_0,w} \zeta_{0,1} + \partial_w \partial_\theta \left[ \partial_\theta \bar{\theta} \zeta_{0,0} + \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right] - 2 + (z - \bar{w}) \partial_\theta \frac{1}{\partial_\theta \bar{w}} \in M_{-\omega} \mathcal{R}_2, \]

while for \( k = 2, 3, 4, \ldots \),

(3.24) \[ \sum_{j=0}^{k} \frac{1}{j!} (\partial_w \partial_\theta)^j M_{\delta_0,w} \zeta_{2,k-j} + \frac{1}{k!} (\partial_w \partial_\theta)^{k-1} \left( k(z - \bar{w}) \partial_\theta \frac{1}{\partial_\theta \bar{w}} + \partial_w \partial_\theta \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right) \in M_{-\omega} \mathcal{R}_2. \]

As for (3.21), the result is, for \( k = 0 \),

\[ M_{\delta_0,w} N \left[ \partial_\theta \bar{\theta} \zeta_{0,0} + \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right] \in M_{-\omega} \mathcal{R}_2. \]

which we rewrite as

(3.25) \[ N \left[ \partial_\theta \bar{\theta} \zeta_{0,0} + \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right] \in M_{-\omega} \mathcal{R}_2. \]

and, for \( k = 1 \),

\[ M_{\delta_0,w} N M_{\delta_0,w} \zeta_{2,1} + \partial_w \partial_\theta M_{\delta_0,w} N M_{\delta_0,w} \zeta_{2,0} \]

\[ - M_{\delta_0,w} \partial_w \partial_\theta N M_{\delta_0,w} \zeta_{2,0} + M_{\delta_0,w} N \partial_w \partial_\theta M_{\delta_0,w} \zeta_{2,0} \]

\[ = - \partial_w \partial_\theta M_{\delta_0,w} N \left( |z - \bar{w}|^2 \right) + M_{\delta_0,w} N \partial_w \partial_\theta M_{\delta_0,w} \zeta_{2,0} \]

\[ - M_{\delta_0,w} N \partial_w \partial_\theta \zeta_{2,0} + M_{\delta_0,w} N \left( \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right) \in M_{-\omega} \mathcal{R}_2, \]

which we may rewrite as

\[ N M_{\delta_0,w} \zeta_{2,1} + M_{\delta_0,w}^{-1} \partial_w \partial_\theta M_{\delta_0,w} N \left( \partial_\theta \bar{\theta} \zeta_{2,0} + \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right) \]

\[ - \partial_w \partial_\theta N \zeta_{2,0} + M_{\delta_0,w} N \left( \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right) + N \zeta_{2,0} \partial_\theta \frac{1}{\partial_\theta \bar{w}} \in M_{-\omega} \mathcal{R}_2, \]

or, more appealingly,

(3.26) \[ N \left[ M_{\delta_0,w} \zeta_{2,1} + \partial_w \partial_\theta \left( \partial_\theta \bar{\theta} \zeta_{2,0} + \frac{|z - \bar{w}|^2}{\partial_\theta \bar{w}} \right) - 2 + \frac{(z - \bar{w}) \partial_\theta \frac{1}{\partial_\theta \bar{w}}}{\partial_\theta \bar{w}} \right] \]

\[ + M_{\delta_0,w}^{-1} \partial_w \partial_\theta M_{\delta_0,w} \zeta_{2,0} \in M_{-\omega} \mathcal{R}_2. \]
where \([A, B] := AB - BA\) is the usual commutator. More generally, for \(k = 2, 3, \ldots\),

\[
(3.27) \quad \sum_{j_0=0}^{k} \sum_{j_1=0}^{j_0} \sum_{j_2=0}^{j_1} \frac{(-1)^{j_2-j_1}}{j_0! (j_2-j_1)! (j_3-j_2)!} \times (\partial_w \partial_{\bar{w}})^{j_0-j_1} \mathcal{M}_{\partial_{\bar{w}}}((\partial_w \partial_{\bar{w}})^{j_3-j_1} \mathcal{N}(\partial_w \partial_{\bar{w}})^{j_1} \mathcal{M}_{\partial_{\bar{w}}} \bar{g}_{2k-j_3})
\]

\[
\in - \sum_{j_0=0}^{k} \sum_{j_1=0}^{j_0} \sum_{j_2=0}^{j_1} \frac{(-1)^{j_2-j_1}}{j_0! (j_2-j_1)! (j_3-j_2)!} \times \left[ (\partial_w \partial_{\bar{w}})^{k-j_1} \mathcal{M}_{\partial_{\bar{w}}}((\partial_w \partial_{\bar{w}})^{j_3-j_1} \mathcal{N}(\partial_w \partial_{\bar{w}})^{j_1} \mathcal{M}_{\partial_{\bar{w}}} (z-w)^2 \frac{1}{\partial_{\bar{w}} \partial_{w}}) + \mathcal{M}_{z-w} \mathfrak{R}_2 \right].
\]

We observe that (3.22) and (3.25) can be combined in a single condition:

\[
(\partial_{\bar{w}} \bar{w}) \bar{g}_{2, 0} + \mathcal{M}_{z-w} N \left[ \frac{|z-w|^2}{\partial_{\bar{w}} \partial_{w}} \right] \in \mathcal{M}_{z-w}^2 \mathfrak{R}_2,
\]

which we may rewrite as

\[
(3.28) \quad (\partial_{\bar{w}} \bar{w}) \bar{g}_{2, 0} + \mathcal{M}_{z-w} N \left[ \frac{|z-w|^2}{\partial_{\bar{w}} \partial_{w}} \right] \in \mathcal{M}_{z-w}^2 \mathfrak{R}_2.
\]

In the same vein, (3.23) and (3.26) combine in the condition

\[
(3.29) \quad \mathcal{M}_{\partial_{\bar{w}}} \left[ \partial_{\bar{w}} \bar{w} \right] (\partial_{\bar{w}} \bar{w}) \bar{g}_{2, 0} + \partial_{\bar{w}} \bar{w} \left[ \frac{|z-w|^2}{\partial_{\bar{w}} \partial_{w}} \right] - 2 (z-w) \partial_{\bar{w}} \partial_{w} \frac{1}{\partial_{\bar{w}} \partial_{w}}
\]

\[
\in \mathcal{M}_{z-w}^2 \mathfrak{R}_2.
\]

Now, for instance, (3.22) and (3.25) should determine \(\bar{g}_{2, 0}\) uniquely, given that \(\bar{g}_{2, 0}\) is biana
tylic in \((z, \bar{w})\). Next, since

\[
(3.30) \quad \left. \frac{1}{\partial_{\bar{w}} \bar{w}} \right|_{\partial_{\bar{w}} \bar{w}} = \partial_{\bar{w}} \theta,
\]

we see that

\[
(3.31) \quad \partial_{\bar{w}} \left. \frac{1}{\partial_{\bar{w}} \bar{w}} \right|_{\partial_{\bar{w}} \bar{w}} = \left[ \partial_{\bar{w}} \theta \right]^{-1} \partial_{\bar{w}} \theta = \partial_{\bar{w}} \log(\partial_{\bar{w}} \theta),
\]

so that

\[
(3.32) \quad \frac{|z-w|^2}{\partial_{\bar{w}} \partial_{w}} = |z-w|^2 \partial_{\bar{w}} \theta \in |z-w|^2 \partial_{\bar{w}} \partial_{w} Q(z, w) + \mathcal{M}_{z-w}^2 \mathcal{M}_{z-w} \mathfrak{R}_1,
\]

and

\[
(3.33) \quad \partial_{\bar{w}} \left. \frac{1}{\partial_{\bar{w}} \bar{w}} \right|_{\partial_{\bar{w}} \bar{w}} = - \frac{\partial_{\bar{w}} \partial_{\bar{w}} Q(z, w)}{[\partial_{\bar{w}} \partial_{\bar{w}} Q(z, w)]^2} + \mathcal{M}_{z-w} \mathfrak{R}_1.
\]

By (3.22),

\[
(3.34) \quad \mathcal{M}_{\partial_{\bar{w}}} \bar{g}_{2, 0} \in \mathfrak{M}_{\partial_{\bar{w}}}^2 \mathcal{M}_{z-w} \mathfrak{R}_2 = \mathcal{M}_{z-w}^2 \mathfrak{R}_2.
\]

If we put

\[
(3.35) \quad \bar{g}_{2, 0}(z, w) := -\frac{|z-w|^2}{\partial_{\bar{w}} \partial_{w}} \bar{Q}(z, w),
\]

then it is easy to check that (3.28) is fulfilled. This means that the choice of \(\bar{g}_{2, 0}\) is correct.
By (3.13) and (3.30),

\[
\frac{|z - w|^2}{\partial_{\bar{w}}w} = |z - w|^2 \partial_{\bar{w}}\theta = \sum_{j=0}^{\infty} \frac{1}{(j + 1)!} (w - z)(w - z)^j \partial^j_{\bar{w}}\partial_{\bar{w}}Q(z, w). \tag{3.36}
\]

If we use (3.15) together with Taylor expansion, we get

\[
\log(\partial_w \theta) = \log(\partial_{\bar{z}}\partial_{\bar{w}}Q(z, w)) + \sum_{k=1}^{\infty} \left( \frac{-1}{k} \sum_{j=1}^{\infty} \frac{1}{(j + 1)!} (w - z)^j \partial^j_{\bar{z}}\partial_{\bar{w}}Q(z, w) \right)^k,
\]

which together with (3.31) leads to

\[
\partial_{\theta} \frac{1}{\partial_{\theta}w} = \partial_{\bar{w}}\log(\partial_w \theta)
\]

\[
\in \partial_{\bar{w}}\log[\partial_{\bar{z}}\partial_{\bar{w}}Q(z, w)] + \frac{1}{2} (w - z)\partial_{\bar{z}}\partial_{\bar{w}}\left[ \log[\partial_{\bar{z}}\partial_{\bar{w}}Q(z, w)] \right] + \mathcal{M}^2_{\bar{z} \bar{w}} R_1.
\]

Let us now check what (3.23) tells us. A calculation shows that

\[
(\partial_{\theta}w)\xi_{2,0} + \frac{|z - w|^2}{\partial_{\theta}w} + (z - w)|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) \in \mathcal{M}^2_{\bar{z} \bar{w}} R_2,
\]

and so

\[
\partial_{\theta}\partial_{\theta}w \left( (\partial_{\theta}w)\xi_{2,0} + \frac{|z - w|^2}{\partial_{\theta}w} + (z - w)|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) \right) \in \mathcal{M}^2_{\bar{z} \bar{w}} R_2.
\]

If we use (3.16), we now see that

\[
\partial_{\theta}\partial_{\theta}w \left( (\partial_{\theta}w)\xi_{2,0} + \frac{|z - w|^2}{\partial_{\theta}w} + (z - w)|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) \right)
\]

\[
= \frac{1}{\partial_{\bar{w}}w} \partial_{\bar{w}}w \left( \partial_{\bar{w}}w - \partial_{w}\theta \partial_{\bar{w}}w \right)(z - w)|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w)
\]

\[
\in \frac{1}{\partial_{\bar{w}}w} \partial_{\bar{w}}w \left( -2|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) \right) + \mathcal{M}^2_{\bar{z} \bar{w}} R_2
\]

\[
= 2(z - w) \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) - 2|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) + \mathcal{M}^2_{\bar{z} \bar{w}} R_2.
\]

It now follows from (3.40) that

\[
\partial_{\theta}\partial_{\theta}w \left( (\partial_{\theta}w)\xi_{2,0} + \frac{|z - w|^2}{\partial_{\theta}w} + 2(z - w) \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) \right)
\]

\[
\in \frac{2(z - w)}{\partial_{\bar{w}}w} \partial_{\bar{w}}w \partial_{\bar{w}}Q(z, w) - 2|z - w|^2 \partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) + \mathcal{M}^2_{\bar{z} \bar{w}} R_2.
\]

so that, in particular,

\[
\partial_{\theta}\partial_{\theta}w \left( (\partial_{\theta}w)\xi_{2,0} + \frac{|z - w|^2}{\partial_{\theta}w} \right) \in \mathcal{M}^2_{\bar{z} \bar{w}} R_2.
\]

In view of (3.43), (3.38), we see that (3.23) gives

\[
\frac{\xi_{2,1}}{\partial_{\theta}w} - 2 + (z - w) \partial_{\bar{z}}\partial_{\bar{w}}Q(z, w) \in \mathcal{M}^2_{\bar{z} \bar{w}} R_2,
\]

which suggests that we should look for \( \xi_{2,1} \) of the form

\[
\xi_{2,1}(z, w) := 2\partial_{\bar{z}}\partial_{\bar{w}}Q(z, w) - (\bar{z} - \bar{w})\partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) + (z - w)\partial^2_{\bar{z}}\partial_{\bar{w}}Q(z, w) + |z - w|^2 M_1(z, w),
\]
where $M_1(z, w)$ is holomorphic in $(z, \bar{w})$. We find that

$$
\frac{\partial^2}{\partial w \partial \theta} \in 2 - (z - \bar{w}) \frac{\partial^2}{\partial w \partial \theta} Q(z, w) - 2(w - z) \frac{\partial^2}{\partial z \partial w} Q(z, w) + |z - w|^2 \frac{M_1(z, w)}{\partial_z \partial_w Q(z, w)} - \frac{1}{2} |z - w|^2 \frac{[\partial^2 \partial_w Q(z, w)][\partial_z \partial_w Q(z, w)]}{[\partial_z \partial_w Q(z, w)]^2} + M_{z-w}^2 R_2.
$$

We may now add up the expression on the left hand side of (3.23), using (3.45), (3.43), (3.38):

$$
M_{\theta \bar{w}} \frac{\partial^2}{\partial w \partial \theta} \frac{\partial^2}{\partial w \partial \theta} Q(z, w) + |z - w|^2 \frac{M_1(z, w)}{\partial_z \partial_w Q(z, w)} - \frac{1}{2} |z - w|^2 \frac{[\partial^2 \partial_w Q(z, w)][\partial_z \partial_w Q(z, w)]}{[\partial_z \partial_w Q(z, w)]^2}
$$

which we simplify to

$$
M_{\theta \bar{w}} \frac{\partial^2}{\partial w \partial \theta} \frac{\partial^2}{\partial w \partial \theta} Q(z, w) + |z - w|^2 \frac{M_1(z, w)}{\partial_z \partial_w Q(z, w)} - \frac{1}{2} |z - w|^2 \frac{[\partial^2 \partial_w Q(z, w)][\partial_z \partial_w Q(z, w)]}{[\partial_z \partial_w Q(z, w)]^2}
$$

It is now immediate that

$$
N[M_{\theta \bar{w}} \frac{\partial^2}{\partial w \partial \theta} \frac{\partial^2}{\partial w \partial \theta} Q(z, w) + |z - w|^2 \frac{M_1(z, w)}{\partial_z \partial_w Q(z, w)} - \frac{1}{2} |z - w|^2 \frac{[\partial^2 \partial_w Q(z, w)][\partial_z \partial_w Q(z, w)]}{[\partial_z \partial_w Q(z, w)]^2}]
$$

We want to implement this into (3.26). Then we also need to know that

$$
[\partial_{\theta} \partial_{\bar{w}}, M_{\theta \bar{w}}] = M_{\theta \bar{w}} \frac{\partial^2}{\partial w \partial \theta} + M_{\theta \bar{w}} \frac{\partial^2}{\partial z \partial w} + M_{\theta \bar{w}} \frac{\partial^2}{\partial \theta \partial w},
$$

which together with (3.39) gives us that

$$
M_{\theta \bar{w}}^{-1} [\partial_{\theta} \partial_{\bar{w}}, M_{\theta \bar{w}}] N[\frac{\partial^2}{\partial w \partial \theta} \frac{\partial^2}{\partial w \partial \theta} Q(z, w) + |z - w|^2 \frac{M_1(z, w)}{\partial_z \partial_w Q(z, w)} - \frac{1}{2} |z - w|^2 \frac{[\partial^2 \partial_w Q(z, w)][\partial_z \partial_w Q(z, w)]}{[\partial_z \partial_w Q(z, w)]^2}]
$$

$$
\in -(z - \bar{w}) \frac{[\partial^2 \partial_w Q(z, w)][\partial_z \partial_w Q(z, w)]}{[\partial_z \partial_w Q(z, w)]^2} + M_{z-w} R_2.
$$
Finally, we see that

\[
(3.51) \quad N\left[M_{\theta, \theta} \partial_\theta \partial_\theta + \partial_\theta \partial_\theta \left( \frac{(\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2} \right) - 2 + (\hat{\theta} \partial_\theta) \partial_\theta \right] + M_{\theta, \theta} \left[ \frac{(\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2} \right] 
\]

\[
eq (\hat{\theta} \partial_\theta) \partial_\theta \left[ \frac{1}{2} \frac{(\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2} \right] - \frac{1}{2} \frac{(\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2} \right] + M_{\theta, \theta} \left[ \frac{1}{2} \frac{(\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2} \right] + M_{\theta, \theta} \left[ \frac{1}{2} \frac{(\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2} \right]
\]

So, the condition (3.26) means that

\[
(3.52) \quad M_1 = -\frac{3}{2} \hat{\theta} \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta + \frac{[\partial \theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta - \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2] \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta + \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2) \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta
\]

In conclusion, we obtain

\[
(3.53) \quad g_{2,1}(z, w) := 2 \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta + \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2) \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta + \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2)
\]

We have also computed the third term \( g_{2,2} \), but we will omit the long computations. To make the expression more compact, we use the notation \( b(z, w) = \partial \theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta \). The result is

\[
(3.54) \quad M_2 = +\frac{3}{2} \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta + \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2) \partial_\theta \partial_\theta (\partial \theta \partial_\theta) \partial_\theta \partial_\theta + \frac{1}{2} (\hat{\theta} \partial \theta) \partial_\theta \partial_\theta^2)
\]

4. From local to global

In this section we show that the local Bergman kernels actually provide asymptotics for the kernel \( K_{2, m} \). We fix a domain \( \Omega \) and let \( \chi, r \) and \( Q \) be as in the beginning of section 2. Suppose that we have a local poly-Bergman kernel mod(\( m^{-k-1} \)):

\[
K_{2, m}^k(z, w) = g_{2,2}(z, w)e^{mQ(z, w)}
\]

with \( z, w \in D(\zeta, r) \). We will use the integral operator

\[
K_{2, m}^k(f)(\xi) = \int f(\xi)K_{2, m}^k(z, \xi)\chi(\xi)e^{-mQ(z, \xi)}dA(\xi).
\]
Let use write \( K_{2,m}(w) := K_{2,m}(w, z) \), with a similar notation for kernels \( K^k_{q,m} \). It is easily verified that
\[
P_{2,m}[K^k_{2,m}; \chi](z) = P_{2,m}[K_{2,m}; \chi](w),
\]
where \( P_{2,m} \) denotes the projection within \( L^2(\Omega, e^{-|z|^2}) \) to the space \( A^2_{2,m} \).

We would like to estimate
\[
\left| K_{2,m}(z, w) - K^k_{2,m}(z, w) \right| \leq \left| K_{2,m}(w, z) - I^k_{q,m}[K_{2,m}; \chi](w) \right| + \left| P_{2,m}[K^k_{2,m}; \chi](z) - K^k_{2,m}(z, w) \chi(z) \right|
\]

For the first term on the right hand side, we use that \( K^k_{2,m} \) is a local Bergman kernel \( mod(m^{-k-1}) \), i.e. it \( \mathcal{O}_2^k \) satisfies (3.6). This can be rephrased as
\[
\left| K_{2,m}(w, z) - I^k_{m}[K_{2,m}; \chi](w) \right| \leq C m^{-k-\frac{3}{2}} e^{\frac{1}{2}|z|^2} \| K_{2,m} \|_m.
\]

Here, and later, \( C \) with a possible subindex denotes a constant with respect to the parameter \( m \) that can refer to a different value line by line. The extra \(-1/2 \) in the exponent comes from applying the Taylor expansion (2.1) and Cauchy-Schwarz inequality to the integral corresponding to the error term.

We now proceed to estimate \( \| K_{2,m} \|_m = K_{2,m}(z, z)^{1/2} \). This amounts to controlling pointwise values of a bianalytic functions in terms of weighted \( L^2 \)-norms. With analytic Bergman kernels, one can use the fact that \( \log |f(z)| \) is subharmonic for any analytic function \( f \). This is not true for more general polyanalytic functions, so a different strategy has to be used. We will use the notation \( \mathbb{D} \) for the unit disk, \( \mathbb{T} \) for the unit circle and \( ds \) for the arc length measure on \( \mathbb{T} \) divided by \( 2\pi \).

**Proposition 4.1.** Let \( \Psi \leq 0 \) be subharmonic on \( \mathbb{D} \) and let \( u(z) \) be a bianalytic function on \( \mathbb{D} \). Then
\[
|u(0)|^2 e^{\Psi(0)} \leq \left[ 8 + 12|\Psi(0)|^2 \right] \int_{\mathbb{D}} |u|^2 e^{\Psi} dA
\]

We start by proving a lemma. We write the bianalytic function \( u \) as \( u(z) = u_0(z) + cz + u_2(z) \), where \( c \in \mathbb{C} \) and \( u_0 \) and \( u_2 \) are analytic.

**Lemma 4.2.**
\[
e^{\Psi(0)} \int_0^1 \left| u_0(0) + r^2 u_2(0) + cr \int_{\mathbb{T}} \Psi(r\zeta) ds(\zeta) \right|^2 r dr \leq \frac{1}{2} \int_{\mathbb{D}} |u(z)|^2 e^{\Psi(z)} dA(z).
\]

**Proof.** We set \( \Psi_r(\zeta) := \Psi(r\zeta) \), and define the functions \( G_r \) by
\[
\log G_r := \frac{1}{2} \int_{\mathbb{T}} \frac{1}{1 - rz} \Psi_r(\zeta) ds(\zeta).
\]

They satisfy
\[
\log |G_r| = \text{Re} \log G_r = \frac{1}{2} P[\Psi_r](z), \quad z \in \mathbb{D},
\]
where \( P[\Psi_r](z) \) denotes the Poisson extension of \( \Psi_r \) to \( \mathbb{D} \). In particular,
\[
|G_r|^2 = e^{\Psi_r}.
\]
on $T$. Now
\begin{equation}
\int_{T} u(r\zeta)G_{r}(\zeta)\, ds(\zeta) = \int_{T} [u_{0}(r\zeta) + cr\zeta + r^{2}u_{2}(r\zeta)]G_{r}(\zeta)\, ds(\zeta) = u_{0}(0)G_{r}(0) + crG_{r}(0) + r^{2}u_{2}(0)G_{r}(0) = G_{r}(0)\left[u_{0}(0) + r^{2}u_{2}(0) + cr\frac{G_{r}^{\prime}(0)}{G_{r}(0)}\right].
\end{equation}

Here
\[\frac{G_{r}^{\prime}(z)}{G_{r}(z)} = [\log G_{r}(z)]^{\prime} = \frac{1}{2} \int_{T} \frac{2\zeta}{(1 - z\zeta)^{2}} \Psi_{r}(\zeta)\, ds(\zeta).\]

Therefore,
\begin{equation}
\frac{G_{r}^{\prime}(0)}{G_{r}(0)} = \int_{T} \zeta \Psi_{r}(\zeta)\, ds(\zeta).
\end{equation}

Applying Cauchy-Schwarz inequality, we get
\begin{equation}
|G_{r}(0)|^{2} \left|u_{0}(0) + r^{2}u_{2}(0) + cr\frac{G_{r}^{\prime}(0)}{G_{r}(0)}\right|^{2} \leq \int_{T} |u(r\zeta)G_{r}(\zeta)|^{2}\, ds(\zeta) = \int_{T} |u(r\zeta)|^{2} e^{\Psi_{r}(\zeta)}\, ds(\zeta).
\end{equation}

We multiply this inequality by $r$ and integrate:
\begin{equation}
\int_{0}^{1} |G_{r}(0)|^{2} \left|u_{0}(0) + r^{2}u_{2}(0) + cr\frac{G_{r}^{\prime}(0)}{G_{r}(0)}\right|^{2} \, r\, dr \leq \frac{1}{2} \int_{D} |u(z)|^{2} e^{\Psi(z)}\, dA(z).
\end{equation}

Subharmonicity of $\Psi$ means that $e^{\Psi(0)} \leq |G_{r}(0)|^{2}$, which gives finally
\begin{equation}
e^{\Psi(0)} \int_{0}^{1} \left|u_{0}(0) + r^{2}u_{2}(0) + cr\frac{G_{r}^{\prime}(0)}{G_{r}(0)}\right|^{2} \, r\, dr \leq \frac{1}{2} \int_{D} |u(z)|^{2} e^{\Psi(z)}\, dA(z).\end{equation}

\textbf{Proof of proposition 4.1.} We apply the lemma to the function $v(z) = zu(z) = zu_{0}(z) + c|z|^{2} + |z|^{2}u_{2}(z)$.

We have $v_{0}(z) = zu_{0}(z), v_{0}(z) = 0, v_{2}(z) = c + zu_{2}(z)$. So, $v_{0}(0) = 0, v_{2}(0) = c$, and therefore
\begin{equation}
|c|^{2} e^{\Psi(0)} \int_{0}^{1} r^{5} \, dr \leq \frac{1}{2} \int_{D} |z|^{5} |u(z)|^{2} e^{\Psi(z)}\, dA(z) \leq \frac{1}{2} \int_{D} |u(z)|^{2} e^{\Psi(z)}\, dA(z).
\end{equation}

This gives an estimate for $c$: $|c|^{2} \leq 3e^{-\Psi(0)} \int_{D} |u|^{2} e^{\Psi}\, dA$.

We use this to estimate the third term under the integral in inequality (4.7).
\begin{equation}
\int_{0}^{1} \left|c^{2} r^{3} \frac{G_{r}^{\prime}(0)}{G_{r}(0)}\right|^{2} \, dr = |c|^{2} \int_{0}^{1} r^{3} \left|\int_{T} \zeta \Psi_{r}(r\zeta)\, ds(\zeta)\right|^{2} \, dr \leq \frac{|c|^{2} |\Psi(0)|^{2}}{4} \leq \frac{3}{4} |\Psi(0)|^{2} e^{-\Psi(0)} \int_{D} |u|^{2} e^{\Psi}\, dA.
\end{equation}
Notice that the first inequality here followed from the assumption \( \Psi \leq 0 \) and subharmonicity of \( \Psi \). An application of triangle inequality now shows that

\[
(4.10) \quad \int_0^1 |u_0(0) + r^2 u_2(0)|^2 r dr \\
\leq e^{-\Psi(0)} \int_D |u|^2 e^\Psi dA + 2e^{-\Psi(0)} \int_0^1 |u|^2 r^3 \left| \frac{\mathcal{G}(0)}{\mathcal{G}_r(0)} \right|^2 dr \\
\leq e^{-\Psi(0)} \int_D |u|^2 e^\Psi dA + \frac{3}{2} |\Psi(0)|^2 e^{-\Psi(0)} \int_D |u|^2 e^\Psi dA \\
= \left[ 1 + \frac{3}{2} |\Psi(0)|^2 \right] e^{-\Psi(0)} \int_D |u|^2 e^\Psi dA.
\]

We expand the left hand side:

\[
(4.11) \quad \int_0^1 \left[ |u_0(0)|^2 + |u_2(0)|^2 + 2r^2 \text{Re}\left[ u_0(0)u_2(0) \right] \right] r dr \\
= \frac{1}{2} |u_0(0)|^2 + \frac{1}{6} |u_2(0)|^2 + \frac{1}{2} \text{Re}\left[ u_0(0)u_2(0) \right] \\
= \frac{1}{2} |u_0(0)|^2 + \frac{1}{6} |u_2(0)|^2 + \frac{1}{2} \left( \frac{\sqrt{3}}{2} u_0(0) + \frac{3}{2} \sqrt{3} u_2(0) \right)^2 \\
- \frac{3}{8} |u_0(0)|^2 - \frac{1}{6} |u_2(0)|^2 \geq \frac{1}{8} |u_0(0)|^2.
\]

Finally, combining with (4.10),

\[
|u_0(0)|^2 \leq e^{-\Psi(0)} \left[ 8 + 12 |\Psi(0)|^2 \right] \int_D |u|^2 e^\Psi dA.
\]

\[\square\]

Given a positive measure \( \mu \) on \( \mathbb{D} \), we denote by \( \mathbf{G}(\mu) \) the Green’s potential of \( \mu \):

\[
\mathbf{G}[\mu](z) := \int \log \left| \frac{z - w}{1 - z\bar{w}} \right|^2 d\mu(w)
\]

**Proposition 4.3.** Let \( \Psi \) be subharmonic and \( u \) bianalytic on \( \mathbb{D} \). Then

\[
|u(0)|^2 e^{\Psi(0)} \leq \left[ 8 + 12 |\mathbf{G}[\Delta \Psi](0)|^2 \right] \int_D |u|^2 e^\Psi dA.
\]

**Proof.** We can decompose \( \Psi = \mathbf{G}[\Delta \Psi] + h \), where \( h \) is a harmonic function. For the harmonic function \( h \), we associate an analytic function \( G \) such that \( h = |G|^2 \). Notice also that the Green’s potential satisfies \( \mathbf{G}[\Delta \Psi] \leq 0 \), so that the proposition 4.1 is available. We get

\[
(4.12) \quad |u(0)|^2 e^{\Psi(0)} \leq |u(0)|G(0)|^2 e^{\mathbf{G}[\Delta \Psi](0)} \\
\leq \left[ 8 + 12 |\mathbf{G}[\Delta \Psi](0)|^2 \right] \int_D |u(\bar{w})G(w)|^2 e^{\mathbf{G}[\Delta \Psi](w)} dA(w) \\
= \left[ 8 + 12 |\mathbf{G}[\Delta \Psi](0)|^2 \right] \int_D |u(\bar{w})|^2 e^{\Psi(\bar{w})} dA(w).
\]

\[\square\]
Proposition 4.4. Let \( m \geq 0 \) and let \( u \) be bianalytic in \( \mathbb{D}(z, m^{-1/2}) \). Let \( Q \) be a \( C^{1,1} \)-smooth real-valued function on \( \mathbb{D}(z, m^{-1/2}) \) satisfying \( \|Q\|_{C^{1,1}} \leq A < \infty \). Then,

\[
|u(z)|^2 e^{-mQ(z)} \leq m(8 + 48A^2)e^A \int_{\mathbb{D}(z, m^{-1/2})} |u(\xi)|^2 e^{-mQ(\xi)} dA(\xi).
\]

Proof. We assume \( z = 0 \) without loss of generality. We set \( \Psi_m(\xi) = |\xi|^2 - mQ(m^{-1/2} \xi) \) and \( u_m(\xi) = u(m^{-1/2} \xi) \) for \( \xi \in \mathbb{D} \). It is easy to check that \( \Delta \Psi_m \geq 0 \) and \( |G[\Delta \Psi_m](0)| \leq 2A \). We now apply the previous proposition:

\[
|u(0)|^2 e^{-mQ(0)} = |u_m(0)|^2 e^{|\Psi_m(0)|} \int_{\mathbb{D}} |u_m(\xi)|^2 e^{-mQ(\xi)} dA(\xi) \leq (8 + 48A^2)e^A \int_{\mathbb{D}} |u_m(\xi)|^2 e^{-mQ(\xi)} dA(\xi) = m(8 + 48A^2)e^A \int_{\mathbb{D}(0, m^{-1/2})} |u(\omega)|^2 e^{-mQ(\omega)} dA(\omega) \]

This proposition shows that

\[
|K_{2,m}(z, z)|^2 e^{-mQ(z)} \leq Cm \int_{\Omega} |K_{2,m}(w, z)|^2 e^{-mQ(w)} dA(w) = CmK_{2,m}(z, z),
\]

where the constant \( C \) is of the form given in the proposition. This implies \( K_{2,m}(z, z) \leq C e^{-mQ(z)} \). We can now estimate the first term in (4.1) as

\[
|K_{2,m}(w, z) - f_{2,m}^k(z)|^2 \leq C_m m^{-1} e^{-mQ(z) + Q(\omega)}.
\]

We turn to the second term in (4.1). The expression in question, which we denote by \( u_0 \), is the norm minimal solution of the equation

\[
\tilde{\partial}^2 u = \tilde{\partial}^2 [K_{2,m}^k(z)] \chi(z)
\]

We iterate the \( \tilde{\partial} \)-estimates of Hörmander to get an estimate for \( u_0 \).

Proposition 4.5. Let \( \Omega \) be a domain in \( \mathbb{C} \) and \( \phi : \Omega \to \mathbb{R} \) a \( C^4 \)-smooth function which satisfies \( \Delta \phi > 0 \) and \( \Delta(\phi + \log \Delta \phi) > 0 \). Let \( f \in L^2_{\text{loc}}(\Omega) \). The norm minimal solution \( u \) of the problem \( \tilde{\partial}^2 u = f \) satisfies

\[
\int_{\Omega} |u|^2 e^{-\phi} \leq \int_{\Omega} |f|^2 e^{-\phi} \frac{e^{-\phi}}{\Delta \phi \cdot \Delta(\phi + \log \Delta \phi)},
\]

provided that the right hand side is finite.

Proof. Apply the \( L^2 \)-estimates for \( \tilde{\partial} \)-operator with the weights \( \phi \) and \( \phi + \log \Delta \phi \).
for big enough \( m \). The second inequality rests also on the fact that \( \Delta Q(z) \geq c > 0 \) for \( z \in D(0, \frac{1}{3} r) \). In this integral, the function \( K_{m,n}^k \) is bianalytic, so at least one \( \partial \) will hit \( \chi \). We can therefore apply a Taylor series argument as in the proof of Proposition 2.1 to get
\[
\mu_0(z) |e^{-Q(z)}| \leq C_3 m^{\frac{5}{4}} e^{mQ(z)} e^{-bm},
\]
for some \( \delta > 0 \). We have now obtained the estimate
\[
\left| P_{2,m}[K_{2,m,w}^k(z)] - K_{2,m,w}^k(z) \chi(z) \right| \leq C_4 m^{\frac{1}{4}} e^{m(Q(z)+Q(w))} e^{-bm}
\]
for the second term in (4.1). By combining (4.16) and (4.19), we get

**Theorem 4.6.** Let \( \Omega \) be a domain in \( \mathbb{C} \) and \( Q \) a positive \( C^4 \)-smooth function on \( \Omega \). Let \( K_{2,m}^k(z,w) \) be a local bianalytic Bergman kernel \( \mod (m^{-k-1}) \) produced by the algorithm in Section 2. Take \( z_0 \in \Omega \) and let \( r \) and \( Q \) satisfy the conditions in the beginning of section 2. Then, there exists \( m_0 \) so that for \( m \geq m_0 \),
\[
\left| K_{2,m}^k(z,w) - K_{2,m}^k(z,w) \right| e^{-\frac{1}{2} m(Q(z)+Q(w))} = O(m^{-k-1}), \quad z, w \in D(z_0, \frac{1}{3} r).
\]

The constant of the error term is uniform for all such \( z \) and \( w \).

Recall the theorem 1.1 from the introduction. We have \( L_1^2(x) = 2 - \chi \), so the special case of theorem for \( q = 2 \) follows easily from Theorem 4.6 and the computation of the terms \( \delta_{2,1} \) and \( \delta_{2,1} \) in Section 2.

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