ON THE UNIQUENESS THEOREM OF HOLMGREN

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Abstract. We review the classical Cauchy-Kovalevskaya theorem and the related uniqueness theorem of Holmgren, in the simple setting of powers of the Laplacian and a smooth curve segment in the plane. As a local problem, the Cauchy-Kovalevskaya and Holmgren theorems supply a complete answer to the existence and uniqueness issues. Here, we consider a *global uniqueness problem* of Holmgren's type. Perhaps surprisingly, we obtain a connection with the theory of quadrature identities, which demonstrates that rather subtle algebraic properties of the curve come into play. For instance, if Ω is the interior domain of an ellipse, and I is a proper arc of the ellipse $\partial\Omega$, then there exists a nontrivial biharmonic function u in Ω which is three-flat on I (i.e., all partial derivatives of u of order ≤ 2 vanish on I) *if and only if* the ellipse is a circle. Another instance of the same phenomenon is that if Ω is bounded and simply connected with C^∞-smooth Jordan curve boundary, and if the arc $I \subset \partial\Omega$ is nowhere real-analytic, then we have local uniqueness already with sub-Cauchy data: if a function is biharmonic in $\mathcal{O} \cap \Omega$ for some planar neighborhood $\mathcal{O}$ of I, and is three-flat on I, then it vanishes identically on $\mathcal{O} \cap \Omega$, provided that $\mathcal{O} \cap \Omega$ is connected.

Finally, we consider a three-dimensional setting, and analyze it partially using analogues of the *square* of the standard 2×2 Cauchy-Riemann operator. In a special case when the domain is of periodized cylindrical type, we find a connection with massive Laplacians [the Helmholz operator with imaginary wave number] and Bers' theory of pseudoanalytic functions.

1. Introduction

1.1. **Basic notation.** Let
$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \mathrm{d}A(z) := \mathrm{d}x\mathrm{d}y,$$
denote the Laplacian and the area element, respectively. Here, $z = x + \mathrm{i}y$ is the standard decomposition into real and imaginary parts. We let $\mathbb{C}$ denote the complex plane. We also need the standard complex differential operators
$$\bar\partial_z := \frac{1}{2}\Big(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\Big), \quad \partial_z := \frac{1}{2}\Big(\frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial y}\Big),$$
so that Δ factors as $\Delta = 4\partial_z\bar\partial_z$. We sometimes drop indication of the differentiation variable z. A function u on a domain is *harmonic* if $\Delta u = 0$ on the domain. Similarly, for a positive integer N, the function u is *N-harmonic* if $\Delta^N u = 0$ on the domain in question.

1.2. **The theorems of Cauchy-Kovalevskaya and Holmgren for powers of the Laplacian.** Let Ω be a bounded simply connected domain in the plane $\mathbb{C}$ with smooth boundary. We let ∂_n denote the operation of taking the normal derivative. For $j = 1, 2, 3, \ldots$, we let ∂_n^j denote the j-th order normal derivative. Here, we understand those higher derivatives in terms of higher derivatives of the restriction of the function to the line normal to the boundary at the given

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boundary point. We consider the Cauchy-Kovalevskaya for powers of the Laplacian Δ^N, where $N = 1, 2, 3 \ldots$.

Theorem 1.1. (Cauchy-Kovalevskaya) *Suppose I is a real-analytic nontrivial arc of $\partial\Omega$. Then if f_j, for $j = 1, \ldots, 2N$, are real-analytic functions on I, there is a function u with $\Delta^N u = 0$ in a (planar) neighborhood of I, having $\partial_n^{j-1} u|_I = f_j$ for $j = 1, \ldots, 2N$. The solution u is unique among the real-analytic functions.*

Holmgren's theorem (1901, [12]) gives uniqueness under less restrictive assumptions on the data and the solution.

Theorem 1.2. (Holmgren) *Suppose I is a real-analytic nontrivial arc of $\partial\Omega$. Then if u is smooth on a planar neighborhood $\mathcal{O}$ of I and $\Delta^N u = 0$ holds on $\mathcal{O} \cap \Omega$, with $\partial_n^{j-1} u|_I = 0$ for $j = 1, \ldots, 2N$, then $u(z) \equiv 0$ on $\mathcal{O} \cap \Omega$, provided that the open set $\mathcal{O} \cap \Omega$ is connected.*

As local statements, the theorems of Cauchy-Kovalevskaya and Holmgren complement each other and supply a complete answer to the relevant existence and uniqueness issues. However, it is often given that the solution u is *global*, that is, it solves $\Delta^N u = 0$ throughout Ω. It is then a reasonable question to ask whether this changes anything. For instance, in the context of Holmgren's theorem, may we reduce the boundary data information on I while retaining the assertion that u vanishes identically? We may, e.g., choose to require a lower degree of flatness along I:

$$\partial_n^{j-1} u|_I = 0 \quad \text{for} \quad j = 1, \ldots, R, \tag{1.1}$$

where $1 \leq R \leq 2N$. We call (1.1) a condition of vanishing *sub-Cauchy data*.

Problem 1.3. (Global Holmgren problem) Suppose u is smooth in $\Omega \cup I$ and solves $\Delta^N u = 0$ on Ω and has the flatness given by (1.1) on I, for some $R = 1, \ldots, 2N$. For which values of R does it follow that $u(z) \equiv 0$ on Ω?

Digression on the global Holmgren problem I. When $R = 2N$, we see that $u(z) \equiv 0$ follows from Holmgren's theorem, by choosing a suitable sequence of neighborhoods $\mathcal{O}$. Another instance is when $R = N$ and $I = \partial\Omega$. Indeed, in this case, we recognize in (1.1) the vanishing of Dirichlet boundary data for the equation $\Delta^N u = 0$, which necessarily forces $u(z) \equiv 0$ given that we have a global solution. When $R < N$ and $I = \partial\Omega$, it is easy to add additional smooth non-trivial Dirichlet boundary data to (1.1) and obtain a nontrivial solution to $\Delta^N u = 0$ on Ω with (1.1). So for $I = \partial\Omega$, we see that the assumptions imply $u(z) \equiv 0$ if and only if $N \leq R \leq 2N$. It remains to analyze the case when $I \neq \partial\Omega$. Then either $\partial\Omega \setminus I$ consists of a point, or it is an arc. When $I \neq \partial\Omega$, we cannot expect that (1.1) with $R = N$ will be enough to force u to vanish on Ω, Indeed, if $\partial\Omega \setminus I$ is a nontrivial arc, we may add nontrivial smooth Dirichlet data on $\partial\Omega \setminus I$ (1.1) and by solving the Dirichlet problem we obtain a nontrivial function u with $\Delta^N u = 0$ on Ω having (1.1) with $R = N$. Similarly, when $\partial\Omega \setminus I$ consists of a single point, we may still obtain a nontrivial solution u by supplying distributional Dirichlet boundary data which are supported at that single point. So, to have a chance to get uniqueness, we must require that $N < R \leq 2N$. As the case $R = 2N$ follows from Holmgren's theorem, the interesting interval is $N < R < 2N$. For $N = 1$, this interval is *empty*. However, for $N > 1$ it is nonempty, and the problem becomes interesting.
Digression on the global Holmgren problem II. Holmgren's theorem has a much wider scope than what is presented here. It applies a wide range of linear partial differential equations with real-analytic coefficients, provided that the given arc I is non-characteristic (see [15]; we also

refer the reader to the related work of Hörmander [13]). So the results obtained here suggest that we should replace Δ^N by a more general linear partial differential operator and see to what happens in the above global Holmgren problem. Naturally, the properties of the given linear partial differential operator and the geometry of the arc I will both influence the the answer.

1.3. **Higher dimensions and nonlinear partial differential equations.** The global Holmgren problem makes sense also in $\mathbb{R}^n$, and it is natural to look for a solution there as well. Moreover, if we think of the global Holmgren problem as asking for uniqueness of the solution for given (not necessarily vanishing) sub-Cauchy data, the problem makes sense also for non-linear partial differential equations. We should note here that things can get quite tricky in the nonlinear context. See, e.g., Métivier's example of non-uniqueness [17] (which is based on on Hörmander's linear first order non-uniqueness example in the C^∞ context [13]).

We refer the interested reader to Gårding's booklet for a survey of Holmgren's theorem with relevant historical references.

In Section 5, we analyze the biharmonic equation in three dimensions with respect to the global Holmgren problem. Along the way, we obtain a factorization of the biharmonic operator Δ^2 as the product of two 3×3 differential operator matrices which are somewhat analogous to the *squares* of the Cauchy-Riemann operators $\partial_z, \bar\partial_z$ from the two-dimensional setting.

1.4. **The local Schwarz function of an arc.** If an arc I is real-analytically smooth, there exists an open neighborhood $\mathcal{O}_I$ of the arc and a holomorphic function $S_I : \mathcal{O}_I \to \mathbb{C}$ such that $S_I(z) = \bar z$ holds along I. This function S_I is called the *local Schwarz function*. In fact, the existence of a local Schwarz function is equivalent to real-analytic smoothness of the arc. It is possible to ask only for a so-called one-sided Schwarz function, which need not be holomorphic in all of $\mathcal{O}_I$ but only in $\mathcal{O}_I \cap \Omega$ (the side which belongs to Ω). Already the existence of a one-sided Schwarz function is very restrictive on the local geometry of I [21]. To ensure uniqueness of the local Schwarz function S_I (including the one-sided setting), we shall *assume that both $\mathcal{O}_I$ and $\mathcal{O}_I \cap \Omega$ are connected open sets*.

1.5. **A condition which gives uniqueness for the global Holmgren problem.** As before, we let Ω be a bounded simply connected domain in the plane. We have obtained the following criterion. In the statement, "nontrivial" means "not identically equal to 0". Moreover, as above, *we assume that the set $\mathcal{O}_I \cap \Omega$* – the domain of definition of the (one-sided) Schwarz function – *is connected*.

Theorem 1.4. *Suppose there exists a nontrivial function $u : \Omega \to \mathbb{C}$ with $\Delta^N u = 0$ on Ω, which extends to a C^{2N-1}-smooth function on $\Omega \cup I$, where I is a real-analytic arc of $\partial\Omega$. If R is an integer with $N < R \le 2N$, and if u has the flatness given by* (1.1) *on I, then there exists a nontrivial function of the form*

$$\Psi(z, w) = \psi_N(z)w^{N-1} + \psi_{N-1}(z)w^{N-2} + \cdots + \psi_1(z), \tag{1.2}$$

where each $\psi_j(z)$ is holomorphic in Ω for $j = 1, \ldots, N$, such that

$$\Psi(z, w) = \mathrm{O}(|w - S_I(z)|^{R-N}) \quad \text{as } w \to S_I(z), \tag{1.3}$$

for $z \in \Omega \cap \mathcal{O}_I$.

The above theorem asserts that $w = S_I(z)$ is the solution (root) of a polynomial equation [over the ring of holomorphic functions on Ω]

$$\Psi(z, w) = \psi_N(z)w^{N-1} + \psi_{N-1}(z)w^{N-2} + \cdots + \psi_1(z) = 0, \tag{1.4}$$

and that $\Psi(z,w)$ has the indicated additional flatness along $w = S_I(z)$ if $N+1 < R$. *An equivalent way to express the flatness condition* (1.3) *is to say that* $w = S_I(z)$ *solves simultaneously the system of equations*

$$\partial_w^{j-1}\Psi(z,w) = \frac{(N-1)!}{(N-j)!}\psi_N(z)w^{N-j} + \cdots + (j-1)!\psi_j(z) = 0, \qquad j = 1,\ldots,R-N. \tag{1.5}$$

The equation (1.4) results from considering $j = 1$ in (1.5). Let J, $1 \le J \le N$, be the largest integer such that the holomorphic function $\psi_J(z)$ is nontrivial. Since the expression $\Psi(z,w)$ is nontrivial, such an integer J must exist. As a polynomial equation in w, (1.4) will have at most $J-1$ roots for any fixed $z \in \Omega$. Counting multiplicities, the number of roots is constant and equal to $J-1$, for points $z \in \Omega$ where $\psi_J(z) \neq 0$. At the exceptional points where $\psi_J(z) = 0$, the number of roots is smaller. With the possible exception of branch points, where some of the roots coalesce, the roots define locally well-defined holomorphic functions in $\Omega \setminus \mathrm{Z}(\psi_J)$, where

$$\mathrm{Z}(\psi_J) := \{z \in \Omega : \ \psi_J(z) = 0\}.$$

If we take the system (1.5) into account, we see that $J > R-N$. Indeed, we may effectively rewrite (1.5) in the form

$$\partial_w^{j-1}\Psi(z,w) = \frac{(J-1)!}{(J-j)!}\psi_J(z)w^{J-j} + \cdots + (j-1)!\psi_j(z) = 0, \qquad j = 1,\ldots,R-N, \tag{1.6}$$

and if $J \le R-N$, we may plug in $j = J$ into (1.6), which would result in

$$\partial_w^{J-1}\Psi(z,w) = (J-1)!\psi_J(z) = 0,$$

which cannot be solved by $w = S_I(z)$ [except on the zero set $\mathrm{Z}(\psi_J)$], a contradiction. We think of (1.4) as saying that $w = S_I(z)$ is an algebraic expression over the ring of holomorphic functions on Ω. In particular, the local Schwarz function S_I extends to a multivalued holomorphic function in $\Omega \setminus \mathrm{Z}(\psi_J)$ with branch cuts. So in particular S_I makes sense not just on $\Omega \cap \mathcal{O}_I$ [this is an interior neighborhood of the arc I], but more generally in $\Omega \setminus \mathrm{Z}(\psi_J)$, if we allow for multivaluedness and branch cuts. The condition that $w = S_I(z)$ solves (1.4) is therefore rather restrictive. To emphasize the implications of the above theorem, we formulate a "negative version".

Corollary 1.5. *Let I be a real-analytically smooth arc of $\partial\Omega$, and suppose that R is an integer with $N < R \le 2N$. Suppose in addition that the local Schwarz function S_I does not solve the system* (1.5) *on $\mathcal{O}_I \cap \Omega$ for any nontrivial function $\Psi(z,w)$ of the form* (1.2). *Then every function function u on Ω, which extends to a C^{2N-1}-smooth function on $\Omega \cup I$, with $\Delta^N u = 0$ on Ω and flatness given by* (1.1) *on I, must be trivial: $u(z) \equiv 0$.*

In particular, for $N = 2$ and $R = 3$, the condition (1.3) says that $w = S_I(z)$ solves the *linear equation*

$$\psi_2(z)w + \psi_1(z) = 0,$$

with solution

$$w = S_I(z) = -\frac{\psi_1(z)}{\psi_2(z)},$$

which expresses a *meromorphic function in* Ω. We formulate this conclusion as a corollary.

Corollary 1.6. *Let I be a real-analytically smooth arc of $\partial\Omega$, and suppose that the local Schwarz function S_I does not extend to a meromorphic function on Ω. Then every function function u on Ω, which extends to a C^3-smooth function on $\Omega \cup I$, with $\Delta^2 u = 0$ on Ω and flatness given by*

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0,$$

must be trivial: $u(z) \equiv 0$.

Remark 1.7. Corollary 1.6 should be compared with what can be said in the analogous situation in three dimensions (see Theorem 5.5 below).

It is well-known that having a local Schwarz function which extends meromorphically to Ω puts a strong rigidity condition on the arc I. For instance, if Ω is the domain interior to an ellipse, and I is any nontrivial arc of $\partial\Omega$ [i.e., of positive length], then S_I extends to a meromorphic function in Ω if and only if the ellipse is a circle. This means that the Global Holmgren Problem gives uniqueness in this case, with $N = 2$ and $R = 3$, unless the ellipse is circular. We formalize this as a corollary.

Corollary 1.8. *Suppose Ω is the domain interior to an ellipse, and that I is a nontrivial arc of the ellipse $\partial\Omega$. Suppose u is C^3-smooth in $\Omega \cup I$, and $\Delta^2 u = 0$ on Ω. If u has*

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0,$$

then $u(z) \equiv 0$ unless the ellipse is a circle.

Remark 1.9. The smoothness condition in Theorem 1.4 and Corollary 1.5 is somewhat excessive. For instance, in Corollaries 1.6 and 1.8, the C^3-smoothness assumption may be reduced to C^2-smoothness. The additional smoothness makes for an easy presentation by avoiding technicalities.

1.6. **Uniqueness results for arc-flat biharmonic functions for smooth arcs that are nowhere real-analytic.** In the global Holmgren problem, we asked when vanishing Cauchy data may be relaxed to vanishing sub-Cauchy data (1.1) if we know that we have a global solution in the domain Ω. In the local problem, no improvement was possible because the Cauchy-Kovalevskaya theorem produces nontrivial solutions if we supply nontrivial (real-analytic) Cauchy data on the arc I. However, there is a way to suppress the effect: just assume the arc is C^∞-smooth and nowhere real-analytic. By this we mean that I has no nontrivial subarc which is real-analytic. Then the Cauchy-Kovalevskaya theorem cannot be applied as stated. We recall the standing assumption that Ω is bounded, simply connected, and has a C^∞-smooth Jordan curve boundary.

Theorem 1.10. *Suppose $I \subset \partial\Omega$ is an arc that is nowhere real-analytic. Let $\mathcal{O}$ be an open planar neighborhood of I, such that $\mathcal{O} \cap \Omega$ is connected. Then every function function u on $\mathcal{O} \cap \Omega$, which extends to a C^3-smooth function on $(\mathcal{O} \cap \Omega) \cup I$, with $\Delta^2 u = 0$ on $\mathcal{O} \cap \Omega$ and flatness given by*

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0,$$

must be trivial: $u(z) \equiv 0$ *on* $\mathcal{O} \cap \Omega$.

1.7. **Meromorphic Schwarz function and construction of arc-flat biharmonic functions.** Here, we study the necessity of the Schwarz function condition in Corollary 1.6.

Theorem 1.11. *Suppose $\partial\Omega$ is a C^∞-smooth Jordan curve, and that $I \subset \partial\Omega$ is a real-analytically smooth arc, such that the complementary arc $\partial\Omega \setminus I$ is nontrivial as well. If the local Schwarz function S_I extends to a meromorphic function in Ω with finitely many poles, then there exists a nontrivial function u on Ω, which extends C^∞-smoothly to $\Omega \cup I$, with $\Delta^2 u = 0$ on Ω and flatness given by*

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0.$$

Remark 1.12. When $\Omega = \mathbb{D}$, the open unit disk, the Schwarz function for the boundary is $S_{\mathbb{T}}(z) = 1/z$, which is a rational function, and in particular, meromorphic in $\mathbb{D}$. So if I is a nontrivial arc of the unit circle $\mathbb{T} = \partial\mathbb{D}$, and $\mathbb{T} \setminus I$ is a nontrivial arc as well, then Theorem 1.11 tells us that there exists a nontrivial biharmonic function u on $\mathbb{D}$ which is C^∞-smooth on $\mathbb{D} \cup I$ and has the flatness

$$u|_I = 0, \;\; \partial_n u|_I = 0, \;\; \partial_n^2 u|_I = 0.$$

In this case, an explicit function u can be found, which works for any nontrivial arc $I \subset \mathbb{T}$ with $I \neq \mathbb{T}$. Indeed, we may use a suitable rotation of the function

$$u(z) = \frac{(1-|z|^2)^3}{|1-z|^4},$$

which is biharmonic with the required flatness except for a boundary singularity at $z = 1$. This shows that the circle is exceptional in Corollary 1.8. We should mention here that the above kernel $u(z)$ appeared possibly for the first time in [1], and then later in [5] and [19]. Elias Stein pointed out that very similar kernels in the upper half plane appear in connection with the theory of *axially symmetric potentials* [24].

Remark 1.13. Corollary 1.6 and Theorem 1.11 settle completely the issue of the Global Holmgren problem for Δ^2 with the flatness condition (1.3) [for $R = 3$], in the case when the meromorphic extension of the Schwarz function S_I to Ω has finitely many poles. Most likely this [technical] finiteness condition may be removed. Moreover, it seems likely that there should exists an analogue of Theorem 1.11 which applies to $N > 2$. More precisely, suppose that $N < R \le 2N$, and that the local Schwarz function $w = S_I(z)$ solves a polynomial equation system of equations (1.6) where the highest order nontrivial coefficient $\psi_J(z)$ has only finitely many zeros in $\mathbb{D}$, and that $I \subset \partial\Omega$ is a nontrivial real-analytically smooth arc whose complementary arc is nontrivial as well. Then there should exist a nontrivial function u on Ω which is C^{2N-1}-smooth on $\Omega \cup I$ with $\Delta^N u = 0$ on Ω having the flatness given by (1.3) on I.

As a corollary to Corollary 1.6 and Theorem 1.11, we obtain a complete resolution for real-analytically smooth boundaries.

Corollary 1.14. *Suppose $\partial\Omega$ is a real-analytically smooth Jordan curve, and that $I \subset \partial\Omega$ is a an arc, such that the complementary arc $\partial\Omega \setminus I$ is nontrivial as well. Then there exists a nontrivial function u on Ω, which extends C^2-smoothly to $\Omega \cup I$, with $\Delta^2 u = 0$ on Ω and flatness given by*

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0,$$

if and only if the local Schwarz function S_I extends to a meromorphic function in Ω.

Remark 1.15. In the context of Corollary 1.14, the condition that the local Schwarz function extend to a meromorphic function in Ω is the same as asking that Ω be a *quadrature domain* (see Subsection 4.1).

2. The proof of Theorem 1.4 and its corollaries

2.1. **Almansi expansion.** It is well-known that a function u which is *N-harmonic on* Ω, that is, has $\Delta^N u = 0$ on Ω, has an *Almansi expansion*

$$u(z) = u_1(z) + |z|^2 u_2(z) + \cdots + |z|^{2N-2} u_N(z), \tag{2.1}$$

where the functions u_j are all harmonic in Ω; the "coefficient functions" u_j are all uniquely determined by the given function u. On the other hand, every function u of the form (2.1), where the functions u_j are harmonic, is N-harmonic.

Proof of Theorem 1.4. The function u is N-harmonic in Ω, and hence it has an Almansi representation (2.1). Next, for $j = 1, 2, 3, \ldots$, we we consider the function

$$U(z) := \partial_z^N u(z),$$

where ∂_z is the complex differentiation operator defined in Subsection 1.1. From the flatness assumption on u, we know that

$$\bar\partial_z^{j-1} U(z) = 0, \qquad z \in I, \quad j = 1, \ldots, R - N. \tag{2.2}$$

Since

$$\bar\partial_z^N U(z) = \bar\partial_z^N \partial_z^N u(x) = 4^{-N} \Delta^N u(z) = 0, \qquad z \in \Omega,$$

the Almansi representation for U has the special form

$$U(z) = U_1(z) + \bar z U_2(z) + \cdots + \bar z^{N-1} U_N(z),$$

where the functions U_j, $j = 0, \ldots, N-1$ are all holomorphic in Ω, and uniquely determined by the function U. As u is assumed C^{2N-1}-smooth on $\Omega \cup I$, the function U is C^{N-1}-smooth on $\Omega \cup I$. In particular,

$$\bar\partial_z^{j-1} U(z) = \bar\partial_z^{j-1} \sum_{k=1}^N \bar z^{k-1} U_k(z) = \sum_{k=j}^N \frac{(k-1)!}{(k-j)!} U_k(z) \tag{2.3}$$

is C^{2N-j}-smooth on $\Omega \cup I$ for $j = 1, \ldots, N$. By plugging in $j = N$ into (2.3), we find that U_N is continuous on $\Omega \cup I$. Next, if we plug in $j = N - 1$, we find that U_{N-1} is continuous on $\Omega \cup I$. Proceeding iteratively, we see that all the functions U_k are continuous on $\Omega \cup I$ ($k = 1, \ldots, N$). In terms the Almansi representation for U, the condition (2.2) reads

$$\bar\partial_z^{j-1} U(z) = \sum_{k=j}^N \frac{(k-1)!}{(k-j)!} \bar z^{k-j} U_k(z) = 0, \qquad z \in I, \quad j = 1, \ldots, R - N. \tag{2.4}$$

We now define the function $\Psi(z, w)$. We declare that $\psi_j(z) := U_j(z)$, so that the function $\Psi(z, w)$ is given by

$$\Psi(z, w) := \sum_{k=1}^N \psi_k(z) w^{k-1} = \sum_{k=1}^N U_k(z) w^{k-1}.$$

By differentiating iteratively with respect to w, we find that

$$\partial_w^{j-1} \Psi(z, w) = \partial_w^{j-1} \sum_{k=1}^N \psi_k(z) w^{k-1} = \sum_{k=j}^N \frac{(k-1)!}{(k-j)!} \psi_k(z) w^{k-j} = \sum_{k=j}^N \frac{(k-1)!}{(k-j)!} U_k(z) w^{k-j},$$

so that

$$\partial_w^{j-1} \Psi(z, w)\Big|_{w := S_I(z)} = \sum_{k=j}^N \frac{(k-1)!}{(k-j)!} U_k(z) [S_I(z)]^{k-j}, \tag{2.5}$$

and according to (2.4), the right hand side expression in (2.5) vanishes on the arc I for $j = 1, \ldots, R - N$, as $S_I(z) = \bar z$ there. But the right hand side of (2.5) is holomorphic on $\Omega \cap \mathcal{O}_I$ and extends continuously to $(\Omega \cup I) \cap \mathcal{O}_I$ and apparently vanishes on I for $j = 1, \ldots, R - N$, so by the boundary uniqueness theorem for holomorphic functions (e.g., Privalov's theorem), the right hand side of (2.5) must vanish on $\Omega \cap \mathcal{O}_I$:

$$\partial_w^{j-1} \Psi(z, w)\Big|_{w := S_I(z)} = 0, \qquad z \in \Omega \cap \mathcal{O}_I, \quad j = 1, \ldots, R - N.$$

This is the system of equations (1.5), which by Taylor's formula is equivalent to the flatness condition (1.3).

It remains to be established that the function $\Psi(z,w)$ is nontrivial. Since, by construction, $\Psi(z,\bar z)=U(z)$, it is enough to show that U is nontrivial. We know by assumption that u is nontrivial, and that $\partial_z^N u=U$ while u has the flatness (1.1) along I. If U is trivial, i.e., $U(z)\equiv 0$, then $\partial_z^N u=0$ which is an elliptic equation of order N and since $R>N$, the flatness (1.1) entails that $u(z)\equiv 0$, by Holmgren's theorem (for the elliptic operator ∂^N). This contradicts the nontriviality of u, and therefore refutes the putative assumption that U was trivial. The proof is complete. □

Proof of Corollary 1.5. This is just the negative formulation of Theorem 1.4. □

Proof of Corollary 1.6. In this case where $N=2$, the equation (1.4) is linear, so by Theorem 1.4 with $N=2$ and $R=3$, the existence of a nontrivial biharmonic function on Ω with flatness (1.3) along I forces the local Schwarz function S_I to extend meromorphically to Ω. □

Proof of Corollary 1.8. It is well-known that the Schwarz function for an non-circular ellipse develops a branch cut along the segment between the focal points (cf. [8], [22]), so it cannot in particular be meromorphic in Ω. So, in view of Corollary 1.6, we must have $u(z)\equiv 0$, as claimed. □

3. Arcs that are nowhere real-analytic

3.1. **Arc-flat biharmonic functions.** We suppose u is C^3-smooth on $\Omega\cup I$, where $I\subset\partial\Omega$ is a nontrivial arc, and that u solves the sub-Cauchy problem ($\mathcal{O}$ stands for an open connected planar neighborhood of I and we assume $\mathcal{O}\cap\Omega$ is connected)

$$\Delta^2 u|_{\mathcal{O}\cap\Omega}=0,\quad u|_I=\partial_n u|_I=\partial_n^2 u|_I=0.$$

We make no particular assumptions on the arc I, but we know $\partial\Omega$ is a C^∞-smooth Jordan curve. As in the proof of Theorem 1.4, we form the associated function

$$U(z):=\partial_z^2 u(z),\qquad z\in\mathcal{O}\cap\Omega,$$

which is bianalytic:

$$\bar\partial_z^2 U(z)=\partial_z^2\bar\partial_z^2 u(z)=\frac{1}{16}\Delta^2 u(z)=0,\qquad z\in\mathcal{O}\cap\Omega.$$

As such, it has an Almansi-type decomposition $U(z)=U_1(z)+\bar z U_2(z)$, where U_j is holomorphic on $\mathcal{O}\cap\Omega$, for $j=1,2$. Since $U_2=\bar\partial_z U$, the smoothness assumptions on U entail that U_2 extends continuously to $(\mathcal{O}\cap\Omega)\cup I$, and, consequently, U_1 extends continuously to $(\mathcal{O}\cap\Omega)\cup I$ as well.

Proof of Theorem 1.10. We argue by contradiction, and assume that there exists a *nontrivial* biharmonic function u on $\mathcal{O}\cap\Omega$ which extends to a C^3-smooth function on $\mathcal{O}\cap\Omega$ with the given flatness on I. As above, we form the function $U(z):=\partial_z^2 u(z)$, which is bianalytic $\mathcal{O}\cap\Omega$ and C^1-smooth on $(\mathcal{O}\cap\Omega)\cup I$. We put $U_2(z):=\bar\partial_z U(z)$, which is then a holomorphic function on $\mathcal{O}\cap\Omega$ which extends continuously to $(\mathcal{O}\cap\Omega)\cup I$. The function $U_1(z):=U(z)-\bar z U_2(z)$ is also holomorphic on $\mathcal{O}\cap\Omega$ and continuous on $(\mathcal{O}\cap\Omega)\cup I$. So the bianalytic function U has the decomposition $U(z)=U_1(z)+\bar z U_2(z)$, where U_j, $j=1,2$, are both holomorphic on $\mathcal{O}\cap\Omega$ and continuous on $(\mathcal{O}\cap\Omega)\cup I$. From the sub-Cauchy flatness on I, we have that

$$U(z)=U_1(z)+\bar z U_2(z)=0,\qquad z\in I. \tag{3.1}$$

We suppose for the moment that the the analytic function U_2 – which is continuous on $(\mathcal{O}\cap\Omega)\cup I$ – is nontrivial; then there must exist a nontrivial subarc $I' \subset I$ where $U_2(z) \neq 0$ for $z \in I'$. The function

$$S_{I'}(z) := -\frac{U_1(z)}{U_2(z)}$$

then has the following properties: (i) it is holomorphic in $\mathcal{O}' \cap \Omega$, for some neighborhood $\mathcal{O}$ of I', and (ii) it extends to a continuous function on $(\mathcal{O}' \cap \Omega) \cup I'$, with boundary values $S_I(z) = \bar{z}$ on I'. We conclude that $S_{I'}$ is a local one-sided Schwarz function. By Sakai's theorem [21], a C^∞-smooth nowhere real-analytic arc cannot have a one-sided local Schwarz function. This gives us a contradiction, and the assumption that U_2 was nontrivial must be incorrect. So $U_2(z) \equiv 0$ on $\mathcal{O}' \cap \Omega$, and by (3.1), $U_1(z) = 0$ on I, and the uniqueness theorem for analytic functions gives that $U_1(z) \equiv 0$. We conclude that $U(z) \equiv 0$, and the function u solves the partial differential equation $\partial_{\bar{z}}^2 u(z) = 0$ on $\mathcal{O} \cap \Omega$, which means that the complex conjugate of u solves the same equation that U did, with even flatter data on the arc I. A repetition of the above argument then gives that $u(z) \equiv 0$, as claimed. □

4. Quadrature domains and the construction of arc-flat biharmonic functions

4.1. **Quadrature domains.** As before, Ω is a bounded simply connected domain in $\mathbb{C}$. For the moment, we assume in addition that the boundary $\partial\Omega$ is a real-analytically smooth Jordan curve. As before, $I \subset \partial\Omega$ is a nontrivial arc. Then the local Schwarz function S_I extends to a local Schwarz function for the whole boundary curve; we write $S_{\partial\Omega}$ for the extension. In [2], Aharonov and Shapiro show that in this setting, the following two conditions are equivalent:

(i) the Schwarz function $S_{\partial\Omega}$ extends to a meromorphic function in Ω,
(ii) the domain Ω is a quadrature domain.

Here, the statement that Ω is a *quadrature domain* means that for all harmonic functions h on Ω that are area-integrable ($h \in L^1(\Omega)$),

$$\int_\Omega h \mathrm{d}A = \langle h, \alpha \rangle_\Omega,$$

for some distribution α with *finite support contained inside* Ω. The notation $\langle \cdot, \cdot \rangle_\Omega$ is the dual action which extends (to the setting of distributions) the standard integral

$$\langle f, g \rangle_\Omega = \int_\Omega f g \mathrm{d}A$$

when $fg \in L^1(\Omega)$. It was also explained in [2] that the conditions (i)-(ii) are equivalent a third condition:

(iii) any conformal map $\varphi : \mathbb{D} \to \Omega$ [with $\varphi(\mathbb{D}) = \Omega$] is a *rational function*.

It is easy to see that the condition (iii) entails that the boundary curve $\partial\Omega$ is *algebraic*. Let us try to understand why the implication (i) $\Longrightarrow$ (iii) holds. So, we assume the Schwarz function extends to a meromorphic function in Ω, and form the function

$$\Psi(\zeta) := \begin{cases} S_{\partial\Omega}(\varphi(\zeta)), & \zeta \in \bar{\mathbb{D}}, \\ \overline{\varphi(1/\bar{\zeta})}, & \zeta \in \mathbb{D}_e, \end{cases}$$

where $\mathbb{D}_e := \{\zeta \in \mathbb{C} : |\zeta| > 1\}$ is the "exterior disk", and φ is any [surjective] conformal map $\mathbb{D} \to \Omega$. By the assumed real-analyticity of $\partial\Omega$, the conformal map φ extends holomorphically (and conformally) across the circle $\mathbb{T} = \partial\mathbb{D}$, see, e.g. [20]. In particular, $\Psi(\zeta)$ is well-defined on $\mathbb{T}$, and is holomorphic in $\mathbb{C} \setminus \mathbb{T}$. As the two definitions in $\mathbb{C} \setminus \mathbb{T}$ agree [in the limit sense] along $\mathbb{T}$, Morera's theorem gives that Ψ extends holomorphically across $\mathbb{T}$. But then Ψ is a rational function, as it has only finitely many poles and is holomorphic everywhere else on the Riemann sphere $\mathbb{C}\cup\{\infty\}$. If we put

$$\varphi_{\text{ext}}(\zeta) := \overline{\Psi(1/\bar{\zeta})},$$

then φ_{ext} is a rational function, which agrees with φ on $\mathbb{D}$. This establishes assertion (iii).

4.2. **Real-analytic arcs with one-sided meromorphic Schwarz function.** We return to the previous setting of a real-analytic arc $I\subset\partial\Omega$, where Ω is a bounded simply connected domain whose boundary $\partial\Omega$ is a C^∞-smooth Jordan curve. We shall assume that the local Schwarz function extends to a meromorphic function in Ω with finitely many poles. In this more general setting, the surjective conformal mapping $\varphi:\mathbb{D}\to\Omega$ extends analytically across the arc $\tilde{I}:=\varphi^{-1}(I)$: the extension is given by

$$\varphi_{\text{ext}}(\zeta) := \overline{S_I\circ\varphi(1/\bar{\zeta})},\qquad \zeta\in\mathbb{D}_e.$$

The extension is then meromorphic in $\mathbb{D}\cup\mathbb{D}_e\cup\tilde{I}$, with finitely many poles; we denote it by φ as well.

Proof of Theorem 1.11. We assume for simplicity that the arc I is open, i.e. does not contain its endpoints. Also, without loss of generality, we may assume that the origin 0 is in Ω. We let $\varphi:\mathbb{D}\to\Omega$ be a surjective conformal mapping with $\varphi(0)=0$, which by the above argument extends meromorphically to $\mathbb{D}\cup\mathbb{D}_e\cup\tilde{I}$, with finitelywith finitely many poles. Here, $\tilde{I}\subset\mathbb{T}$ be the arc of the circle for which $\varphi(\tilde{I})=I\subset\partial\Omega$. We let F be a the function

$$F(\zeta,\xi) := \frac{1}{\varphi(\zeta)}\int_0^\zeta \frac{1+\bar{\xi}\eta}{1-\bar{\xi}\eta}\varphi'(\eta)\mathrm{d}\eta, \tag{4.1}$$

where $|\xi|=1$ is assumed. For fixed $\xi\notin\tilde{I}$, the function $F(\cdot,\xi)$ is well-defined and holomorphic in a neighborhood of $\mathbb{D}\cup\tilde{I}$. Moreover, $F(\zeta,\xi)$ enjoys an estimate in terms of a (radial) function of $|\zeta|$ which is independent of the parameter $\xi\in\mathbb{T}$.

Next, we let proceed by considering functions real-valued v_1,v_2 that are harmonic in $\mathbb{D}$ (to be determined shortly), and associate holomorphic functions V_1,V_2 with $\operatorname{Im}V_1(0)=\operatorname{Im}V_2(0)=0$ and $\operatorname{Re}V_j=v_j$ for $j=1,2$. Then $2\partial_\zeta v_2(\zeta)=V_2'(\zeta)$, for $j=1,2$. We form the associated function

$$v(\zeta) := v_1(\zeta)+|\varphi(\zeta)|^2 v_2(\zeta). \tag{4.2}$$

The functions v_1,v_2 are real-valued and harmonic, and we calculate that

$$\Delta v = \Delta[v_1+|\varphi|^2v_2] = \Delta[|\varphi|^2v_2] = 4|\varphi'|^2\Big\{v_2+2\operatorname{Re}\Big[\frac{\varphi}{\varphi'}\partial_\zeta v_2\Big]\Big\}, \tag{4.3}$$

and

$$2\partial_\zeta\frac{1}{\varphi'(\zeta)}\partial_\zeta[v(\zeta)] = [V_1'/\varphi']'(\zeta)+\overline{\varphi(\zeta)}\Big\{2V_2'(\zeta)+\varphi(\zeta)[V_2'/\varphi']'(\zeta)\Big\}. \tag{4.4}$$

Now, we require of v_1,v_2 that the function v gets to have vanishing second order derivatives along $\tilde{I}$ in the following sense:

$$\Delta v|_{\tilde{I}}=0,\quad \partial_\zeta\frac{1}{\varphi'}\partial_\zeta[v]\Big|_{\tilde{I}}=0. \tag{4.5}$$

Since

$$v_2+2\operatorname{Re}\Big[\frac{\varphi}{\varphi'}\partial_\zeta v_2\Big] = \operatorname{Re}\Big\{V_2+\frac{\varphi}{\varphi'}V_2'\Big\} = \operatorname{Re}\Big\{\frac{(\varphi V_2')'}{\varphi'}\Big\},$$

the first condition in (4.5) may be expressed as

$$\operatorname{Re}\left\{\frac{(\varphi V_2')'}{\varphi'}\right\} = 0 \quad \text{on } \tilde{I}. \tag{4.6}$$

If we let V_2 be the holomorphic function with $V_2(0) = 0$ whose derivative is given by

$$V_2'(\zeta) = \int_{\mathbb{T}} F(\zeta, \xi)\mathrm{d}\nu(\xi), \tag{4.7}$$

where F is as in (4.1) and ν is a *real-valued Borel measure supported on the complementary arc* $\mathbb{T} \setminus \tilde{I}$, then condition (4.6) is automatically met, so that the first requirement in (4.5) is satisfied. It remains to meet the second requirement of (4.5) as well. In view of (4.4), and the uniqueness theorem for holomorphic functions, we may write the second requirement in the form

$$[V_1'/\varphi']'(\zeta) + \bar{\varphi}(1/\bar{\zeta})\Big\{2V_2'(\zeta) + \varphi(\zeta)[V_2'/\varphi']'(\zeta)\Big\} = 0,$$

which is the same as

$$\left[\frac{V_1'}{\varphi'}\right]'(\zeta) + \frac{\bar{\varphi}(1/\bar{\zeta})}{\varphi(\zeta)}\,\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{[\varphi(\zeta)]^2 V_2'(\zeta)}{\varphi'(\zeta)}\right\} = 0.$$

We think of this is as a second order linear differential equation in V_1, with a nice holomorphic solution V_1 in a neighborhood of $\mathbb{D} \cup \tilde{I}$ unless the finitely many poles in $\mathbb{D}$ of the function

$$\frac{\bar{\varphi}(1/\bar{\zeta})}{\varphi(\zeta)} = \frac{S_I(\varphi(\zeta))}{\varphi(\zeta)}$$

are felt. In order to suppress those poles, we may ask that the function V_2' should have a sufficiently deep zero at each of those poles in $\mathbb{D}$. This amounts to asking that

$$V_2^{(j)}(\zeta) = \int_{\mathbb{T}} \partial_\zeta^{j-1} F(\zeta, \xi)\mathrm{d}\nu(\xi) = 0 \qquad j = 1, \ldots, j_0(\zeta), \tag{4.8}$$

for a finite collection of points ζ in the disk $\mathbb{D}$. Taking real and imaginary parts in (4.8), we still are left with a finite number of linear conditions, and the space of real-valued Borel measures supported in $\mathbb{T} \setminus \tilde{I}$ is infinite-dimensional. So, clearly, there exists a nontrivial ν that satisfies (4.8). If we like, we may even find such a ν with C^∞-smooth density. Then the function V_2 is nonconstant, and its real part is nonconstant as well.

Finally, we turn to the issue of the biharmonic function u on Ω that we are looking for. We put $\tilde{u}(z) := v \circ \varphi^{-1}(z)$ and observe that with the choice of the Borel measure ν, the function $\tilde{u}$ is real-valued with

$$\Delta \tilde{u}|_I = 0, \quad \partial_z^2 \tilde{u}|_I = 0,$$

by (4.5). This means that all partial derivatives of $\tilde{u}$ of order 2 vanish along I, which says that both $\partial_x \tilde{u}$ and $\partial_y \tilde{u}$ have gradient vanishing along I. So both $\partial_x \tilde{u}$ and $\partial_y \tilde{u}$ are *constant* on I. If we repeat this argument, we see that there exists an affine function $A(z) := A_0 + A_1 x + A_2 y$ such that $u := \tilde{u} - A$ has the required flatness along I. Since by construction $\tilde{u}$ cannot itself be affine, this completes the proof of the theorem. □

Proof of Corollary 1.14. In view of Remark 1.9, the forward implication follows from Corollary 1.6. In the reverse direction, we appeal to Theorem 1.11 and use the observation that the local Schwarz function S_I is automatically holomorphic in a neighborhood of the entire boundary $\partial\Omega$, so it can only have finitely many poles in Ω. □

5. The biharmonic equation in three dimensions and the global Holmgren problem

5.1. **Matrix-valued differential operators.** In $\mathbb{C} \cong \mathbb{R}^2$, we may identify a complex-valued function $u = u_1 + \mathrm{i}u_2$, where u_1, u_2 are real-valued, with a column vector:

$$u \sim \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

In the same fashion, we identify the differential operators ∂_z and $\bar\partial_z$ with 2×2 matrix-valued differential operators

$$2\partial_z \sim \begin{pmatrix} \partial_x & \partial_y \\ -\partial_y & \partial_x \end{pmatrix}, \qquad 2\bar\partial_z \sim \begin{pmatrix} \partial_x & -\partial_y \\ \partial_y & \partial_x \end{pmatrix},$$

so that

$$\Delta = 4\partial_z\bar\partial_z \sim \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix},$$

which identifies the Laplacian Δ with its diagonal lift. Along the same lines, we see that

$$4\partial_z^2 \sim \begin{pmatrix} \partial_x^2-\partial_y^2 & 2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2-\partial_y^2 \end{pmatrix}, \qquad 4\bar\partial_z^2 \sim \begin{pmatrix} \partial_x^2-\partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2-\partial_y^2 \end{pmatrix},$$

and the main identity which we have used in this paper is simply that

$$\begin{pmatrix} \partial_x^2-\partial_y^2 & 2\partial_x\partial_y \\ -2\partial_x\partial_y & \partial_x^2-\partial_y^2 \end{pmatrix}\begin{pmatrix} \partial_x^2-\partial_y^2 & -2\partial_x\partial_y \\ 2\partial_x\partial_y & \partial_x^2-\partial_y^2 \end{pmatrix} = \begin{pmatrix} \Delta^2 & 0 \\ 0 & \Delta^2 \end{pmatrix}. \tag{5.1}$$

While it seems unclear what should be the canonical analogue of the Cauchy-Riemann operators $\partial_z, \bar\partial_z$ in the three-dimensional setting, it turns out to be possible to find suitable analogues of their squares! Indeed, there is a three-dimensional analogue of the factorization (5.1). We write $x = (x_1, x_2, x_3)$ for a point in $\mathbb{R}^3$, and let ∂_j denote the partial derivative with respect to x_j, for $j = 1, 2, 3$, and let

$$\Delta := \partial_1^2 + \partial_2^2 + \partial_3^2$$

be the three-dimensional Laplacian. We then define the 3×3 matrix-valued differential operators

$$\mathbf{L} := \begin{pmatrix} \partial_1^2-\partial_2^2-\partial_3^2 & 2\partial_1\partial_2 & 2\partial_1\partial_3 \\ -2\partial_1\partial_2 & \partial_1^2-\partial_2^2+\partial_3^2 & -2\partial_2\partial_3 \\ -2\partial_1\partial_3 & -2\partial_2\partial_3 & \partial_1^2+\partial_2^2-\partial_3^2 \end{pmatrix}$$

and

$$\mathbf{L}' := \begin{pmatrix} \partial_1^2-\partial_2^2-\partial_3^2 & -2\partial_1\partial_2 & -2\partial_1\partial_3 \\ 2\partial_1\partial_2 & \partial_1^2-\partial_2^2+\partial_3^2 & -2\partial_2\partial_3 \\ 2\partial_1\partial_3 & -2\partial_2\partial_3 & \partial_1^2+\partial_2^2-\partial_3^2 \end{pmatrix}.$$

Proposition 5.1. *The matrix-valued partial differential operators* $\mathbf{L}, \mathbf{L}'$ *commute and factor the bilaplacian:*

$$\mathbf{L}\mathbf{L}' = \mathbf{L}'\mathbf{L} = \begin{pmatrix} \Delta^2 & 0 & 0 \\ 0 & \Delta^2 & 0 \\ 0 & 0 & \Delta^2 \end{pmatrix}.$$

Proof. We first observe that it enough to check $\mathbf{L}\mathbf{L}'$ equals the diagonally lifted bilaplacian, because $\mathbf{L}'\mathbf{L}$ amounts to much the same computation (after all, $\mathbf{L}'$ equals $\mathbf{L}$ after the change of variables $x_1 \mapsto -x_1$). The entry in the $(1, 1)$ corner position of the product equals

$$(\partial_1^2-\partial_2^2-\partial_3^2)^2 + 4(\partial_1\partial_2)^2 + 4(\partial_1\partial_3)^2 = (\partial_1^2+\partial_2^2+\partial_3^2)^2 = \Delta^2.$$

Similarly, the entry in the $(1, 2)$ position equals

$$(\partial_1^2 - \partial_2^2 - \partial_3^2)(-2\partial_1\partial_2) + 2\partial_1\partial_2(\partial_1^2 - \partial_2^2 + \partial_3^2) + 2\partial_1\partial_3(-2\partial_2\partial_3) = 0,$$

and the entry in the $(1, 3)$ position equals

$$(\partial_1^2 - \partial_2^2 - \partial_3^2)(-2\partial_1\partial_3) + 2\partial_1\partial_2(-2\partial_2\partial_3) + 2\partial_1\partial_3(\partial_1^2 + \partial_2^2 - \partial_3^2) = 0.$$

Furthermore, the entry in the $(2, 1)$ position equals

$$-2\partial_1\partial_2(\partial_1^2 - \partial_2^2 - \partial_3^2) + (\partial_1^2 - \partial_2^2 + \partial_3^2)(2\partial_1\partial_2) - 2\partial_2\partial_3(2\partial_1\partial_3) = 0,$$

the entry in the $(2, 2)$ position equals

$$-2\partial_1\partial_2(-2\partial_1\partial_2) + (\partial_1^2 - \partial_2^2 + \partial_3^2)^2 - 2\partial_2\partial_3(-2\partial_2\partial_3) = (\partial_1^2 + \partial_2^2 + \partial_3^2)^2 = \Delta^2,$$

and the entry in the $(2, 3)$ position equals

$$-2\partial_1\partial_2(-2\partial_1\partial_3) + (\partial_1^2 - \partial_2^2 + \partial_3^2)(-2\partial_2\partial_3) - 2\partial_2\partial_3(\partial_1^2 + \partial_2^2 - \partial_3^2) = 0.$$

Finally, the entry in the $(3, 1)$ position equals

$$-2\partial_1\partial_3(\partial_1^2 - \partial_2^2 - \partial_3^2) - 2\partial_2\partial_3(2\partial_1\partial_2) + (\partial_1^2 + \partial_2^2 - \partial_3^2)(2\partial_1\partial_3) = 0,$$

the entry in the $(3, 2)$ position equals

$$-2\partial_1\partial_3(-2\partial_1\partial_2) - 2\partial_2\partial_3(\partial_1^2 - \partial_2^2 + \partial_3^2) + (\partial_1^2 + \partial_2^2 - \partial_3^2)(-2\partial_2\partial_3) = 0,$$

and the entry in the $(3, 3)$ corner position equals

$$4(\partial_1\partial_3)^2 + 4(\partial_2\partial_3)^2 + (\partial_1^2 + \partial_2^2 - \partial_3^2)^2 = (\partial_1^2 + \partial_2^2 + \partial_3^2)^2 = \Delta^2.$$

This completes the proof. □

5.2. **An Almansi-type expansion.** We need to have an Almansi-type representation of the biharmonic functions. We formulate the result in general dimension n.

Definition 5.2. We say that the domain $\Omega \subset \mathbb{R}^n$ is *x_1-contractive* if $x = (x_1, \ldots, x_n) \in \Omega$ implies that $(tx_1, x_2, \ldots, x_n) \in \Omega$ for all $t \in [0, 1]$. Moreover, we say that $\Omega \subset \mathbb{R}^n$ is *strongly x_1-contractive* if, in addition, $x = (x_1, \ldots, x_n) \in \partial\Omega$ with $x_1 \neq 0$ implies that $(tx_1, x_2, \ldots, x_n) \in \Omega$ for all $t \in [0, 1[$.

Proposition 5.3. *If $\Omega \subset \mathbb{R}^n$ is x_1-contractive, and if $u : \Omega \to \mathbb{R}$ is biharmonic, i.e., solves $\Delta^2 u = 0$, then $u(x) = v(x) + x_1 w(x)$, where v, w are harmonic in Ω.*

Proof. By calculation, we have that

$$\Delta[x_1 w] = (\partial_1^2 + \cdots + \partial_n^2)[x_1 w] = 2\partial_1 w + x_1 \Delta w,$$

so that if w is harmonic, $\Delta[x_1 w] = 2\partial_1 w$, and hence, $\Delta^2[x_1 w] = 2\Delta\partial_1 w = 2\partial_1\Delta w = 0$. It is now clear that any function of the form $u(x) = v(x) + x_1 w(x)$, with v, w both harmonic, is biharmonic.

We turn to the reverse implication. So, we are given a biharmonic function u on Ω, and attempt to find the two harmonic functions v, w so that $u(x) = v(x) + x_1 w(x)$. We first observe that $h := \Delta u$ is a harmonic function, and that if v, w exist, we must have that $h = \Delta[v + x_1 w] = \Delta[x_1 w] = 2\partial_1 w$. Let $x' := (x_2, \ldots x_n) \in \mathbb{R}^{n-1}$, so that $x = (x_1, x')$. By calculation, then,

$$\Delta \int_0^{x_1} h(t_1, x')\mathrm{d}t_1 = \partial_1 h(x) + \int_0^{x_1} \Delta' h(t_1, x')\mathrm{d}t_1 = \partial_1 h(x) - \int_0^{x_1} \partial_1^2 h(t_1, x')\mathrm{d}t_1 = \partial_1 h(0, x'),$$

where we used that h was harmonic, and let Δ' denote the Laplacian with respect to $x' = (x_2, \ldots, x_n)$. Next, we apply the results on P-convexity in Sections 10.6-10.8 of [14] to the slice

$\Omega' := \Omega \cap (\{0\} \times \mathbb{R}^{n-1})$, and obtain a solution F to the Poisson equation $\Delta' F(x') = \partial_1 h(0, x')$ on Ω'. We now declare w to be the function

$$w(x) = w(x_1, x') := \frac{1}{2}\Big\{\int_0^{x_1} h(t_1, x')\mathrm{d}t_1 - F(x')\Big\},$$

which is well-defined since Ω was assumed x_1-contractive. In view of the above calculation, w is harmonic in Ω, and we quickly see that $2\partial_1 w = h$, so that $\Delta[x_1 w] = h$. Finally we put $v := u - x_1 w$ which is harmonic in Ω by construction. □

Remark 5.4. If Ω is strongly x_1-contractive, and if $I \subset \partial\Omega$ is a relatively open patch on the boundary $\partial\Omega$ – which is assumed C^∞-smooth – and u is C^4-smooth on $\Omega \cup I$, then the above proof can be made to produce a decomposition $u = v + x_1 w$, where v, w are harmonic in Ω and C^2-smooth on $\Omega \cup I$.

5.3. **Application of the matrix-valued differential operators.** We return to three dimensions and assume u is biharmonic in a bounded convex domain Ω is $\mathbb{R}^3$ which is strongly x_1-contractive. We assume that the boundary $\partial\Omega$ is C^∞-smooth, and that $I \subset \partial\Omega$ is a nontrivial open patch. We may lift u to a vector-valued function in the following three ways:

$$u^{\langle 1\rangle} := u \oplus 0 \oplus 0 = \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix}, \quad u^{\langle 2\rangle} := 0 \oplus u \oplus 0 = \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix}, \quad u^{\langle 3\rangle} := u \oplus 0 \oplus 0 = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}.$$

We assume that *all partial derivatives of u of order ≤ 2 vanish on I*, and that u is C^4-smooth on $\Omega \cup I$. Since u is biharmonic in Ω, we apply Proposition 5.3 to decompose $u^{\langle 1\rangle}, u^{\langle 2\rangle}, u^{\langle 3\rangle}$:

$$u^{\langle 1\rangle} = v^{\langle 1\rangle} + x_1 w^{\langle 1\rangle}, \quad u^{\langle 2\rangle} = v^{\langle 2\rangle} + x_1 w^{\langle 2\rangle}, \quad u^{\langle 3\rangle} = v^{\langle 3\rangle} + x_1 w^{\langle 3\rangle},$$

with obvious interpretation of $v^{\langle j\rangle}, w^{\langle j\rangle}$ as vector-valued functions. In view of Remark 5.4, the functions v, w are both C^2-smooth in $\Omega \cup I$. Moreover, by the flatness assumption on u,

$$\mathbf{L}'[u^{\langle j\rangle}] = \mathbf{L}'[v^{\langle j\rangle}] + \mathbf{L}'[x_1 w^{\langle j\rangle}] = 0 \quad \text{on } I, \qquad j = 1, 2, 3.$$

We let $\mathbf{R}$ denote the matrix-valued operator

$$\mathbf{R} := \begin{pmatrix} \partial_1^2 & -\partial_1\partial_2 & -\partial_1\partial_3 \\ \partial_1\partial_2 & -\partial_2^2 & -\partial_2\partial_3 \\ \partial_1\partial_3 & -\partial_2\partial_3 & -\partial_3^2 \end{pmatrix},$$

and observe that $\mathbf{L}'[h] = 2\mathbf{R}[h]$ holds for all harmonic 3-vectors h. In a similar fashion, we calculate that $\mathbf{L}'[x_1 h] = 2\mathbf{C}[h] + 2x_1\mathbf{R}[h]$ for harmonic 3-vectors h, where $\mathbf{C}$ is the matrix-valued differential operator

$$\mathbf{C} := \begin{pmatrix} \partial_1 & -\partial_2 & -\partial_3 \\ \partial_2 & \partial_1 & 0 \\ \partial_3 & 0 & \partial_1 \end{pmatrix}.$$

In particular, for $j = 1, 2, 3$,

$$0 = \mathbf{L}'[u^{\langle j\rangle}] = 2\mathbf{R}[v^{\langle j\rangle}] + 2\mathbf{C}[w^{\langle j\rangle}] + 2x_1\mathbf{R}[w^{\langle j\rangle}] \quad \text{on } I,$$

which we may write in the form

$$x_1\mathbf{R}[w^{\langle j\rangle}] = -\mathbf{R}[v^{\langle j\rangle}] - \mathbf{C}[w^{\langle j\rangle}] \quad \text{on } I, \text{ for } j = 1, 2, 3. \tag{5.2}$$

Let $\mathbf{H}[f]$ be the *Hessian matrix operator*:

$$\mathbf{H} := \begin{pmatrix} \partial_1^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_2 & \partial_2^2 & \partial_2\partial_3 \\ \partial_1\partial_3 & \partial_2\partial_3 & \partial_3^2 \end{pmatrix},$$

where the similarity with $\mathbf{R}$ is apparent. The system (5.2) amounts to the 3×3 matrix equation

$$x_1\mathbf{H}[w] = -\mathbf{H}[v] - \mathbf{B}[w] \quad \text{on } I, \tag{5.3}$$

where

$$\mathbf{B} := \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_1 & 0 \\ \partial_3 & 0 & \partial_1 \end{pmatrix}.$$

By the flatness assumption on I, a calculation gives that

$$0 = \Delta u = \Delta(v + x_1 w) = \Delta(x_1 w) = x_1\Delta w + 2\partial_1 w = 2\partial_1 w \quad \text{on } I,$$

so we may reduce the system (5.3) to

$$x_1\mathbf{H}[w] = -\mathbf{H}[v] - \mathbf{A}[w] \quad \text{on } I, \tag{5.4}$$

where

$$\mathbf{A} := \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & 0 & 0 \\ \partial_3 & 0 & 0 \end{pmatrix}.$$

We have arrived at the following.

Theorem 5.5. *In the above setting, we have that*

$$\partial_1 w = 0 \quad \textit{and} \quad x_1\mathbf{H}[w] = -\mathbf{H}[v] - \mathbf{A}[w] \quad \textit{on } I.$$

In particular, if the Hessian $\mathbf{H}[w]$ *is nonsingular on the patch* I*, then the matrix field*

$$X_1 := -(\mathbf{H}[w])^{-1}(\mathbf{H}[v] + \mathbf{A}[w]),$$

defined in $\Omega \cup I$ *wherever the Hessian* $\mathbf{H}[w]$ *is nonsingular, has the property that* $X_1 = x_1\mathbf{I}$ *holds on the patch* I*, where* $\mathbf{I}$ *denotes the unit* 3×3 *matrix.*

Unfortunately, the determinant of the Hessian of a harmonic function may vanish identically (see Lewy [16]). However, if the determinant vanishes then the given harmonic function is rather special, connected with the theory of minimal surfaces (Lewy [16]). Quite possibly the system (5.3) should give a lot of information anyway also in this case. We mention here Lewy's observation that unless the harmonic function is affine, the corresponding Hessian has rank at least 2.

Remark 5.6. Theorem 5.5 is a three-dimensional analogue of Corollary 1.6. It should be mentioned that part of the assertion of Theorem 5.5 is the equality

$$\nabla[w + \partial_1 v] + x_1\nabla[\partial_1 w] = 0 \quad \text{on } I,$$

so that x_1 multiplied by one gradient of a harmonic function equals the gradient of another harmonic function.

Remark 5.7. Holmgren's theorem for the Laplacian Δ asserts that in the above three-dimensional setting that if the harmonic function vanishes along with its normal derivative along the (relatively open) patch $I \subset \partial\Omega$, then the function vanishes identically [observe that the vanishing of the normal derivative is the same as the vanishing of the gradient here]. In two dimensions, the corresponding statement is a special instance of Privalov's uniqueness theorem, as the gradient of the harmonic function is identified with a holomorphic function. Privalov's uniqueness theorem allows the arc to be replaced by a (closed) set of positive linear measure (see [3] for a collection of related results), and it might seem reasonable to suspect that a corresponding positive area criterion would be possible in three dimensions. This was disproved by a counterexample of Bourgain and Wolff [6], [25] (see also Wang [23]). Our understanding remains rather incomplete.

5.4. **Domains with toric product structure.** Theorem 5.5, and especially Remark 5.6, tells us about the rigidity imposed by the global Holmgren problem in three dimensions with $R = 3$ (sub-Cauchy data involving partial derivatives of order ≤ 2). It is, however, not clear what the result really says about the patch I and the domain Ω. Here, we will try to analyze a rather special situation in three dimensions where we can apply "dimension reduction" which leads us to analyze two-dimensional problems in place of three-dimensional problems. The appearance of a flat direction should make it easier to have nontrivial biharmonic functions with sub-Cauchy flat boundary data. Indeed, this is precisely what we find.

To begin with, We let $\Omega_* \subset \mathbb{R}^2$ be a bounded simply connected domain with smooth boundary, and consider the cylindrical domain $\Omega := \Omega_* \times \mathbb{R} \subset \mathbb{R}^3$. This domain is unfortunately unbounded. To remedy this, we fix a positive real parameter α and consider

$$\Lambda := \{(0,0)\} \times \alpha\mathbb{Z} = \{(0,0,\alpha n) : n \in \mathbb{Z}\},$$

which is then a discrete additive subgroup of $\mathbb{R}^3$. We will consider Ω/Λ, where two points $x, y \in \Omega$ are equated if $x - y \in \Lambda$. Geometrically, Ω/Λ looks like a doughnut. The functions on Ω/Λ can be understood as the functions on Ω which are periodic with respect to Λ. We let $I_* \subset \partial\Omega_*$ be an arc, and consider the patch $I := I_* \times \mathbb{R}$. The global Holmgren problem now asks whether a biharmonic function on Ω/Λ which is, say, C^4-smooth on $\Omega \cup I$, with vanishing sub-Cauchy data

$$u|_I = 0, \quad \partial_n u|_I = 0, \quad \partial_n^2 u|_I = 0,$$

must necessarily vanish identically.

In the theorem below, we use the notation

$$\Delta' := \partial_1^2 + \partial_2^2$$

for the two-variable Laplacian. In the proof, it is convenient to work with complex-valued functions, so in the assumptions we may as well assume each of the functions u, U to be complex-valued.

Theorem 5.8. *The following two conditions are equivalent:*
(i) *There exists a nontrivial biharmonic function on Ω/Λ which extends to a C^4-smooth function on $(\Omega \cup I)/\Lambda$ with vanishing sub-Cauchy data*

$$u|_I = \partial_n u|_I = \partial_n^2 u|_I = 0.$$

(ii) *There exists an integer $n \in \mathbb{Z}$ and a nontrivial function U of two variables that is C^4-smooth on $\Omega_* \cup I_*$ which solves the sub-Cauchy problem*

$$(\Delta' - 4\pi^2\alpha^{-2}n^2)^2 U = 0 \quad \text{on} \quad \Omega_*, \quad \text{and} \quad U|_{I_*} = \partial_n U|_{I_*} = \partial_n^2 U|_{I_*} = 0.$$

Proof. We first consider the implication (ii) $\implies$ (i). By (ii), we have a nontrivial function U with the given properties, and consider the function

$$u(x) = e^{i2\pi n\alpha^{-1}x_3} U(x_1, x_2), \qquad x = (x_1, x_2, x_3) \in \Omega,$$

which is Λ-periodic and also biharmonic:

$$\Delta^2 u(x) = (\Delta' + \partial_3^2)^2[e^{i2\pi n\alpha^{-1}x_3} U(x_1, x_2)] = e^{i2\pi n\alpha^{-1}x_3}(\Delta' - 4\pi^2\alpha^{-2}n^2)^2 U(x_1, x_2) = 0.$$

The vanishing of the sub-Cauchy data along I is immediate.

Second, we consider the implication (i) $\implies$ (ii). Given the nontrivial biharmonic function u, we form, for a suitable integer $n \in \mathbb{Z}$, the function

$$U(x_1, x_2) := \int_0^\alpha e^{-i2\pi n\alpha^{-1}t} u(x_1, x_2, t)\mathrm{d}t.$$

The choice of n should be such that the resulting function U is nontrivial. The nontriviality of u ensures the existence of such an n, by standard Fourier analysis. Next, by the biharmonicity of u, we have

$$(\Delta' - \kappa^2)^2 U(x_1, x_2) = \int_0^\alpha e^{-i2\pi n\alpha^{-1}t}\{(\Delta')^2 - 2\kappa^2\Delta' + \kappa^2\} u(x_1, x_2, t)\mathrm{d}t \tag{5.5}$$
$$= -\int_0^\alpha e^{-i2\pi n\alpha^{-1}t}\{\partial_3^4 + 2\partial_1^2\partial_3^2 + 2\partial_2^2\partial_3^2 + 2\kappa^2\partial_1^2 + 2\kappa^2\partial_2^2 - \kappa^4\} u(x_1, x_2, t)\mathrm{d}t,$$

where we write $\kappa := 2\pi\alpha^{-1}n$. A calculation involving integration by parts shows that

$$\int_0^\alpha e^{-i2\pi n\alpha^{-1}t}\partial_1^a\partial_2^b\partial_3^c u(x_1, x_2, t)\mathrm{d}t = (-1)^c \int_0^\alpha \partial_1^a\partial_2^b u(x_1, x_2, t)\partial_t^c[e^{-i2\pi n\alpha^{-1}t}]\mathrm{d}t$$
$$= (i2\pi\alpha^{-1}n)^c\partial_1^a\partial_2^b \int_0^\alpha e^{-i2\pi n\alpha^{-1}t} u(x_1, x_2, t)\mathrm{d}t = (i\kappa)^c\partial_1^a\partial_2^b U(x_1, x_2),$$

for $a, b, c = 0, 1, 2, \ldots$. By combining this with (5.5), it follows that

$$(\Delta' - 4\pi^2\alpha^{-2}n^2)^2 U(x_1, x_2) = 0,$$

as claimed. The smoothness and the vanishing of the sub-Cauchy data along I_* are rather immediate. □

5.5. **The structure of homogeneous solutions to the square of the massive Laplacian.** Theorem 5.8 allows us to reduce the three-dimensional situation to a two-dimensional one. Accordingly, we no longer need to separate Δ and Δ', and write $\Delta := \partial_1^2 + \partial_2^2$ for the two-dimensional Laplacian. The partial differential operators

$$\Delta - \kappa^2 \tag{5.6}$$

for real κ are very similar to the classical Helmholtz equation, only there, there would be a plus sign instead of the minus. Instead the operators (5.6) appear in the context of massive Gaussian free fields (see, e. g., [18]), so we call $\Delta - \kappa^2$ [with $\kappa \neq 0$] the *massive Laplacian*.

Proposition 5.9. *Suppose $U : \Omega_* \to \mathbb{C}$ solves $(\Delta - \kappa^2)^2 U = 0$ on Ω_*. If the domain $\Omega_* \subset \mathbb{R}^2$ is x_1-contractive, then $U = V + x_1 W$, where V, W solve $(\Delta - \kappa^2)V = 0$ and $(\Delta' - \kappa^2)W = 0$ on Ω_*. If $\Omega_* \subset \mathbb{R}^2$ is strongly x_1-contractive and U is C^4-smooth on $\Omega_* \cup I_*$, then V, W are both C^2-smooth on $\Omega_* \cup I_*$.*

Remark 5.10. A direct calculation shows that

$$(\Delta-\kappa^2)(x_1W)=2\partial_1W+x_1(\Delta-\kappa^2)W. \tag{5.7}$$

It follows that if $(\Delta-\kappa^2)V=0$ and $(\Delta'-\kappa^2)W=0$ hold on Ω_*, then

$$(\Delta-\kappa^2)(V+x_1W)=2\partial_1W\quad\text{on}\ \ \Omega_*, \tag{5.8}$$

which leads to

$$(\Delta-\kappa^2)^2(V+x_1W)=2(\Delta'-\kappa^2)\partial_1W=2\partial_1(\Delta-\kappa^2)W=0\quad\text{on}\ \ \Omega_*. \tag{5.9}$$

Proof of Proposition 5.9. We put $H:=(\Delta-\kappa^2)U$, which solves $(\Delta-\kappa^2)H=0$, and let $F(x_2)$ be one of the solutions to the ordinary differential equation

$$F''(x_2)-\kappa^2F(x_2)=\partial_1H(0,x_2).$$

The function W is now declared to be

$$W(x_1,x_2):=\frac{1}{2}\bigg\{\int_0^{x_1}H(t_1,x_2)\mathrm{d}t_1-F(x_2)\bigg\},$$

because then

$$\begin{aligned}2(\Delta-\kappa^2)W(x_1,x_2)&=\partial_1H(x_1,x_2)+\int_0^{x_1}\partial_2^2H(t_1,x_2)\mathrm{d}t_1-F''(x_2)+\kappa^2F(x_2)-\kappa^2\int_0^{x_1}H(t_1,x_2)\mathrm{d}t_1\\&=\partial_1H(x_1,x_2)+\int_0^{x_1}\Big\{-\partial_1^2H(t_1,x_2)+\kappa^2H(t_1,t_2)\Big\}\mathrm{d}t_1-F''(x_2)+\kappa^2F(x_2)-\kappa^2\int_0^{x_1}H(t_1,x_2)\mathrm{d}t_1\\&=\partial_1H(x_1,x_2)-\partial_1H(x_1,x_2)+\partial_1H(0,x_2)-F''(x_2)+\kappa^2F(x_2)=0.\end{aligned}$$

The function V is declared to be $V:=U-x_1W$, and using (5.7), we calculate that

$$(\Delta-\kappa^2)V=(\Delta-\kappa^2)[U-x_1W]=H-(\Delta-\kappa^2)[x_1W]=H-2\partial_1W=0.$$

Next, we should verify that the smoothness carries over. If $(0,y_2)\in I_*$, then the function $\partial_1H(0,x_2)$ is C^1-smooth near $x_2=y_2$, and $F(x_2)$ is C^2-smooth near $x_2=y_2$. At any other point of I_*, the function $F(x_2)$ is real-analytically smooth. Also, as

$$\int_0^{x_1}H(t_1,x_2)\mathrm{d}t_1=x_1\int_0^1H(tx_1,x_2)\mathrm{d}t,$$

the integral of the left hand side expresses a C^2-smooth function near any point of I_*. In conclusion, W is C^2-smooth on $\Omega_*\cup I_*$, and hence V is C^2-smooth on $\Omega_*\cup I_*$ as well. The proof is complete. □

5.6. **Operators of Cauchy-Riemann type.** Let $\mathbf{D}_{\langle\boldsymbol{\kappa}\rangle},\mathbf{D}'_{\langle-\boldsymbol{\kappa}\rangle}$ be the matrix differential operators

$$\mathbf{D}_{\langle\boldsymbol{\kappa}\rangle}:=\begin{pmatrix}\partial_1-\kappa_1&\partial_2-\kappa_2\\-\partial_2-\kappa_2&\partial_1+\kappa_1\end{pmatrix},\qquad\mathbf{D}'_{\langle-\boldsymbol{\kappa}\rangle}:=\begin{pmatrix}\partial_1+\kappa_1&-\partial_2+\kappa_2\\\partial_2+\kappa_2&\partial_1-\kappa_1\end{pmatrix},$$

which are perturbations of the Cauchy-Riemann matrix operators, and κ_1,κ_2 are constants. As can be seen from the setup in [18], the operator $\Delta-\kappa^2$ admits the 2×2 matrix factorization

$$\mathbf{D}_{\langle\boldsymbol{\kappa}\rangle}\mathbf{D}'_{\langle-\boldsymbol{\kappa}\rangle}=\mathbf{D}'_{\langle-\boldsymbol{\kappa}\rangle}\mathbf{D}_{\langle\boldsymbol{\kappa}\rangle}=\begin{pmatrix}\Delta-\kappa^2&0\\0&\Delta-\kappa^2\end{pmatrix}, \tag{5.10}$$

if $\kappa^2=\kappa_1^2+\kappa_2^2$. In this setup, we should think that $\boldsymbol{\kappa}=(\kappa_1,\kappa_2)$ is a vector, and κ^2 is its length squared. We lift the functions U,V,W to the matrices $U\mathbf{I},V\mathbf{I},W\mathbf{I}$ – where $\mathbf{I}$ now stands for the unit 2×2 matrix – and apply the operator $\mathbf{D}^2_{\langle\kappa\rangle}$ to the decomposition $U = V + x_1 W$ of Proposition 5.9. Then the given sub-Cauchy flatness of U on I_* leads $\partial_1 W = 0$ on I_* and to the identity

$$\mathbf{D}_{\langle\kappa\rangle}[W + \partial_1 V] + x_1\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W] = 0 \quad \text{on} \quad I_*.$$

Assuming that the matrix $\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W]$ is nonsingular on I_*, we find – as in Theorem 5.5 – a matrix field X_1 with $X_1 = x_1\mathbf{I}$ on the arc I_* such that

$$\mathbf{D}_{\langle\kappa\rangle}[W + \partial_1 V] + \mathbf{D}_{\langle\kappa\rangle}[\partial_1 W]X_1 = 0 \quad \text{on} \quad \Omega_* \cup I_*, \tag{5.11}$$

except where $\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W]$ is singular, that is. It is of interest to analyze the properties of X_1. We apply the matrix differential operator $\mathbf{D}'_{\langle-\kappa\rangle}$ to both sides of (5.11):

$$\mathbf{D}'_{\langle-\kappa\rangle}\mathbf{D}_{\langle\kappa\rangle}[W + \partial_1 V] + \mathbf{D}'_{\langle-\kappa\rangle}[(\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W])X_1] = 0, \tag{5.12}$$

In view of (5.10), the first term on the left hand side vanishes, and the second may be expanded using the product rule, which is a little involved in this case. More precisely, if A, B are (smooth) 2×2 matrix fields, the product rule says that

$$\mathbf{D}'_{\langle-\kappa\rangle}(AB) = (\mathbf{D}'_{\langle-\kappa\rangle}A)B + (A^{\mathrm{t}}\mathbf{D}_{\langle0\rangle})^{\mathrm{t}}\, B,$$

where the superscript "t" means taking the *transpose* of the matrix (also in the differential operator context). Since V, W solve the massive Laplace equation, (5.12) gives that the matrix field X_1 solves

$$[(\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W])^{\mathrm{t}}\mathbf{D}_{\langle0\rangle}]^{\mathrm{t}}\, X_1 = 0.$$

We formalize our observations in a theorem.

Theorem 5.11. *Suppose $\Omega_* \subset \mathbb{R}^2$ is a bounded simply connected domain with C^∞-smooth Jordan curve boundary, and that Ω_* is strongly x_1-contractive. Suppose moreover that U is C^4-smooth on $\Omega \cup I_*$, where I_* is a nontrivial arc of $\partial\Omega_*$. Let the function U solve the sub-Cauchy problem*

$$(\Delta - \kappa^2)^2 U = 0 \quad on \quad \Omega_*, \quad and \quad U|_{I_*} = \partial_n U|_{I_*} = \partial_n^2 U|_{I_*} = 0.$$

Then $U = V + x_1 W$, where V, W are C^2-smooth on $\Omega \cup I_*$, and solve $(\Delta - \boldsymbol{\kappa}^2)V = (\Delta - \kappa^2)W = 0$ on Ω_*. Suppose in addition that there exists vector $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ with $\kappa_1^2 + \kappa_2^2 = \kappa^2$ such that the 2×2 matrix $\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W]$ is nonsingular along I_*. Let $\Sigma_1 \subset \Omega_*$ denote the closed set where the matrix $\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W]$ is singular. Then there exists a 2×2 matrix field X_1 defined in $(\Omega_* \setminus \Sigma_1) \cup I_*$, with $X_1 = x_1\mathbf{I}$ on I_*, which solves the differential equation

$$[(\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W])^{\mathrm{t}}\mathbf{D}_{\langle0\rangle}]^{\mathrm{t}}\, X_1 = 0 \quad on \quad \Omega_* \setminus \Sigma_1.$$

Remark 5.12. (i) We expect that the expression

$$(\mathbf{D}_{\langle\kappa\rangle}[\partial_1 W])^{\mathrm{t}}\mathbf{D}_{\langle0\rangle}]^{\mathrm{t}}\, X_1$$

defines a (singular) matrix-valued distribution supported on the set Σ_1. Still, we do not fully understand what the condition asserted by the theorem means in terms of the geometry of Ω_*.
(ii) When $\kappa = 0$, the assertion of the theorem is that the function $(x_1, x_2) \mapsto x_1$ has a complex-valued extension which is holomorphic away from the zero set of $\nabla[\partial_1 W]$. Using this extension, we easily reconstruct the Schwarz function, because $z + \bar{z} = 2\,\mathrm{Re}\, z = 2x_1$ holds for complex $z = x_1 + \mathrm{i}x_2$.

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