The Lambert W function and the spectrum of some multidimensional time-delay systems

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Keywords: Lambert W function, time-delay, simultaneously triangularizable, matrix function

Abstract

In this note we find an explicit expression for the eigenvalues of a *retarted time-delay system* with one delay, for the special case that the system matrices are *simultaneously triangularizable*, which includes the case where they commute. Using matrix function definitions we define a matrix version of the Lambert Wfunction, from which we form the expression. We prove by counter-example that some expressions in other publications on Lambert $W$for time-delay systems do not always hold.

1 Introduction

In this work we consider retarded linear single-delay time-delay system (TDS) described by

$$\Sigma = \begin{cases} \dot{x}(t) = Ax(t) + Bx(t-\tau),\ t > 0 \\ x(t) = \varphi(t),\ t \in [-\tau, 0] \end{cases} \tag{1}$$

where $A, B \in \mathbb{C}^{n\times n}$. The corresponding characteristic equation is

$$0 = \det\left(-sI + A + Be^{-s\tau}\right).$$

We denote the set of all solutions of the characteristic equation, i.e. the eigenvalues of Σ, with σ_Σ. Unlike delay free systems, time-delay systems have a countably infinte number of eigenvalues. This is one of the difficult aspects of time-delay systems from a computational point of view. In the literature, there are several results on qualitative properties of the spectrum. For instance, we know from [9] (Lemma 4.1) that the eigenvalues are aligned along curves (called root-chains) in the complex plane and that given any vertical line in the complex plane, there are only a finite number of eigenvalues to the right of this line.

There are efficient methods to approximate some of the eigenvalues numerically. The eigenvalues with smallest magnitude can be approximated using discretization of the infinitesimal generator [3] or using discretization of the solution operator [8]. In practice, e.g. in the package DDE-BIFTOOL [8], the accuracy of these approximations is improved by Newton-iterations.

In the present paper we deal with explicit expressions of the eigenvalues for a special class of time-delay systems, based on a matrix version of the *Lambert W function*. It is well known that the spectrum of a scalar single delay system can be computed using the Lambert W function, see e.g. [7].

In [1] and the derivative works [22] and [23] a nice generalization for multidimensional systems has been given using a matrix version of the *Lambert W function*. Unfortunately, the result in [1] does not hold in the stated generality. The goal of this paper therefore is to give sufficient conditions on the system matrices for the formula in [1] to hold. As the weakest sufficient condition, we obtain simultaneous triangularizability of the matrices A and B. Independently of our work, similar observations have been made recently in [19], where basically the same spectral results are obtained without explicit use of a matrix version of the Lambert W function. Here we establish these results for the representation in [1]. Moreover, we present explicit counterexamples, which prove that in general the formula may be wrong. This is important, since the results of [1] have found interest and applications e.g. in [14], [21], [13], [6], [5], [16], [2], [10], [17] and [4]. Even if most of the conclusions drawn in these papers still seem to be valid, since mainly the scalar case is considered, it is worthwhile to clarify the range of applicability of the formula.

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We proceed as follows. In Section 2 we define a matrix version of the Lambert W function. Our definition is more general than the one used in previous papers, where only diagonalizable matrices are considered. Then, in Section 3, we consider cases of different generality to show that the formula in [1] is correct if the system matrices A and B are simultaneously triangularizable. In particular, this includes the case where the system matrices commute as well as the pure delay case. Moreover, we explore uncontrollability properties of the pairs (A, B) and (B, A), which guarantee that the formula gives at least some correct eigenvalues. In Section 4 we present an example of non-commuting matrices A and B, where the formula does not hold. Thus we disprove results stated in some of our references.

2 The Lambert W function

For scalar arguments $z \in \mathbb{C}$, the *Lambert W function* is defined as the (multivalued) inverse of the function $z \mapsto ze^z$. It has a countably infinite number of branches

$$W_k(z) \in \{w \in \mathbb{C} : z = we^w\} , \quad k \in \mathbb{Z} ,$$

which can be defined by the branchcuts in [15] and [7]. With exception for the point $z = -e^{-1}$, where the principal branch W_0 is not differentiable, each of these branches is locally analytic. Hence, we may define the Lambert W function in a standardized way, given e.g. in [12] or [11]. We first define Lambert W for matrices in Jordan canonical form, i.e.

$$J = \mathrm{diag}(J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \ldots, J_{n_s}(\lambda_s)) ,$$

where $J_n(\lambda)$ is the n-by-n Jordan block belonging to eigenvalue λ with multiplicity n. Then

$$W_k(J) = \mathrm{diag}(W_{k_1}(J_{n_1}(\lambda_1)), \ldots, W_{k_s}(J_{n_s}(\lambda_s))) .$$

Note that we are allowed to pick a different branch for each Jordan block. If J has s Jordan blocks and the index set for the branches of the scalar Lambert W function is $\mathbb{Z}$, then the index set for the branches of $W_k(J)$ is $\mathbb{Z}^s$. For Jordan blocks of dimension 1, i.e. single eigenvalues, we can use the scalar Lambert W function. For larger blocks, we define the Lambert W function (for a fixed branch) of a Jordan block by the standard definition of matrix functions (e.g. see (6.1.8) in [12]), i.e.

$$W_k(J_n(\lambda)) = \\ = \begin{pmatrix} W_k(\lambda) & W_k'(\lambda) & \cdots & \frac{1}{(n-1)!}W_k^{(n-1)}(\lambda) \\ 0 & W_k(\lambda) & & \\ \vdots & 0 & \ddots & \\ 0 & 0 & & W_k(\lambda) \end{pmatrix} .$$

If $k = 0$ we have to assume additionally that $\lambda \neq -e^{-1}$ (since $W_0'(-e^{-1})$ is not defined).
We complete the definition of the Lambert W function for arbitrary matrices, by noting that all matrices can be brought to Jordan canonical form by a similarity transformation $A = SJS^{-1}$. Thus we may set

$$W_k(A) = SW_k(J)S^{-1} ,$$

where for the principal branch $k = 0$ we from now on tacitly assume that $-e^{-1}$ is not an eigenvalue corresponding to a Jordan-block of dimension larger than 1, i.e.

$$\mathrm{rank}\ (A + e^{-1}I) = \mathrm{rank}\ (A + e^{-1}I)^2 . \qquad (2)$$

Remark 1. The limitation (2) lessens the elegance of the matrix Lambert W function slightly. This point was brought to our knowledge by Robert Corless.

Example 2. We illustrate the definition of the Lambert W function for a 2×2-Jordan block with $\lambda \neq -e^{-1}$.
Let $J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, then $W_k(J) = \begin{bmatrix} W_k(\lambda) & W_k'(\lambda) \\ 0 & W_k(\lambda) \end{bmatrix}$. We verify that indeed $J = W_k(J)e^{W_k(J)}$. To this end we note that by differentiating the equation $\lambda = W_k(\lambda)e^{W_k(\lambda)}$, we obtain $1 = W_k'(\lambda)e^{W_k(\lambda)} + W_k'(\lambda)\lambda$. Thus we have

$$\begin{aligned} W_k(J)e^{W_k(J)} &= e^{W_k(\lambda)} \begin{bmatrix} W_k(\lambda) & W_k'(\lambda) \\ 0 & W_k(\lambda) \end{bmatrix} \begin{bmatrix} 1 & W_k'(\lambda) \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \lambda & \lambda W_k'(\lambda) + e^{W_k(\lambda)}W_k'(\lambda) \\ 0 & \lambda \end{bmatrix} = J . \end{aligned}$$

On the other hand, if $\lambda = -e^{-1}$, then $W_0(\lambda) = -1$ and the ansatz $W = \begin{bmatrix} -1 & w \\ 0 & -1 \end{bmatrix}$ yields $We^W = -e^{-1}I$ for all $w \in \mathbb{C}$. Hence $\sigma(W) = \{-1\}$ implies $We^W = -e^{-1}I$. We conclude that $W_0(J)$ is not defined in this case, while $W_0(-e^{-1}I)$ is not unique.

3 Main results

With the help of the Lambert W function we can easily express the spectrum of triangular systems.

Lemma 3. *If A and B are both upper or both lower triangular matrices, then*

$$\sigma_\Sigma = \bigcup_k \sigma\left(\frac{1}{\tau} W_k(B\tau e^{-A\tau}) + A\right) . \tag{3}$$

Proof. We exploit the fact that the determinant of a triangular matrix is the product of the diagonal elements. The characteristic equation is hence

$$0 = \det\left(-sI + A + Be^{-s\tau}\right) = \prod_j (-s + a_{jj} + b_{jj}e^{-s\tau}).$$

Clearly, $-s + a_{jj} + b_{jj}e^{-s\tau} = 0$ for some j, if and only if s is an eigenvalue. It follows that

$$(s - a_{jj})\tau e^{(s-a_{jj})\tau} = b_{jj}\tau e^{-a_{jj}\tau},$$

which, for any branch W_k results in

$$s = \tfrac{1}{\tau} W_k(b_{jj}\tau e^{-a_{jj}\tau}) + a_{jj},$$

The expression holds for all choices j, hence

$$\begin{aligned} s \in \sigma_\Sigma &= \bigcup_{k,j} \tfrac{1}{\tau} W_k(b_{jj}\tau e^{-a_{jj}\tau}) + a_{jj} \\ &= \bigcup_k \sigma\left(\tfrac{1}{\tau} W_k(B\tau e^{-A\tau}) + A\right) , \end{aligned}$$

completing the proof. □

Lemma 3 can easily be extended to the case where A and B are simultaneously triangularizable in the following sense.

Definition 4. The matrix pair $A, B \in \mathbb{C}^{n\times n}$ is called *simultaneously triangularizable* if there is a regular $S \in \mathbb{C}^{n\times n}$ and upper triangular matrices T_A and T_B such that

$$A = S^{-1}T_A S \quad \text{and} \quad B = S^{-1}T_B S .$$

Assuming simultaneous triangularizability, we can introduce new variables $\xi = Sx$, such that system (1) can be written as a cascade of inhomogenous scalar equations

$$\dot{\xi}_j(t) = \alpha_j \xi_j(t) + \beta_j \xi_j(t-h) + \gamma_j(t) ,$$

where γ_j is a linear combination of the functions $\xi_1, \ldots, \xi_{j-1}$. The spectrum of the whole system is the union of the spectra of these scalar equations. We thus obtain the most general case for the formula to hold.

Theorem 5. *If A and B are simultaneously triangularizable, then (3) holds.*

Proof. The characteristic equation is invariant under simultaneous similarity transformation i.e.

$$\det(-sI + A + Be^{-s\tau}) = \det(-sI + T_a + T_b e^{-s\tau}) .$$

Moreover, the exponentiation operator and Lambert W commute with similarity transformation, i.e., $W(S^{-1}CS) = S^{-1}W(C)S$. This implies that (3) is invariant under simultaneous similarity transformation of A and B. Hence we can assume without loss of generality that A and B are both upper triangular and apply Lemma 3. □

We mention some interesting special cases.

Corollary 6. *If A and B commute, then (3) holds.*

Proof. This follows from Theorem 5 and the fact that commutativity implies simultaneous triangularizability (cf. [18]). □

This result for $\tau = 1$ is also stated (without proof) in [7].
It implies that (3) also holds in the *pure delay case.*

Corollary 7. *If $A = 0$ then $\sigma_\Sigma = \bigcup_k \sigma(\frac{1}{\tau} W_k(\tau B)) = \frac{1}{\tau} \bigcup_k W_k(\tau\sigma(B))$.*

Proof. The matrices B and 0 commute, which allows us to apply the previous corollary. □

Finally we note two partial results.

Lemma 8. *Assume that the pair (A, B) is not controllable and $\sigma_u(A)$ denotes the corresponding set of uncontrollable eigenvalues of A. Then*

$$\sigma_u(A) \subset \sigma_\Sigma \cap \bigcup_k \sigma\left(\frac{1}{\tau}W_k(B\tau e^{-A\tau}) + A\right) .$$

Proof. By the Kalman decomposition (e.g. [20, Lemma 3.3.3]) there exists a nonsingular S so that

$$S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} , \quad S^{-1}BS = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix}$$

where $\sigma(A_{22}) = \sigma_u(A)$. We can assume A and B to be in this form. Hence $\sigma(A_{22}) \subset \sigma_\Sigma$. Now we consider $\sigma(\frac{1}{\tau}W_k(B\tau e^{-A\tau}) + A)$. Here $B\tau e^{-A\tau}$ has the form $\begin{bmatrix} X & Y \\ 0 & 0 \end{bmatrix}$, and $W = \begin{bmatrix} W_k(X) & Ye^{-W_k(X)} \\ 0 & 0 \end{bmatrix}$ satisfies $We^W = B\tau e^{-A\tau}$. Thus $\sigma(A_{22}) \subset \sigma\left(\frac{1}{\tau}W_k(B\tau e^{-A\tau}) + A\right)$ for some branch of the Lambert W function. □

Analogously, by Corollary 7, we get the following variant.

Lemma 9. *Assume that the pair (B, A) is not controllable and $\sigma_u(B)$ denotes the corresponding uncontrollable eigenvalues of B. Then*

$$\frac{1}{\tau}W_k(\tau\sigma_u(B)) \subset \sigma_\Sigma \cap \bigcup_k \sigma\left(\frac{1}{\tau}W_k(B\tau e^{-A\tau}) + A\right) .$$

The proof is omitted for brevity.

4 A counter-example

To demonstrate that formula (3) is not applicable to arbitrary TDS, we pick the following non simultaneously triangularizable (and non commuting) pair of matrices

$$A = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,$$

for some $\alpha \in \mathbb{R}$, $\alpha > 0$. We now find an explicit expression for the eigenvalues. The characteristic equation is given by

$$0 = \det\left(-sI + A + Be^{-s\tau}\right) = s^2 - \alpha e^{-s\tau} . \quad (4)$$

Eigenvalues s are thus characterized by

$$\alpha = s^2 e^{s\tau} \quad \Longleftrightarrow \quad \pm\frac{1}{2}\tau\sqrt{\alpha} = \frac{1}{2}s\tau e^{\frac{1}{2}s\tau} . \quad (5)$$

In particular, $s_0 = \frac{2}{\tau}W_0\left(\pm\frac{1}{2}\tau\sqrt{\alpha}\right)$ is an eigenvalue, where W_0 denotes the principal branch of the Lambert W function. The example becomes explicitly tractable, if we pick $\tau = 1$, $\alpha = \pi^2$ and make use of the fact that $W_0(-\frac{1}{2}\pi) = \frac{1}{2}\pi i$. Hence we obtain $s_0 = \pi i$.

By formula (3) we would have

$$\sigma_\Sigma = \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right) , \quad (6)$$

where again $\tau = 1$. It is clear from Fig. 1 that this expression is not consistent with (5).

To prove this strictly, we first find an s such that $s \in \sigma_\Sigma$, but not $s \in \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right)$, and hence prove that $\sigma_\Sigma \not\subset \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right)$. Secondly, we find an s such $s \in \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right)$ but not $s \in \sigma_\Sigma$, and prove that $\sigma_\Sigma \not\supset \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right)$.

We note that $Be^{-A} = \begin{pmatrix} -\alpha & 1 \\ 0 & 0 \end{pmatrix}$, and

$$W_k(Be^{-A}) = \begin{pmatrix} W_k(-\alpha) & -\frac{1}{\alpha}W_k(-\alpha) \\ 0 & 0 \end{pmatrix} .$$

By (6), eigenvalues would be characterized via

$$0 = s^2 - sW_k(-\alpha) + W_k(-\alpha) , \quad (7)$$

or more explicitly,

$$s = \frac{W_k(-\alpha) \pm \sqrt{W_k(-\alpha)^2 + 4W_k(\alpha)}}{2} . \quad (8)$$

In particular, for $\alpha = \pi^2$ and some $k \in \mathbb{Z}$, the eigenvalue $s_0 = \pi i$ would have to satisfy (7). Thus

$$\begin{aligned} 0 &= (i\pi)^2 - (i\pi)W_k(-\pi^2) + W_k(-\pi^2) \\ &= W_k(-\pi^2)(1 - i\pi) - \pi^2 . \end{aligned}$$

Hence we conclude $W_k(-\pi^2) = \frac{\pi^2}{1-i\pi}$. This is not fulfilled for any branch k since

$$-\pi^2 = \frac{\pi^2}{1-i\pi} e^{\frac{\pi^2}{1-i\pi}} \qquad \Longleftrightarrow \qquad i\pi - 1 = e^{\frac{\pi^2}{1-i\pi}} \ .$$

Taking absolute values, we get $\sqrt{\pi^2+1} = e^{\frac{\pi^2}{1+\pi^2}}$ which contradicts $\pi > e$. Hence $\sigma_\Sigma \not\subset \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right)$.

Vice versa, we can also produce an explicit example, where $\sigma_\Sigma \not\supset \bigcup_k \sigma\left(W_k(Be^{-A}) + A\right)$. Let $\alpha = \frac{1}{2}\pi$. For the principal branch of W equation (8) reduces to

$$s = \frac{i\pi \pm \sqrt{-\pi^2 + 8\pi i}}{4} \ .$$

It remains to show that s does not always satisfy the characteristic equation $s^2 = \frac{\pi}{2} e^{-s}$ from (4).
Setting $a+bi = \pm\sqrt{-\pi^2+8\pi i}$ with $a > 0$ we find that $a^2 + b^2 \geq 2ab = 8\pi$, whence also $b > 0$. This implies $|s|^2 > (a^2+b^2)/4 \geq 2\pi$ and $\frac{\pi}{2}|e^{-s}| < \frac{\pi}{2}|e^{-\pi}| < \frac{\pi}{2}$, i.e. $s^2 \neq \frac{\pi}{2} e^{-s}$.

Actually, the spectra are disjoint (Fig. 1), which is consistent with Lemma 8 and the controllability of (A, B).

One may ask, whether our counter-example hinges on the controllability of (A, B). It is, in fact, an immediate consequence of the Kalman decomposition that any non simultaneously triangularizable pair of 2×2 matrices is controllable. We may, however, embed our example in a higher-dimensional uncontrollable system $\tilde{\Sigma}$, setting

$$\tilde{A} = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix},$$

so that

$$\det\left(-sI + \tilde{A} + \tilde{B}e^{-s}\right) = (1-s)\det\left(-sI + A + Be^{-s}\right),$$

i.e. $\sigma_{\tilde{\Sigma}} = \sigma_\Sigma \cup \{1\}$, and (for all branches W_k)

$$\begin{aligned} &\det\left(-sI + W_k(\tilde{B}e^{-\tilde{A}}) + \tilde{A}\right) \\ &= (1-s)\det\left(-sI + W_k(Be^{-A}) + A\right) \ , \end{aligned}$$

i.e. $\sigma\left(W_k(\tilde{B}e^{-\tilde{A}}) + \tilde{A}\right) = \sigma\left(W_k(Be^{-A}) + A\right) \cup \{1\}$. Our conclusions thus hold for this system

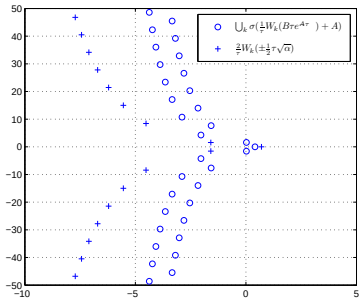

Figure 1: Counter-example from Section 4 with $\tau = \alpha = 1$

as well, where in accordance with Lemma 8 the uncontrollable eigenvalue 1 is contained in $\sigma_{\tilde{\Sigma}} \cap \sigma\left(W_k(\tilde{B}e^{-\tilde{A}}) + \tilde{A}\right)$.

5 Conclusions

Defining a general matrix-version of the Lambert W function, we have provided a formula for the eigenvalues of a time-delay system, if the system matrices commute, and the more general case where the matrices are simultaneously triangularizable. If the pair of system matrices is not controllable, then at least the uncontrollable eigenvalues are given correctly by the formula. For the general case, where the matrices are not simultaneously triangularizable we have given explicit examples where the formula fails.

Acknowledgement: The authors acknowledge the comments of Robert Corless, as well as interesting suggestions by the anonymous referees.

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