A Quadratic Eigenproblem in the Analysis of a Time Delay System

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In this work we solve a *quadratic eigenvalue problem* occurring in a method to compute the set of delays of a *linear time delay system* (TDS) such that the system has an imaginary eigenvalue. The computationally dominating part of the method is to find all eigenvalues z of modulus one of the quadratic eigenvalue problem

$$\left(z^2 A_m \otimes I + z\left(\sum_{k=0}^{m-1} I \otimes A_k e^{-i\varphi_k} + A_k \otimes I e^{i\varphi_k}\right) + I \otimes A_m\right) u = 0,$$

where $\varphi_1, \ldots, \varphi_{m-1} \in \mathbb{R}$ are free parameters and u a vectorization of a Hermitian rank one matrix.

Because of its origin in the vectorization of a *Lyapunov type matrix equation*, the quadratic eigenvalue problem is, even for moderate size problems, of very large size. We show one way to treat this problem by exploiting the Lyapunov type structure of the quadratic eigenvalue problem when constructing an iterative solver. More precisely, we show that the *shift-invert operation* for the companion form of the quadratic eigenvalue problem can be efficiently computed by solving a *Sylvester equation*. The usefulness of this exploitation is demonstrated with an example.

1 Introduction

In this work we consider linear time-delay systems with multiple constant delays. These types of systems occur in models where there is a non-negligible delay, normally originating from some physical limitation, e.g, finite switching times in controllers, unavailability of the current state of the system, etc. The stability properties of the system are highly dependent on the size and the relations of the delays. We are therefore led to analyze the stability region in delay space.

The analysis of the stability region in delay space and delay-dependent stability conditions has, for special cases, received a lot of attention the last decades, for instance the treatment of two delay scalar equation [7] and references therein. The multi-dimensional one-delay case is treated by Louisell in [9]. Chen et al [1] compute the stability margin for the commensurate case, i.e., the case where the delays are integer multiples of each other. The large amount of literature about conservative bounds of the stability region, often using *linear matrix inequalities*, is well described in [11] which also contain more thorough descriptions of methods to analyze stability. For more special cases, see the survey article [13]. The only existing method finding the boundaries of the stability region for multi-dimensional, incommensurate case is presented in the recent paper by Sipahi and Olgac [14].

Here, similar to [14], we address the general problem to find the conservative bounds for the stability region for the multi-dimensional multi-delay case. But the approach here is completely different. In [14] the characteristic equation is solved by making a Rekasius substitution and treating the resulting (high order) parameterized polynomial with a Routh's array. In the method presented here, we define a Lyapunov type matrix operator which, under simple conditions, share roots with the characteristic equation. Moreover, the operator turns out to have a particularly simple structure on the boundary of the stability region, which makes it possible to rewrite it into a quadratic eigenvalue problem. This allows us to apply the rich theories on eigenvalue problems, and in particular computationally efficient iterative methods for eigenvalue problems.

This contribution is an extension of the method presented in [8] in the way that we provide a way to solve the occurring quadratic eigenvalue problem more efficiently. An approach similar to [8] is taken in [2]. A Lyapunov-exploitation similar to the one done here, in a completely different context, is done in [4].

We note that no direct computational comparison with [14] should be done, as the two methods solve slightly different problems. Here we find delays for which the system has an imaginary eigenvalue, which generates potential boundaries for the stability region, whereas Sipahi and Olgac present a method called CTCR (cluster treatment of characteristic roots) which finds all stability regions in delay space.

This contribution is organized as follows. Some basic concepts and definitions of time delay systems are introduced in Section 2. The main theorem from which one can compute the critical delays containing the quadratic eigenproblem, as well as the exploitation of the structure of the quadratic eigenproblem is described in Section 3.

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2 Basic concepts

The retarded linear m-delay TDS is described by

$$\Sigma = \begin{cases} \dot{x}(t) = \sum_{k=0}^{m} A_k x(t-h_k),\ t>0 \\ x(t) = \varphi(t),\ t \in [-h_m, 0] \end{cases} \tag{1}$$

with $h_0 = 0 < h_1 < \ldots < h_m$, $x : [-h_m, \infty) \mapsto \mathbb{R}^n$ and $A_k \in \mathbb{R}^{n\times n}$. We will sometimes denote the system Σ with $\Sigma(h_1, \ldots, h_m)$ in order to stress the dependence on the delays.

Definition 2.1 The *characteristic eigenvalue problem* of Σ is

$$\mathbb{M}(s)v := \left(-sI_n + \sum_{k=0}^{m} A_k e^{-h_k s} \right) v = 0,\ \|v\| = 1, \tag{2}$$

where $v \in \mathbb{C}^n$ is called *eigenvector* and $s \in \mathbb{C}$ an *eigenvalue*. The set of all eigenvalues $\sigma(\Sigma)$ is called the *spectrum*.

Similar to the delay-free case, a system is exponentially stable if and only if all eigenvalues lie in the open left complex half-plane, i.e. $\sigma(\Sigma) \subset \mathbb{C}_-$. An essential difference is that, unlike the (delay-free) dynamical systems, the spectrum contains a countably infinite number of eigenvalues. Fortunately, it can be proven (see for instance [5]) that there are only a finite number of eigenvalues in $\mathbb{C}_+$.

From continuity it is clear that the TDS at the boundary of any stability region in the delay-parameter space $h_1, h_2, \ldots, h_m$, has at least one purely imaginary eigenvalue. This justifies the following definitions, inspired by the use of the word *critical* in for instance [12] and [3].

Definition 2.2

1. Σ is called *critical*[1] if and only if $\sigma(\Sigma) \cap i\mathbb{R} \neq \emptyset$.
2. The set of all points in delay-parameter space $(h_1, h_2, \ldots, h_m)$ for which $\Sigma(h_1, \ldots, h_m)$ is critical are called the *critical curves* ($m = 2$) and *critical surfaces* ($m > 2$).[2]

The stability region in the delay-parameter space is bounded by critical surfaces. The rest of this article will, for that reason, deal with the computation of critical surfaces.

3 Results

The computation of the critical surfaces is done by introducing the following operator of Lyapunov-type.

Definition 3.1 Let

$$\mathbb{L}(X, s) := \mathbb{M}(s)X + X\mathbb{M}(s)^* = \sum_{k=0}^{m} \left(A_k X e^{-h_k s} + X A_k^T e^{-h_k \bar{s}} \right) - 2X \mathrm{Re}\, s, \tag{3}$$

where * denotes complex conjugate transpose.

Note that for the critical case $\mathrm{Re}\, s = 0$, the linear term disappears and the $\mathbb{L}$ operator reduces to a sum of exponential functions.

The following theorem characterizes eigenpairs using the Lyapunov operator.

Theorem 3.2 ([8]) *Given $s \in \mathbb{C}$ and $v \in \mathbb{C}^n$, $v^*v = 1$ the following are equivalent.*

$$\mathbb{M}(s)v = 0 \tag{4}$$

$$\mathbb{L}(vv^*, s) = 0 \wedge v^*\mathbb{M}(s)v = 0 \tag{5}$$

P r o o f. The forward implication is trivial from definitions, i.e., (2) and (3). The backward implication is clear from the following set of equalities.

$$\mathbb{M}(s)v = \mathbb{M}(s)vv^*v = \left(\mathbb{L}(vv^*, s)vv^* - vv^*\mathbb{M}(s)^*vv^* \right) v = 0$$

□

It is shown in other work by the author (see [8]) that the method of Chen, et al [1] is in fact the application of Theorem 3.2 to a commensurate time delay system.

[1] In the context of nonlinear differential equations a critical system is normally called *non-hyperbolic* (at a fixed point).

[2] The critical curves (surfaces) are called *offspring curves* and *kernel curves* in [14] and *Hopf bifurcation curves (surfaces)* in [6].

3.1 Quadratic eigenproblem

As stated earlier, we will now consider the critical case, i.e $s = i\omega$. By making the substitutions $z = e^{i\omega}$ and $\varphi_k = \omega h_k$, $k = 0, \dots, m-1$ equation (5) reduces to a quadratic relation which turns out to be a quadratic eigenvalue problem. This is demonstrated in the following theorem which allows parameterization of the $(m-1)$-dimensional critical surface.

Theorem 3.3 ([8]) *Let $\bar{h} = (h_1, \dots, h_m)$ be a point in delay-parameter space. $\bar{h}$ lies on the critical surface if and only if there are some $\varphi_k \in [-\pi, \pi], k = 1, \dots, m-1$, $z \in \mathbb{C}$ on the unit circle, $\omega \in \mathbb{R}$ such that z and v is a solution to the equation*

$$z^2 vv^* A_m^T + z\left(\sum_{k=0}^{m-1} A_k vv^* e^{-i\varphi_k} + vv^* A_k^T e^{i\varphi_k}\right) + A_m vv^* = 0 \tag{6}$$

$$\omega = -iv^*\left(A_m z^{-1} + \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}\right)v, \tag{7}$$

and $h_m = \frac{\operatorname{Arg} z + 2p_m\pi}{\omega}$, $h_k = \frac{\varphi_k + 2p_k\pi}{\omega}$, $k = 1, \dots, m-1$ for some $p_k \in \mathbb{Z}, k = 1 \dots m$.

P r o o f. This follows from Theorem 3.2 by letting $\varphi_k = \omega h_k$ and $z = e^{i\varphi}$. □

From this theorem we can parameterize the critical surface using the $m-1$ free parameters $\varphi_k \in [-\pi, \pi]$, $k = 1, \dots, m-1$. We now note that (6) can be vectorized into the following vector equation

$$0 = \left(z^2 A_m \otimes I + z\left(\sum_{k=0}^{m-1} I \otimes A_k e^{-i\varphi_k} + A_k \otimes I e^{i\varphi_k}\right) + I \otimes A_m\right)u$$

where $u = \operatorname{vec} vv^* = \bar{v} \otimes v$ and $\otimes$ denotes the Kronecker product. This is a *quadratic eigenvalue problem*. For a survey on quadratic eigenvalue problems see [15]. Quadratic eigenvalue problems can be solved using the companion form or generalization thereof (see [10]), which transforms it to a (generalized) eigenvalue problem. For the scalar case, i.e $n = 1$ the quadratic eigenvalue problem is a quadratic (scalar) equation, and we can find an explicit parametrization of the critical surface (see [8] for details).

3.2 Exploiting the structure

The first companion form of the quadratic eigenvalue problem (6) is

$$\begin{pmatrix} 0 & I \\ K & G \end{pmatrix}\begin{pmatrix} u \\ zu \end{pmatrix} = z\begin{pmatrix} I & 0 \\ 0 & -M \end{pmatrix}\begin{pmatrix} u \\ zu \end{pmatrix}, \text{ where } M = I \otimes A_m, G = \left(\sum_{k=0}^{m-1} I \otimes A_k e^{-i\varphi_k} + A_k \otimes I e^{i\varphi_k}\right), K = A_m \otimes I.$$

By doing a Caley-transformation $z = \frac{1+i\sigma}{1-i\sigma}$ we transform the eigenvalues z on the unit circle to eigenvalues σ on the real line. The transformed eigenproblem is,

$$\begin{pmatrix} -I & I \\ K & G+M \end{pmatrix}\begin{pmatrix} u \\ zu \end{pmatrix} = i\sigma\begin{pmatrix} I & I \\ K & G-M \end{pmatrix}\begin{pmatrix} u \\ zu \end{pmatrix}.$$

If we neglect the degenerate case, the generalized eigenvalues are the same as the eigenvalues of the matrix

$$\begin{pmatrix} I & I \\ K & G-M \end{pmatrix}^{-1}\begin{pmatrix} I & -I \\ K & G+M \end{pmatrix} = \begin{pmatrix} (G-M-K)^{-1} & 0 \\ 0 & (G-M-K)^{-1} \end{pmatrix}\begin{pmatrix} G-M & I \\ K & I \end{pmatrix}\begin{pmatrix} I & -I \\ K & G+M \end{pmatrix}.$$

When computing the matrix-vector product of the matrix above, the main computational effort lies in the solution of the linear system $(G-M-K)x = b$. This system is of dimension $2n^2 \times 2n^2$ which is computationally demanding even for moderately sized problems. If we perform a matrification of this linear system (i.e the inverse of the operation vectorization) and use the definition of M, G and K, we identify that the solution of the linear system can be computed by solving a *Lyapunov equation*. More precisely,

$$b = (G-M-K)x = \left(I \otimes \left(-A_m + \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}\right) + \left(-A_m + \sum_{k=0}^{m-1} A_k e^{i\varphi_k}\right) \otimes I\right)x$$

is equivalent to

$$B = A_s X + X A_s^* \text{ where } A_s := -A_m + \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}, x = \operatorname{vec} X, b = \operatorname{vec} B.$$

Similar steps can be done to compute the inverse-matrix-vector product. In this case we also arrive at a Lyapunov equation but with a sign switch in A_s, matrix $A_s = A_m + \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}$. Similarly, for the shift-invert operation we end up with the a *Sylvester equation*.

4 Examples

To demonstrate the efficiency of the exploitation we consider the partial differential delay equation from [8], with different space-discretization. The solution of the corresponding quadratic eigenvalue problem for different system dimension n is solved using Matlab 6.5 (R13) using a Dual Core AMD Athlon 4400+. Figure 1 shows the compational time required for three different methods. The first method constructs the complete companion-matrix and solves the full eigenproblem using the function `eig`. The second method uses sparse representation of the matrices and the sparse eigensolver `eigs` to find the eigenvalues of interest. The third and computationally most efficient method uses the matrix-vector described in Section 3.2 to find the eigenvalues of interest. The solving of the Lyapunov equation is accelerated by first computing the Schur-decomposition of A_s.

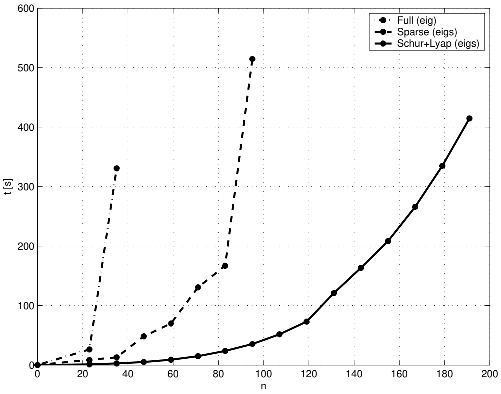

Fig. 1 Computational time to solve (6) for different system dimensions using different methods

5 Conclusions

We summarize previous results [8] regarding computing critical delays, i.e. the delays such that there exists an imaginary eigenvalue, for multiple delay time delay systems, and provide a way to more efficiently solve the quadratic eigenproblem occurring in this method. This is done by exploiting the Lyapunov-structure of the quadratic eigenproblem. The efficiency of the acceleration is demonstrated with an example.

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References

[1] J. Chen and G. Gu and C.N. Nett. A new method for computing delay margins of linear delay systems. *Systems and Control Letters*, 26(2):107-117, 1995
[2] A. Ergenc and N. Olgac and H. Fazelinia. Extended Kronecker summation for determining the kernel and offspring of LTI systems with multiple delays. Submitted to *SIAM Journal on Control and Optimization*
[3] K. Gu and S. Niculescu. Additional dynamics in transformed time-delay systems. *IEEE transactions on automatic control*, 24(3):572-575, 2000.
[4] M. Gu, E. Mengi, M.L. Overton, J. Xia and J. Zhu, Fast methods for estimating the distance to uncontrollability, *SIAM Journal on Matrix Anal. Appl.*, 2005, accepted
[5] Jack Hale. *Theory of functional differential equations*. Springer-Verlag, 1977.
[6] J. Hale. Dynamics and delays. In *Delay differential equations and dynamical systems*, Lecture notes in mathematics 1475, pages 16-30. Springer, Berlin, 1991.
[7] J. Hale and W. Huang. Global geometry of the stable regions for two delay differential equations. *Journal of Mathematical analysis and applications* pages 344-362, 1993.
[8] E. Jarlebring. Computing critical delays for time delay systems using polynomial eigenproblems. *Automatica*. Submitted.
[9] James Louisell. A matrix method for determining the imaginary axis eigenvalues of a delay system. *IEEE Trans. Autom. Control* 12:20082012, 2001.
[10] D. S. Mackey and N. Mackey and C. Mehl and V. Mehrmann. Vector spaces of linearizations of matrix polynomials. *SIAM J. of Matrix Analysis and Appl.* To appear.
[11] S. Niculescu. *Delay effects on stability. A robust control approach.* Number 269 in Lecture notes in control and information sciences. Springer, 2001.
[12] E. Plischke. *Transient effects of linear dynamical systems*. PhD thesis, Universität Bremen, 2005.
[13] R. Sipahi and N. Olgac. A comparative survey in determining the imaginary characteristic roots of LTI time delayes systems. In *IFAC 2005 World Congress, Praha, Czech republic*, 2005.
[14] R. Sipahi and N. Olgac. Complete stability robustness of third-order LTI multiple time-delay systems. *Automatica*, 41:1414-1422, 2005.
[15] K. Tisseur and F. Meerbergen. The quadratic eigenvalue problem. *SIAM Rev*, 43(2):235-286, 2001.