On Critical Delays for Linear Neutral Delay Systems

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***Abstract*— In this work we address the problem of finding the critical delays of a linear neutral delay system, i.e., the delays such that the system has a purely imaginary eigenvalue. Even though neutral delay systems exhibit some discontinuity properties with respect to changes in the delays an essential part in a non-conservative stability analysis with respect to changes in the delays, is the computation of the critical delays.**

We generalize previous results on critical delays and stability switches for retarded time-delay systems, under some minor assumptions on the delay system.

The work starts with stating a general equivalence theorem between the spectrum and a matrix function condition. We show how this theorem can be applied to the commensurate time-delay system to compute the critical delays. It turns out that the resulting method is closely related to parts of the results of Fu, Niculescu and Chen[6]. For the incommensurate case we present a scheme which allows the computation of the critical curves, i.e., the points in delay-space for which the system has a purely imaginary eigenvalue.

We apply the method to previously investigated examples, in order to provide a verification of the results, as well as to an example for which the stability picture is, to our knowledge, not yet known.

I. INTRODUCTION

Time-delay systems are natural models of many phenomenas in engineering, biology and physics. Some applications occur in the topics related to electric circuits, finite switch-time controllers, networks with communication limitations, population dynamics, traffic dynamics and congestion control, machine-tool cutting, simulation and control of chemical processes. For a more thorough list of applications of time-delay systems see [16].

In this work we treat *neutral linear time-delay systems* with m discrete delays, i.e.,

$$\Sigma = \begin{cases} \displaystyle\sum_{k=0}^{m} B_k \dot{x}(t-h_k) = \sum_{k=0}^{m} A_k x(t-h_k),\ t>0 \\ x(t) = \varphi(t),\ t \in [-h_{max}, 0] \end{cases} \tag{1}$$

where $A_k, B_k \in \mathbb{R}^{n\times n}$ and h_{max} is the largest delay. Without the loss of generality, we assume that $h_0 = 0$ and $h_m = h_{max}$. The stability of the neutral system, can be determined from the solutions of the characteristic equation, i.e., the nontrivial solution of

$$\mathbb{M}(s)v = 0,$$

where $\mathbb{M}(s) = -s\mathbb{B}(s) + \mathbb{A}(s)$,

$$\mathbb{A}(s) = \sum_{k=0}^{m} A_k e^{-h_k s} \text{ and } \mathbb{B}(s) = \sum_{k=0}^{m} B_k e^{-h_k s}.$$

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We call $s \in \mathbb{C}$ an eigenvalue and $v \in \mathbb{C}^n$ an eigenvector of the system. If the spectrum, i.e., the set of all eigenvalues, is to the left of some vertical line in the open left complex half-plane, then the system is exponentially stable.

It is known (e.g. [1] and [14]) that the spectrum of neutral delay systems exhibits some discontinuity properties. Even though each eigenvalue-path is continuous, an infinitesimal change in the delays may cause the system to turn unstable. These discontinuities of the cluster points of the real part of the spectrum, are closely related to the essential spectrum of the system, i.e., the solutions of $\det(\mathbb{B}(s)) = 0$. These discontinuity properties can be characterized by *strongly exponential stability* (e.g., [9] and [15]).

One way for a stability-switch of a neutral delay system to occur is that the rightmost eigenvalue goes from the left complex half-plane into the right complex half-plane by passing the imaginary axis. In this work we find conditions on the delay parameters such that the system has a purely imaginary eigenvalue. This is clearly a first step in a non-conservative stability analysis of neutral time-delay systems.

Unlike many (in practice) conservative stability-checking methods formulated using *linear matrix inequalities* and the theory of *Lyapunov-Krasovskii* (e.g., [5], [10], [19] and [7]), we aim to find non-conservative stability conditions.

Explicit non-conservative stability conditions for neutral systems have only received moderate amount of attention in literature. For the single delay case, a method to compute the imaginary eigenvalues as well as the critical delays is presented in [12]. Another analysis using a Rekasius substitution is done in [18] and [20]. A frequency sweeping approach using a frequency dependent matrix pencil is described in [2]. Gu, Niculescu and Chen [8] analyzes a geometric approach for finding the crossing curves of scalar problems corresponding to two-delay systems and systems without *delay cross-terms*. More recently, for the commensurate multi-dimensional system, the matrix-pencil based method in [6] provides a way to compute the stability margin. This is a generalization of [3] to neutral systems. In [17] it is shown how one can apply these results to lossless propagation systems.

Our main results are divided into two parts: the results for commensurate delays and the results on incommensurate delays. For the commensurate case we find a matrix condition similar to that of [6]. For the incommensurate case, we present a method to compute the critical curves in delay-space, generalizing the method for retarded systems [11]. Both of these results are shown by first proving an equivalence relation between the spectrum of the time-delay system and a matrix-condition.

II. RESULTS

We exclude the following case in order to simplify the analysis.

Assumption 1 We assume that the problem is well posed in the sense that $\mathbb{B}(s)v \neq 0$ for all eigenpairs v, s. In other words, we assume that $\mathbb{A}(s)v = 0$ and $\mathbb{B}(s)v = 0$ do not have a solution s,v in common.

The assumption corresponds to the case where the discrete operator corresponding to the neutral part, i.e., $\mathbb{B}(s)$, has an eigenpair in common with the discrete operator corresponding to the retarded part, i.e., $\mathbb{A}(s)$. This does not cause a major restriction to the results as it is easy to identify and verify, e.g., Assumption 1 is fulfilled for retarded systems as $\mathbb{B}(s)v = v \neq 0$. Other examples on how to check whether it is fulfilled are shown in the examples section.

Note that this assumption is not equivalent to *strong stability* (e.g., [9]), which is a common assumption in stability analysis of neutral time-delay systems. Assumption 1 is only motivated by the fact that it is a natural assumption for the proofs.

In order to make a rigorous but simple analysis, we state, in a general form, an equivalence theorem which is used in the analysis of the incommensurate as well as the commensurate case. To enable us to compactly state this theorem, we introduce the following matrix function, defined from the characteristic matrix function $\mathbb{M}(s) = -s\mathbb{B}(s) + \mathbb{A}(s)$.

Definition 2 Let

$$\begin{aligned}\mathbb{L}(X, s) &:= \mathbb{M}(s)X\mathbb{B}(s)^* + \mathbb{B}(s)X\mathbb{M}(s)^* = \\ &= \mathbb{A}(s)X\mathbb{B}(s)^* + \mathbb{B}(s)X\mathbb{A}(s)^* - 2\mathbb{B}(s)X\mathbb{B}(s)^*\mathrm{Re}\ s.\end{aligned} \tag{2}$$

Using this definition we can compactly state the fundamental equivalence theorem of this work.

Theorem 3 Given $s \in \mathbb{C}$ and $v \in \mathbb{C}^n$ such that $\mathbb{B}(s)v = w \neq 0$, then the following statements are equivalent

$$\mathbb{M}(s)v = 0 \tag{A}$$

$$\mathbb{L}(vv^*, s) = 0 \wedge w^*\mathbb{M}(s)v = 0 \tag{B}$$

Proof: The implication $(A) \Rightarrow (B)$ is clear from the definition. The implication $(B) \Rightarrow (A)$ holds from the equality

$$\mathbb{L}(vv^*, s)w = \mathbb{M}(s)vv^*\mathbb{B}(s)^*w + \mathbb{B}(s)vv^*\mathbb{M}(s)^*w.$$

■

We will now focus on finding the solutions of (B) for the critical case, i.e., the case where there is a purely imaginary eigenvalue. For this, we note that clearly $\mathrm{Re}\ s = 0$ and the non-exponential term of $\mathbb{L}$ disappears.

A. Commensurate delays

We now assume that the delays are integer multiples of some delay h, i.e., $h_j = hj$ for $j = 0, \ldots, m$. We substitute $\zeta = e^{-i\omega h}$, and rephrase the theorem. Note that results very similar to this theorem are contained in [6].

Theorem 4 The commensurate m delay time-delay system fulfilling Assumption 1 has the critical delay h if and only if there is an $\zeta \in \partial\mathbb{D}$, $\omega \in \mathbb{R}$ and $v \in \mathbb{C}^n$ such that

$$\sum_{j,k}^{m} \left(A_j vv^* B_k^T + B_j vv^* A_k^T\right) \zeta^{m+j-k} = 0 \tag{3}$$

and

$$\omega = -i\frac{w^* \sum_j^m A_j \zeta^j v}{w^* w}$$

and

$$h = \frac{-\mathrm{Arg}\ \zeta + 2p\pi}{\omega}$$

for some $p \in \mathbb{Z}$, where

$$w = \sum_j^m B_j \zeta^j v.$$

Proof: This follows from Theorem 3, if we let $\zeta = e^{-i\omega h}$. ■

Now we note that (3) can be vectorized, i.e., stacking the column of the matrix-equation on top of each other, into

$$\begin{aligned}&\sum_{j,k=0}^{m} \left(B_k \otimes A_j + A_k \otimes B_j\right) \zeta^{m+j-k} u = \\ &\sum_{q=0}^{2m} \sum_{j=\max(q-m,0)}^{\min(m,q)} \left(B_{j-q+m} \otimes A_j + A_{j-q+m} \otimes B_j\right) \zeta^q u\end{aligned} \tag{4}$$

where $u = \mathrm{vec}\ \ vv^*$. This is polynomial eigenvalue problem. There are numerical methods methods for computing its' solution[1].

The most common way of solving polynomial eigenproblems is by transforming it to an eigenvalue problem, constructing the *companion matrices* (for generalizations of the companion form see [13]). Generally a polynomial eigenproblem

$$\sum_{k=0}^{N} C_k \zeta^k u = 0$$

fulfills the equation

$$\zeta \begin{pmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & C_N \end{pmatrix} w = \begin{pmatrix} 0 & I & & \\ & \ddots & \ddots & \\ & & 0 & I \\ -C_0 & \cdots & -C_{N-2} & -C_{N-1} \end{pmatrix} w, \tag{5}$$

where $w = (u^T, \zeta u^T, \zeta^2 u^T, \cdots \zeta^{N-1} u^T)^T$, which is a generalized eigenvalue problem, solvable on a computer (for moderate sized problem) to sufficient accuracy.

[1] As of version 7.1.0 (R14) of Matlab, the command `polyeig` is available for solving dense polynomial eigenproblems.

Here $N = 2m$ and $C_q = \sum_{j=\max(q-m,0)}^{\min(m,q)} B_{j-q+m} \otimes A_j + A_{j-q+m} \otimes B_j$ for $q = 0, \ldots, 2m$.

Remark 5 Again, a very similar matrix condition is contained in [6]. Note however that the matrices Q_k in [6] are not identical to C_q, even though some numerical experiments (not reported here) indicate that they share eigenvalues on the unit circle, the exact relation is not obvious.

B. *Incommensurate parametrization*

For incommensurate time-delay systems, our aim is to visualize for what points in delay-space the system has a purely imaginary eigenvalue. For this we introduce the free variable $\vec{\varphi} = (\varphi_0, \ldots, \varphi_{m-1})$, where we for notational purposes let $\varphi_0 = 0$. If we have an m-delay system we aim to parameterize a $(m-1)$-dimensional hypersurface using the free variable $\vec{\varphi}$.

To simplify the notation we introduce $A(\vec{\varphi}) := \sum_{k=0}^{m-1} A_k e^{-i\varphi_k}$ and $B(\vec{\varphi}) := \sum_{k=0}^{m-1} B_k e^{-i\varphi_k}$. We are now ready to state our main theorem for the incommensurate time-delay system.

Theorem 6 Let $\bar{h} = (h_1, \ldots, h_m)$ be a point in delay-parameter space. For the neutral TDS fulfilling Assumption 1, the point $\bar{h}$ lies on the critical surface only if there are some $\varphi_k \in [-\pi, \pi], k = 1, \ldots, m-1$, $z \in \mathbb{C}$ on the unit circle, $\omega \in \mathbb{R}$ such that

$$\begin{aligned} z^2 \quad & (A_m vv^* B(\vec{\varphi})^* + B_m vv^* A(\vec{\varphi})^*) + \\ z \quad & (A(\vec{\varphi}) vv^* B(\vec{\varphi})^* + A_m vv^* B_m^* + \\ & B(\vec{\varphi}) vv^* A(\vec{\varphi})^* + B_m vv^* A_m^*) + \\ & A(\vec{\varphi}) vv^* B_m^* + B(\vec{\varphi}) vv^* A_m^* = 0, \end{aligned} \tag{6}$$

$$\omega = -i \frac{w^* \left(A_m z + A(\vec{\varphi})\right) v}{w^* w}, \tag{7}$$

$$w = (B_m z + B(\vec{\varphi})) v \tag{8}$$

and $h_m = \frac{-\mathrm{Arg}\ z + 2p_m\pi}{\omega}$, $h_k = \frac{\varphi_k + 2p_k\pi}{\omega}$, $k = 1, \ldots, m-1$ for some $p_k \in \mathbb{Z}, k = 1 \ldots m$.

Proof: This follows from the Theorem 3 by choosing $z = e^{-i\omega h_m}$, $\varphi_0 = 0$ and $\varphi_k = h_k \omega$ for $k = 1, \ldots, m-1$. ∎

Similar to the previous section we note that (6) is a matrix equation which can be vectorized into

$$(z^2 M(\vec{\varphi}) + zC(\vec{\varphi}) + K(\vec{\varphi}))u = 0, \tag{9}$$

where $u = \mathrm{vec}\ \ vv^*$, $M(\vec{\varphi}) = B(-\vec{\varphi}) \otimes A_m + A(-\vec{\varphi}) \otimes B_m$, $C(\vec{\varphi}) = B(-\vec{\varphi}) \otimes A(\vec{\varphi}) + B_m \otimes A_m + A(-\vec{\varphi}) \otimes B(\vec{\varphi}) + A_m \otimes B_m$ and $K(\vec{\varphi}) = B_m \otimes A(\vec{\varphi}) + A_m \otimes B(\vec{\varphi})$. Equation (9) is a polynomial eigenproblem of degree two, i.e., a quadratic eigenproblem (for details on the quadratic eigenproblem see [21]). Again, the solutions of the quadratic eigenproblems can be computed from a corresponding companion form, e.g.

$$z \begin{pmatrix} I & 0 \\ 0 & M(\vec{\varphi}) \end{pmatrix} \begin{pmatrix} u \\ zu \end{pmatrix} = \begin{pmatrix} 0 & I \\ -K(\vec{\varphi}) & -C(\vec{\varphi}) \end{pmatrix} \begin{pmatrix} u \\ zu \end{pmatrix}. \tag{10}$$

Note that given a free parameter $\vec{\varphi}$ we can use Theorem 6 to compute a point on the critical curves, and that if we let $\vec{\varphi}$ run over the whole domain, Theorem 6 will generate all critical points. In practice we typically let the free parameter $\vec{\varphi}$ run over a finite number of grid points with a grid size small enough that we can convince ourselves of the continuity of the critical curves. We outline a numerical procedure for the two-delay case in pseudo-code

1. FOR $\varphi = -\pi : \Delta : \pi$
2.   Find eigenpairs (z_k, u_k) of (10)
3.   FOR $k = 1 : \mathrm{length}(z)$
4.   IF z_k is on unit circle
5.   Compute v_k such that $u_k = \mathrm{vec}\ \ v_k v_k^*$
6.   Compute ω_k using (7)
7.   Accept critical points (h_1, h_2)

$$\begin{aligned} h_1 &= \frac{\varphi + 2p\pi}{\omega_k}, \ p = -p_{max}, \ldots, p_{max} \\ h_2 &= \frac{-\mathrm{Arg}\ z_k + 2q\pi}{\omega_k}, \ q = -p_{max}, \ldots, p_{max} \end{aligned}$$

8.   END
9.   END
10. END

In step 1, Δ is the stepsize of the parameter φ. In step 7, p_{max} is the number of branches which should be included in the computation. Step 7 is not computationally demanding. We can therefore select p_{max} so large that the computation contains all relevant branches. The generalization to more than two delays is straighforward. It involves a nesting of the outer iteration (step 1) with for-loops of the new free variables φ_k and computing the other delays in step 7 similar to h_1.

III. EXAMPLES

To increase the understanding of Theorem 6 and Theorem 4 we apply them to previously well investigated time-delay systems.

We also apply the theorems to scalar time-delay systems and arrive at analytic expressions for the critical delays which are (to the author‘s knowledge) not known.

The result of the application of the numerical scheme from the previous section are presented in Example 12.

Example 7 (Classical) For the retarded time-delay system

$$\dot{x}(t) = a_0 x(t) + a_1 x(t-h)$$

we have that $\mathbb{B}(s) = 1$ and $\mathbb{A}(s) = a_0 + a_1 e^{-hs}$. Note that we have no free variables for a single delay system. Assumption 1 is always fulfilled because $\mathbb{B}(s) = 0$ has no solution. In the notation of Theorem 6 we have that $A(\vec{\varphi}) = a_0$,$B(\vec{\varphi}) = 1$, $A_m = a_1$ and $B_m = 0$, and the quadratic eigenvalue problem (9) is

$$z^2 a_1 + 2za_0 + a_1 = 0.$$

It has the solutions

$$z = \frac{-a_0 \pm \sqrt{a_0^2 - a_1^2}}{a_1} = \frac{-a_0 \pm i\sqrt{a_1^2 - a_0^2}}{a_1}.$$

The solution z is of unit magnitude if and only if $a_0^2 \leq a_1^2$. If this is not the case, there are no critical delays. The critical frequencies are

$$\omega = -i\frac{a_0 + a_1 z}{1} = \pm\sqrt{a_1^2 - a_0^2}.$$

From Theorem 6 we now have the critical delays

$$\begin{aligned} h &= -\frac{\mathrm{atan}\binom{\pm\mathrm{sgn}(a_1)\sqrt{a_1^2-a_0^2}}{-\mathrm{sgn}(a_1)a_0} + 2p\pi}{\pm\sqrt{a_1^2 - a_0^2}} = \\ &= -\frac{\mathrm{sgn}(a_1)\mathrm{atan}\binom{\sqrt{a_1^2-a_0^2}}{-\mathrm{sgn}(a_1)a_0} \mp 2p\pi}{\sqrt{a_1^2 - a_0^2}}, \end{aligned}$$

where $\mathrm{atan}\binom{a}{b}$ denotes the four-quadrant inverse tangent, i.e., $\mathrm{atan}\binom{a}{b} = \mathrm{Arg}\ (b + ia)$, corresponding to the matlab command `atan2`.

Using the formula

$$\mathrm{atan}\begin{pmatrix}\sqrt{a^2 - b^2}\\ -\mathrm{sgn}(a)b\end{pmatrix} = \mathrm{acos}\left(-\frac{b}{a}\right)$$

we arrive at the final expression

$$h = \frac{-\mathrm{sgn}(a_1)}{\sqrt{a_1^2 - a_0^2}}\left(\mathrm{acos}\left(-\frac{a_0}{a_1}\right) + 2p\pi\right)$$

for any $p \in \mathbb{Z}$.

For the case that $a_1 < 0$, which is necessary to have delay-dependent stability, this is a classical result. See for instance [4] or [16, Section 3.4.1].

Example 8 ([6]) For the neutral time-delay system

$$\dot{x}(t) + b_1\dot{x}(t - h) = a_0 x(t) + a_1 x(t - h)$$

we have that $\mathbb{B}(i\omega) = 1 + b_1 e^{-ih\omega}$ and $\mathbb{A}(i\omega) = a_0 + a_1 e^{-ih\omega}$. If we assume that $a_1 \neq b_1 a_0$, then $\mathbb{A}(i\omega) = 0$ and $\mathbb{B}(i\omega) = 0$ do not have any roots in common and Assumption 1 holds. Moreover, we have that $A(\vec{\varphi}) = a_0$,$B(\vec{\varphi}) = 1$, $A_m = a_1$ and $B_m = b_1$, and the quadratic eigenvalue problem (9) is now

$$z^2(a_1 + b_1 a_0) + 2z(a_0 + a_1 b_1) + a_0 b_1 + a_1 = 0$$

with the solutions

$$\begin{aligned} z &= \frac{-(a_0 + a_1 b_1) \pm \sqrt{(a_0 + a_1 b_1)^2 - (a_0 b_1 + a_1)^2}}{a_0 b_1 + a_1} = \\ &= \frac{-(a_0 + a_1 b_1) \pm i\sqrt{(a_0 b_1 + a_1)^2 - (a_0 + a_1 b_1)^2}}{a_0 b_1 + a_1}. \end{aligned} \quad (11)$$

The time-delay system has critical delays if and only if $(a_0 b_1 + a_1)^2 > (a_0 + a_1 b_1)^2$, which implies that $b_1 \neq \pm 1$. The crossing frequencies are

$$\begin{aligned} \omega &= \frac{a_0 + a_1 z}{i(1 + b_1 z)} = -\frac{a_0 + a_1 \mathrm{Re}\ z}{b_1 \mathrm{Im}\ z} = \\ &\pm \frac{a_1^2 - a_0^2}{\sqrt{(a_0 b_1 + a_1)^2 - (a_0 + a_1 b_1)^2}} = \\ &= \mp\sqrt{\frac{a_1^2 - a_0^2}{1 - b_1^2}}. \end{aligned} \quad (12)$$

This formula is found in [6]. The critical delays are

$$\begin{aligned} h &= \frac{-\mathrm{Arg}\ z + 2p\pi}{\omega} = \\ &- \rho\sqrt{\frac{1 - b_1^2}{a_1^2 - a_0^2}}\left(\mathrm{acos}\left(-\frac{a_0 + a_1 b_1}{a_0 b_1 + a_1}\right) + 2p\pi\right) \end{aligned} \quad (13)$$

where $\rho = \mathrm{sgn}(a_0 b_1 + a_1)$. A similar formula is contained in [16, Section 3.4.2].

Example 9 For the neutral two-delay system

$$b_1\dot{x}(t - h_1) + b_2\dot{x}(t - h_2) = x(t)$$

we have that $A(\varphi) = 1$, $A_m = 0$, $\mathbb{B}(\varphi) = b_1 e^{-i\varphi}$, $B_m = b_2$. Assumption 1 is always fulfilled as $\mathbb{A}(s) = 1 \neq 0$. The quadratic eigenproblem corresponding to (9) is

$$z^2 b_2 + z 2 b_1 \cos(\varphi) + b_2 = 0$$

and

$$\begin{aligned} z &= \frac{-b_1\cos(\varphi) \pm \sqrt{b_1^2\cos^2(\varphi) - b_2^2}}{b_2} \\ &= \frac{-b_1\cos(\varphi) \pm i\sqrt{b_2^2 - b_1^2\cos^2(\varphi)}}{b_2}. \end{aligned} \quad (14)$$

The parametrization yields proper critical delays only if we require that φ fulfills $b_2^2 \geq b_1^2\cos^2(\varphi)$. The crossing frequencies are

$$\begin{aligned} \omega &= \frac{1}{i(b_1 e^{-i\varphi} + (-b_1 \pm i\sqrt{b_2^2 - b_1^2\cos^2(\varphi)}))} = \\ &= \frac{1}{b_1\sin(\varphi) \mp \sqrt{b_2^2 - b_1^2\cos^2(\varphi)}}. \end{aligned} \quad (15)$$

Hence, a parametrization of the critical delays are the functions

$$\begin{aligned} h_1 &= \frac{\varphi + 2p\pi}{\omega} = \\ &\left(b_1\sin(\varphi) \mp \sqrt{b_2^2 - b_1^2\cos^2(\varphi)}\right)(\varphi + 2p\pi) \end{aligned}$$

$$\begin{aligned} h_2 &= \frac{-\mathrm{Arg}\ z + 2q\pi}{\omega} = \\ &\left(b_1\sin(\varphi) \mp \sqrt{b_2^2 - b_1^2\cos^2(\varphi)}\right)\Bigg(2q\pi + \\ &\mp \mathrm{sgn}(b_2)\mathrm{acos}\left(-\frac{b_1\cos(\varphi)}{b_2}\right)\Bigg), \end{aligned}$$

for any $p, q \in \mathbb{Z}$.

Example 10 We now consider the neutral system corresponding to Example 8 where the two delays are not necessarily equal, i.e.,

$$\dot{x}(t) + b_1\dot{x}(t - h_1) = a_0 x(t) + a_2 x(t - h_2).$$

We have that $b(\varphi) = 1 + b_1 e^{-i\varphi}$, $b_m = 0$, $a(\varphi) = a_0$ and $a_m = a_2$. The quadratic eigenproblem/equation corresponding to (9) is

$$z^2 a_2(1+b_1 e^{i\varphi}) + 2z a_0(1+b_1\cos(\varphi)) + a_2(1+b_1 e^{-i\varphi}) = 0.$$

After many simple manipulations, which we leave out for brevity, we arrive at an expression for the critical frequencies.

$$\omega(\varphi) = \frac{a_0 + a_2 \text{Re } z}{b_1 \sin(\varphi)} = \tag{16}$$

$$= \frac{1}{1+2b_1\cos(\varphi)+b_1^2}\Bigg(b_1\sin(\varphi) + \\ \mp \sqrt{(a_2^2-a_0^2)(1+2b_1\cos(\varphi)+b_1^2)+b_1^2a_0^2\sin^2(\varphi)}\Bigg) \tag{17}$$

It is clear that even for examples like this, which may seem simple, the explicit real trigonometric expression (17) is too large to easily identify properties of the critical frequencies.

For brevity we only express the critical delays using the complex expression.

$$\begin{aligned} h_1 =& \frac{\varphi+2p\pi}{\omega(\varphi)} = \frac{b_1\sin(\varphi)(\varphi+2p\pi)}{a_0+a_2\text{Re } z} \\ h_2 =& \frac{-\text{Arg } z+2q\pi}{\omega(\varphi)} = \frac{b_1\sin(\varphi)(-\text{Arg } z+2q\pi)}{a_0+a_2\text{Re } z} \end{aligned}$$

Example 11 (From [10] and [19]) With this example we show how one can find the critical delays of some multi-dimensional systems analytically. The commonly occurring example,

$$\dot{x}(t) - 0.1\dot{x}(t-h_1) = \\ \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix} x(t) + \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} x(t-h_2),$$

can be decoupled because all matrices are triangular. Hence, the spectrum is the union of the spectrum of the two decoupled systems

$$\begin{aligned} \dot{y}_1(t) - 0.1\dot{y}_1(t-h_1) &= -2y_1(t) - y_1(t-h_2) \\ \dot{y}_2(t) - 0.1\dot{y}_2(t-h_1) &= -0.9y_2(t) - y_2(t-h_2) \end{aligned}$$

If we let $h_1 = h_2$ we can apply the result of Example 8. Here the system corresponding to y_1 does not have any critical delays. For y_2 we have $a_0 = -0.9$, $b_0 = 1$, $b_1 = -0.1$, $a_1 = -1$ $\rho = \text{sgn}(a_0 b_1 + a_1) = \text{sgn}(-0.1a_0 - 1) = -1$. From (13) the critical delays are

$$h = \sqrt{\frac{1-0.1^2}{1-0.9^2}}\left(\text{acos}\left(\frac{-0.8}{0.91}\right)+2p\pi\right) = \\ = 3\sqrt{\frac{11}{19}}\left(\text{acos}\left(\frac{-80}{91}\right)+2p\pi\right), \tag{18}$$

which is an exact expression. For $p = 0$ we have $h \approx 6.0372$ and $\omega \approx \pm 0.4381$.

For the case that $h_1 \neq h_2$ we can apply the formula from Example 10 (or directly apply the numerical scheme) to produce the critical curves. The results are shown in Figure 1.

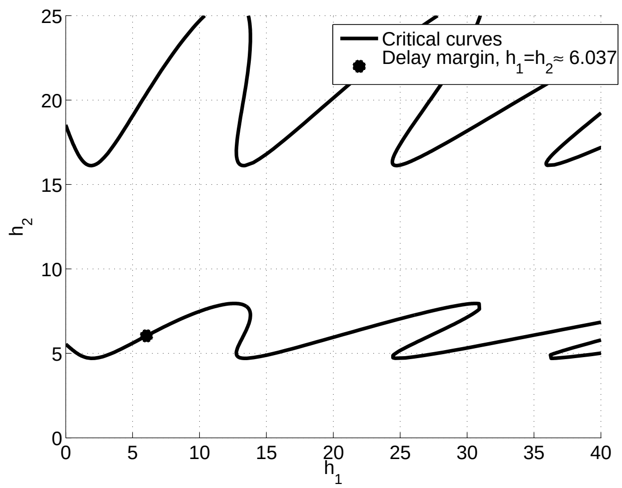

Fig. 1. Critical curves for Example 11

Example 12 In this example we apply the numerical scheme to an example for which we believe there is no simple analytical expression.

Consider the delay-free-feedback version of the example in [15, Section 5],

$$\dot{x}(t) + B_1\dot{x}(t-h_1) + B_2\dot{x}(t-h_2) = A_0 x(t),$$

where

$$\begin{aligned} B_1 &= -\begin{pmatrix} 0 & 0.2 & -0.4 \\ -0.5 & 0.3 & 0 \\ 0.2 & 0.7 & 0 \end{pmatrix}, \\ B_2 &= -\begin{pmatrix} -0.3 & -0.1 & 0 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.4 \end{pmatrix}, \\ A_0 &= \begin{pmatrix} -4.8 & 4.7 & 3 \\ 0.1 & 1.4 & -0.4 \\ 0.7 & 3.1 & -1.5 \end{pmatrix} + B^T K, \\ B &= \begin{pmatrix} 0.3 & 0.7 & 0.1 \end{pmatrix}^T, \\ K &= \begin{pmatrix} -2.593 & 1.284 & 1.826 \end{pmatrix}^T. \end{aligned}$$

Assumption 1 holds because A_0 is not singular, and hence $\mathbb{A}(s)v = A_0 v = 0$ has no solutions. The critical curves computed by the numerical scheme are given in Figure 2.

IV. CONCLUSIONS AND FUTURE WORKS

We have presented a new method to compute the delays of neutral multiple-delay time-delay system such that it has a purely imaginary eigenvalue. This is done by, in a general setting, stating an equivalence theorem between the spectrum and the solutions of a constructed matrix condition.

For the case that the delays are commensurate, we show that this equivalence theorem reduces to results which are

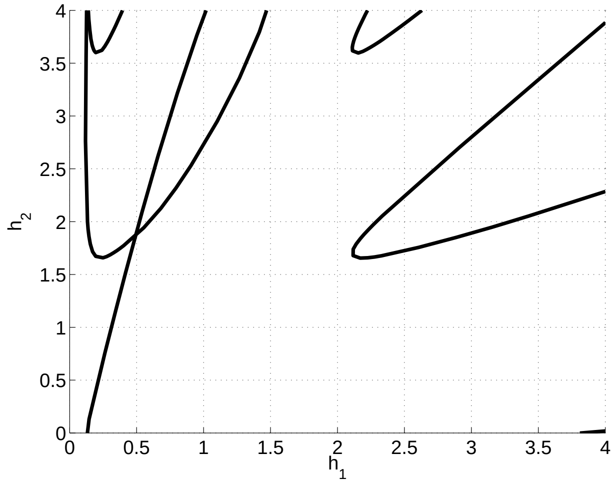

Fig. 2. Critical curves for Example 12

very similar to parts of the method of Fu, Niculescu and Chen[6]. For the incommensurate case, we find a way to parametrize the curves along which the system has a purely imaginary eigenvalue.

It is worth noting that most of the results are based on a well-posedness assumption which is easy to check for many problems. Even though this assumption is well suited for this work, the work could possible be brought closer to other characterizations of neutral systems by finding a connection to the concept of strong stability.

In the examples section we show how analytical expressions can be found for some problems and that for systems of larger dimensions a numerical scheme can be applied to compute the critical curves.

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