Implicit time integration using summation-by-parts operators and weak SAT penalty conditions

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First order initial value problem:

$$\begin{aligned} u_t + F(t,u) &= 0, \quad 0 < t \leq T \\ u(0) &= f \end{aligned}$$

Global discrete solution vector:

$$\vec{U} = (U_o \ U_1 \ \dots \ U_N)^T \approx (u(0) \ u(t_1) \ \dots \ u(T))^T$$

$SBP - SAT$ discretization:

$$(P^{-1}Q \otimes I)\vec{U} + \begin{pmatrix} F(t_0, U_0) \\ \vdots \\ F(t_N, U_N) \end{pmatrix} = P^{-1}\sigma \vec{e_0} \otimes (U_0 - f)$$

Example in one dimension: $u_t + \lambda u = 0 \quad \rightarrow$

$$P^{-1}Q\vec{U} + \lambda\vec{U} = P^{-1}\sigma(U_0 - f)\vec{e_0}. \qquad (1)$$

σ is called the *Simultaneous-Approximation-Term* (SAT) penalty coefficient, and the first derivative operator $P^{-1}Q$ has the *summation-by-parts* (SBP) properties:

- $Q + Q^T = diag(-1, 0, \dots, 0, 1)$.
- $P = P^T > 0$ defines a quadrature rule.

Multi-stage version ($r + 1$ stages): replace $\vec{U}_k$ with $\vec{U}_k = (U_k^0, \dots, U_k^r)$.

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The classical *SBP* operators are based on central finite difference schemes, e.g.

$$P = \triangle t \begin{bmatrix} \frac{1}{2} & & \\ & 1 & \\ & & \ddots \end{bmatrix} \quad Q = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & & \\ -\frac{1}{2} & 0 & \frac{1}{2} & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

Also: generalized SBP operators can be constructed based on *any* quadrature rule P (Férnandes et al 2014). This leads to schemes very similar to the classical fully implicit Runge-Kutta methods.

Properties:

- The summation-by-parts properties lead *automatically* to clean, optimally sharp energy estimates. E.g. (1) $\rightarrow$ ($\sigma = -1$)

$$|U_N|^2 + 2Re(\lambda)||\vec{U}||_P^2 = |f|^2 - |U_0 - f|^2.$$

- A-stability, L-stability. For diagonal P also B-stability.
- Convergence: Same order as the quadrature rule P (a consequence of dual consistency).
- Stiff convergence: Same order as the operator $P^{-1}Q$ (limited by the boundary accuracy).

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Ongoing/future research:

- Efficient implementation for stiff problems in fluid dynamics.
- SBP-SAT discretizations of second order initial value problems, e.g. the wave equation.

References:

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