Exchange and exclusion for non-abelian anyons

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$$0 \to \infty \to$$ 

Main references:

[LQ] D.L., Viktor Qvarfordt, arXiv:2009.12709

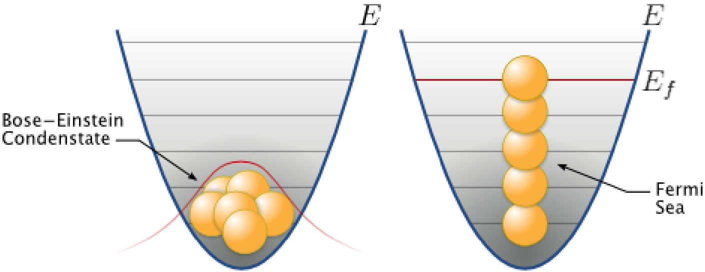
[Q] Viktor Qvarfordt, MSc thesis, KTH/SU, 2017

[L] D.L., lecture notes 2017-19, arXiv:1805.03063, LMU

Outline

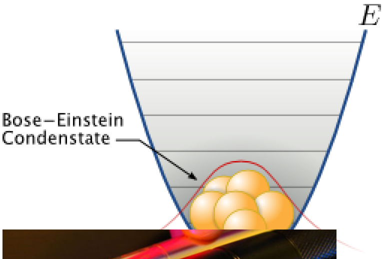
1. Quantum statistics in 2D vs. 3D
2. Anyon models: algebraic - geometric - magnetic
3. Exchange vs. exclusion ("statistical repulsion")
4. Examples: Abel, Fibonacci and Ising
5. (Algebraic anyon models)

Quantum statistics (in 3D)

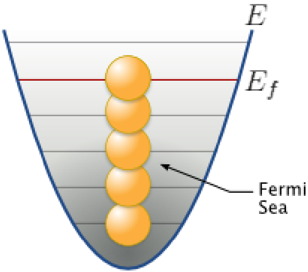


Cold **bosons** Cold **fermions**

Quantum statistics (in 3D)

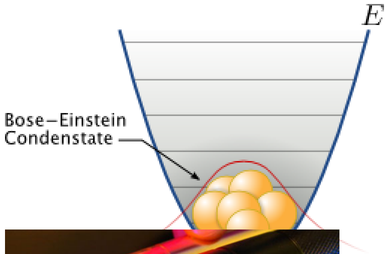


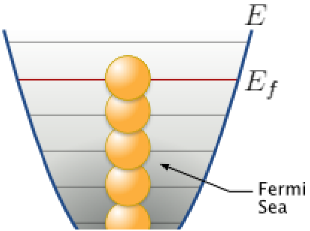
force carriers (fluffy/degenerate)

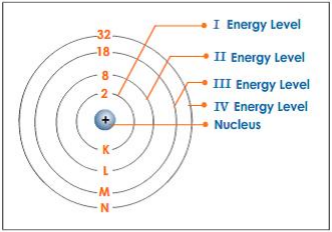


Cold **fermions**

Quantum statistics (in 3D)

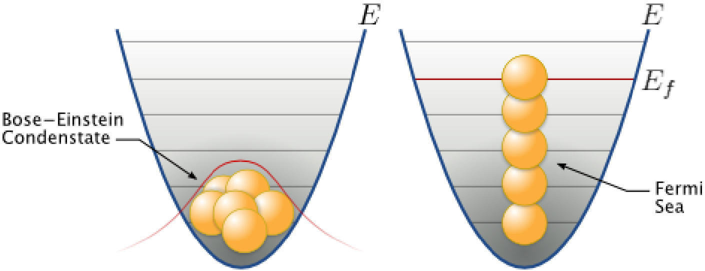






force carriers (fluffy/degenerate) matter (stable/non-degenerate)

Quantum statistics done 'wrong'



Cold **bosons** Cold **fermions**

Observable: $|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N)|^2, \quad \mathbf{x}_j \in \mathbb{R}^3$ ✓

Exchange symmetry: representation $\rho\colon S_N \to \mathrm{U}(1)$

$+1 : \rho = 1 \quad \Rightarrow$ **bosons** (Bose–Einstein statistics)

$-1 : \rho = \mathrm{sign} \quad \Rightarrow$ **fermions** (Fermi–Dirac statistics)

Quantum statistics done 'right'

The **configuration space** of N **distinguishable** particles: $(\mathbb{R}^d)^N$
The configuration space of N **identical** particles in $\mathbb{R}^d$: [Gibbs]

$$\mathcal{C}^N := \left((\mathbb{R}^d)^N \setminus \triangle^N \right) \Big/ S_N \cong \{N\text{-point subsets of } \mathbb{R}^d\}$$

Distinct points by removal of the **diagonals**:

$$\triangle^N := \{(\mathbf{x}_1, \ldots, \mathbf{x}_N) \in (\mathbb{R}^d)^N : \exists\, j \neq k \text{ s.t. } \mathbf{x}_j = \mathbf{x}_k\}$$

Exchanges of particles are continuous **loops** in $\mathcal{C}^N$:

$$\{\text{loops in } \mathcal{C}^N \text{ modulo homotopy}\} = \pi_1(\mathcal{C}^N) = \begin{cases} 1, & d = 1, \\ B_N, & d = 2, \\ S_N, & d \geq 3. \end{cases}$$

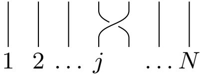
[Leinaas, Myrheim '77; Goldin, Menikoff, Sharp '81; Wilczek '82]

The braid group

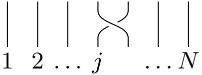
B_N is the **braid group** on N strands:

$$B_N = \left\langle \sigma_1, \ldots, \sigma_{N-1} : \sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1},\ \sigma_j \sigma_k = \sigma_k \sigma_j \right\rangle_{|j-k|>1}$$

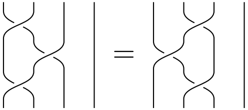
σ_j :



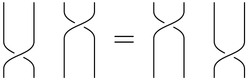
σ_j^{-1} :



Examples in B_4:



$\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$



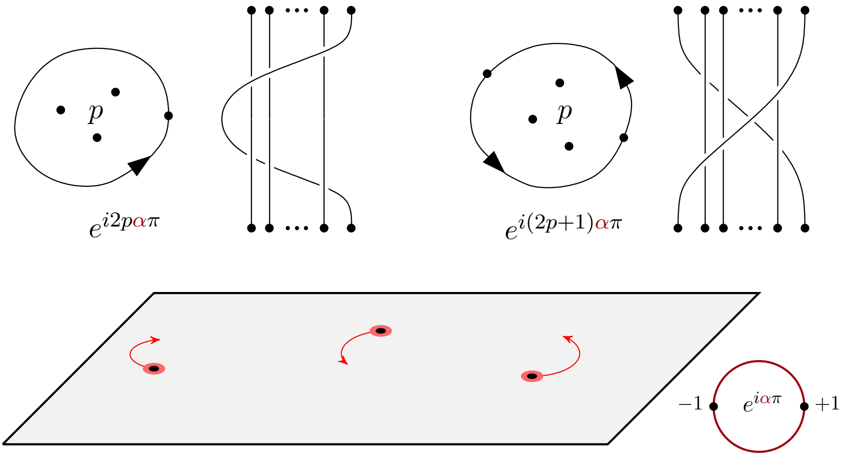
$\sigma_1 \sigma_3 = \sigma_3 \sigma_1$

If we add the relations $\sigma_j^2 = 1$ we obtain the **permutation group** S_N

Quantum statistics in 2D

Different in 2D!

Quantum statistics in 2D



Exchange symmetry $\rho\colon B_N \to \mathrm{U}(1)$ any phase $\Rightarrow$ "**anyons**"

Abelian vs. non-abelian representations

An N-**anyon wave function** is *l*ocally a map $\Psi \colon \Omega \subseteq \mathcal{C}^N \to \mathcal{F}$,
$\mathcal{F}$ Hilbert space of 'internal degrees of freedom' on which B_N acts:

$$\rho \colon B_N \to \mathrm{U}(\mathcal{F})$$

$$\langle \Phi, \Psi \rangle = \int_{\mathcal{C}^N} \langle \Phi(X), \Psi(X) \rangle_{\mathcal{F}} \, dX, \quad \|\Psi\|^2 = \int_{\mathcal{C}^N} |\Psi|^2_{\mathcal{F}} = 1$$

Irreducible abelian anyons: $\mathcal{F} = \mathbb{C}$,

$$\rho(\sigma_j) = e^{i\alpha\pi}$$

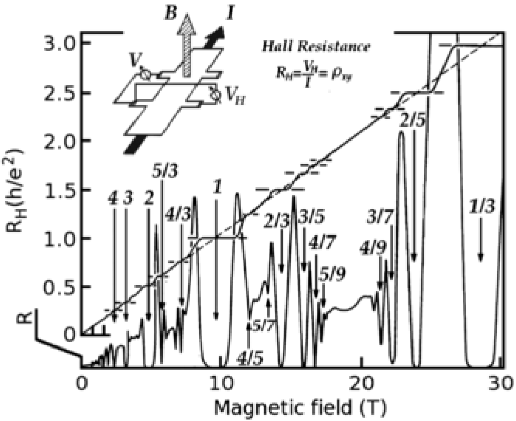
Reducible abelian anyons: $\mathcal{F} = \mathbb{C}^D$, $D > 1$,

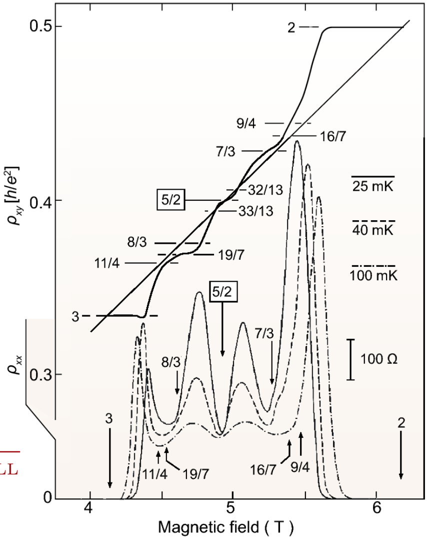
$$\rho(\sigma_j) = S^{-1} \mathrm{diag}(e^{i\alpha_1\pi}, \ldots, e^{i\alpha_D\pi}) S$$

Non-abelian anyons: $\mathcal{F} = \mathbb{C}^D$, $D > N-3$,

$$\rho(\sigma_j)\rho(\sigma_k) \neq \rho(\sigma_k)\rho(\sigma_j) \qquad \text{for some } j \neq k.$$

Application: Fractional quantum Hall effect at $\nu = 5/2$





conductance$_\perp$ $\sim$ filling factor $= \frac{N}{\dim_{\mathrm{LL}}}$

$\frac{1}{\rho_{xy}} \left[\frac{e^2}{h} \right] \sim \nu = \frac{p}{q}$, usually q **odd**

[Willett, Eisenstein, Störmer, Tsui, Gossard, English '87]

Application: Topological quantum computing

PRL **103**, 160501 (2009) PHYSICAL REVIEW LETT

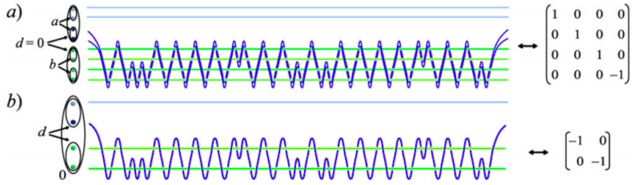


FIG. 2 (color online). "Effective qubit" gate construction for $\mathfrak{su}(2)_3$ anyons. Part (a) shows a braid in which a pair of anyons from the control qubit (blue) weaves around pairs of anyons in the target qubit (green). When either qubit is in the state $|0\rangle$, this braid produces the identity operation. When both control and target qubits are in the state $|1\rangle$, the braid consists of weaving a

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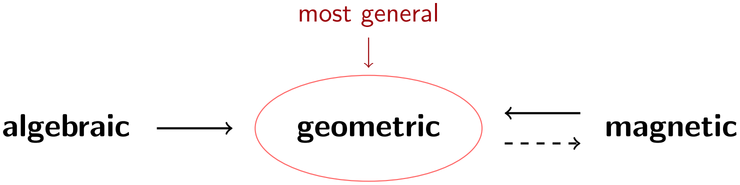
[Hormozi, Bonesteel, Simon]

Anyon (/plekton/nonabelion) models

algebraic **geometric** **magnetic**

Anyon (/plekton/nonabelion) models



$$\rho\colon B_N \to \mathrm{U}(D)$$

Anyon (/plekton/nonabelion) models

most computational — most general — most realistic

algebraic ⟶ **geometric** ⟵ (⇢) **magnetic**

statistics transmutation

$$\rho\colon B_N \to \mathrm{U}(D)$$

algebraic:

Moore, Seiberg, Witten '89
Fröhlich et al. '90
Kitaev '97-'06-
+Freedman, Wang '02-
Bonderson '07
+Gurarie, Nayak '11
...
DL, Qvarfordt '17-

geometric:

Leinaas, Myrheim '77
Goldin, Menikoff, Sharp '81
...
Dowker '85
Mueller, Doebner, '93
Mund, Schrader '95
Dell'Antonio, Figari, Teta '97
Harrison, Keating, Robbins '11
Maciazek, Sawicki '19

magnetic:

Wilczek '82, Wu '84
+Arovas, Schrieffer '84
Moore, Read '91, +Rezayi '99
Verlinde '91, Lee, Oh '94 (NACS)
...
Mancarella, Trombettoni, Mussardo '13
DL, Solovej '13,'14
+Rougerie, Larson, Seiringer, Correggi, Duboscq '15-
Yakaboylu et al '19-

Geometric vs. magnetic anyon models

Lengthy discussion on mathematical definitions...

$$T = \sum_{j=1}^{N} \mathbf{p}_j^2$$

...$\Rightarrow$ $\hat{T}_\rho$ quantization of kinetic energy, labeled by $\rho \colon B_N \to \mathrm{U}(D)$

For abelian anyons, $\hat{T}_\rho$ is equivalent to the **magnetic** operator

$$\hat{T}_\alpha = \sum_{j=1}^{N} \left(-i\nabla_{\mathbf{x}_j} + \alpha \sum_{k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_k)^\perp}{|\mathbf{x}_j - \mathbf{x}_k|^2} \right)^2$$

acting on the **bosonic** Hilbert space $L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$.

Geometric anyon models: definition

Free anyons: demand that *locally*, i.e. on any topologically *trivial* open subset $\Omega \subseteq \mathcal{C}^N$, the particles behave like usual free non-relativistic *distinguishable* particles (Schrödinger rep.)

$$\hat{T}_\Omega = \sum_{j=1}^{N} (-i\nabla_{\mathbf{x}_j})^2 \qquad \text{on} \qquad \Psi \in C_c^\infty(\Omega; \mathcal{F}) \subseteq L^2(\Omega; \mathcal{F}),$$

with some fiber (local/internal) Hilbert space $\mathcal{F} \cong \mathbb{C}^D$.

Fiber bundles: *globally* on $\mathcal{C}^N$ we should consider a hermitian vector bundle $E \to \mathcal{C}^N$ with fiber $\mathcal{F}$, endowed with a (locally) *flat* connection $\mathcal{A}$. **Wave functions** Ψ are L^2-sections of this bundle.

Theorem: There is a 1-to-1 correspondence between such flat bundles and representations $\rho\colon \pi_1(\mathcal{C}^N) = B_N \to \mathrm{U}(\mathcal{F})$.

Definition: A **geometric N-anyon model** is such a rep. $\rho \Rightarrow \hat{T}_\rho$

Geometric anyon models: alt. definition

Consider the **covering space**, i.e. the space of paths from a fixed base point modulo homotopy equivalences,

$$\tilde{\mathcal{C}}^N \to \mathcal{C}^N \qquad \text{with fiber } B_N.$$

An N**-anyon wave function** $\Psi \in L^2_\rho$ is a ρ-equivariant function

$$\Psi \colon \tilde{\mathcal{C}}^N \to \mathcal{F}, \quad \Psi(\gamma.\tilde{X}) = \rho([\gamma])\Psi(\tilde{X}), \quad \gamma \text{ loop in } \mathcal{C}^N,$$

$$\langle \Phi, \Psi \rangle_{L^2_\rho} := \int_{\mathcal{C}^N} \langle \Phi(\tilde{X}), \Psi(\tilde{X}) \rangle_{\mathcal{F}} \, dX.$$

The **Sobolev space** H^1_ρ is the closure of smooth ρ-equivariant functions $\Psi \colon \tilde{\mathcal{C}}^N \to \mathcal{F}$, with the projection of $\operatorname{supp} \Psi$ to $\mathcal{C}^N$ compact, w.r.t.

$$\langle \Phi, \Psi \rangle_{H^1_\rho} := \int_{\mathcal{C}^N} \left(\langle \Phi(\tilde{X}), \Psi(\tilde{X}) \rangle_{\mathcal{F}} + \langle \nabla\Phi(\tilde{X}), \nabla\Psi(\tilde{X}) \rangle_{\mathcal{F}^{2N}} \right) dX.$$

The associated operator is $\hat{T}_\rho \geq 0$ (Friedrichs extension)

Magnetic anyon models & transmutability

Sections Ψ are ρ-equivariant functions $\Psi_\rho \colon \tilde{\mathcal{C}}^N \to \mathcal{F}$,

$$\Psi_\rho(\gamma.\tilde{X}) = \rho([\gamma])\Psi_\rho(\tilde{X}), \quad \gamma \text{ loop in } \mathcal{C}^N.$$

Local **gauge transformation** $\Psi \to u\Psi$ where $u \colon \Omega \to \mathrm{U}(\mathcal{F})$:

$$\hat{T}_\Omega = -(\nabla + \mathcal{A})^2, \qquad \mathcal{A}(\tilde{X}) := u(\tilde{X})^{-1}\nabla u(\tilde{X})$$

If $u_\rho \colon \tilde{\mathcal{C}}^N \to \mathrm{U}(\mathcal{F})$ is a *global* section of the associated principal bundle,

$$u_\rho(\gamma.\tilde{X}) = \rho([\gamma])u_\rho(\tilde{X}), \quad \gamma \text{ loop in } \mathcal{C}^N,$$

then we have a tranformation to *trivial* bundle $\Psi_1 \in L^2_{\mathrm{sym}}(\mathbb{R}^{2N};\mathcal{F})$:

$$\Psi_\rho = u_\rho \Psi_1 \quad \Leftrightarrow \quad \Psi_1 = u_\rho^{-1}\Psi_\rho$$

Definition: A **transmutable N-anyon model** is an N-anyon model $\rho \colon B_N \to \mathrm{U}(\mathcal{F})$ such that its corresponding flat principal bundle $P \to \mathcal{C}^N$ is topologically trivial.

Magnetic anyon models & transmutability

So, *transmutable* models ρ may equivalently be described using **bosons** (or fermions) with gauge potentials $\mathcal{A}\colon \mathbb{R}^{2N} \setminus \mathbb{A}^N \to \mathfrak{u}(\mathcal{F})$.

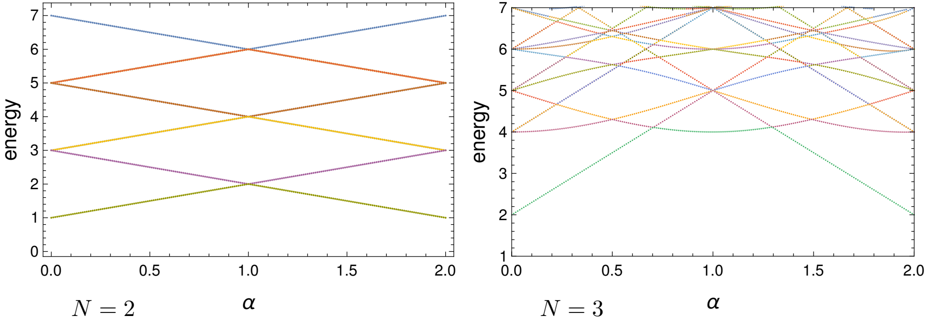
Obstacle: Only some of rep. theory of B_N and topology known.

Theorem: Any *abelian* model is transmutable. [Dowker '85, Mund, Schrader '95]

Theorem: Any rep $\rho\colon B_N \to \mathrm{U}(D)$, $N > 6$, is abelian if $D < N - 2$. [Formanek '96]

- In the non-abelian case typically $D \sim c^N$ for some $c > 0$.
- NACS: $\rho \sim \rho_1^{\otimes N}$ is transmutable.
- Transmutability of a bundle $E \to \mathcal{C}^N$ improves with $E \oplus E$...

How do anyons actually behave?



$N = 2$ $N = 3$

[Leinaas, Myrheim '77; Wilczek et al '82,'85;
Murthy, Law, Brack, Bhaduri, '91; Sporre, Verbaarschot, Zahed, '91,'92;
Correggi et al '19; Yakaboylu et al '19]

Exchange vs. exclusion

Bosons, $\rho(\sigma_j) = +1$,

$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = +\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$, $j \neq k$,

may be **independent identically distributed** in a single state ψ_1:

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_N) = \prod_{j=1}^{N} \psi_1(\mathbf{x}_j).$$

Fermions, $\rho(\sigma_j) = -1$,

$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = -\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_N)$, $j \neq k$,

obey **Pauli's exclusion principle:**

$$\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N) = 0 \quad \text{if } \mathbf{x}_j = \mathbf{x}_k,\ j \neq k,$$

or, generally, $\Psi \in \bigwedge^N L^2(\mathbb{R}^2)$, spanned by **"Slater determinants"**

$$(\psi_1 \wedge \ldots \wedge \psi_N)(\mathbf{x}_1, \ldots, \mathbf{x}_N) := \frac{1}{\sqrt{N!}} \det \left[\psi_j(\mathbf{x}_k) \right]_{j,k}.$$

Exchange vs. exclusion

Consider $\hat{T} = -\Delta = -\sum_{j=1}^{N} \Delta_{\mathbf{x}_j}$ on the unit cube $[0,1]^{2N}$.
Ground-state energy $E_N = \inf \operatorname{spec} \hat{T}$ for bosons:

$$E_N = \lambda_1 N = \ 0 \quad \text{alt.} \quad 2\pi^2 N$$

Ground-state energy for fermions (**Weyl's law**):

$$E_N = \sum_{j=1}^{N} \lambda_j \sim 2\pi N^2 + o(N^2)$$

$\Rightarrow$ **Thomas-Fermi approximation**:

$$E_N = \inf_{\Psi \in H^1_{\mathrm{asym}} : \int_{\mathbb{R}^{2N}} |\Psi|^2 = 1} \langle \Psi, \hat{T}\Psi \rangle \approx \inf_{\varrho \geq 0 : \int_{\mathbb{R}^2} \varrho = N} \int_{\mathbb{R}^2} 2\pi \varrho(\mathbf{x})^2 \, d\mathbf{x},$$

What about anyons?

[Canright, Johnson '94 "Fractional statistics: α to β"]

Exchange vs. exclusion: anyons

Take a **local approach** to exchange and exclusion. [DL, Solovej '13]

Statistical repulsion manifests in three ways (at least):

1. effective *scalar* pairwise repulsion $\Rightarrow \Psi \to 0$ at $\triangle^N$
2. local exclusion principle: $E_N \geq \pi^2(N-1)_+$
3. degeneracy pressure, ex. Thomas-Fermi or Lieb-Thirring

Exchange vs. exclusion: anyons

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3. degeneracy pressure, ex. Thomas-Fermi or Lieb-Thirring

Given $\rho = \rho_N$, consider '**exchange operator**' $(2p+1$ braidings):

$$U_p := \rho(\sigma_1\sigma_2\ldots\sigma_p\sigma_{p+1}\sigma_p\ldots\sigma_2\sigma_1), \quad p \in \{0,1,\ldots,N-2\}$$

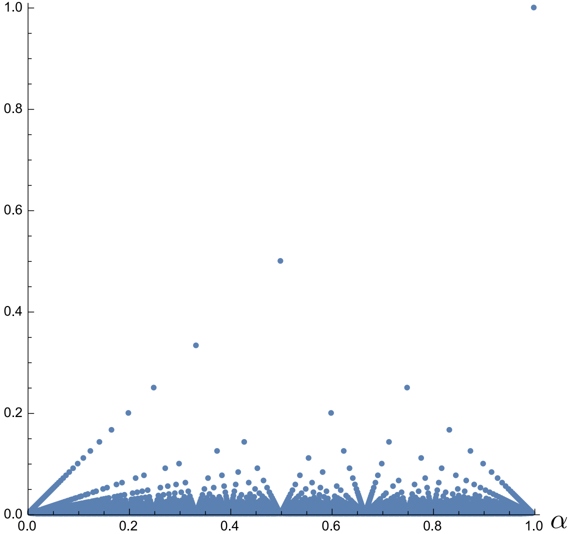
and '**exchange parameters**' for p enclosed or n involved particles

$$\beta_p := \min\{\beta \in [0,1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_p\}$$

$$\alpha_n := \min_{p\in\{0,1,2,\ldots,n-2\}} \beta_p, \quad n \in \{0,1,\ldots,N\}.$$

The odd-numerator Thomae/'popcorn' function

$$\alpha_{N\to\infty} = \inf_{p,q\in\mathbb{Z}} |(2p+1)\alpha - 2q| \quad \text{(abelian)}$$



The homogeneous ideal anyon gas

Theorem (D.L., Solovej, Larson, Seiringer, Qvarfordt)

For any sequence of N-anyon models $\rho_N \colon B_N \to \mathrm{U}(\mathcal{F}_N)$ with n-anyon exchange parameters $\alpha_n = \alpha_n(N) \in [0,1]$, $2 \leq n \leq N$, we have the uniform bounds

$$\frac{1}{4}C(\rho_N)\big(1 - O(N^{-1})\big) \leq E_N/N^2 \leq 2\pi^2\big(1 + O(N^{-1/2})\big),$$

where

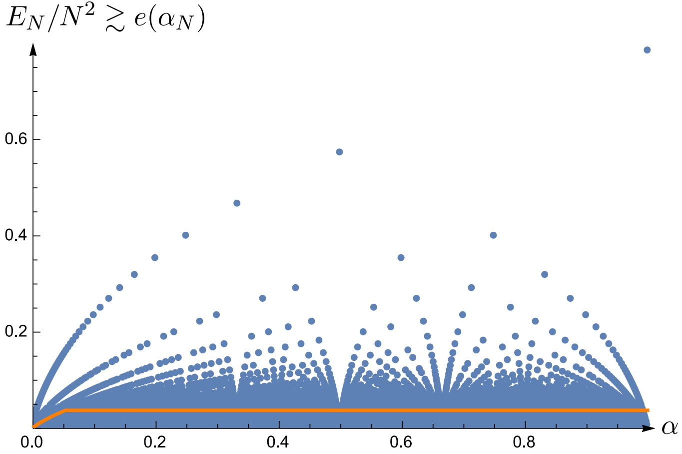
$$C(\rho_N) := \max\{c(\alpha_2), e(\alpha_N)\}$$

$$c(\alpha_2) := \frac{1}{4}\min\{e(\alpha_2), 0.147\},$$

and $e(\alpha)$ is a 2-particle energy, $\alpha \in [0,1]$

$$\alpha/3 \leq e(\alpha) \leq 4\pi\alpha(1+\alpha), \quad e(\alpha) = 4\pi\alpha + O(\alpha^{4/3}).$$

The homogeneous ideal anyon gas



Numerical lower bounds for $e(\alpha) = 4\pi\alpha + O(\alpha^{4/3})$,
at $\alpha = \alpha_{N\to\infty}$ versus $c(\alpha_2) = \frac{1}{4}\min\{e(\alpha_2), 0.147\}$

Statistical repulsion $\Leftarrow$ Poincaré inequality

Hardy inequality for fermions in $\mathbb{R}^d$: [Hoffmann-Ostenhof^2, Laptev, Tidblom '08]

$$\hat{T}_{\rho=\mathrm{sign}} \geq \frac{d^2}{N} \sum_{1 \leq j < k \leq N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

Poincaré for fermions: $u(-\omega) = -u(\omega)$, $\omega \in \mathbb{S}^{d-1}$ relative angles

$$\int_{\mathbb{S}^{d-1}} |\nabla_\omega u|^2 \, d\omega \;\geq\; (d-1) \int_{\mathbb{S}^{d-1}} |u|^2 \, d\omega$$

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^{2\pi} |u'|^2 \, d\varphi \;\geq\; \int_0^{2\pi} |u|^2 \, d\varphi$$

Statistical repulsion $\Leftarrow$ Poincaré inequality [2 anyons]

Poincaré for 2D fermions: $u(\varphi + \pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **abelian anyons**: $u(\varphi + \pi) = e^{i\pi\alpha} u(\varphi)$, $\alpha \in (-1, 1]$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \alpha^2 \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **non-abelian anyons**: $u(\varphi + \pi) = U_0 u(\varphi)$, $U_0 \in \mathrm{U}(\mathcal{F})$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \beta_0^2 \int_0^\pi |u|^2 \, d\varphi$$

$$\beta_0 := \min\{\beta \in [0,1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_0\}$$

$\Rightarrow$ **statistical repulsion** for a pair of anyons if $\beta_0 > 0$

Statistical repulsion $\Leftarrow$ Poincaré inequality $[2+p$ anyons$]$

Poincaré for 2D fermions: $u(\varphi+\pi) = -u(\varphi)$

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **abelian anyons**: $u(\varphi+\pi) = e^{i\pi(2p+1)\alpha} u(\varphi)$,

$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \min_{q\in\mathbb{Z}} |(2p+1)\alpha - 2q|^2 \int_0^\pi |u|^2 \, d\varphi$$

Poincaré for **non-abelian anyons**: $u(\varphi+\pi) = U_p u(\varphi)$, $U_p \in \mathrm{U}(\mathcal{F})$

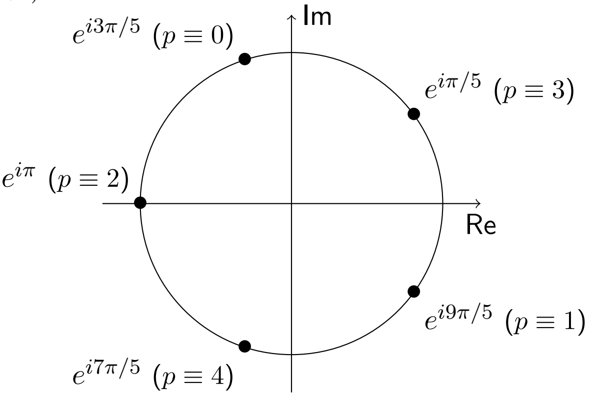
$$\int_0^\pi |u'|^2 \, d\varphi \;\geq\; \beta_p^2 \int_0^\pi |u|^2 \, d\varphi$$

$$\beta_p := \min\{\beta \in [0,1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_p\}$$

$\Rightarrow$ **statistical repulsion** (Hardy, extensivity, LT) for anyons if $\beta_p > 0$

Abelian anyons, ex. $\alpha = 3/5$

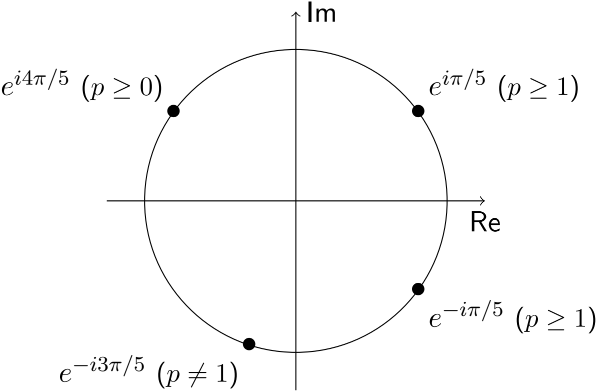
$U_p = e^{(2p+1)i\pi\alpha}$



$\Rightarrow$ **Poincaré inequality** with $\beta_p = \begin{cases} 3/5, & p = 0, 4, 5, \ldots \\ 1/5, & p = 1, 3, \ldots \\ 1, & p = 2, 7, \ldots \end{cases}$

Fibonacci anyons: Exchange eigenvalues

$$U_p \sim U_{\tau,1,\tau}^{\oplus \mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \mathrm{Fib}(p)},$$
$$\mathrm{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \ \ \mathrm{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$

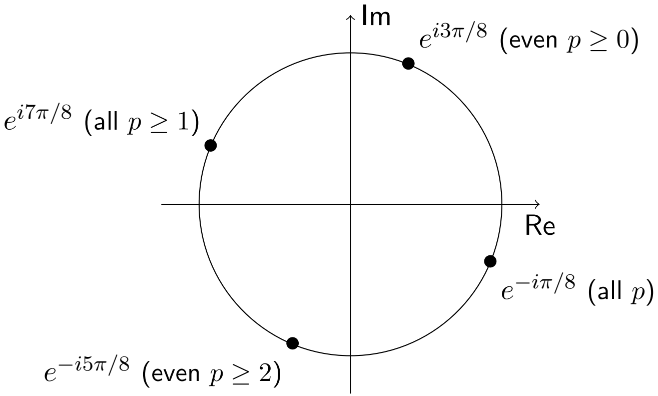


$\Rightarrow$ **Poincaré inequality** with $\beta_0 = 3/5$ and $\beta_{p\geq 1} = 1/5$

Ising anyons: Exchange eigenvalues

$$U_{p=2n+1} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{p=2n} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$



$\Rightarrow$ **Poincaré inequality** with $\beta_{p\geq 0} = 1/8$

Further references on the math-phys of anyons

Introduction and reviews of some recent (abelian) results:

D. L., *Methods of modern mathematical physics: Uncertainty and exclusion principles in quantum mechanics*, lecture notes for a master-level course given at KTH in 2017 and LMU Munich in 2019, `arXiv:1805.03063` (revision underway)

N. Rougerie, *Some contributions to many-body quantum mathematics*, habilitation thesis, 2016, `arXiv:1607.03833`

D. L., *Many-anyon trial states*, Phys. Rev. A 96 (2017) 012116, `arXiv:1608.05067`

M. Correggi, R. Duboscq, D. L., N. Rougerie, *Vortex patterns in the almost-bosonic anyon gas*, EPL 126 (2019) 20005, `arXiv:1901.10739`

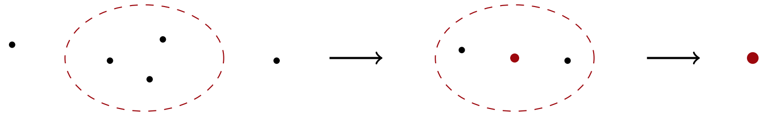
Thanks!

Funbo runestone, Uppsala

Bonus Part

Algebraic anyon models

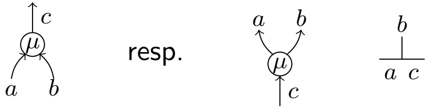
Braided fusion categories... Idea: **zoom out**



Labels / topological charges / particle types:

$$\mathcal{L} = \{a, b, c, \ldots\} = \{1, a, \bar{a}, b, \bar{b}, \ldots\}$$

Fusion / splitting diagrams:



Span spaces $V_{ab}^{c} \cong V_{c}^{ab}$ of dimension N_{ab}^{c} = number of ways fusion/splitting can occur.

Algebraic anyon models: Fusion

Fusion algebra: $a, b \in \mathcal{L}$,

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^{c}\, c$$

The model turns out to be abelian if there is a unique result of fusion,

$$a \times b = c$$

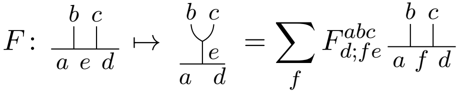
Typically, $N_{ab}^{c} \in \{0, 1\}$, i.e. *multiplicity-free* models.

We write $c \in a \times b$ if $N_{ab}^{c} \neq 0$. Sums only over allowed indices.

Algebraic anyon models: Fusion

Associativity of fusion: $(a \times b) \times c = a \times (b \times c)$

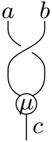
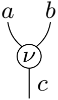
$\Rightarrow$ **F operator** (isomorphism on 2-split diagrams):



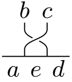
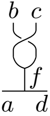
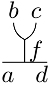
Algebraic anyon models: Braiding

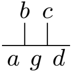
Commmutativity of fusion: $a \times b = b \times a$

$\Rightarrow$ **R operator** (isomorphism on 1-split diagrams): $R_c^{ab} \in \mathrm{U}(V_c^{ab})$

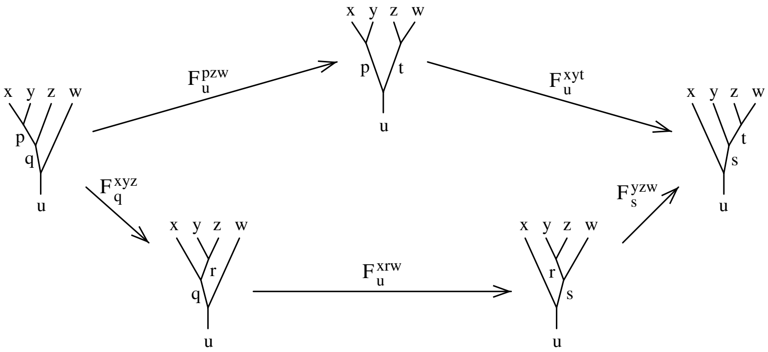
$R^{ab}:$  $\mapsto$  $= \sum_\nu \left[R_c^{ab}\right]_{\nu\mu}$  .

$\Rightarrow$ **B operator** (isomorphism on 2-split diagrams): $B := FRF^{-1}$

 $= \sum_f \left(F^{-1}\right)^{acb}_{d;fe}$  $= \sum_f R_f^{bc} \left(F^{-1}\right)^{acb}_{d;fe}$ 

$= \sum_g \sum_f F^{abc}_{d;gf}\, R_f^{bc} \left(F^{-1}\right)^{acb}_{d;fe}$ 

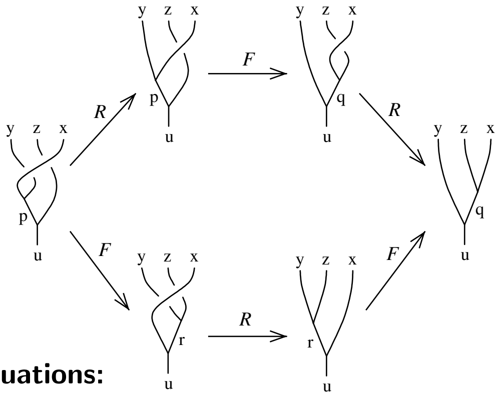
Algebraic anyon models: Consistency conditions



Pentagon equation:

$$F^{aby}_{e;xu} F^{ucd}_{e;yv} = \sum_{w \in \mathcal{L}} F^{abc}_{v;wu} F^{awd}_{e;xv} F^{bcd}_{x;yw},$$

Algebraic anyon models: Consistency conditions

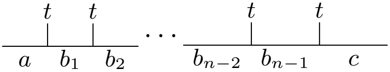


Hexagon equations:

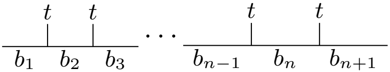
$$R_p^{yx} F_{u;qp}^{yxz} R_q^{zx} = \sum_{r \in \mathcal{L}} F_{u;rp}^{xyz} R_u^{rx} F_{u;qr}^{yzx},$$

$$\left(R_p^{yx}\right)^{-1} F_{u;qp}^{yxz} \left(R_q^{zx}\right)^{-1} = \sum_{r \in \mathcal{L}} F_{u;rp}^{xyz} \left(R_u^{rx}\right)^{-1} F_{u;qr}^{yzx},$$

Standard splitting spaces:

$V_c^{a,t^n} = \mathrm{Span}\Big\{$ 

: for all possible $b_j \Big\}$

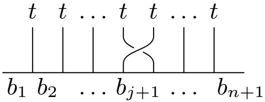
$V_*^{*,t^n} = \mathrm{Span}\Big\{$ 

: for all possible $b_j \Big\}$

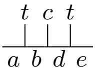
$$V_*^{*,t^n} = \bigoplus_{\substack{a\in\mathcal{L} \\ c\in a\times t^n}} V_c^{a,t^n}$$

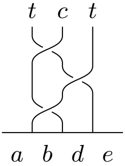
Defines a representation $\rho_n \colon B_n \to \mathrm{U}(V_*^{*,t^n})$:

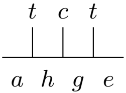
$\rho_n(\sigma_j)$:

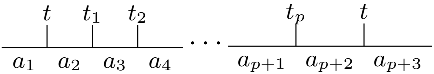


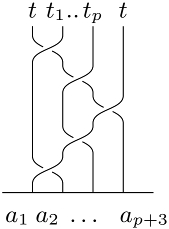
Algebraic anyon models: Exchange operators

$U_{t,c,t}:$ 

$\mapsto$ 

$= \sum_{f,g,h} B^{act}_{d;fb}\, B^{ftt}_{e;gd}\, B^{atc}_{g;hf}$ 

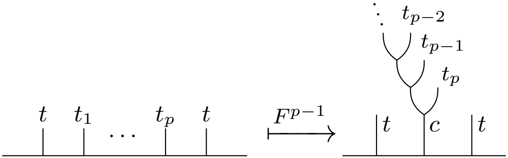
$U_{t,\{t_1,\ldots,t_p\},t}:$ 

$\mapsto$ 

If the anyon type $t = t_1 = \ldots = t_p$ is understood: $U_p := U_{t,t^p,t}$

$$U_p = \rho_n(\sigma_1\sigma_2\ldots\sigma_p\sigma_{p+1}\sigma_p\ldots\sigma_2\sigma_1)$$

Exchange in algebraic anyon models



Theorem ([DL, Qvarfordt])

The exchange operator of two t's around $t_1, \ldots, t_p$ is given by

$$U_{t,\{t_1,\ldots,t_p\},t} \sim \bigoplus_{c \in t_1 \times \ldots \times t_p} U_{t,c,t}$$

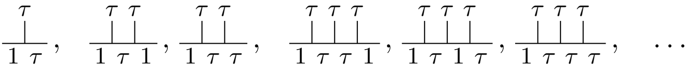
where c is a possible result of the fusion $t_1 \times t_2 \times \ldots \times t_p$, counted with multiplicity.

Fibonacci anyons

Model: $\mathcal{L} = \{1, \tau\}$,

$$\tau \times \tau = 1 + \tau$$

N-particle basis states:



$$\tau^N = \mathrm{Fib}(N-1)1 + \mathrm{Fib}(N)\tau,$$

where $\mathrm{Fib}(0) = 0$, $\mathrm{Fib}(1) = 1$, $\mathrm{Fib}(n) = \mathrm{Fib}(n-2) + \mathrm{Fib}(n-1)$.

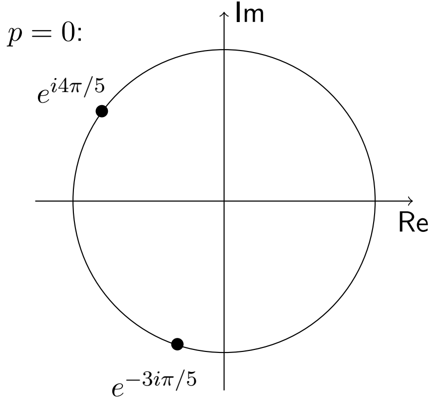
$$F_1^{\tau\tau\tau} = 1 \qquad \text{and} \qquad F_\tau^{\tau\tau\tau} = \begin{bmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{bmatrix}, \ \phi = \frac{1+\sqrt{5}}{2},$$

$$R_1^{\tau\tau} = e^{4\pi i/5}, \qquad R_\tau^{\tau\tau} = e^{-3\pi i/5}.$$

An **N-anyon Fibonacci model**: $\mathcal{F} = V_*^{1,\tau^N} \cong \mathbb{C}^D$, $D = \mathrm{Fib}(N+1)$

Fibonacci anyons: Exchange eigenvalues

$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus \mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \mathrm{Fib}(p)},$$
$$\mathrm{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \ \ \mathrm{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$



Fibonacci anyons: Exchange eigenvalues

$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus \mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \mathrm{Fib}(p)},$$
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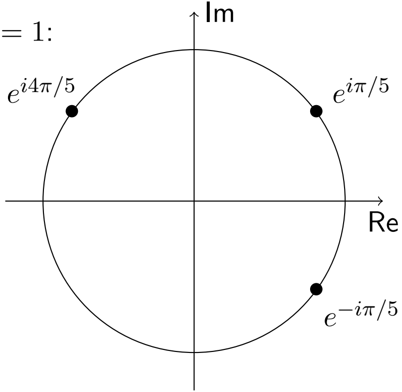
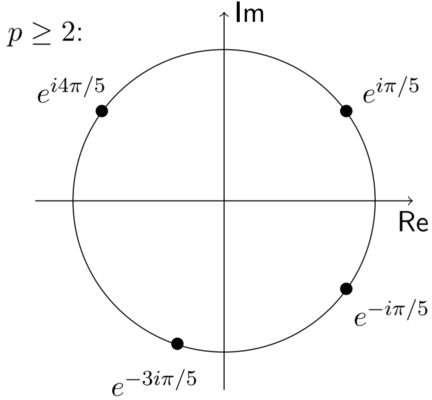
$p = 1$:



Fibonacci anyons: Exchange eigenvalues

$$U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus \mathrm{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \mathrm{Fib}(p)},$$
$$\mathrm{spec}(U_{\tau,1,\tau}) = \{e^{4\pi i/5}, e^{-3\pi i/5}\}, \;\; \mathrm{spec}(U_{\tau,\tau,\tau}) = \{e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5}\}$$

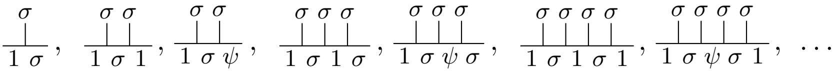


$\Rightarrow$ **Poincaré inequality** with $\beta_0 = 3/5$ and $\beta_{p\geq 1} = 1/5$

Ising anyons

Model: $\mathcal{L} = \{1, \psi, \sigma\}$,

$$\sigma \times \sigma = 1 + \psi, \qquad \sigma \times \psi = \sigma, \qquad \psi \times \psi = 1$$



$$\sigma^{2n+1} = 2^n \sigma, \qquad \sigma^{2n} = 2^{n-1}(1+\psi)$$

$$F_{\sigma}^{\sigma\sigma\sigma} = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad F_{\psi;\sigma\sigma}^{\sigma\psi\sigma} = F_{\sigma;\sigma\sigma}^{\psi\sigma\psi} = -1,$$

$$R_1^{\sigma\sigma} = e^{-\pi i/8}, \quad R_{\psi}^{\sigma\sigma} = e^{3\pi i/8}, \quad R_{\sigma}^{\sigma\psi} = R_{\sigma}^{\psi\sigma} = -i, \quad R_1^{\psi\psi} = -1.$$

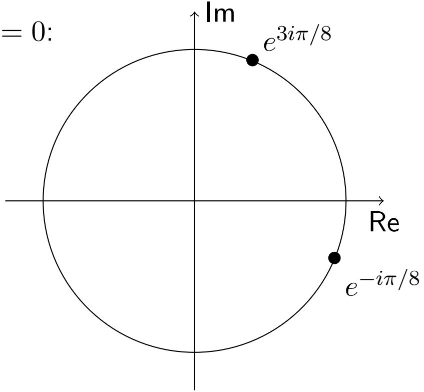
An **N-anyon Ising model**: $\hat{T}_{\rho_N}$ with $\mathcal{F} = V_*^{1,\sigma^N} \cong \mathbb{C}^D$, $D = 2^{\lfloor N/2 \rfloor}$

Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$

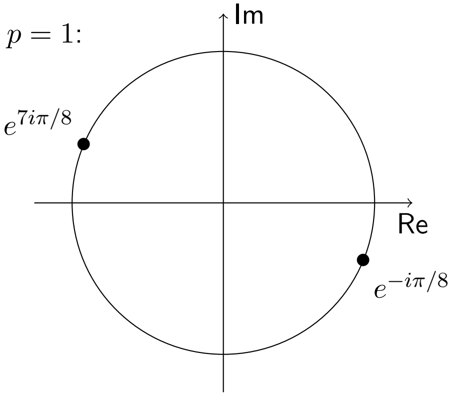
$p = 0$:



Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

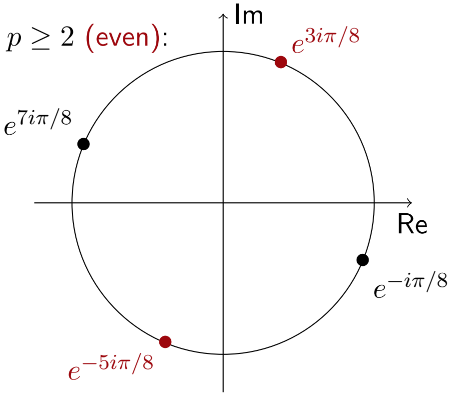
$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$



Ising anyons: Exchange eigenvalues

$$U_{\sigma,\sigma^{2n+1},\sigma} \sim U_{\sigma,\sigma,\sigma}^{\oplus 2^n}, \quad U_{\sigma,\sigma^{2n},\sigma} \sim U_{\sigma,1,\sigma}^{\oplus 2^{n-1}} \oplus U_{\sigma,\psi,\sigma}^{\oplus 2^{n-1}},$$

$\mathrm{spec}(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,1,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}$, $\mathrm{spec}(U_{\sigma,\psi,\sigma}) = \{e^{-5\pi i/8}, e^{7\pi i/8}\}$



$\Rightarrow$ **Poincaré inequality** with $\beta_{p\geq 0} = 1/8$