Emergence of anyons from polarons and angulons

Douglas Lundholm
Uppsala University

MCMB, Beirut, February 2021

$0 \to \infty \to$ 

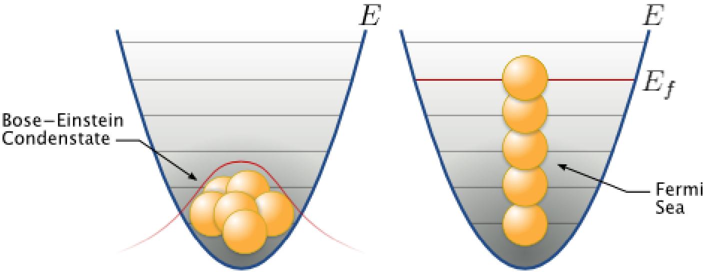
Main references:

[LR]	D.L., N. Rougerie, PRL 2016
[YGLRLS]	E. Yakaboylu, A. Ghazaryan, D.L., N. Rougerie, M. Lemeshko, R. Seiringer, PRB 2020
[BLLY]	M. Brooks, M. Lemeshko, D.L., E. Yakaboylu, PRL 2021

Outline

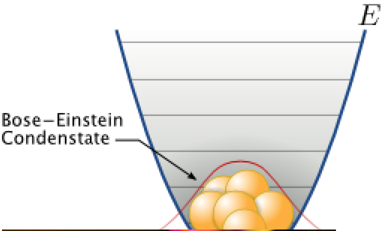
1. Quantum statistics in 2D vs. 3D
2. Illustrative model
3. Statistics transmutation bosons/fermions $\to$ anyons
4. Polarons ($\mathbb{R}^2$)
5. Angulons ($\mathbb{S}^2$)

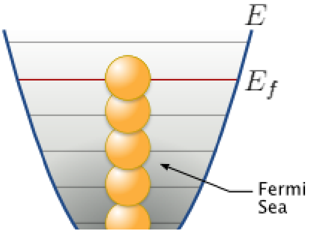
Quantum statistics in 3D

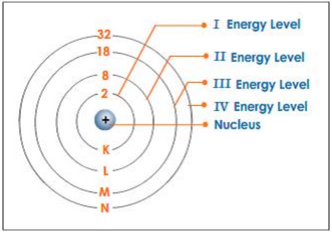


Cold **bosons**   Cold **fermions**

Quantum statistics in 3D



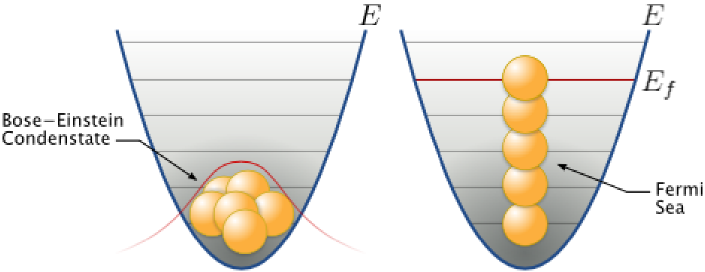




force carriers (fluffy/degenerate)

matter (stable/non-degenerate)

Quantum statistics in 3D



Cold **bosons** Cold **fermions**

Observable: $|\Psi(\mathbf{x}_1, \ldots, \mathbf{x}_j, \ldots, \mathbf{x}_k, \ldots, \mathbf{x}_N)|^2, \qquad \mathbf{x}_j \in \mathbb{R}^3$

Exchange symmetry: representation $\rho \colon S_N \to \mathrm{U}(1)$

$+1 : \rho = 1 \quad \Rightarrow$ **bosons** (Bose–Einstein statistics) $L^2_{\mathrm{sym}}(\mathbb{R}^{3N})$

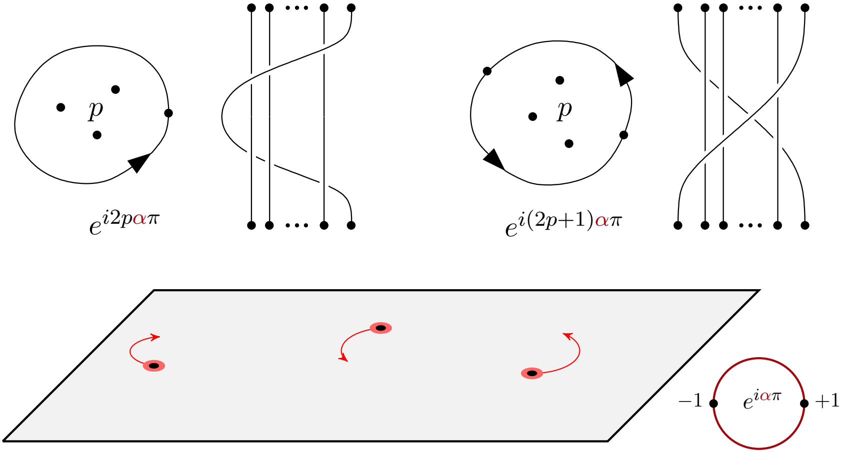
$-1 : \rho = \mathrm{sign} \quad \Rightarrow$ **fermions** (Fermi–Dirac statistics) $L^2_{\mathrm{asym}}(\mathbb{R}^{3N})$

Quantum statistics in 2D

Different in 2D!

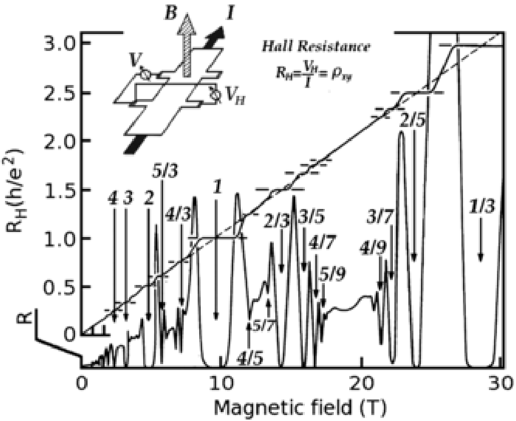
[Leinaas, Myrheim '77; Goldin, Menikoff, Sharp '81; Wilczek '82]

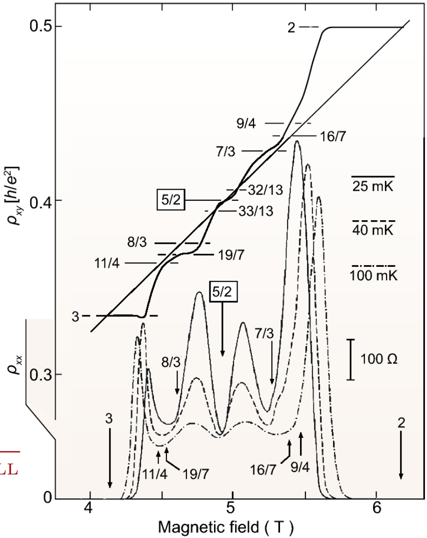
Quantum statistics in 2D



Exchange symmetry $\rho\colon B_N \to \mathrm{U}(1)$ any phase $\Rightarrow$ "**anyons**"

Application 1: Fractional quantum Hall effect



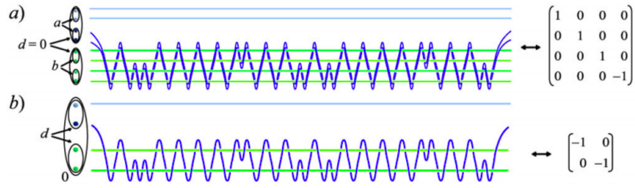


conductance$_\perp$ $\sim$ filling factor $= \frac{N}{\dim_{\mathrm{LL}}}$

$\frac{1}{\rho_{xy}}\left[\frac{e^2}{h}\right] \sim \nu = \frac{p}{q}$, usually q **odd**

[TSG'82, L'83, H'84, ASW'84] [Willett, Eisenstein, Störmer, Tsui, Gossard, English '87]

Application 2: Topological quantum computing



PRL **103,** 160501 (2009) PHYSICAL REVIEW LETT

FIG. 2 (color online). "Effective qubit" gate construction for $\mathfrak{su}(2)_3$ anyons. Part (a) shows a braid in which a pair of anyons from the control qubit (blue) weaves around pairs of anyons in the target qubit (green). When either qubit is in the state $|0\rangle$, this braid produces the identity operation. When both control and target qubits are in the state $|1\rangle$, the braid consists of weaving a

follows this b
around pairs
Fig. 2(a)], the
resulting two-
equivalent to a
We now tur
rule for combi
2. This implie
qubits shown i
The unitary op
shown in Fig.
While it is in

[GMS'85, Kitaev, ...] [Hormozi, Bonesteel, Simon] $\rho\colon B_N \to \mathrm{U}(D)$ **non-abelian anyons**

Hamiltonians for anyons ($\mathrm{U}(1)$ bundles)

Starting from the classical Hamiltonian on $\mathbb{R}^{2N} \times \mathbb{R}^{2N}$

$$H(\mathrm{x}, \mathrm{p}) = \sum_{j=1}^{N} \left[\mathbf{p}_j^2 + V(\mathbf{x}_j)\right]$$

$\Rightarrow$ 'free' **bosons**/**fermions** on $\mathcal{H}_0 = L^2_{\mathrm{sym}}(\mathbb{R}^{2N})$ or $L^2_{\mathrm{asym}}(\mathbb{R}^{2N})$

$$H_0 = \sum_{j=1}^{N} \left[-\nabla^2_{\mathbf{x}_j} + V(\mathbf{x}_j)\right]$$

$\Rightarrow$ 'magnetic' **bosons**/**fermions** with $\mathcal{A}\colon \mathbb{R}^{2N} \to \mathbb{C}^N$ a connection

$$H_0^{\mathcal{A}} = \sum_{j=1}^{N} \left[-(\nabla_{\mathbf{x}_j} + \mathcal{A}_j)^2 + V(\mathbf{x}_j)\right]$$

Hamiltonians for anyons ($\mathrm{U}(1)$ bundles)

$\Rightarrow$ 'free' **anyons** with statistics parameter $\alpha \in \mathbb{R}$

$$H_0^{\alpha i \mathbf{A}} = \sum_{j=1}^{N} \left[-(\nabla_{\mathbf{x}_j} + \alpha i \mathbf{A}_j)^2 + V(\mathbf{x}_j) \right]$$

where $\mathbf{A} \colon \mathbb{R}^{2N} \setminus \triangle \to \mathbb{R}^{2N}$ is *locally flat*,

$$\mathbf{A}_j(\mathrm{x}) := \sum_{k \neq j} (\mathbf{x}_j - \mathbf{x}_k)^{-\perp} = (Z/|Z|)^{-1} \nabla_{\mathbf{x}_j} (Z/|Z|),$$

$$(x, y)^{-\perp} := \frac{(-y, x)}{x^2 + y^2}, \qquad Z(\mathrm{x}) := \prod_{j<k} (z_j - z_k)$$

with magnetic field (point-flux attachment)

$$\mathbf{B}_j(\mathrm{x}) = \operatorname{curl} \mathbf{A}_j(\mathrm{x}) = 2\pi \sum_{k \neq j} \delta_{\mathbf{x}_k}(\mathbf{x}_j).$$

Regularized: $\tilde{H}_0^{\alpha i \mathbf{A}} := |Z|^{-\alpha} H_0^{\alpha i \mathbf{A}} |Z|^{\alpha} = H_0^{\alpha i \mathbf{A} - \alpha \mathbf{A}^{\perp}}$

Hamiltonians for anyons ($\mathrm{U}(1)$ bundles)

$\Rightarrow$ 'free' **anyons** with statistics parameter $\alpha \in \mathbb{R}$

$$H_0^{\alpha i \mathbf{A}} = \sum_{j=1}^{N} \left[-(\nabla_{\mathbf{x}_j} + \alpha i \mathbf{A}_j)^2 + V(\mathbf{x}_j) \right]$$

where $\mathbf{A} \colon \mathbb{R}^{2N} \setminus \triangle \to \mathbb{R}^{2N}$ is *locally flat*,

$$\mathbf{A}_j(\mathrm{x}) := \sum_{k \neq j} (\mathbf{x}_j - \mathbf{x}_k)^{-\perp} = (Z/|Z|)^{-1} \nabla_{\mathbf{x}_j} (Z/|Z|),$$

$$(x,y)^{-\perp} := \frac{(-y,x)}{x^2+y^2}, \qquad Z(\mathrm{x}) := \prod_{j<k} (z_j - z_k)$$

with magnetic field (point-flux attachment)

$$\mathbf{B}_j(\mathrm{x}) = \operatorname{curl} \mathbf{A}_j(\mathrm{x}) = 2\pi \sum_{k \neq j} \delta_{\mathbf{x}_k}(\mathbf{x}_j).$$

Regularized: $\tilde{H}_0^{\alpha i \mathbf{A}} := |Z|^{-\alpha} H_0^{\alpha i \mathbf{A}} |Z|^{\alpha} = H_0^{\alpha i \mathbf{A} - \alpha \mathbf{A}^{\perp}}$

Main results

Refined perspectives:

- emergent anyons via statistics transmutation
- coherent state of composite bosons/fermions
- deformation of quantum system / interpolation between integer-flux bundles
- computational framework for spectral estimation

Ex:

$$H_0^{\alpha i\mathbf{A}} = e^{-\alpha/2}|Z|^\alpha \sum_{n=0}^{\infty} \frac{(\alpha/2)^n}{n!} \underbrace{Z^{-2n} H_0 Z^{2n}}_{\tilde{H}_0^{2ni\mathbf{A}}} |Z|^{-\alpha}$$

and

$$\tilde{H}_0^{\alpha i\mathbf{A}} = H_0 + \frac{\alpha}{2}\left(\tilde{H}_0^{2i\mathbf{A}} - H_0\right)$$

An illustrative model for statistics transmutation

Add a collective degree of freedom: $[a, a^\dagger] = 1$, $\mathcal{N} = a^\dagger a$, $|n\rangle$

$$\boxed{H_\omega := H_0 + \omega a^\dagger a + \gamma\omega(F a^\dagger + F^{-1} a) + \gamma^2 \omega}$$

Two parameters: $\omega > 0$, $\gamma \in \mathbb{R}$

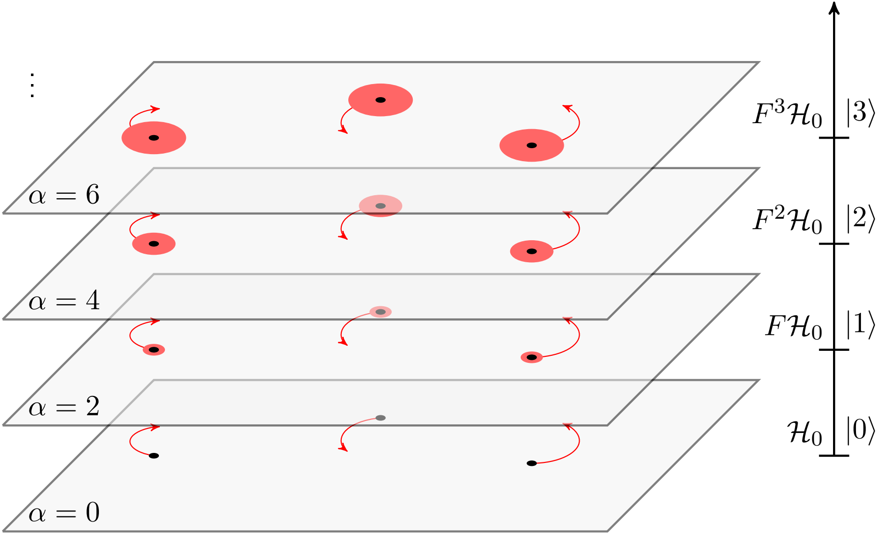
Two model choices:

1. $F = (Z/|Z|)^2$ **flux** attachment
2. $F = Z^2$ **vortex** attachment

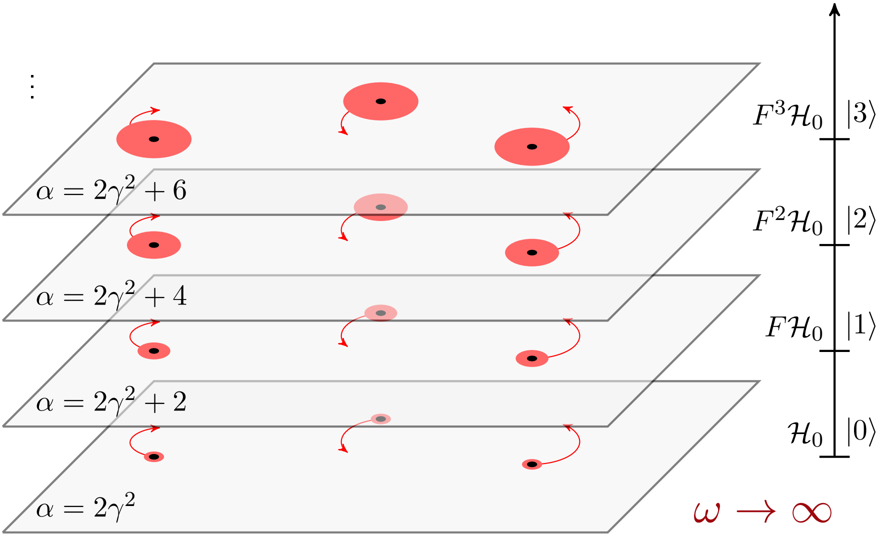
$\Rightarrow$ Hilbert spaces of **composite bosons/fermions**: $\mathcal{H}^n = F^n \mathcal{H}_0 |n\rangle$
$\alpha = 2n$, $n = 0, 1, 2, \ldots$

Now take the 'adiabatic' limit $\omega \to \infty$ with γ fixed.
Claim: in the bottom of the spectrum we obtain anyons with
$\alpha = 2\gamma^2 + 2n$

Composite bosons/fermions: ladder of integer bundles



Emergent anyons: ladder of fractional bundles



Algebraic transmutation: H_ω

We diagonalize H_ω by taking the similarity

$$S := F^{\mathcal{N}} = \exp\big[(\log F)a^\dagger a\big]$$

and unitary shift transformation

$$U := e^{-\gamma(a^\dagger - a)}$$

Make use of some algebra

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \dots$$

$$S^{-1}aS = a\exp[\log F] = Fa \qquad U^{-1}aU = a - \gamma$$

$$S^{-1}a^\dagger S = a^\dagger \exp[-\log F] = F^{-1}a^\dagger \qquad U^{-1}a^\dagger U = a^\dagger - \gamma$$

$$H'_\omega := U^{-1}S^{-1}H_\omega SU = H'_0 + \underbrace{\omega(a^\dagger - \gamma)(a-\gamma) + \gamma\omega(a^\dagger + a - 2\gamma) + \gamma^2\omega}_{\omega a^\dagger a}$$

Algebraic transmutation: H_0

Emergent gauge field: $\mathbf{F}_j := \nabla_{\mathbf{x}_j} \log F = F^{-1} \nabla_{\mathbf{x}_j} F$

$$S^{-1} \nabla_{\mathbf{x}_j} S = \exp[-\log F \mathcal{N}] \nabla_{\mathbf{x}_j} \exp[\log F \mathcal{N}] = \nabla_{\mathbf{x}_j} + \mathbf{F}_j \mathcal{N}$$

$$\begin{aligned} \Delta'_{\mathbf{x}_j} &:= U^{-1} (\nabla_{\mathbf{x}_j} + \mathbf{F}_j \mathcal{N})^2 U \\ &= \Delta_{\mathbf{x}_j} + (\nabla_{\mathbf{x}_j} \cdot \mathbf{F}_j + 2\mathbf{F}_j \cdot \nabla_{\mathbf{x}_j}) U^{-1} \mathcal{N} U + \mathbf{F}_j^2 U^{-1} \mathcal{N}^2 U \end{aligned}$$

$$H'_\omega = \sum_{j=1}^{N} [-\Delta'_{\mathbf{x}_j} + V(\mathbf{x}_j)] + \omega \mathcal{N}$$

Coherent state:

$$|-\gamma\rangle := U|0\rangle = e^{-\gamma(a^\dagger - a)}|0\rangle = e^{-\gamma^2/2} e^{-\gamma a^\dagger} e^{\gamma a}|0\rangle = e^{-\gamma^2/2} \sum_{n=0}^{\infty} \frac{(-\gamma)^n}{\sqrt{n!}} |n\rangle$$

$$\langle -\gamma | \mathcal{N} | -\gamma \rangle = \gamma^2, \qquad \langle -\gamma | \mathcal{N}^2 | -\gamma \rangle = \gamma^2 + \gamma^4,$$

$$\langle n | H'_\omega | n \rangle = H_0^{(\gamma^2 + n)\mathbf{F}} - \gamma^2 (1 + 2n) \mathbf{F}^2 + \omega n$$

Algebraic transmutation: emergent anyons

$$\langle n|H'_\omega|n\rangle = H_0^{(\gamma^2+n)\mathbf{F}} - \gamma^2(1+2n)\mathbf{F}^2 + \omega n$$

$$\langle 0|H'_\omega|0\rangle = e^{-\gamma^2}\sum_{n=0}^{\infty}\frac{\gamma^{2n}}{n!}H_0^{n\mathbf{F}}$$

Case 1: $F = (Z/|Z|)^2 = \exp\left[2\sum_{j<k} i \arg z_{jk}\right]$

$$\mathbf{F} = 2i\mathbf{A} \;\Rightarrow\; \mathbf{F}^2 = -4\sum_j \mathbf{A}_j^2$$

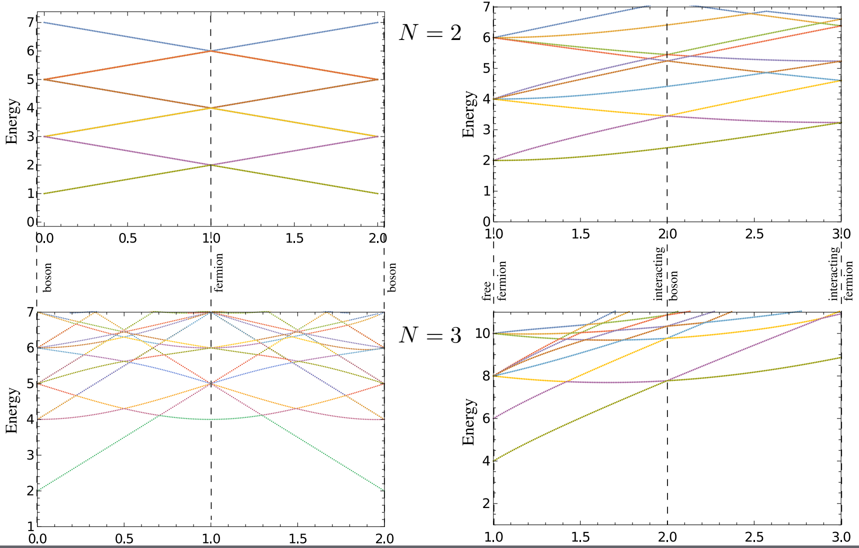
$\Rightarrow$ **interacting anyons** with $\alpha = 2\gamma^2 + 2n$

Case 2: $F = Z^2 = \exp\left[2\sum_{j<k}(i\arg z_{jk} + \ln|z_{jk}|)\right]$

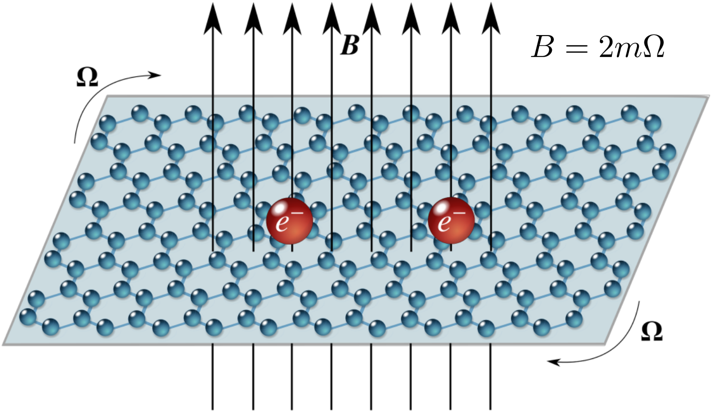
$$\mathbf{F} = 2i\mathbf{A} - 2\mathbf{A}^\perp \;\Rightarrow\; \mathbf{F}^2 = -4\mathbf{A}^2 + 4\mathbf{A}^2 = 0$$

$\Rightarrow$ **free & regularized anyons** with $\alpha = 2\gamma^2 + 2n$

Computation of spectrum: ‘free’ vs interacting (harm. osc.)



Application 3: Polarons



$$H_\omega = \frac{1}{2m}\sum_{j=1}^{N} \mathbf{p}_j^2 + W(\mathrm{x}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \lambda_{\mathbf{k}}(\mathrm{x}) \left(e^{-i\beta_{\mathbf{k}}(\mathrm{x})} b_{\mathbf{k}}^\dagger + h.c. \right)$$

Application 3: Polarons

For $N = 2$ with relative coordinates (r, φ) and radial interaction:

$$J_z = L_z + \Lambda_z, \qquad L_z = -i\partial_\varphi, \qquad \Lambda_z = \sum_{k,\mu} \mu b^\dagger_{k\mu} b_{k\mu}$$

$$\Psi = \psi_{\mathrm{A}}(r,\varphi) S(\varphi) U(r) |0\rangle$$

$$S(\varphi) = \exp\left[-i\varphi\Lambda_z\right], \qquad U(r) = \exp\left[-\sum_{k,\mu} \frac{\lambda_{k\mu}(r)}{\omega_{k\mu}} (b^\dagger_{k\mu} - b_{k\mu})\right]$$

Fixed total angular momentum but shift in relative:

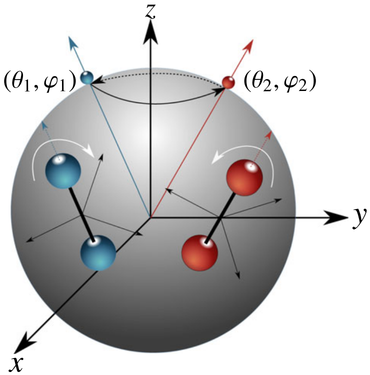
$$\mathbb{Z} \ni j = \langle J_z \rangle_\Psi = \langle L_z \rangle_\Psi + \langle \Lambda_z \rangle_\Psi \;\Rightarrow\; \langle L_z \rangle_\Psi = j - \langle \Lambda_z \rangle_\Psi$$

$$\mathcal{A}(r,\varphi) = -\langle 0 | U^{-1} \Lambda_z U | 0 \rangle \frac{1}{r} \mathbf{e}_\varphi \quad \text{i.e.} \quad \alpha(r) = -\langle \Lambda_z \rangle_{\text{coherent state}\,\gamma(r)}$$

Can be computed for suitable interaction, $\alpha \sim$ const.(Ω)

Application 4: Angulons

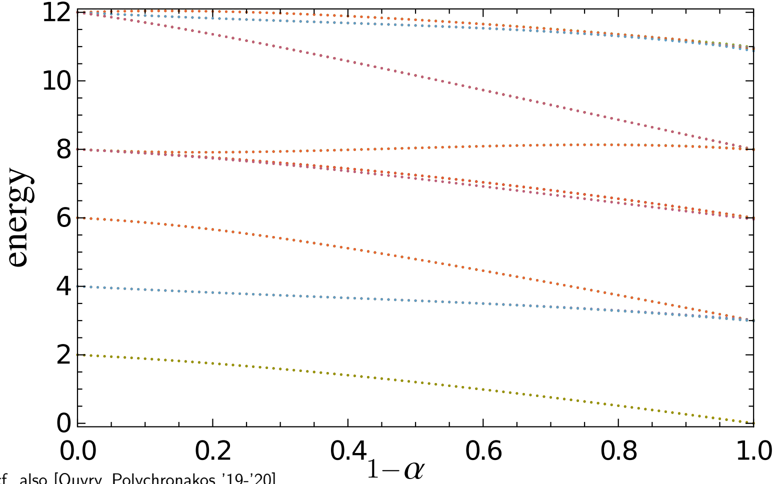
Molecular orientation



$$H_\omega = \sum_{j=1}^{N} \mathbf{L}_j^2 + W(\mathrm{x}) + \sum_{\mathbf{k}} \omega_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} + \sum_{\mathbf{k}} \lambda_{\mathbf{k}}(\mathrm{x}) \left(e^{-i\beta_{\mathbf{k}}(\mathrm{x})} b_{\mathbf{k}}^\dagger + h.c. \right)$$

Application 4: Angulons

Spectrum of $N = 2$ anyons on $\mathbb{S}^2$ with monopole field $2B = (N-1)\alpha$



cf. also [Ouvry, Polychronakos '19-'20]

Conclusions

Moral of the story (QM):

If we don't know with certainty which collective state (here $\mathcal{H}^n$, $n \in \mathbb{N}$) has been assumed by the system,

- allow for superpositions of all possibilites (here $\oplus_n \mathcal{H}^n$),
- take coherent states of such superpositions (maximal simultaneous information; here $|-\gamma\rangle$), and
- find if their distribution is determined by (/correlated with) some other collective degree of freedom (here Ω) to high certainty.

The result may have a useful alternative representation (here anyons $\mathcal{H}^\alpha$, $\alpha = 2\gamma^2 \in \mathbb{R}$).

Further references on the math-phys of anyons

D. L., N. Rougerie, *Emergence of fractional statistics for tracer particles in a Laughlin liquid*, PRL 116 (2016) 170401, arXiv:1601.02508

M. Correggi, R. Duboscq, D. L., N. Rougerie, *Vortex patterns in the almost-bosonic anyon gas*, EPL 126 (2019) 20005, arXiv:1901.10739

E. Yakaboylu, A. Ghazaryan, D. L., N. Rougerie, M. Lemeshko, R. Seiringer, *Quantum impurity model for anyons*, PRB 102 (2020) 144109, arXiv:1912.07890

M. Brooks, M. Lemeshko, D.L., E. Yakaboylu, *Molecular Impurities as a Realization of Anyons on the Two-Sphere*, PRL 126 (2021) 015301, arXiv:2009.05948

D. L., V. Qvarfordt, *Exchange and exclusion in the non-abelian anyon gas*, 2020, arXiv:2009.12709

Thanks!

Funbo runestone, Uppsala