REAL ANALYSIS — LECTURE NOTES

DOUGLAS LUNDHOLM

ABSTRACT. Lecture notes and studying suggestions for 1MA226 Real Analysis given at Uppsala University 2020-21. The material is complementary to Rudin, Principles of mathematical analysis, 3rd ed.
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1. Introduction [L1]

In this course there will be a strong focus on theory, i.e. definitions, theorems and proofs. These things take time, effort and *focus* to absorb, so it is strongly advised to take your own peaceful time to *read*, think and digest. The lectures will serve to guide you through the reading material and offer points of discussion.

As the official course literature we will use the classic textbook of Rudin [Rud76], while these notes will contain complementary material, reading suggestions, excersises, corrections, etc. Some might find these notes to be a bit "dryer" than Rudin, while some may even find Rudin a bit too dry. As an alternative/complement book some students may find [Abb15] helpful (see in particular the epilogues to each chapter which include some historic notes).

1.1. **Lecture plan (OBS: preliminary).**

L1 29/10 Introduction, Preliminaries
L2 2/11 Definition and properties of real numbers
L3 6/11 Cauchy sequences, upper and lower limits
L4 11/11 Bolzano-Weierstrass theorem
L5 12/11 " / Problem session
L6 16/11 Topology in $\mathbb{R}^n$, Metric spaces and their topology
L7 19/11 Compactness, Heine-Borel lemma
L8 26/11 Continuous functions

L9 27/11 Baire's theorem
L10 2/12 " / Problem session
L11 3/12 Normed vector spaces, series
L12 9/12 Differentiable functions: mean-value theorem and its consequences
L13 10/12 Taylor series
L14 21/1 Riemann integral
L15 25/1 " / Problem session
L16 27/1 Sequences and series of functions: uniform convergence
L17 1/2 "
L18 4/2 Equicontinuous families, Arzelà-Ascoli theorem
L19 8/2 Stone-Weierstrass theorem, power series
L20 11/2 " / Problem session
L21 15/2 Banach's fixed point theorem and applications
L22 19/2 "
L23 1/3 Inverse and implicit function theorems
L24 5/3 "
L25 11/3 Summary/Repetition
— 15/3 Exam (don't forget to register!)

1.2. **Notes concerning notation/style.**

We use **bold face** whenever a new term or concept is introduced, and it will also appear in the index of these notes.

An asterisk (*) denotes optional (non-examinable) material for the curious student.

We will read : (colon) as "such that" (also abbreviated s.t.), and thus usually define sets

$$\{x \in X : P(x)\},$$

which reads "the set of all x in X such that $P(x)$". Beware that it is also common to write

$$\{x \in X \mid P(x)\}.$$

where | could generally mean "subject to" (/ conditioned by / sv: betingat av).

Let us use the directed notation := for "defined to equal" and :⇔ "defined to be equivalent".

Conventions:

$\mathbb{N} = \{0, 1, 2, \ldots\}$	natural numbers
$\mathbb{N}^+ = \{1, 2, \ldots\}$	positive integers
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$	integers
$\mathbb{Q} = \{p/q : p \in \mathbb{Z}, q \in \mathbb{N}^+\}$	rational numbers
$\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$	positive rational numbers
$\mathbb{Q}_+ = \{x \in \mathbb{Q} : x \geq 0\}$	non-negative rational numbers
$\mathbb{R} = (-\infty, +\infty)$	real numbers
$\mathbb{R}^+ = (0, +\infty)$	positive real numbers
$\mathbb{R}_+ = [0, +\infty)$	non-negative real numbers
$\mathbb{R}_\infty = [-\infty, +\infty]$	extended real numbers

Abbreviations:
ex: example,
iff: if and only if (sv: omm, om och endast om),
WLOG: without loss of generality (sv: utan inskränkning),
STS: suffices to show (sv: RAV, räcker att visa)
sv: svenska/Swedish translation
corr: corrections

Acknowledgments. Significant parts of these lecture notes have been based on a course on the foundations of analysis given by Lars Svensson at KTH Stockholm around 2004, whose clarity, insight and uplifting spirit is gratefully acknowledged.

2. Some preliminaries and notation [L1-2]

We assume that the reader is familiar with basic notions in calculus in one and several variables, as well as linear algebra (first-year university courses). For convenience and reference, let us recall a few fundamental mathematical concepts and algebraic notions.

2.1. **Sets.** Sets may be constructed using certain axioms (Zermelo-Fraenkel; see Section 2.9). We will use the following notations:

- $\emptyset$ denotes the **empty set**.
- $x \in A$: x is an element of the set A
- $x \notin A$: x is not an element of the set A
- $\exists x$: there exists an x
- $\nexists x$: there exists no x
- $\exists! x$: there exists a unique x
- $\forall x$: for all x
- We say that A is a **subset** of B and write $A \subseteq B$ if $\forall x\ (x \in A \Rightarrow x \in B)$.
- We say that A is a **strict subset** of B and write $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$ (these are in analogy to $\leq$ and $\lneq$, while e.g. Rudin uses $\subset$ instead of $\subseteq$).
- $x \in A \cap B$ ("intersection of A and B") if $x \in A$ and $x \in B$.
- $x \in A \cup B$ ("union of A and B") if $x \in A$ or $x \in B$.
- $x \in A \setminus B$ ("A minus B") if $x \in A$ but $x \notin B$.
- $\mathcal{P}(A)$ denotes the set of all subsets of the set A (**power set** / sv: potensmängd).
- $\neg P$ ("not P") denotes the logical negation of P.

We may write & for "and" ($\wedge$ is also common but has many other uses), and $\vee$ for "or".

2.2. **Relations.** A **relation** R on a set M is a subset of the cartesian product $M \times M$, i.e.

$$R \subseteq M \times M.$$

If $(x, y) \in R$ we may also write xRy. Furthermore, R is called:

- **reflexive** if $\quad xRx \quad \forall x \in M$
- **symmetric** if $\quad xRy \Rightarrow yRx \quad \forall x, y \in M$
- **antisymmetric** if $\quad xRy\ \&\ yRx \Rightarrow x = y \quad \forall x, y \in M$
- **transitive** if $\quad xRy\ \&\ yRz \Rightarrow xRz \quad \forall x, y, z \in M$
- **connex** if $\quad \forall x, y \in M \quad xRy$ or yRx

We also define:

- **Partial order**: A reflexive, antisymmetric and transitive relation.
- **Total order**: A connex partial order.
- **Equivalence relation**: A reflexive, symmetric and transitive relation.

To every total order $\leq$ there is a **strict total order** $<$ defined by $a < b$ iff $a \leq b$ and $a \neq b$ (or, equivalently, iff $\neg(b \leq a)$), and vice versa, $a \leq b \ :\Leftrightarrow\ \big(a < b$ or $a = b\big)$.

Relations between two different sets A and B may also be defined, to be subsets of $A \times B$.

2.3. **Functions.** A **function** or **map** $f\colon A \to B$ may be regarded as a subset of $A \times B$, i.e. a certain type of relation on $A \times B$. Namely, we usually call this set the **graph** of f. Let us denote it $\operatorname{graph}(f) \subseteq A \times B$, with the property that to each $x \in A$ there is a *unique*

$y \in B$ such that $(x, y) \in \text{graph}(f)$, and we denote this y by $f(x)$. The set A is called the **domain** (sv: definitionsmängd) of f, while B is the **codomain** (sv: målmängd). The set

$$f(A) := \{y \in B : \exists x \in A \text{ s.t. } y = f(x)\}$$

is the **range** or **image** (sv: bild/värdemängd) of f. For $f : A \to B$ we may also denote A by $\text{dom}(f)$ and $f(A)$ by $\text{im}(f)$.

Remark 2.1. It may be is useful in practice to consider A (source set/sv: källmängd) and B (target set/sv: målmängd) as fixed sets of a given context (ex: $A = \mathbb{R}^n$ and $B = \mathbb{R}^m$) and then

$$f : A \supseteq \text{dom}(f) \to \text{im}(f) \subseteq B$$

is a (surjective) map from its domain to its range, which both could be *strict* subsets of A resp. B. Recall that $\text{dom}(f)$ might not be given explicitly but taken to be the largest set in a given context (ex. $A = \mathbb{R}$ in one-variable calculus) for which the expressions for $f(x)$ are defined. Strictly speaking, such f is called a **partial function** from A to B: for all $x \in A$ there exists *at most* one $y \in B$ s.t. $(x, y) \in \text{graph}(f)$.

We may write $f : A \to B$, or $A \xrightarrow{f} B$, or more explicitly

$$\begin{aligned} f : A &\to B \\ x &\mapsto f(x) \end{aligned}$$

A function $f : X \to Y$ is called **injective** (or **one-to-one**) if it preserves inequality, i.e.

$$x \neq x' \;\Rightarrow\; f(x) \neq f(x') \qquad \forall\, x, x' \in X,$$

and **surjective** (or **onto**) if its range equals its codomain, $f(X) = Y$, i.e.

$$\forall y \in Y\ \exists x \in X : f(x) = y.$$

Functions that are both injective and surjective are called **bijective**.

If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$ then their **composition** is $X \xrightarrow{g \circ f} Z$,

$$(g \circ f)(x) := g\big(f(x)\big), \qquad x \in X.$$

Note:

- If f and g are injective then $g \circ f$ is injective:
  if $x \neq x'$ then $f(x) \neq f(x')$ and thus $g(f(x)) \neq g(f(x'))$.
- If f and g are surjective then $g \circ f$ is surjective:
  if $z \in Z$ then $\exists\ y \in Y$ s.t. $g(y) = z$, and thus $\exists\ x \in X$ s.t. $f(x) = y$, implying $g(f(x)) = z$.
- Hence, if f and g are bijective then $g \circ f$ is bijective.

2.4. **Infinite unions and intersections.** Let I and X be sets and assume that $I \xrightarrow{f} \mathcal{P}(X)$ is a function. Let us denote

$$f(i) =: A_i, \qquad \text{i.e. } A_i \subseteq X \text{ if } i \in I.$$

Thus I may be called an **index set**. Now define the **union** of all A_i's by

$$\bigcup_{i \in I} A_i := \{x \in X : \exists i \in I \text{ s.t. } x \in A_i\}.$$

In particular, if $I = \emptyset$ then $\bigcup_{i \in \emptyset} A_i = \emptyset$.

We also define the **intersection** of all A_i's by

$$\bigcap_{i\in I} A_i := \{x \in X : \forall i\,(i \in I \Rightarrow x \in A_i)\}\,.$$

Remark 2.2. If $I = \emptyset$ we have (a bit surprisingly) that

$$\bigcap_{i\in\emptyset} A_i = X,$$

because for each i it holds that $i \in I$ is false, i.e. that the implication $(i \in I \Rightarrow x \in A_i)$ is true.

The **complement** A_i^c of A_i in X is $X \setminus A_i$, and we have the following relationships:

$$\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c, \tag{2.1}$$

$$\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c. \tag{2.2}$$

Remark 2.3. When taking the complement of a set it is important to know what it is taken *with respect to*. Often one assumes it to be understood which "background" set is being considered. For example, it is in principle erroneous to write

$$A^c = \{x : x \notin A\}.$$

If the "background" is X, i.e. $A \subseteq X$, one should (despite sometimes being more sloppy) instead write

$$A^c = \{x \in X : x \notin A\} = X \setminus A.$$

2.5. **Pullback and pushforward.** Assume that $f\colon X \to Y$, then we have two induced maps, the **pushforward**

$$f_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$

and the **pullback**

$$f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X),$$

defined

$$f_*(A) := \{f(x) \in Y : x \in A\} =: f(A), \qquad \text{if } A \subseteq X,$$
$$f^*(B) := \{x \in X : f(x) \in B\} =: f^{-1}(B), \qquad \text{if } B \subseteq Y.$$

The latter notations are usually the ones used in practice (and correct also if f is invertible).

2.5.1. *f^* respects unions and intersections.* Let I be an index set and assume that for each $i \in I$ we have a set $B_i \subseteq Y$ (we also assume that $I \neq \emptyset$). Note then that

$$f^*\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f^*(B_i). \tag{2.3}$$

Namely, if x is an element of the l.h.s. then by definition $f(x)$ is an element of $\bigcap_{i\in I} B_i \subseteq B_j\ \forall j \in I$. Thus $x \in f^*(B_j)$ for all $j \in I$, so $x \in \bigcap_{j\in I} f^*(B_j)$.

Conversely, if x is an element of the r.h.s. then $x \in f^*(B_i)\ \forall i \in I$, so $f(x) \in B_i\ \forall i \in I$. Thus $f(x) \in \bigcap_{i\in I} B_i$ and $x \in f^*\left(\bigcap_{i\in I} B_i\right)$.

In a similar way one may verify that

$$f^*\left(\bigcup_{i\in I} B_i\right) = \bigcup_{i\in I} f^*(B_i). \tag{2.4}$$

2.5.2. *f_* respects unions but not intersections.* Let $A_i \subseteq X$, $i \in I$. That

$$f_*\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f_*(A_i) \tag{2.5}$$

is left as an exercise. However, that f_* does not preserve intersections may be seen by a simple counterexample:

Example 2.4. Let $X = \mathbb{Z}$ and $Y = \mathbb{N}$ and define $f\colon X \to Y$ by $f(x) = x^2$. Let $A = \{x \in \mathbb{Z} : x < 0\}$, $B = \{x \in \mathbb{Z} : x > 0\}$. Then $f_*(A) = f_*(B) = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \ldots\}$, and thus $f_*(A) \cap f_*(B) = \mathbb{N} \setminus \{0\} \neq \emptyset$, however $f_*(A \cap B) = f_*(\emptyset) = \emptyset$.

Exercise 2.1. *Prove* (2.4)-(2.5) *and come up with other counterexamples for f_*.*

2.6. **Some basic algebraic notions.** We can't do analysis without some algebra. In fact we will see that a large part of analysis is about linear algebra (usually in infinite-dimensional vector spaces). Let us recall a few basic notions here.

2.6.1. *Binary compositions.* A **binary composition** $*$ on a set M is a map

$$\begin{array}{ccc} M \times M & \to & M \\ (x, y) & \mapsto & x * y \end{array}$$

The composition is called

- **associative** if $(x * y) * z = x * (y * z) \quad \forall x, y, z \in M$,
- **commutative** if $x * y = y * x \quad \forall x, y \in M$.

2.6.2. *Unit.* If there exists an element $e \in M$ such that

$$e * x = x * e = x \quad \forall x \in M$$

then e is called a **unit**. Units are unique because, if e and e' are units, then $e = e * e' = e'$.

2.6.3. *Inverse.* An element $x \in M$ has a **left inverse** y if $y * x = e$, **right inverse** z if $x * z = e$, and **inverse** y if $y * x = x * y = e$. If $*$ is associative and if y and z are inverses to x (actually it is sufficient that y is right inverse to x and z left inverse to x) then

$$z = z * e = z * (x * y) = (z * x) * y = e * y = y. \tag{2.6}$$

In other words, inverses are unique whenever $*$ is associative.

2.6.4. *Distributivity.* Let $*$ and $\diamond$ be two binary compositions on M. We say that $*$ is **distributive** over $\diamond$ if

$$x * (y \diamond z) = (x * y) \diamond (x * z) \quad \forall x, y, z \in M \tag{2.7}$$

and

$$(y \diamond z) * x = (y * x) \diamond (z * x) \quad \forall x, y, z \in M. \tag{2.8}$$

2.6.5. *Basic algebraic structures.* From these notions we can define a variety of basic mathematical structures as follows:

Monoid: Set with an associative binary composition and a unit.
Group: Monoid where every element has an inverse.
Abelian group: Commutative group.
Ring: A set R with two binary compositions, called **addition** $(+)$ and **multiplication** $(\cdot)$, and such that $(R,+)$ is an abelian group and $(R,\cdot)$ is a monoid, and where multiplication is distributive over addition. Furthermore, it should hold that $0\cdot x = x\cdot 0 = 0$ for all $x\in R$, where 0 denotes the additive unit in R and is called **zero** (sv: nollan). The additive inverse to $x\in R$ is denoted $-x$.
Ring with unit: Ring where multiplication has a unit, usually denoted 1 and called **one** (sv: ettan) or **the identity**.
Commutative ring: Ring with commutative multiplication.
Field (sv: kropp): Commutative ring with unit, where every nonzero element has a multiplicative inverse.
Ordered field: A field $\mathbb{F}$ on which a strict total order $<$ is defined, such that
(i) if $x,y,z\in\mathbb{F}$ and $y<z$ then $x+y<x+z$
(ii) if $x,y\in\mathbb{F}$, $x>0$ and $y>0$ then $xy>0$.
If $x>0$ we call x **positive**, and if $x<0$ we call x **negative**.
Vector space over a field: A set V is a vector space over the field $\mathbb{F}$ if we have defined two maps

$$\begin{array}{rcl} V\times V & \xrightarrow{+} & V \\ (v,w) & \mapsto & v+w \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathbb{F}\times V & \xrightarrow{\cdot} & V \\ (\alpha,v) & \mapsto & \alpha v \end{array}$$

such that $(V,+)$ is an abelian group, and the following holds for all $\alpha,\beta\in\mathbb{F}$ and $v,w\in V$:
(i) $0_{\mathbb{F}}v = 0_V$ $\quad(0_{\mathbb{F}}, 0_V$ additive units in $\mathbb{F}$ resp. $V)$,
(ii) $1_{\mathbb{F}}v = v$ $\quad(1_{\mathbb{F}}$ multiplicative unit in $\mathbb{F})$,
(iii) $(\alpha+\beta)v = (\alpha v)+(\beta v)$,
(iv) $\alpha(v+w) = (\alpha v)+(\alpha w)$,
(v) $(\alpha\beta)v = \alpha(\beta v)$.

2.6.6. *Typical examples.*

1. If A is a set then we let $F = \mathrm{Fun}(A,A)$ denote the set of all functions $f\colon A\to A$. We can then introduce the binary composition

$$\begin{array}{rcl} F\times F & \xrightarrow{\circ} & F \\ (f,g) & \mapsto & f\circ g \end{array}$$

where $f\circ g$ is the usual composition of the functions f and g. If id denotes the identity on A, $\mathrm{id}(x)=x\ \forall x\in A$, then we have that $(F,\circ)$ is a monoid with unit id.
2. If

$$G = \{g\in F : g \text{ is a bijection} : A\to A\}$$

then $(G,\circ)$ is a group.
3. Typical examples of rings are $\mathbb{Z}$ (the integers), polynomials in one or several variables with coefficients in a ring, as well as square $(n\times n)$ matrices. If R is a ring, then

we denote $R^{n\times m}$ the set of matrices of size $n \times m$ (n rows, m columns) with entries in R. Hence, $R^{n\times n}$ is a ring, and if $\mathbb{F}$ is a field, then $\mathbb{F}^{n\times m}$ is a vector space over $\mathbb{F}$.

4. Typical examples of fields are $\mathbb{Q}$ (the rational numbers), $\mathbb{R}$ (the real numbers) and $\mathbb{C}$ (the complex numbers), as well as rational functions (quotients of polynomials over nonzero polynomials) in one or several variables with coefficients in some field. If $\mathbb{F}$ is a field then $\mathbb{F}^{n\times n}$ is not a field unless $n = 1$ (why?). Note that $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields, while $\mathbb{C}$ is not ordered (cf. Exercise 3.5).
5. Let M be an arbitrary set and $\mathbb{F}$ a field. Let [corr]

$$V = \mathrm{Fun}(M; \mathbb{F}) := \{f : f \text{ is a function from } M \text{ to } \mathbb{F}\} \subseteq \mathcal{P}(M \times \mathbb{F})$$

the set of $\mathbb{F}$-valued functions on M, and define addition

$$\begin{array}{ccc} V \times V & \xrightarrow{+} & V \\ (f, g) & \mapsto & f + g \end{array}$$

via $(f + g)(m) := f(m) + g(m)$, for $m \in M$, and scalar multiplication

$$\begin{array}{ccc} \mathbb{F} \times V & \xrightarrow{\cdot} & V \\ (\alpha, f) & \mapsto & \alpha \cdot f = \alpha f \end{array}$$

via $(\alpha \cdot f)(m) := \alpha \cdot \big(f(m)\big)$. You may then verify (exercise) that V becomes a vector space over $\mathbb{F}$.

Now introduce the **support** (sv: stödet) of a function $f\colon M \to \mathbb{F}$ by

$$\operatorname{supp} f := \{m \in M : f(m) \neq 0\},$$

and let

$$V_{\text{fin}} := \{f \in V : \operatorname{supp} f \text{ is finite}\}.$$

It is a suitable exercise to verify that if $\alpha \in \mathbb{F}$ and $f, g \in V_{\text{fin}}$ then $\alpha f + g \in V_{\text{fin}}$, i.e. V_{fin} is a subspace of V (cf. below).

It is possible to prove (the axiom of choice is needed) that every vector space can be realized (or represented) as V_{fin} for some set M. In addition M is a basis of V_{fin}, in the sense that the functions $\hat{m}\colon M \to \mathbb{F}$, $m \in M$, defined by

$$\hat{m}(x) = \begin{cases} 1 & \text{if } x = m, \\ 0 & \text{if } x \neq m, \end{cases}$$

form a basis of V_{fin}.

2.6.7. *Linear maps.* A map $L\colon V \to W$, where V and W are vector spaces over $\mathbb{F}$, is called **linear** if

$$L(\alpha x + \beta y) = \alpha L(x) + \beta L(y) \qquad \forall\ \alpha, \beta \in \mathbb{F},\ x, y \in V.$$

The **kernel**, ker, and the **image**, im, of a map $L\colon V \to W$ are defined by

$$\ker L := \{x \in V : L(x) = 0\},$$
$$\operatorname{im} L := \{L(x) \in W : x \in V\}.$$

A subset U of a vector space V is called a **subspace** of V if it is closed under the operations:

$$\alpha, \beta \in \mathbb{F},\ x, y \in U \quad \Rightarrow \quad \alpha x + \beta y \in U.$$

We note (exercise) that $\ker L$ and $\operatorname{im} L$ of a linear map L are subspaces of V resp. W.

If $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{F}$ and $v_1, v_2, \dots, v_n \in V$ then $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$ is called a **linear combination** of the vectors $v_1, v_2, \dots, v_n$. The set (subspace) of linear combinations

$$\{\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n \in V : \alpha_1 \in \mathbb{F}, \dots, \alpha_n \in \mathbb{F}\}$$

is called the **linear span** of $\{v_1, \dots, v_n\}$ and is denoted $\mathrm{Span}_{\mathbb{F}}\{v_1, \dots, v_n\}$ (we drop the subscript $\mathbb{F}$ if it is understood) or $\mathbb{F}v_1 + \mathbb{F}v_2 + \dots + \mathbb{F}v_n$.

2.7. **Quotients.** Let M be a set and $\sim$ an equivalence relation on M. Then we can define the **equivalence class** $[x]$ of $x \in M$ by

$$[x] := \{x' \in M : x \sim x'\}.$$

One verifies (exercise) that

$$x \sim y \;\Rightarrow\; [x] = [y]$$

and that

$$x \nsim y \;\Leftrightarrow\; \neg(x \sim y) \;\Rightarrow\; [x] \cap [y] = \emptyset,$$

in other words the equivalence classes give rise to a partition of M into disjoint subsets. Conversely, one may observe that any partition of M into disjoint subsets gives rise to an equivalence relation (defined $x \sim y$ iff they are in the same subset). Given M and $\sim$, the set of equivalence classes is denoted

$$M/_{\sim} = \big\{[x] \in \mathcal{P}(M) : x \in M\big\},$$

and is also called M **modulo** $\sim$, or the **quotient** of M by $\sim$.

Let $*$ be a binary composition on M and $\sim$ an equivalence relation. We say that $\sim$ **respects** $*$ if

$$x \sim x' \;\&\; y \sim y' \;\Rightarrow\; x * y \sim x' * y' \qquad \forall\, x, y \in M.$$

Then we can define an induced binary composition $\tilde{*}$ on $M/_{\sim}$ through

$$\begin{array}{ccc} M/_{\sim} \times M/_{\sim} & \xrightarrow{\tilde{*}} & M/_{\sim} \\ ([x],[y]) & \mapsto & [x * y] \end{array}$$

This is well defined since if $[x] = [x']$ and $[y] = [y']$, i.e. $x \sim x'$ and $y \sim y'$, then $x*y \sim x'*y'$, i.e. $[x * y] = [x' * y']$.

Exercise 2.2. *Show that if $*$ is commutative, associative, has a unit, etc., and $\sim$ respects $*$, then $\tilde{*}$ is commutative, associative, has a unit, etc.*

2.8. **Cardinality.**

Definition 2.5. We say that two sets X and Y have the same **cardinality** iff there exists a bijection $f\colon X \to Y$. We then write $\mathrm{card}\, X = \mathrm{card}\, Y$ or $|X| = |Y|$ (the latter notation $|X| = \mathrm{card}\, X$ is more common for finite sets). Furthermore, we write $\mathrm{card}\, X \leq \mathrm{card}\, Y$ iff there exists an injection $f\colon X \to Y$.

Exercise 2.3. *Show that*

1) *if* $\mathrm{card}\, X = \mathrm{card}\, Y$ *then* $\mathrm{card}\, Y = \mathrm{card}\, X$
2) *if* $\mathrm{card}\, X = \mathrm{card}\, Y$ *and* $\mathrm{card}\, Y = \mathrm{card}\, Z$ *then* $\mathrm{card}\, X = \mathrm{card}\, Z$
3) $\mathrm{card}\, X = \mathrm{card}\, X$

In other words, the cardinality behaves like an equivalence relation among sets (despite not being able to form the set of all sets).

A set X is called **countable** if $\operatorname{card} X = \operatorname{card} \mathbb{N}$.

Theorem 2.6. *Assume that $A_1, A_2, \ldots$ are countable. Then $\bigcup_{n=1}^{\infty} A_n$ is countable.*

Proof. Write

$$\begin{aligned} A_1 &= \{a_{1,0}, a_{1,1}, a_{1,2}, \ldots\} \\ A_2 &= \{a_{2,0}, a_{2,1}, a_{2,2}, \ldots\} \\ A_3 &= \{a_{3,0}, a_{3,1}, a_{3,2}, \ldots\} \\ &\vdots \end{aligned}$$

and form the zig-zag snaking sequence

$$\{a_{1,0}, a_{1,1}, a_{2,0}, a_{3,0}, a_{2,1}, a_{1,2}, a_{1,3}, a_{2,2}, a_{3,1}, \ldots\},$$

which, after skipping any duplicates (it is a set and not a list), is in bijection to $\cup_n A_n$. □

Example 2.7. In particular, $\mathbb{Q} = \cup_{n \in \mathbb{N}^+}(-Q_n \cup Q_n)$ can be arranged in this way by increasing denominator,

$$\begin{aligned} Q_1 &= \{0/1, 1/1, 2/1, \ldots\}, \\ Q_2 &= \{0/2, 1/2, 2/2, \ldots\}, \\ Q_3 &= \{0/3, 1/3, 2/3, \ldots\}, \end{aligned}$$

etc., so $\operatorname{card} \mathbb{Q} = \operatorname{card} \mathbb{N}$.

Exercise 2.4. a) *Show that $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ have the same cardinality.*
b) *Show that the real interval $(0, 1)$ and $\mathbb{R}$ have the same cardinality.*

Note that $\operatorname{card} X \leq \operatorname{card} \mathcal{P}(X)$, because we can map $f \colon X \ni x \mapsto \{x\} \in \mathcal{P}(X)$ injectively (the image of f is the set of all **singleton subsets** of X). The following fundamental theorem of set theory shows that $\operatorname{card} X \neq \operatorname{card} \mathcal{P}(X)$ (i.e. the power set of X has *strictly* higher cardinality than X).

Theorem 2.8 (**Cantor's theorem**)**.** *Let X be a set. There exists no surjection from X to $\mathcal{P}(X)$.*

Proof. Let $X \xrightarrow{f} \mathcal{P}(X)$ be an arbitrary function and form the set

$$C = \{x \in X : x \notin f(x)\}.$$

Then there exists no $c \in X$ such that $f(c) = C$. Because if $C = f(c)$ were to hold then by definition of C we find

$$c \in C \Rightarrow c \notin C,$$

and

$$c \notin C \Rightarrow c \in C.$$

This contradiction proves the theorem. □

Example 2.9. Starting from $X = \emptyset$ we may form

$$\begin{aligned} \mathcal{P}(\emptyset) &= \{\emptyset\}, \\ \mathcal{P}\big(\mathcal{P}(\emptyset)\big) &= \big\{\emptyset, \{\emptyset\}\big\}, \\ \mathcal{P}\Big(\mathcal{P}\big(\mathcal{P}(\emptyset)\big)\Big) &= \Big\{\emptyset, \{\emptyset\}, \big\{\{\emptyset\}\big\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\}, \end{aligned}$$

and so on. Writing $\mathcal{P}^0(\emptyset) = \emptyset$, $\mathcal{P}^1(\emptyset) = \{\emptyset\}$ and $\mathcal{P}^{n+1}(\emptyset) = \mathcal{P}(\mathcal{P}^n(\emptyset))$ we have $|\mathcal{P}^0(\emptyset)| = 0$ and $|\mathcal{P}^n(\emptyset)| = \underbrace{2\hat{\ }(2\hat{\ }\cdots 2)}_{n-1}$ if $n \geq 1$. More generally, if X is some finite set then $|\mathcal{P}(X)| = 2^{|X|}$.

Given X and Y such that $\operatorname{card} X \leq \operatorname{card} Y$ and $\operatorname{card} Y \leq \operatorname{card} X$ we would like to conclude that $\operatorname{card} X = \operatorname{card} Y$, and indeed this is true as the following (surprisingly deep) theorem shows:

Theorem 2.10 (**Schröder-Bernstein's theorem**). *If $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} X$ are injective then there exists a bijection $X \xrightarrow{h} Y$.*

For a short proof we will use a first instance of a fixpoint theorem:

***Theorem 2.11** (**Tarski's fixpoint theorem**). *Assume that $F\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is monotone increasing, i.e.*

$$A \subseteq B \quad \Rightarrow \quad F(A) \subseteq F(B).$$

Then there exists $M \subseteq X$ such that $F(M) = M$.

**Proof.* Let $\mathcal{I} = \{A \subseteq X : A \subseteq F(A)\}$. Since $\emptyset \subseteq F(\emptyset)$ we have $\emptyset \in \mathcal{I}$ and thus $\mathcal{I}$ is non-empty. Now form the set

$$M = \bigcup_{A \in \mathcal{I}} F(A).$$

If $A \in \mathcal{I}$ then $A \subseteq F(A) \subseteq M$, and by monotonicity $F(A) \subseteq F(M)$. Therefore also $M \subseteq F(M)$, i.e. $M \in \mathcal{I}$. We conclude that $M \subseteq F(M) \subseteq M$. ☐

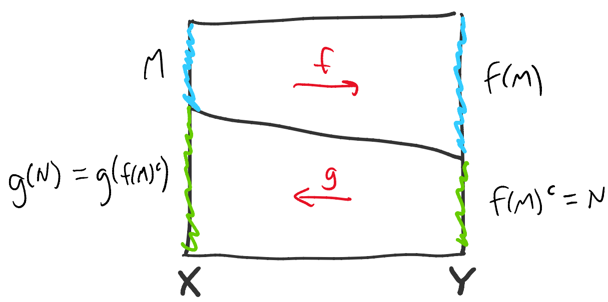
**Proof of Theorem 2.10.* Define $F\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$F(A) := \big(g(f(A)^c)\big)^c$$

(where $f(A)^c = Y \setminus f(A)$ and $g(f(A)^c)^c = X \setminus g(f(A)^c)$). Then, by injectivity of f and g,

$$\begin{aligned} A \subseteq B \ &\Rightarrow \ f(A) \subseteq f(B) \ \Rightarrow \ f(A)^c \supseteq f(B)^c \ \Rightarrow \ g(f(A)^c) \supseteq g(f(B)^c) \\ &\Rightarrow \ g(f(A)^c)^c \subseteq g(f(B)^c)^c. \end{aligned}$$

Hence F is monotone increasing. By Theorem 2.11 there then exists $M \subseteq X$ such that $F(M) = M$. Let $N = f(M)^c$ and consider the figure



We must then have that $M \xrightarrow{f} f(M)$ is bijective, call it $f|_M$ (the restriction of f to M), and so is $N \xrightarrow{g} g(N)$, call it $g|_N$. Now define a map $h\colon X \to Y$ by

$$h(x) := \begin{cases} f|_M(x) & \text{if } x \in M, \\ (g|_N)^{-1}(x) & \text{if } x \in M^c, \end{cases}$$

which is a bijection. □

Exercise 2.5. *Let $p \in \mathbb{R}$ and show that $\mathbb{R}$ and $\mathbb{R} \setminus \{p\}$ have the same cardinality. Also show that if $a < b$ then $[a, b]$ and (a, b) have the same cardinality.*

2.9. ***Rough sketch of ZFC.** When mathematics was put on a sound logical foundation through works of Cantor, Dedekind, Frege, Gödel, Russel, Weierstrass and others (late 1800's to early 1900's), it was eventually understood that sets cannot be arbitrarily defined. Below is a rough sketch of **Zermelo and Fraenkel**'s proposal for an **axiomatic system of sets** (for more details on this topic we refer to courses in mathematical logic, or e.g. https://en.wikipedia.org/wiki/Zermelo%E2%80%93Fraenkel_set_theory):

1. There exists a set $\emptyset$, called the **empty set**, which contains no elements. That is,
$$\forall x \; x \notin \emptyset$$
2. Two sets are **equal** if and only if they have the same elements, i.e.
$$A = B \quad \Leftrightarrow \quad \forall x \; (x \in A \Leftrightarrow x \in B)$$
3. The **union** of two sets is a set, i.e. if A and B are sets then there exists a set C, denoted $A \cup B$, such that
$$\forall x \; \big(x \in A \cup B \Leftrightarrow (x \in A \vee x \in B)\big)$$
4. The **intersection** of two sets is a set, i.e. if A and B are sets then there exists a set C, denoted $A \cap B$, such that
$$\forall x \; \big(x \in A \cap B \Leftrightarrow (x \in A \;\&\; x \in B)\big)$$
5. All subsets of a given set form a new set, called the **power set** (sv: potensmängden), i.e. if A is a set then there exists a set B, denoted $\mathcal{P}(A)$, such that
$$\forall x \; \big(x \in \mathcal{P}(A) \Leftrightarrow x \subseteq A\big)$$
(note that $x \subseteq A$ means by definition that $\forall y \; (y \in x \Rightarrow y \in A)$).
6. If P is a one-variable predicate (property, sv: egenskap/ett-ställigt predikat), i.e. (in formal logic) a well-formulated formula in our formal language with one free variable x, and if A is a set, then there exists a set A_P whose elements are exactly those elements x in A such that $P(x)$. We denote this set
$$A_P = \{x \in A : P(x)\}.$$
7. If A is a set then there exists a set B, denoted $\{A\}$, whose only element is A. That is,
$$\forall x \; \big(x \in \{A\} \Leftrightarrow x = A\big).$$
8. If A and B are sets then there exists a set C, denoted $A \times B$ and called the **cartesian product** of A and B, whose elements consist of all ordered pairs (a, b) with $a \in A$ and $b \in B$.
9. **Axiom of choice**: every surjection has a right inverse.
10. **Axiom of infinity**: There exists a set $\mathbb{N}$, called the **natural numbers**, such that $\emptyset \in \mathbb{N}$ and
$$\forall x \; \big(x \in \mathbb{N} \Rightarrow x \cup \{x\} \in \mathbb{N}\big).$$

Remark to 6.: This axiom protects us from **Russel's paradox**. Namely, Gottlob Frege initially thought that one may form sets more arbitrarily, so that for example

$$R = \{x : x \notin x\}$$

could be a set. But then Bertrand Russel observed that this yields a contradiction:

$$R \in R \Rightarrow R \notin R \qquad \text{and} \qquad R \notin R \Rightarrow R \in R.$$

Remarks to 9.: In accordance with our conventions, we call a function $g\colon B \to A$ a right inverse to $f\colon A \to B$ if $f \circ g = \mathrm{id}_B$, where id_B is the identity on B, $\mathrm{id}_B(y) = y \; \forall y \in B$. g is called left inverse to f if $g \circ f = \mathrm{id}_A$ and inverse to f if it is both left and right inverse to f.

Some mathematicians prefer to leave out the axiom of choice or use weaker versions of it. We will probably not discuss this further but may use **ZFC** and **ZF** to refer to Zermelo-Fraenkel (ZF) with and without the axiom of choice (C).

Remarks to 10.: The set $x' := x \cup \{x\}$ is called the **successor** to x, and we can make the following formal definition, that was proposed by von Neumann, for the natural numbers $\mathbb{N}$:

$$\begin{aligned} 0 &:= \emptyset, \\ 1 &:= 0' = \{0\}, \\ 2 &:= 1' = \{0, 1\}, \\ &\vdots \\ n+1 &:= n' = \{0, 1, \dots, n\}, \\ &\vdots \end{aligned}$$

This also gives a natural total order on $\mathbb{N}$ defined by $\forall n, m \in \mathbb{N}$

$$n \leq m \quad \Leftrightarrow \quad n \subseteq m.$$

Note: we are interested in the existence and order of $\mathbb{N}$, while in practice we forget about this set structure and shall never write $2 \in 5$ or things like that.

3. Definition and properties of the real numbers [L2-5]

3.1. **Reading tip.** First make sure you understand and feel comfortable with the basic terminology of Section 2. Compare to your previous courses and/or other resources such as wikipedia (but keep in mind varying conventions!). Also check that you are comfortable with the summation symbol notation, e.g. $\sum_{j=a}^{b}\sum_{k=j^2}^{j^2+5} e^{j+k}$ [Rud76, Notation 1.34].

Read/compare to [Rud76, Chapter 2 up to 2.15] for properties of sets, functions and cardinality.

It is advised to read [Rud76, Chapter 1] in conjunction with Section 3.2 below. The parts about complex numbers should be no surprise but may be skimmed over as the focus in this course is *real* analysis.

See [Rud76, Chapter 1 Appendix] for details on the construction of the reals, and (optionally) [Abb15, Ch. 1.7] for further discussion on the logical issues involved.

Parts of [Rud76, Chapter 3, up to 3.19] enter already in this section for real sequences and will be generalized to $\mathbb{R}^n$ and metric spaces in Section 4. In particular, recall/practice the proof of [Rud76, Theorem 3.3]!

3.1.1. *Typos in Rudin.*

- in Eq. (4) the square is misplaced
- in Definition 1.5 (ii) it should be $y < z$
- in Theorem 1.21 it should be stressed that $y > 0$
- in Exercise 3.3 the $<$ is misplaced

3.1.2. *Exercises.* Rudin Ch. 1: 1-7,20; Ch. 2: 1,4-5; Ch. 3: 1-5
Exam 2014-04-23: problems 1-2. Exam 2019-06-15: problems 1-2. Exam 2020-06-15: problem 2. Exam 2015-03-21: problem 3.

3.1.3. *Aims.* Concepts discussed in this Section:

- definitions and properties of real numbers
- Cauchy sequences (in $\mathbb{R}$)
- upper and lower limits

Learning outcomes: After this Section you should be able to

- describe the construction and properties of real numbers
- apply the theory to solve mathematical problems including the construction of simple proofs

3.2. **Constructing numbers.** Our route to constructing numbers will be the following:

$$\begin{array}{rll} & \text{ZF(C)} & \\ \text{deeper} \to & \downarrow & \\ & \mathbb{N} & = \{0,1,2,\ldots\} \\ \text{simple} \to & \downarrow & \\ & \mathbb{Q}_+ & = \{p/q : p,q \in \mathbb{N}, q \neq 0\} \\ \text{deeper} \to & \downarrow & \\ & \mathbb{R}_+ & = \{x \in \mathbb{R} : x \geq 0\} \\ \text{simple} \to & \downarrow & \\ & \mathbb{R} & \supseteq \mathbb{Q} \supseteq \mathbb{Z} \supseteq \mathbb{N} \end{array}$$

$\mathbb{R}$ may also be defined by the axioms we wish that it should have, or as a type of completion of $\mathbb{Q}$.

Note that the deeper steps are "transcendent" in the sense that we achieve a lift from something finite to countably infinite, and then to uncountably infinite.

3.2.1. *Construction of* $\mathbb{Z}$ *and* $\mathbb{Q}$. Starting from $\mathbb{N}$ with its usual laws for addition $(+)$ and multiplication $(\cdot)$, we may define $\mathbb{Z}$ and $\mathbb{Q}$ simply using equivalence relations and quotients.

As a preparation we first consider the integers $\mathbb{Z}$. Our motivation is to study pairs of natural numbers:

$$\mathbb{N} \times \mathbb{N} \ni (x, y) \quad \underline{\text{should correspond to}} \quad x - y \in \mathbb{Z}. \tag{3.1}$$

Thus, if (a, b) corresponds to $a - b \in \mathbb{Z}$ and (c, d) to $c - d \in \mathbb{Z}$ then it makes sense to demand that

$$(a, b) + (c, d) := (a + c, b + d) \quad \underline{\text{should correspond to}} \quad (a - b) + (c - d) = (a + c) - (b + d). \tag{3.2}$$

Furthermore, the product $(a - b)(c - d) = ac + bd - (ad + bc)$ in $\mathbb{Z}$ may be represented in $\mathbb{N} \times \mathbb{N}$ as

$$(a, b) \cdot (c, d) := (ac + bd, ad + bc). \tag{3.3}$$

However, we have several ways of writing the same number $x - y \in \mathbb{Z}$ as a pair of natural numbers $x, y \in \mathbb{N}$. To make things consistent we also want that pairs be identified $x - y = x' - y'$ iff $x + y' = x' + y$. Note that the latter can be formulated completely within $(\mathbb{N}, +)$. Thus, define an equivalence relation $\sim$ on $\mathbb{N} \times \mathbb{N}$ by

$$(x, y) \sim (x', y') \quad \Leftrightarrow \quad x + y' = x' + y. \tag{3.4}$$

We may check that with this definition, $\sim$ respects not only addition but also multiplication, namely if $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$ are pairs in $\mathbb{N} \times \mathbb{N}$, then

$$(a + c, b + d) \sim (a' + c', b' + d') \tag{3.5}$$

and

$$(ac + bd, ad + bc) \sim (a'c' + b'd', a'd' + b'c'). \tag{3.6}$$

Namely the latter translates within $(\mathbb{N}, +, \cdot)$ to

$$ac + bd + a'd' + b'c' = a'c' + b'd' + ad + bc \tag{3.7}$$

which $\underline{\text{should correspond to}}$

$$ac + bd - ad - bc = a'd' + b'c' - a'c' - b'd' \quad \Leftrightarrow \quad (a - b)(c - d) = (a' - b')(c' - d'). \tag{3.8}$$

Thus we define the set of all integers $\mathbb{Z}$ as the quotient by $\sim$ (identifying equivalent pairs)

$$\boxed{\mathbb{Z} := \mathbb{N} \times \mathbb{N}/_{\sim},}$$

and corresponding laws for addition and multiplication

$$\begin{array}{rcl} \mathbb{Z} \times \mathbb{Z} & \to & \mathbb{Z} \\ \big([(a, b)], [(c, d)]\big) & \stackrel{+}{\mapsto} & [(a + c, b + d)], \\ \big([(a, b)], [(c, d)]\big) & \stackrel{\cdot}{\mapsto} & [(ac + bd, ad + bc)]. \end{array}$$

Eventually we may conclude that this construction yields our usual laws for the integers, and with the injection $\mathbb{N} \ni x \mapsto [(x,0)] \in \mathbb{Z}$ we may also *define* our usual way of writing $x - y := [(x,y)] \in \mathbb{Z}$ for any $x, y \in \mathbb{N}$. This gives rise to an **embedding**, or identification,

$$\mathbb{N} \hookrightarrow \mathbb{Z}. \tag{3.9}$$

A total order on $\mathbb{Z}$ is given by

$$[(x,y)] \leq [(x',y')] \quad \Leftrightarrow \quad x + y' \leq x' + y. \tag{3.10}$$

Exercise 3.1. *Fill in the details in this construction! That is,*

a) *Verify that* (3.4) *defines an equivalence relation.*
b) *Verify* (3.5) *and* (3.6) *(note that you are not allowed to use subtraction freely yet!).*
c) *Verify that addition and multiplication defined on* $\mathbb{Z}$ *as above is well defined, commutative, associative, and that multiplication is distributive over addition. Also check that* $[(0,0)]$ *is additive unit (sv: nolla) and* $[(1,0)]$ *is multiplicative unit (sv: etta).*
d) *Verify that* (3.10) *is a well-defined total order.*

Similarly, for the rational numbers $\mathbb{Q}$, our motivation is to study pairs of integers:

$$\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \ni (p,q) \quad \underline{\text{should correspond to}} \quad \frac{p}{q} \in \mathbb{Q},$$

where we would like that $p/q = p'/q'$ iff $pq' = p'q$. The latter is formulated completely within $(\mathbb{Z}, +, \cdot)$ that we constructed above. Thus, define an equivalence relation $\sim$ on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$

$$(p,q) \sim (p',q') \quad \Leftrightarrow \quad pq' = p'q. \tag{3.11}$$

Then define the quotient

$$\boxed{\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})/\sim},$$

with suitably defined operations for addition and multiplication,

$$\begin{array}{rcl} \mathbb{Q} \times \mathbb{Q} & \to & \mathbb{Q} \\ \big([(a,b)],[(c,d)]\big) & \overset{+}{\mapsto} & [(ad+bc, bd)], \\ \big([(a,b)],[(c,d)]\big) & \overset{\cdot}{\mapsto} & [(ac, bd)]. \end{array}$$

Finally we may identify our usual way of writing $p/q := [(p,q)] \in \mathbb{Q}$ for any $p, q \in \mathbb{Z}$, $q \neq 0$. Again, we have an embedding $\mathbb{Z} \ni p \mapsto [(p,1)] \in \mathbb{Q}$ and, recalling (3.9),

$$\mathbb{N} \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q}.$$

Exercise 3.2. *Again, fill in the details in this construction, repeating the steps of Exercise 3.1.*

3.2.2. *Dedekind's construction of* $\mathbb{R}$. Let

$$\mathbb{Q}_+ := \{p/q \in \mathbb{Q} : p, q \in \mathbb{N}, q \neq 0\}$$

denote the **non-negative rational numbers** with their natural total order induced from $\mathbb{N}$,

$$p/q \leq p'/q' \quad \Leftrightarrow \quad pq' \leq p'q, \tag{3.12}$$

as well as the induced strict total order $p/q < p'/q'$ iff $pq' < p'q$.

Definition 3.1. A proper subset $S \subsetneq \mathbb{Q}_+$ is called a (Dedekind) **cut** (sv: snitt) if

(i) $\forall x \in S\ \forall y \in \mathbb{Q}_+\ (y \leq x \Rightarrow y \in S)$ (closed/filling all rationals to the left)

(ii) $\forall x \in S\ \exists y \in S\ x < y$ (open to the right)

Thus, intuitively, a cut is similar to a half-open interval "$[0, r+\varepsilon)$" of rational numbers.

Example 3.2. To each $r \in \mathbb{Q}_+$ we may define the cut

$$\hat{r} := \{x \in \mathbb{Q}_+ : x < r\},$$

so that for example $\hat{0} = \emptyset$ is a cut (the **zero cut**). However not all cuts are of this form! For example we have also the cut

$$\{x \in \mathbb{Q}_+ : x^2 < 2\},$$

which can be shown to be of a different type (recall that $\sqrt{2} \notin \mathbb{Q}$).

Definition 3.3. Let us define the set of all **non-negative real numbers** as the set of all cuts,

$$\boxed{\mathbb{R}_+ := \{S \in \mathcal{P}(\mathbb{Q}_+) : S \text{ is a cut}\}.}$$

Furthermore, addition in $\mathbb{R}_+$ is defined by

$$\begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R}_+ & \xrightarrow{+} & \mathbb{R}_+ \\ (S,T) & \mapsto & S + T := \{x + y \in \mathbb{Q}_+ : x \in S, y \in T\}, \end{array}$$

and multiplication by

$$\begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R}_+ & \xrightarrow{\cdot} & \mathbb{R}_+ \\ (S,T) & \mapsto & S \cdot T := \{xy \in \mathbb{Q}_+ : x \in S, y \in T\}. \end{array}$$

We may also define an order $\leq$ on cuts via inclusion

$$S \leq T :\Leftrightarrow S \subseteq T. \tag{3.13}$$

One may verify that indeed $S + T$ and $S \cdot T$ are cuts, and $\leq$ a total order, and show that addition and multiplication are commutative and associative. Furthermore, $\hat{0} = \emptyset$ is the **additive unit**, while

$$\hat{1} = \{x \in \mathbb{Q}_+ : x < 1\}$$

is the **multiplicative unit**. One could also formally define (positive) **infinity** as the improper cut

$$\infty = \mathbb{Q}_+.$$

As always the total order $\leq$ induces a strict total order

$$S < T :\Leftrightarrow S \subsetneq T. \tag{3.14}$$

We also note that if $r_1, r_2 \in \mathbb{Q}_+$ and $r_1 \neq r_2$ then $\hat{r_1} \neq \hat{r_2}$, in other words $r \mapsto \hat{r}$ gives an injective map or embedding $\mathbb{Q}_+ \hookrightarrow \mathbb{R}_+$.

In this way we have constructed the non-negative real numbers $\mathbb{R}_+$. To construct all of $\mathbb{R}$ we can proceed exactly as we did for $\mathbb{Z}$ by means of pairs of numbers, replacing $(\mathbb{N}, +, \cdot)$ by $(\mathbb{R}_+, +, \cdot)$. Namely, consider the equivalence relation $\sim$ on $\mathbb{R}_+ \times \mathbb{R}_+$:

$$(x,y) \sim (x',y') \quad \Leftrightarrow \quad x + y' = x' + y. \tag{3.15}$$

Definition 3.4. Let us define the set of **real numbers**

$$\boxed{\mathbb{R} := \mathbb{R}_+ \times \mathbb{R}_+/_\sim = \{[(x,y)] : (x,y) \in \mathbb{R}_+ \times \mathbb{R}_+\},}$$

with addition and multiplication

$$\begin{array}{rcl} \mathbb{R}\times\mathbb{R} & \to & \mathbb{R} \\ \big([(x,y)],[(x',y')]\big) & \stackrel{+}{\mapsto} & [(x+x',y+y')], \\ \big([(x,y)],[(x',y')]\big) & \stackrel{\cdot}{\mapsto} & [(xx'+yy',xy'+x'y)]. \end{array}$$

A total order on $\mathbb{R}$ is defined by

$$[(x,y)] \le [(x',y')] \quad \Leftrightarrow \quad x+y' \le x'+y. \tag{3.16}$$

We may verify that $(\mathbb{R},+,\cdot,\le)$ is an ordered field. Alternatively, we could have started from all of $\mathbb{Q}$ and defined cuts there (this is what Rudin does) but some of the process becomes a bit more tedious when there are also negative numbers to worry about. See [Rud76, Chapter 1 Appendix] (with additional references) for further discussion.

The **extended real numbers** are defined with the additional two symbols

$$\boxed{\mathbb{R}_\infty := \mathbb{R} \cup \{-\infty,\infty\},}$$

and a few additional rules like (cf. [Rud76, p.12])

$$x \pm \infty = \pm\infty, \quad x/(\pm\infty) = 0, \quad x\cdot(\pm\infty) = (\pm\operatorname{sign} x)\infty, \tag{3.17}$$

for any $x \in \mathbb{R}$ in the first two, while the third one is only defined if $x \neq 0$,

$$\operatorname{sign} x := \begin{cases} +1, & x>0, \\ -1, & x<0. \end{cases} \tag{3.18}$$

Note that $\mathbb{R}_\infty$ is not a field or ring, but is anyway handy when discussing limits for example. Its order is given by

$$-\infty < x < \infty \qquad \forall x \in \mathbb{R}. \tag{3.19}$$

Exercise 3.3. *Show that an arbitrary union of cuts is also a cut, unless it is all of $\mathbb{Q}_+$ (an improper cut; recall that $\emptyset = \hat{0}$ is also a (proper) cut).*

Exercise 3.4. *Verify the necessary properties for addition and multiplication on $\mathbb{R}_+$. What is the multiplicative inverse S^{-1} of a nonzero cut S?*

***Exercise 3.5.** *[Rudin Ch. 1 Exc. 8] Prove that no order can be defined in the complex field $\mathbb{C}$ that turns it into an ordered field.*

3.3. **Supremum and infimum.**

Definition 3.5. Given a totally ordered set $(M,\le)$, and a subset $A \subseteq M$ we say that:

- $x \in M$ is an **upper bound** to A if $x \ge a$ $\forall a \in A$ (A is bounded from above by x).
- $x \in M$ is a **lower bound** to A if $x \le a$ $\forall a \in A$ (A is bounded from below by x).
- $x \in M$ is a **least upper bound** to A, or its **supremum**, denoted $\sup A$, if it is an upper bound to A and, if y is another upper bound to A, then $y \ge x$ (i.e. either $y = x$ or $y > x$). Equivalently,
  (i) $\forall a \in A$ $a \le x$ (upper bound)
  (ii) $\forall y \in M$ $y < x \Rightarrow \exists a \in A : y < a$ (anything smaller is not an upper bound).

- $x \in M$ is a **greatest lower bound** to A, or its **infimum**, denoted $\inf A$, if it is a lower bound to A and, if y is another lower bound to A, then $y \leq x$ (i.e. either $y = x$ or $y < x$).

A totally ordered set $(M, \leq)$ is said to have the **supremum property** if for every nonempty subset $A \subseteq M$ that is bounded from above there exists $\sup A \in M$ (and if so, it also has the corresponding infimum property; see [Rud76, Theorem 1.11])

Example 3.6. Consider the ordered field $(\mathbb{Q}, \leq)$ and the subset (cut)

$$S = \{p/q \in \mathbb{Q}_+ : p^2/q^2 < 2\}.$$

Then S has an upper bound such as $b = 3/2 \in \mathbb{Q}$ ($b^2 = 9/4 > 2 > p^2/q^2$), however S does not have a least upper bound in $\mathbb{Q}$ (cf. [Rud76, Example 1.1]).

The most distinct feature of $\mathbb{R}$ (apart from being an ordered field) is that it has the supremum and infimum properties.

Theorem 3.7 (Supremum property for $\mathbb{R}$)**.** *Every nonempty subset $A \subseteq \mathbb{R}$ which is bounded from above has a least upper bound* $\sup A \in \mathbb{R}$.

Proof. We assume WLOG that $A \subseteq \mathbb{R}_+$, considered as the set of cuts in $\mathbb{Q}_+$. Denote an upper bound to A by the cut $B \subsetneq \mathbb{Q}_+$, i.e. $T \in A \Rightarrow T \subseteq B$. Define

$$S := \cup A := \{x \in \mathbb{Q}_+ : \exists T \in A \text{ s.t. } x \in T\},$$

i.e. the union of all the cuts in A.

If $A = \{\hat{0}\}$ then $S = \hat{0} = \emptyset$. Otherwise, S contains the elements of at least one nonzero cut. Further, by the existence of $B \supseteq S$ we have $S \neq \mathbb{Q}_+$. Thus by Exercise 3.3, S is a cut.

Also, we verify that $T \leq S$ i.e. $T \subseteq S$ for any $T \in A$, i.e. S is indeed an upper bound for A. Furthermore, we verify that if $T < S$ i.e. $T \subsetneq S$ then there exists some $T' \in A$ and $x \in T'$ such that $x \notin T$. Hence $T \subsetneq \hat{x} \subsetneq T'$, i.e. $T < T' \in A$, and T is not an upper bound for A. Therefore $S = \sup A$. □

Remark 3.8. In the case that $A \neq \emptyset$ is not bounded from above we find that $S = \mathbb{Q}_+ = \infty$, and indeed in this case we can *define* $\sup A := +\infty \in \mathbb{R}_\infty$. Similarly, if A is unbounded from below we define $\inf A := -\infty \in \mathbb{R}_\infty$.

As applications consider [Rud76, Theorems 1.20 and 1.21]:

Theorem 3.9. *The following holds for real numbers:*

(a) *If $x, y \in \mathbb{R}$ and $x > 0$ then there exists $n \in \mathbb{N}^+$ such that $nx > y$. (**Archimedean property** of $\mathbb{R}$.)*
(b) *If $x, y \in \mathbb{R}$ and $x < y$ then there exists $q \in \mathbb{Q}$ such that $x < q < y$. (We say $\mathbb{Q}$ is **dense** in $\mathbb{R}$.)*
(c) *For every $x \in \mathbb{R}^+$ ($x > 0$) and every $n \in \mathbb{N}^+$ there exists a unique $y \in \mathbb{R}^+$ such that $y^n = x$. (We call y the n:th **root** of x.)*

By uniqueness we may denote the n:th root of $x > 0$ by

$$\sqrt[n]{x} = x^{1/n} = \{q \in \mathbb{Q}_+ : q^n < x\} \in \mathbb{R}^+.$$

Exercise 3.6. *Let A and B be nonempty sets of real numbers and assume that $a \leq b$ for all $a \in A$ and $b \in B$. Prove that* $\sup A$ *and* $\inf B$ *are real numbers (i.e. not $\pm\infty$) and that* $\sup A \leq \inf B$.

***Exercise 3.7** (Difficult!). *Given $x \in \mathbb{R}$, prove that*

$$\inf\big\{|(2p+1)x-2q| : p,q \in \mathbb{Z}\big\} = \begin{cases} 1/n, & \text{if } x = m/n \in \mathbb{Q} \text{ is a reduced rational with } m \text{ \underline{odd}}, \\ 0, & \text{otherwise.} \end{cases}$$

This is a variant of ***Thomae's popcorn function****, and appears in some recent problems in quantum mechanics* [Lun19, Section 5.6].

3.4. **Sequences of real numbers.** A function

$$\begin{array}{ccc} \mathbb{N} & \to & \mathbb{R} \\ n & \mapsto & x_n \end{array} \tag{3.20}$$

is called a **sequence** (in $\mathbb{R}$ / of real numbers), and may also be written either

$$x_0, x_1, x_2, \ldots, \qquad \text{or} \qquad x_n,\ n = 0,1,2\ldots, \qquad \text{or} \qquad (x_n)_{n=0}^{\infty}, \tag{3.21}$$

or simply $(x_n)_n$ or (x_n) if it is understood that $n \in \mathbb{N}$. It is equally common and logical to index sequences using $n \in \mathbb{N}^+$, and they may also be interpreted as countably infinite lists.

Remark 3.10. A common convention (used by Rudin for example) is to write sequences using curly brackets, i.e.

$$\{x_0, x_1, x_2, \ldots\} \qquad \text{or} \qquad \{x_n\}_{n=0}^{\infty}, \qquad \text{or simply} \qquad \{x_n\},$$

which will be acceptable due to its widespread use, however strictly speaking we would like to reserve this notation for sets (what will then be the difference between (x_n) and $\{x_n\}$?).

A sequence $x_0, x_1, x_2, \ldots$ is called

- **bounded** if $\exists B \in \mathbb{R}$ s.t. $|x_n| < B \quad \forall n \in \mathbb{N}$
- **bounded from above** if $\exists B \in \mathbb{R}$ s.t. $x_n < B \quad \forall n \in \mathbb{N}$
- **bounded from below** if $\exists B \in \mathbb{R}$ s.t. $x_n > B \quad \forall n \in \mathbb{N}$
- **increasing** if $x_n \leq x_m$ if $n \leq m$
- **decreasing** if $x_n \geq x_m$ if $n \leq m$
- **strictly increasing/decreasing** if $x_n < x_{n+1}$ resp. $x_n > x_{n+1} \quad \forall n \in \mathbb{N}$
- **monotonic** if increasing or decreasing
- **convergent** if $\exists y \in \mathbb{R}$ such that
$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n\big(n \geq N \Rightarrow |x_n - y| < \varepsilon\big).$$
We then say that the sequence (x_n) **converges** to y, or $x_n \to y$ as $n \to \infty$, and call y the **limit** of (x_n). We also write $\lim_{n\to\infty} x_n = y$.
- **Cauchy** if
$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n \geq N\ \forall m \geq M \quad |x_n - x_m| < \varepsilon.$$

Theorem 3.11. *Every bounded monotonic sequence of real numbers is convergent.*

Proof. Let us assume that $x_n \leq x_{n+1} \leq B \in \mathbb{R}$ for all $n \in \mathbb{N}$. Then the set

$$E = \{x_n : n \in \mathbb{N}\} \tag{3.22}$$

is a non-empty and bounded from above subset of $\mathbb{R}$. Hence, by Theorem 3.7, there exists $\sup E = y \in \mathbb{R}$ such that $x_n \leq y\ \forall n \in \mathbb{N}$. Further, if $\varepsilon > 0$ we have $y - \varepsilon < y$, so there exists $N \in \mathbb{N}$ such that $y-\varepsilon < x_N$ (because otherwise we would have found an even smaller upper bound to the sequence). In other words,

$$y - \varepsilon < x_N \leq x_n \leq y \quad \forall\, n \geq N, \tag{3.23}$$

which proves that $x_n \to y$ as $n \to \infty$.

The case that $x_n \geq x_{n+1} \geq B$ is completely analogous. □

Theorem 3.12. *Every Cauchy sequence in $\mathbb{R}$ is bounded.*

Proof. Assume $x_0, x_1, x_2, \dots$ is Cauchy, and choose $\varepsilon = 1$. Then, by definition, there exists $N \in \mathbb{N}$ s.t. $|x_n - x_m| < 1$ if $n, m \geq N$. In particular, we have $|x_n - x_N| < 1$ for all $n \geq N$. Hence

$$-1 < x_n - x_N < 1 \qquad \forall\, n \geq N, \tag{3.24}$$

i.e.

$$x_N - 1 < x_n < x_N + 1 \qquad \forall\, n \geq N. \tag{3.25}$$

This shows that the sequence (x_n) is bounded, because its first part $(x_0, x_1, \dots, x_{N-1})$ is a finite list and thus also bounded (by its maximum resp. minimum). □

Theorem 3.13. *Convergent sequences in $\mathbb{R}$ are Cauchy.*

Proof. Assume that $x_n \to y$ as $n \to \infty$. Take $\varepsilon > 0$, then $\exists\, N \in \mathbb{N}$ s.t. $|x_n - y| < \varepsilon$ if $n \geq N$. Hence, if $n, m \geq N$ we have

$$|x_n - x_m| = |x_n - y + y - x_m| \leq |x_n - y| + |y - x_m| < \varepsilon + \varepsilon, \tag{3.26}$$

by the triangle inequality. This proves that (x_n) is Cauchy. □

Corollary 3.14. *Convergent sequences in $\mathbb{R}$ are bounded.*

We also prove a useful immediate consequence of Theorem 3.11:

Theorem 3.15 (Interval enclosure / Intervallkapsling; [Rud76, Thm. 2.38])**.** *Let*

$$I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}, \quad n \in \mathbb{N}, \tag{3.27}$$

be a sequence of closed intervals, and assume that $I_n \supseteq I_{n+1} \neq \emptyset$ for all $n \in \mathbb{N}$. Then there exists $c \in \mathbb{R}$ which is contained in all I_n, i.e.

$$\bigcap_{n \in \mathbb{N}} I_n \neq \emptyset. \tag{3.28}$$

Proof. We have $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. This means that (a_n) is an increasing and bounded from above sequence, which is convergent by Theorem 3.11:

$$a_n \to a^*, \quad n \to \infty, \tag{3.29}$$

for some $a^* \in \mathbb{R}$. Furthermore, it must hold that $a^* \leq b_n\ \forall n \in \mathbb{N}$.

In the same way we see that

$$b_n \to b^*, \quad n \to \infty, \tag{3.30}$$

for some $b^* \in \mathbb{R}$, for which $a^* \leq b^*$.

It follows that $[a^*, b^*] = \bigcap_{n \in \mathbb{N}} I_n$ and c may be choosen arbitrarily within this interval (or it happens that $c = a^* = b^*$). □

Remark 3.16. Note that it is necessary in the above theorem that the intervals are closed and nonempty, since e.g.

$$\bigcap_{n>0}(0,1/n) = \{x \in \mathbb{R} : \forall n > 0\ (0 < x < 1/n)\} = \emptyset.$$

We ask you to review a few of the well-known properties of limits (see [Rud76, Theorem 3.3] for the proofs (where you may replace complex by real if you want), however try to do them yourself first to practice the concept of limits):

Theorem 3.17 (Limit arithmetic)**.** *If (x_n) and (y_n) are sequences in $\mathbb{R}$ and if $\lim_{n\to\infty} x_n = x \in \mathbb{R}$ and $\lim_{n\to\infty} y_n = y \in \mathbb{R}$ then*

(a) $\lim_{n\to\infty}(x_n + y_n) = x + y$,
(b) $\lim_{n\to\infty} cx_n = cx$ *for any* $c \in \mathbb{R}$,
(c) $\lim_{n\to\infty}(x_n y_n) = xy$,
(d) $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$ *provided that* $x_n \neq 0\ \forall n$ *and* $x \neq 0$.

3.4.1. *Subsequences.* If (x_n) is a sequence in $\mathbb{R}$ and the function

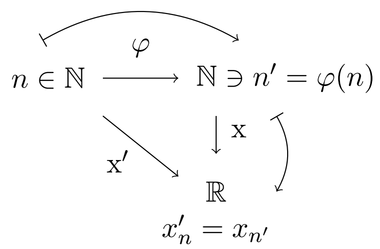
$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\varphi} & \mathbb{N} \\ n & \mapsto & n' \end{array} \tag{3.31}$$

is strictly increasing, i.e.

$$n_1 < n_2 \quad \Rightarrow \quad n'_1 = \varphi(n_1) < \varphi(n_2) = n'_2, \tag{3.32}$$

then we call the sequence $\mathrm{x} \circ \varphi \colon \mathbb{N} \to \mathbb{R}$ a **subsequence** of $\mathrm{x} = (x_n)$. In other words, the sequence $x_0, x_1, x_2, \ldots$ has a subsequence $x_{0'}, x_{1'}, x_{2'}, \ldots$ if for any pair of indices $i < j \Rightarrow i' < j'$ (and we can then write $i' = \varphi(i)$ as above).

Another way to illustrate this concept is with a diagram:



where $(x'_n) = \mathrm{x}' = \mathrm{x} \circ \varphi$ is the subsequence to $\mathrm{x} = (x_n)$.

Example 3.18. Consider the sequence $a = (a_n)$ with subsequence $a' = (a'_n) = (a_{n'})$:

$$\begin{array}{ccccccccc} a_0, & a_1, & a_2, & a_3, & a_4, & a_5, & a_6, & a_7, & \ldots \\ & \parallel & & \parallel & \parallel & & \parallel & & \\ & a'_0 & & a'_1 & a'_2 & & a'_3 & & \ldots \\ & \parallel & & \parallel & \parallel & & \parallel & & \\ & a_{0'} & & a_{1'} & a_{2'} & & a_{3'} & & \ldots \end{array}$$

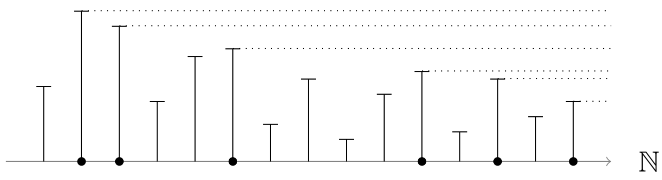
that is, $0' = 1, 1' = 3, 2' = 4, 3' = 6$, etc.

Lemma 3.19. *Every sequence of real numbers contains a monotonic subsequence.*

Proof. Let $x_0, x_1, x_2, \ldots$ be a sequence in $\mathbb{R}$ and define the set

$$P = \{n \in \mathbb{N} : n \leq m \Rightarrow x_n \geq x_m\}. \tag{3.33}$$

A mental image of P is



If P is infinite, i.e. $\exists 0' < 1' < 2' < \ldots$, $i' \in P$, then $x_{0'} \geq x_{1'} \geq x_{2'} \geq \ldots$, in other words we have found a decreasing subsequence.

If P is finite then there exists some $N \in \mathbb{N}$ such that $n \geq N \Rightarrow n \notin P$. That is, $\forall n \geq N\ \exists m > n$ with $x_n < x_m$. But this means that we can construct a strictly increasing subsequence, because if we have chosen $0' < 1' < \ldots < n'$ such that $N \leq 0'$ and $x_{0'} < x_{1'} < \ldots < x_{n'}$ then we know there exists $(n+1)' > n'$ such that $x_{n'} < x_{(n+1)'}$. □

Note also that if a sequence converges to a limit $x_n \to y$, $y \in \mathbb{R}$, then any subsequence $(x_{n'})$ also converges to the same limit $x_{n'} \to y$, as $n \to \infty$.

3.4.2. *Limits in* $\mathbb{R}_\infty$. In the extended real number system $\mathbb{R}_\infty$ we may define the following useful extended notions of limits:

Given a sequence (x_n) in $\mathbb{R}$, if for every $B \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that $n \geq N \Rightarrow x_n \geq B$ (resp. $x_n \leq B$) then we say that x_n **diverges/tends to infinity** (resp. minus infinity) as $n \to \infty$ and write

$$x_n \to \infty \qquad \text{resp.} \qquad x_n \to -\infty,$$

or

$$\lim_{n\to\infty} x_n = +\infty \qquad \text{resp.} \qquad \lim_{n\to\infty} x_n = -\infty.$$

Exercise 3.8. *Extend the limit arithmetic Theorem 3.17 to all the cases that are acceptable within the arithmetic of* $\mathbb{R}_\infty$ *(i.e.* $\infty - \infty$ *and* $0 \cdot \infty$ *are not acceptable, but* $1/\infty = 0$ *is).*

3.4.3. *Upper and lower limits of sequences.* Consider a sequence (x_n) of real numbers, and let $E \subseteq \mathbb{R}_\infty$ be the set of (extended) limits of all possible subsequences of (x_n):

$$E = \{y \in \mathbb{R}_\infty : \exists (x_{n'}) \text{ subsequence of } (x_n) \text{ s.t. } x_{n'} \to y \text{ as } n \to \infty\} \tag{3.34}$$

(note $E \neq \emptyset$ (exercise)). We then define the **upper and lower limits** of (x_n) as

$$\limsup_{n\to\infty} x_n := \sup E, \qquad \text{resp.} \qquad \liminf_{n\to\infty} x_n := \inf E.$$

Example 3.20. Given the sequence $x_n = \frac{n}{n+(-1)^n n+1}$, $n = 1, 2, 3, \ldots$, we have

$$\limsup_{n\to\infty} x_n = +\infty, \qquad \liminf_{n\to\infty} x_n = \frac{1}{2}, \tag{3.35}$$

since for odd n, $x_n = n \to \infty$, and for even n, $x_n = n/(2n+1) = 1/(2+1/n) \to 1/2$, and furthermore any subsequence which mixes odd and even n indefinitely cannot converge (hence these are the only possible limit points).

Note that a sequence (x_n) is convergent with limit $\lim_{n\to\infty} x_n = y$ iff [Rud76, Ex. 3.18c]

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = y \in \mathbb{R}. \tag{3.36}$$

Exercise 3.9. *Show that* $E \neq \emptyset$ *in* (3.34).

3.5. **Bolzano-Weierstrass.** We are reaching the end of our review of the most important properties characterizing $\mathbb{R}$, where we consider a first instance of the **Bolzano-Weierstrass** (BW) theorem and some of its consequences:

Theorem 3.21 (Bolzano-Weierstrass for $\mathbb{R}$)**.** *Every bounded sequence of real numbers contains a convergent subsequence.*

We supply two different proofs:

Proof of BW by means of a monotonic subsequence. If (x_n) is a sequence in $\mathbb{R}$ and if $|x_n| \leq B\ \forall n \in \mathbb{N}$ then by Lemma 3.19 we may extract a monotonic subsequence $(x_{n'})$, which is also bounded. But then it is convergent by Theorem 3.11. □

Proof of BW by means of interval enclosure. Consider a bounded sequence (x_n), say on the finite interval $[a_0, b_0]$:

$$a_0 \leq x_n \leq b_0, \qquad \forall\ n \in \mathbb{N}. \tag{3.37}$$

Split this interval into two:

$$I_0^L = [a_0, \tfrac{1}{2}(a_0 + b_0)], \qquad I_0^R = [\tfrac{1}{2}(a_0 + b_0), b_0],$$

and consider the corresponding sets of indices

$$\mathbb{N}_0^L = \{n \in \mathbb{N} : x_n \in I_0^L\}, \qquad \mathbb{N}_0^R = \{n \in \mathbb{N} : x_n \in I_0^R\}.$$

Since their union is infinite, either $\mathbb{N}_0^L$ or $\mathbb{N}_0^R$ must be infinite, or both. If $\mathbb{N}_0^L$ is infinite then let $0'$ be the smallest number in $\mathbb{N}_0^L$ and set $a_1 = a_0$ and $b_1 = \frac{1}{2}(a_0 + b_0)$. If $\mathbb{N}_0^L$ is finite then instead we let $0'$ be the smallest number in $\mathbb{N}_0^R$ and set $a_1 = \frac{1}{2}(a_0 + b_0)$ and $b_1 = b_0$.

Now assume by induction that we have constructed $0', 1', \ldots, n'$ and $a_0, b_0, a_1, b_1, \ldots, a_n, b_n$. Form the sets

$$\mathbb{N}_n^L = \{i \in \mathbb{N} : a_n \leq x_i \leq \tfrac{1}{2}(a_n + b_n)\}, \qquad \mathbb{N}_n^R = \{i \in \mathbb{N} : \tfrac{1}{2}(a_n + b_n) \leq x_i \leq b_n\}.$$

If $\mathbb{N}_n^L$ is infinite then let its smallest element $> n'$ be denoted $(n+1)'$ and set $a_{n+1} = a_n$ and $b_{n+1} = \frac{1}{2}(a_n + b_n)$. If $\mathbb{N}_n^L$ is finite then instead we let $(n+1)'$ be the smallest number in $\mathbb{N}_n^R$ which is $> n'$ and set $a_{n+1} = \frac{1}{2}(a_n + b_n)$ and $b_{n+1} = b_n$.

We have then the following for every $n \in \mathbb{N}$:

$$x_{n'} \in [a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]. \tag{3.38}$$

By the interval enclosure Theorem 3.15 we have

$$\bigcap_{n\geq 0} [a_n, b_n] \neq \emptyset. \tag{3.39}$$

Furthermore, since we half the considered interval each time, it holds that $|a_{n+1} - b_{n+1}| = |a_n - b_n|/2$, so if $a_n \to a^*$ and $b_n \to b^*$ as $n \to \infty$ then necessarily $a^* = b^*$. In other words $\cap [a_n, b_n]$ contains exactly one element $c \in \mathbb{R}$. We then have that

$$|x_{n'} - c| \leq |b_n - a_n| \leq 2^{-n} |b_0 - a_0| \to 0 \tag{3.40}$$

as $n \to \infty$, i.e. $(x_{n'})$ is convergent. □

We have already shown in Theorem 3.13 that convergent sequences in $\mathbb{R}$ are necessarily Cauchy. Using BW we can also prove the non-trivial converse, that any Cauchy sequence in $\mathbb{R}$ necessarily converges.

Theorem 3.22. *Every Cauchy sequence in $\mathbb{R}$ converges.*

Proof. Let (x_n) be a Cauchy sequence. By Theorem 3.12 it is necessarily bounded, and hence by Theorem 3.21 it contains a convergent subsequence, let us denote it $(x_{n'})$ and $x_{n'} \to y$ as $n \to \infty$ for some $y \in \mathbb{R}$. Take $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ if $n, m \geq N$. By the triangle inequality,

$$|x_n - y| = |x_n - x_{n'} + x_{n'} - y| \leq |x_n - x_{n'}| + |x_{n'} - y|, \tag{3.41}$$

where the first term is smaller than ε for all $n \geq N$ since $n' \geq n$ $\forall n$. Possibly taking an even larger N, we can also ensure by the fact that $x_{n'} \to y$ that the second term is smaller than ε. This proves that $x_n \to y$ as $n \to \infty$. □

3.6. **Uncountability of $\mathbb{R}$.** See [Rud76, Thm. 2.14] (Cantor's diagonal process) concerning the fact that $\mathbb{R}$ is an uncountable set. In a sense, we have added to $\mathbb{Q}$ all the limits of all possible Cauchy sequences, which is of strictly greater cardinality. This will be made precise in the next section, however let us note some similarity to the second proof of BW above:

Consider any real number x in the interval

$$I_0 = [0, 1) = \{x \in \mathbb{R} : 0 \leq x < 1\}.$$

Then either

$$x \in I_0^L := [0, 1/2) \quad \text{or} \quad x \in I_0^R := [1/2, 1).$$

Associate $i_0 \in \{0, 1\}$ with $i_0 = 0$ in the former case and $i_0 = 1$ in the latter. We repeat the process with $I_1 = I_0^L$ in the former case and in the latter we take $I_1 = I_0^R$. That is, we now split this interval I_1 into two and assign a digit $i_1 \in \{0, 1\}$ according to in which half x resides. Repeating this construction yields smaller and smaller intervals I_n (of length 2^{-n}) and results in the **binary representation** of x in terms of a (countable) sequence of bits:

$$x = 0.i_1 i_2 i_3 \ldots, \qquad i_n \in \{0, 1\}.$$

(A computationally more convenient way to repeat each step is to multiply x by 2 and then assign $i_n = 0$ if $x \in I_0$, and $i_n = 1$ if $x \in [1, 2)$. In the latter case we subtract 1 from x. Then repeat the process with n replaced by $n + 1$.)

Conversely, given an arbitrary sequence of bits

$$(i_n) \colon \mathbb{N}^+ \to \{0, 1\},$$

we may associate a sequence of rational numbers

$$x_0 = 0, \qquad x_n = x_{n-1} + 2^{-n} i_n,\ n \in \mathbb{N}^+.$$

This sequence (x_n) in $\mathbb{Q}$ is monotone increasing and bounded, and thus has a limit in $\mathbb{R}$ which is precisely $x = \lim_{n\to\infty} x_n$.

Theorem 3.23 (Cantor's diagonal process; [Rud76, Thm. 2.14])**.** *The set of all sequences $\mathbb{N} \to \{0, 1\}$ is uncountable.*

We would then conclude that $\operatorname{card} \mathbb{R} = \operatorname{card} [0,1) > \operatorname{card} \mathbb{N}$. Note however that with repeating sequences of 1's we have two ways of writing the same limit, e.g.

$$0.0111111\ldots = 0.1000000\ldots,$$

and therefore there are a few details left to be filled in here. (Rudin also avoids this. If you are curious this is discussed further in [Abb15, Chapter 1.6].) We will arrive at an alternative proof on the uncountability of $\mathbb{R}$ (using Baire's theorem) in the next section.

4. Topology of metric spaces [L6-10]

The picture of spaces that we would like to convey in this Section is the following:

$$\mathbb{R}^n \hookrightarrow \text{pos. def. quadratic} \hookrightarrow \text{normed} \hookrightarrow \text{metric} \hookrightarrow \text{topological.}$$

4.1. **Reading tip.** After the preparations and the special case $\mathbb{R}$ that was analyzed in detail in the previous section, here we shall cover the remainder of [Rud76, Chapter 2-3] as well as parts of [Rud76, Chapter 4] (uniform continuity is deferred to Section 6). However the order of the material is a bit different, and we start by recalling properties of $\mathbb{R}^n$ that will be generalized to metric spaces, which is the main focus of Rudin.

The topological notions defined in Rudin Ch 2.15-2.26 are central to the course!

Note regarding 2.18 and 2.43-44: The concept of "**perfect sets**" is not part of the syllabus of the course, however it is recommended to read through Ch 2.43-44 as nice examples of concepts in metric spaces. (And in case there would be a problem on the exam about "perfect sets", the definition of "perfect" would be given in the problem formulation.)

Some further remarks concerning notation, Rudin Ch 2.17-2.18:

Rudin's definition of a **neighborhood** in 2.18(a) is non-standard, and we would simply call this an **open ball**. (Nowadays in most books, a "neighborhood of a point p" is defined to be any subset (alt: open subset) $V \subseteq X$ which contains $B_r(p)$ for some $r > 0$.) In $\mathbb{R}$ we refer to (open/closed/half-open) **intervals**, which may be embedded as **line segments** in $\mathbb{R}^n$. Rudin also talks about k**-cells** which we may call products of intervals or simply **boxes**. Some concepts might be initially defined differently than Rudin (such as closed sets) but then shown to be equivalent.

By the end of this Section we take the opportunity to recall/note a few facts concerning series (in a suitable generalization). This is discussed in Rudin's Chapter 3, however we assume some familiarity with this material, in particular the convergence tests, from previous courses in calculus. We finally provide examples of Banach spaces (not in Rudin(!)).

4.1.1. *Exercises.* Rudin Ch. 2: 6-16, 20-26, 30; Ch. 3: 20-25; Ch. 4: 1-7, 14-19, (23-24)
Exam 2015-03-21: problem 1. Exam 2019-01-14: problem 1. Exam 2019-06-15: problem 4. Exam 2020-03-16: problem 1. Exam 2020-06-15: problem 1. Exam 2020-08-19: problem 1.

4.1.2. *Aims.* Concepts discussed in this Section:

- open and closed sets, compact sets
- Heine-Borel lemma
- continuous functions
- metric spaces and their topology

Learning outcomes: After this Section you should be able to

- explain the basic theory of metric spaces
- apply the theory to solve mathematical problems including the construction of simple proofs

4.2. **Topology in $\mathbb{R}^n$.** In essence, **topology** is about specifying roughly[(1)] how close the points of a set are to each other, so that we can talk about convergence, and also in which ways the points are connected. As a preparation let us begin by recalling notions of topology in $\mathbb{R}^n$.

(1) That is, more qualitatively, while **geometry** would be more about quantitative distances etc.

As usual we let $\mathbb{R}^n$ denote the set of n-**tuples**, i.e. lists $(x_1, x_2, \ldots, x_n)$ with n **components/coordinates** $x_1, \ldots, x_n \in \mathbb{R}$. Addition and scalar multiplication are defined component/coordinate-wise:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) := (x_1 + y_1, \ldots, x_n + y_n),$$
$$t(x_1, \ldots, x_n) := (tx_1, \ldots, tx_n), \quad t \in \mathbb{R},$$

making $\mathbb{R}^n$ an n-dimensional vector space over $\mathbb{R}$.

In $\mathbb{R}^n$ we use balls to discuss topology:

$$B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{y}| < r\}, \qquad \mathbf{x} \in \mathbb{R}^n, \quad r > 0, \tag{4.1}$$

denotes the **open ball** with radius r centered at the point $\mathbf{x}$, where

$$|\mathbf{x}| = |(x_1, \ldots, x_n)| = \sqrt{\sum_{j=1}^{n} x_j^2} \tag{4.2}$$

is the usual **euclidean norm** in $\mathbb{R}^n$. The ball $B_r(\mathbf{x})$ defines a typical **open neighborhood** around the point $\mathbf{x}$. Note that if $\mathbf{y} \in B_r(\mathbf{x})$ is any point in this neighborhood then there is also some (though possibly quite small) ball $B_{r'}(\mathbf{y})$ around $\mathbf{y}$ which also fits in the neighborhood. Namely take any

$$0 < r' < r - |\mathbf{y} - \mathbf{x}|, \tag{4.3}$$

then, by the triangle inequality for the norm (4.2) (we shall come back to this in Section 4.8),

$$\mathbf{z} \in B_{r'}(\mathbf{y}) \;\Rightarrow\; |\mathbf{z} - \mathbf{y}| < r' \;\Rightarrow\; |\mathbf{z} - \mathbf{x}| \leq |\mathbf{z} - \mathbf{y}| + |\mathbf{y} - \mathbf{x}| < r \;\Rightarrow\; \mathbf{z} \in B_r(\mathbf{x}), \tag{4.4}$$

so indeed $B_{r'}(\mathbf{y}) \subseteq B_r(\mathbf{x})$. This means that there is some space to move about in a neighborhood.

In general, an **open set** $V \subseteq \mathbb{R}^n$ is such that at every point $\mathbf{x} \in V$ there exists an open neighborhood $B_r(\mathbf{x})$ around that point which is also contained in V, i.e. $B_r(\mathbf{x}) \subseteq V$ for some $r > 0$.

4.2.1. *Sequences.* A **sequence** in $\mathbb{R}^n$ is a function (let's now use the index $m \in \mathbb{N}$)

$$(\mathbf{x}_m) \colon \mathbb{N} \to \mathbb{R}^n,$$

or equivalently a countably infinite list of finite lists, or n-tuples,

$$\mathbf{x}_0 = (x_{0,1}, x_{0,2}, \ldots, x_{0,n}),\; \mathbf{x}_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,n}),\; \ldots,\; \mathbf{x}_m = (x_{m,1}, x_{m,2}, \ldots, x_{m,n}),\; \ldots$$

Naturally, a sequence is called **bounded** iff its values don't escape to infinity, i.e. iff it fits within some fixed (possibly very large but still finite) ball. WLOG (by taking an even larger ball) we may compare it to a ball centered at the origin $\mathbf{0} = (0, \ldots, 0)$, i.e.

$$(\mathbf{x}_m) \text{ bounded} \quad :\Leftrightarrow \quad \exists R > 0 \text{ such that } \mathbf{x}_m \in B_R(\mathbf{0}) \; \forall m \in \mathbb{N}. \tag{4.5}$$

We say that the sequence $(\mathbf{x}_m)$ **converges** to $\mathbf{y} \in \mathbb{R}^n$,

$$\lim_{m \to \infty} \mathbf{x}_m = \mathbf{y},$$

iff for *any* $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that

$$m \geq N \quad \Rightarrow \quad |\mathbf{x}_m - \mathbf{y}| < \varepsilon. \tag{4.6}$$

This can be formulated in terms of balls that for *any* open ball/neighborhood at $\mathbf{y}$, say $B_\varepsilon(\mathbf{y})$, $\varepsilon > 0$, there is some integer $N \geq 0$ such that the tail $(\mathbf{x}_m)_{m \geq N}$ of the sequence is entirely contained within $B_\varepsilon(\mathbf{y})$. Thus by choosing smaller and smaller $\varepsilon > 0$ ("error margins") we see that by just taking large enough N we can confine the sequence arbitrarily close to $\mathbf{y}$.

We also extend the notion of Cauchy sequences, namely $(\mathbf{x}_m)$ is **Cauchy** iff for *any* $\varepsilon > 0$ there exists $N \in \mathbb{N}$ so that

$$m, k \geq N \quad \Rightarrow \quad |\mathbf{x}_m - \mathbf{x}_k| < \varepsilon. \tag{4.7}$$

Note that there is an inherent symmetry here. Namely this may also be expressed symmetrically as follows: given an arbitrarily small ball at the origin (of radius ε), by taking a tail of the sequence starting at a sufficiently large N, any pair of subsequent values $\mathbf{x}_m, \mathbf{x}_k$ will have their difference within that small ball. Similarly to the case $\mathbb{R}$, the benefit of using the notion of Cauchy sequences is that we will have a notion of convergence of a sequence without specifying a convergence point.

Equivalent to this geometric interpretation using balls, we also have a **coordinate-wise** interpretation. Since for any $j \in \{1, 2, \ldots, n\}$,

$$|x_{m,j} - y_j|^2 \leq |\mathbf{x}_m - \mathbf{y}|^2, \tag{4.8}$$

it is clear that convergence according to (4.6) requires convergence in *each coordinate* in the sense already defined in the previous section for sequences in $\mathbb{R}$.

Proposition 4.1. *Let us denote the projection $P_j \colon \mathbb{R}^n \to \mathbb{R}$ on the j:th coordinate,*

$$P_j(x_1, \ldots, x_n) := x_j, \qquad j \in \{1, 2, \ldots, n\}.$$

Then a sequence $(\mathbf{x}_m)$ in $\mathbb{R}^n$ is:

1. *bounded iff for all $1 \leq j \leq n$, the sequence $\big(P_j(\mathbf{x}_m)\big)_{m\in\mathbb{N}}$ in $\mathbb{R}$ is bounded.*
2. *convergent iff for all $1 \leq j \leq n$, the sequence $\big(P_j(\mathbf{x}_m)\big)_{m\in\mathbb{N}}$ in $\mathbb{R}$ is convergent.*
3. *Cauchy iff for all $1 \leq j \leq n$, the sequence $\big(P_j(\mathbf{x}_m)\big)_{m\in\mathbb{N}}$ in $\mathbb{R}$ is Cauchy.*

Exercise 4.1. *Think through and prove Proposition 4.1. Note that you may also use (prove) a geometric fact that a small enough box fits inside a ball and vice versa. In a simplified form, centered at the origin and rescaled to a cube, this could be stated as follows:*

Proposition 4.2. *For any $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ it holds*

$$\frac{1}{\sqrt{n}}|\mathbf{x}| \leq \max\big(|x_1|, |x_2|, \ldots, |x_n|\big) \leq |\mathbf{x}|. \tag{4.9}$$

4.2.2. *Bolzano-Weierstrass (BW).*

Theorem 4.3 (**Bolzano-Weierstrass** for $\mathbb{R}^n$)**.** *Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.*

Again there are two ways to generalize BW from the case $n = 1$ (Theorem 3.21):

Proof of BW by means of subsequences. Repeat the steps of the one-variable case, using Proposition 4.1. Namely, given a sequence $(\mathbf{x}_m)_{m\in\mathbb{N}}$ which is bounded we may consider the sequence $P_1(\mathbf{x}_m) = x_{m,1}$ which is bounded and pick a subsequence $P_1(\mathbf{x}_{m'}) = x_{m',1}$ which converges (by BW in $\mathbb{R}$, Theorem 3.21) or which is even monotonic (by Lemma 3.19). Now consider the corresponding sequence $(\mathbf{x}_{m'})$ in $\mathbb{R}^n$ and do the same analysis in the second coordinate, $P_2(\mathbf{x}_{m'}) = x_{m',2}$. Again we may pick a subsequence, call it $(x_{m'',2})$

which converges (or is even monotonic). Note that we will also have convergence (resp. monotonicity) of the other sequence $(P_1(\mathbf{x}_{m''})) = (x_{m'',1})$, since it is a subsequence of $(x_{m',1})$. We repeat this process a finite number n of times and arrive at a subsequence $(\mathbf{x}_{m^{(n)}})$ which converges in each coordinate, and therefore converges in $\mathbb{R}^n$. $\square$

For the second approach, split a large enough box into smaller boxes and use the following generalization of the interval enclosure theorem (exercise):

Theorem 4.4 (Box enclosure / Intervallkapsling i $\mathbb{R}^n$; [Rud76, Thm. 2.39])**.** *Let*

$$I_m = \times_{j=1}^n [a_{m,j}, b_{m,j}] = \{\mathbf{x} \in \mathbb{R}^n : a_{m,j} \leq x_j \leq b_{m,j},\ 1 \leq j \leq n\}, \quad m \in \mathbb{N}, \tag{4.10}$$

be a sequence of closed boxes in $\mathbb{R}^n$, and assume that $I_m \supseteq I_{m+1} \neq \emptyset$ for all $m \in \mathbb{N}$. Then there exists $\mathbf{x} \in \mathbb{R}$ which is contained in all I_m, i.e.

$$\bigcap_{m \in \mathbb{N}} I_m \neq \emptyset. \tag{4.11}$$

A third way of deriving BW is via notions of compactness (cf. [Rud76, Thm. 2.42]) that we will return to below.

Our most important corollary to BW is:

Theorem 4.5. *Every Cauchy sequence in $\mathbb{R}^n$ converges.*

Proof. The proof is identical to the case $n = 1$ (recall Theorem 3.22). Namely, any Cauchy sequence $(\mathbf{x}_m)$ in $\mathbb{R}^n$ is bounded (exercise) and therefore contains a convergent subsequence $(\mathbf{x}_{m'})$ by BW, say $\mathbf{x}_{m'} \to \mathbf{y}$ as $m \to \infty$. Then, by the triangle inequality in $\mathbb{R}^n$,

$$|\mathbf{x}_m - \mathbf{y}| = |\mathbf{x}_m - \mathbf{x}_{m'} + \mathbf{x}_{m'} - \mathbf{y}| \leq |\mathbf{x}_m - \mathbf{x}_{m'}| + |\mathbf{x}_{m'} - \mathbf{y}|, \tag{4.12}$$

which can be made arbitrarily small by just taking $m' \geq m \geq N$ large enough. $\square$

Exercise 4.2. *Complete the second proof of BW.*

4.2.3. *Topological notions.* We recall for convenience a few other useful topological notions in $\mathbb{R}^n$ that should be familiar from several-variable calculus:

Definition 4.6.

- An **inner/interior point** $\mathbf{x}$ of a set $A \subseteq \mathbb{R}^n$ is such that $B_r(\mathbf{x}) \subseteq A$ for some $r > 0$.
- A subset $F \subseteq \mathbb{R}^n$ is called **closed** if its complement $F^c = X \setminus F$ is open.
- A subset $B \subseteq \mathbb{R}^n$ is called **bounded** if $\exists R > 0$ s.t. $B \subseteq B_R(\mathbf{0})$.
- A subset $K \subseteq \mathbb{R}^n$ is called **compact** if it is closed and bounded.

Remark 4.7. Our definition of a closed set is the easiest to generalize, however we shall see below that it is equivalent to the one that is often used in metric spaces (also by Rudin).

Theorem 4.8. *In $\mathbb{R}^n$ we have the following:*

(a) *A set $U \subseteq \mathbb{R}^n$ is open iff every point $\mathbf{x} \in U$ is an inner point.*
(b) *A set $F \subseteq \mathbb{R}^n$ is closed iff for any sequence $(\mathbf{x}_m)$ in $\mathbb{R}^n$*

$$F \ni \mathbf{x}_m \to \mathbf{y} \in \mathbb{R}^n \qquad \Rightarrow \qquad \mathbf{y} \in F,$$

i.e. "we cannot converge out of closed sets".
(c) *If U_i, $i \in I$ (some index set), are open sets then also $\bigcup_{i \in I} U_i$ is open. In other words, arbitrary unions of open sets are open.*

(d) *If* $U_1, \dots, U_N$ *are open sets then* $\bigcap_{i=1}^N U_i$ *is open. In other words,* *finite* *intersections of open sets are open.*
(e) *Finite unions of closed sets are closed.*
(f) *Arbitrary intersections of closed sets are closed.*

Exercise 4.3. *Prove Theorem 4.8 (note that they will be generalized below).*
(Hint: For (e) *and* (f) *the relationships for complements* (2.1) *and* (2.2) *are useful.)*

4.3. **Metric spaces.** (cf. Rudin 2.15-17)

Definition 4.9. Let X be a set. A **metric** (or **distance function**) d on X is a map

$$X \times X \xrightarrow{d} \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$$

such that
(i) $d(x, y) = d(y, x) \ \ \forall x, y \in X$ (symmetric)
(ii) $d(x, z) \leq d(x, y) + d(y, z) \ \ \forall x, y, z \in X$ (triangle inequality)
(iii) $d(x, y) = 0 \Leftrightarrow x = y \ \ \forall x, y \in X$ (non-degenerate)

The pair (X, d) is called a **metric space** (and we also call the set X a metric space if d is understood).

Example 4.10. $X = \mathbb{R}^n$ with $d(\mathbf{x}, \mathbf{y}) := |\mathbf{x} - \mathbf{y}|$ is our most familiar example of a metric space (and in fact, as will be discussed further in Section 4.8, any normed vector space is also a metric space). In particular, $\mathbb{R}$ is a metric space with distance function $d(x, y) = |x - y|$.

Example 4.11. Let $X = \mathbb{S}^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere with $d(\mathbf{x}, \mathbf{y})$ defined as the length of the geodesic curve (i.e. the shortest path along a great circle) connecting the points $\mathbf{x} \in \mathbb{S}^2$ and $\mathbf{y} \in \mathbb{S}^2$. Then (X, d) is a metric space.

Example 4.12. Let X be any set and define $d \colon X \times X \to \{0, 1\} \subseteq \mathbb{R}_+$ by

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases} \tag{4.13}$$

This is called the **discrete metric** on X.

Example 4.13. Let $X = \mathbb{Z}^2$ be the integer lattice points in the plane and define the distance $d(\mathbf{x}, \mathbf{y})$ between any two such lattice points to be the minimal number of steps (considering the edges connecting the points to be aligned with the coordiate axes) that need to be traversed to go from $\mathbf{x}$ to $\mathbf{y}$. This defines a metric space.

Example 4.14. Let M be a finite set and $X = \mathcal{P}(M)$ the set of all subsets of M. Define $d \colon X \times X \to \mathbb{N} \subseteq \mathbb{R}_+$ by

$$d(A, B) = |A \triangle B| = \operatorname{card}(A \triangle B), \tag{4.14}$$

where

$$A \triangle B := (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A) \tag{4.15}$$

denotes the **symmetric difference** of the sets A and B in X. Then d may be verified to be a metric on X.

Example 4.15. Any subset $Y \subseteq X$ of a metric space (X, d) is also a metric space with metric on Y given by the restriction of d to $Y \times Y$, i.e. $d|_{Y \times Y} \colon Y \times Y \to \mathbb{R}_+$, where $d|_{Y \times Y}(x, y) = d(x, y)$ for all $x, y \in Y$. This is called the **subspace metric**.

Exercise 4.4. *Verify that the discrete metric* (4.13) *is a metric. Try to realize the discrete metric space* (X, d) *as a metric subspace of* $\mathbb{R}^n$ *with its standard metric, at least for very small (finite) sets* X.

Exercise 4.5. *Verify that* (4.14) *defines a metric on* X.
(Hint: use Venn diagrams for example.)

4.3.1. *Topology in metric spaces.* Inspired by our sense of topology of $\mathbb{R}^n$ using balls, we make the following definitions in metric spaces, simply by measuring distances using the given metric.

Definition 4.16 (Metric topology)**.** Let (X, d) be a metric space.

- A subset $V \subseteq X$ is called **open** iff
$$\forall x \in V \ \exists \varepsilon > 0 \quad \text{s.t.} \quad \forall y \in X \ \big(d(x,y) < \varepsilon \ \Rightarrow \ y \in V\big).$$
- An **open "ball"** at $x \in X$ of radius $r > 0$ is
$$B_r(x) := \{y \in X : d(x,y) < r\}.$$
(Rudin calls this a **neighborhood** $N_r(x)$ of x. We may also just simply call it a **ball** in X and ignore its actual geometry.)
Hence an open set is such that at any one of its points there is enough room to contain also some open "ball" or neighborhood around the point. We may say that the set consists solely of inner points:
- Given a subset $A \subseteq X$, a point $x \in A$ is called an **inner/interior point** of A iff there exists a neighborhood at x which is enclosed in A, i.e. $\exists r > 0$ s.t. $B_r(x) \subseteq A$.
- A subset $F \subseteq X$ is called **closed** iff its complement $F^c = X \setminus F$ is open.
- A subset $B \subseteq X$ is called **bounded** iff it fits in some finite "ball", i.e. if there exists $p \in X$ and $R > 0$ s.t. $B \subseteq B_R(p)$.

Theorem 4.17. *If* (X, d) *is a metric space, then*
(a) *an open "ball"* $B_r(x)$ *is an open set,*
(b) $\emptyset$ *and* X *are open and closed,*
(c) *if* V *and* W *are open then* $V \cap W$ *is open,*
(d) *if* $\mathcal{U}$ *is a set of open sets then their union*
$$\cup\,\mathcal{U} = \{x \in X : \exists V \in \mathcal{U} \ \text{ s.t. } \ x \in V\} \tag{4.16}$$
is also open,
(e) *if* F *and* S *are closed then* $F \cup S$ *is closed,*
(f) *if* $\mathcal{F}$ *is a set of closed sets then their intersection*
$$\cap\,\mathcal{F} = \{x \in X : x \in F \ \forall F \in \mathcal{F}\} \tag{4.17}$$
is also closed.

The proof is left as Exercise 4.7.

Remark 4.18. Note that if the intersection of any pair of open sets is open, then also the intersection of any finite number of open sets is open.

Furthermore, the finiteness of intersections in (c) cannot be relaxed if we want to have a reasonable sense of topology, namely if we consider the intervals $I_n = (-1/n, 1/n) = B_{1/n}(0)$ to be open in the metric space $\mathbb{R}$, then their intersection $\bigcap_{n \geq 1} I_n = \{0\}$ is not open (it is in fact closed, and the only subsets of $\mathbb{R}$ which are both open and closed are $\emptyset$ and $\mathbb{R}$ itself; cf. Exercise 4.14).

We will not leave the realm of metric spaces in this course, however let us simply remark that any family of subsets having the above properties (b), (c) and (d) (the open sets) defines a "topology":

Definition 4.19 (Topology)**.** Let X be a set and let $\mathcal{T} \subseteq \mathcal{P}(X)$ be a set of subsets of X. Then $\mathcal{T}$ is called a **topology** on X, and the elements in $\mathcal{T}$ are called the **open sets** of this topology, if the following holds:

(i) $\emptyset$ and X belong to $\mathcal{T}$,
(ii) $V \in \mathcal{T}$ and $W \in \mathcal{T} \Rightarrow V \cap W \in \mathcal{T}$
(iii) $\mathcal{U} \subseteq \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$.

The pair $(X, \mathcal{T})$ is then called a **topological space**.

Exercise 4.6. *Sketch the "balls" for the examples of metric spaces in Section 4.3.*

Exercise 4.7. *Prove Theorem 4.17 by extending the use of balls from $\mathbb{R}^n$ to metric spaces. For example, for* (c) *one may note that for any $r_1, r_2 > 0$ and $x \in X$*

$$B_{r_1}(x) \cap B_{r_2}(x) = B_{\min(r_1,r_2)}(x). \tag{4.18}$$

Exercise 4.8. *Let B be a bounded subset of a metric space (X, d). Prove that there is a real number M such that $d(x, y) < M$ for every pair of points $x, y \in B$.*

Exercise 4.9. *Define by*

$$\bar{B}_r(x) := \{y \in X : d(x, y) \leq r\} \tag{4.19}$$

*the **closed "ball"** of radius $r \geq 0$ at $x \in X$. Show that indeed it is closed.*

Exercise 4.10. *Prove that for the discrete metric space (X, d) in Example 4.12, every subset $A \subseteq X$ is both open and closed.*

4.3.2. *Sequences in metric spaces.* As usual, a **sequence** in a set X is a map $(x_n): \mathbb{N} \to X$.

Definition 4.20. A sequence $x_0, x_1, x_2, \ldots$ in a metric space (X, d) is called

- **bounded** if there exists $y \in X$ and $R > 0$ such that
$$d(x_n, y) < R \qquad \forall n \in \mathbb{N}$$
- **convergent** if there exists some $y \in X$ such that
$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \; : \; n \geq N \;\Rightarrow\; d(x_n, y) < \varepsilon$$
We then write $x_n \to y$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = y$.
- **Cauchy** if
$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \; : \; n, m \geq N \;\Rightarrow\; d(x_n, x_m) < \varepsilon$$

Another way to formulate Cauchy is using the notion of diameter:

Definition 4.21. The **diameter** of a subset $E \subseteq X$ of a metric space (X, d) is the greatest distance between any two of its elements,

$$\operatorname{diam} E := \sup\{d(x, y) : x \in E, y \in E\} \in \mathbb{R}_\infty. \tag{4.20}$$

Then a sequence (x_n) in X is Cauchy iff

$$\lim_{N\to\infty} \operatorname{diam}\{x_n\}_{n\geq N} = 0. \tag{4.21}$$

Definition 4.22. A metric space is called **complete** (sv: fullständigt) if every Cauchy sequence converges.

Example 4.23. By Theorem 4.5, $\mathbb{R}^n$ with its standard metric is a complete metric space. However, $\mathbb{Q}^n$ (with the subspace metric) is not complete.

Exercise 4.11. *Prove that any Cauchy sequence in a metric space is bounded.*

Exercise 4.12. *Show that convergent sequences in a metric space are Cauchy, but that the converse does not always hold (unless it is complete).*

4.3.3. *Limit points, closure and interior.* (cf. Rudin 2.18-28)

Let us for convenience say that two sets A and B **meet** if their intersection $A \cap B$ is non-empty (note e.g. that A and A^c do not meet). Compare the following definitions to [Rud76, Def. 2.18]:

Definition 4.24. Let (X, d) be a metric space (or topological space) and $A \subseteq X$ a subset.

- A point $p \in X$ is called a **limit point** of A if for each open set V containing p there exists some $q \in V \cap A$, $q \neq p$. In other words, $A \setminus \{p\}$ meets every open neighborhood of p. In a metric space this is equivalent to:
$$\forall r > 0 \; \exists q \in A \quad \text{s.t.} \quad 0 < d(p, q) < r. \tag{4.22}$$
- The **closure** (sv: tillslutningen) $\bar{A}$ is defined to be the intersection of all closed sets that contain A,
$$\bar{A} := \cap\{F \in \mathcal{P}(X) : F \text{ closed}, \; A \subseteq F\}.$$
Hence $A \subseteq \bar{A}$ and $\bar{A}$ is the smallest closed set containing A.
- The **interior** (sv: inre) A° is defined to be the union of all open sets contained in A,
$$A^\circ := \cup\{V \in \mathcal{P}(X) : V \text{ open}, \; V \subseteq A\}.$$
Hence $A^\circ \subseteq A$ and A° is the largest open set contained in A.
- The **boundary** (sv: randen) of A is the set $\partial A := \bar{A} \setminus A^\circ$.
- A is called **dense** (sv: tät) in X if every nonempty open subset of X meets A. In a metric space this is equivalent to:
$$\forall x \in X, r > 0 \; \exists q \in A \quad \text{s.t.} \quad d(x, q) < r. \tag{4.23}$$

Theorem 4.25 (Equivalent formulations)**.** *Let (X, d) be a metric space and $A \subseteq X$ a subset.*

(a) *A point $p \in X$ is a limit point of A iff every open ball/neighborhood at p contains infinitely many points of A. Or, equivalently, iff there exists a sequence (x_n) in $A \setminus \{p\}$ such that $x_n \to p$.*

(b) *The set A is closed iff it contains all of its limit points, i.e. for any $x \in X$ it holds that, if $A \ni x_n \to x$ as $n \to \infty$, then $x \in A$.*

(c) *The closure $\bar{A}$ is the union of A and the set of all limit points of A (hence A is closed iff $A = \bar{A}$).*

(d) *The interior* A° *is the set of all inner points of* A *(hence* A *is open iff* $A = A^\circ$*).*
(e) *The set* A *is dense in* X *iff every point of* X *is either a point of* A *or a limit point of* A *(hence* A *is dense in* X *iff* $\bar{A} = X$*).*

Proof.

(a) If $p \in X$ is a limit point of A then by (4.22) we can for each $n \in \mathbb{N}^+$ find a point $x_n \in A$ s.t. $0 < d(p, x_n) < 1/n$. This sequence (x_n), as well as its intersection with $B_\varepsilon(p)$ for any $\varepsilon > 0$, contains infinitely many points since otherwise the set $\{d(p, x_n)\}_{n\in\mathbb{N}^+}$ of strictly positive numbers would be bounded from below by some $\varepsilon > 0$. The converse is immediate by (4.22), that if a sequence $A \ni x_n \to p$ then p is a limit point of A.

(b) Let A be closed, i.e. A^c open, and let $x \in X$ be the limit of a sequence in A, $A \ni x_n \to x$. If $x \notin A$ then $x \in A^c$, and hence is an inner point of A^c. Thus there is some $r > 0$ and a ball $B_r(x) \subseteq A^c$, but then $B_r(x) \cap A = \emptyset$ which contradicts the assumption that $A \ni x_n \to x$. Therefore $x \in A$.

Conversely, let A contain all its limit points and consider $x \in A^c$. Then since this cannot be a limit point of A, by logical negation there exists a ball $B_r(x)$, $r > 0$, with none of its points in A. That is, $B_r(x) \subseteq A^c$ and therefore x is an inner point of A^c. Hence A^c is open and A is closed.

(c) Denote by A' the set of limit points of A in X. We claim that $A \cup A'$ is a closed set. Namely, if $x \in (A \cup A')^c$ then it is neither in A nor a limit point of A. Thus there exists a ball $B_r(x) \subseteq A^c$. Furthermore, no point of this ball can be a limit point of A, hence $B_r(x) \subseteq A^c \cap (A')^c$, which means that $A^c \cap (A')^c = (A \cup A')^c$ is open.

Since $A \cup A'$ is a closed set, we obtain $\bar{A} \subseteq A \cup A'$ (it will be included in the intersection). Furthermore, for each F s.t. $A \subseteq F$ and F is closed, by (b) we have $A \cup A' \subseteq F$. Thus $A \cup A' \subseteq \bar{A}$ (it remains included in the intersection).

(d) We have immediately that if $x \in A^\circ$ then $x \in V \subseteq A$ for some open V, and therefore x is an interior point of A. And conversely, if x is interior to A then $x \in B_r(x) \subseteq A^\circ$ for some $r > 0$.

(e) If every nonempty open subset V meets A, then for every point $x \in X$, and any $r > 0$, $B_r(x)$ meets A, which implies that either $x \in A$ or x is a limit point of A. Conversely, given an open subset V and $x \in V$, if $x \in X$ is either in A or a limit point of A, then any ball $B_r(x) \subseteq V$ will meet A.

□

Remark 4.26. A note on terminology: A limit point $p \in X$ of a set A concerns an infinity of *different* points of A accumulating to p. However recall that for a sequence (x_n) we may also talk about a point $p \in X$ being the *limit* of the sequence, i.e. $\lim_{n\to\infty} x_n = p$, but this does not necessarily imply that p is a limit *point* of the set $A = \{x_n : n \in \mathbb{N}\}$, as is exemplified by the constant sequence $x_n = p$ $\forall n \in \mathbb{N}$. We can capture this difference by calling p an **accumulation point** for the sequence, and a limit of the sequence, but not a limit point of the corresponding set $A = \{p\}$ (it has no limit points).

Example 4.27. Recall that by Theorem 3.9, $\mathbb{Q}$ is dense in $\mathbb{R}$, namely if we take an open set $V \neq \emptyset$ then there is a point $x \in V$ and an open interval $(x - \varepsilon, x + \varepsilon) \subseteq V$. Thus we may also find some $q \in \mathbb{Q}$ s.t. $x - \varepsilon < q < x + \varepsilon$, i.e. $q \in V$.

Exercise 4.13. *Show that* $\mathbb{Q}^n$ *is dense in* $\mathbb{R}^n$*.*

Remark 4.28. A metric (or topological) space is called **separable** if it contains a countable dense subset. Hence $\mathbb{R}^n$ is separable.

Exercise 4.14. *Show that the only subsets of* $\mathbb{R}$ *which are both open and closed are* $\emptyset$ *and* $\mathbb{R}$.

Exercise 4.15. *Show that if* $A \subseteq B$ *then* $A^\circ \subseteq B^\circ$ *and* $\bar{A} \subseteq \bar{B}$.

4.3.4. *Relative topology.* (cf. Rudin 2.29-30)

Given a subset $Y \subseteq X$ of a metric space (X, d) (or topological space), we can define an induced topology on Y by defining a set $A \subseteq Y$ to be **open relative** to Y iff there exists some open set W in X such that $V = Y \cap W$. That is, the open sets of the **relative topology** (or **subspace topology** inherited from X) are precisely

$$\mathcal{T}_{Y \subseteq X} := \{A \in \mathcal{P}(Y) : \exists W \subseteq X \text{ open s.t. } A = Y \cap W\}. \tag{4.24}$$

Exercise 4.16. *Verify that this defines a topology on* Y *in the sense of Definition 4.19, i.e. that* $\emptyset$ *and* Y *are open, and finite intersections and arbitrary unions of open sets are open.*

Exercise 4.17. *Which sets are both open and closed relative to* $[a, b] \subseteq \mathbb{R}$*? Why?*

4.4. **Compactness and Heine-Borel.** (cf. Rudin 2.31-41)

Recall that closed and bounded sets in $\mathbb{R}^n$ are called compact. We will now revise this definition to one which can be used also without a metric or a notion of boundedness (only a topology, i.e. notion of what is an open set, is required), and even if we have a metric it turns out that this definition better captures the properties we need of compactness in a general space (and in fact, although compact metric spaces are always complete and bounded, not every complete and bounded metric space is compact).

Definition 4.29 (Covering)**.** Let $\mathcal{T} \subseteq \mathcal{P}(X)$ denote the set of all open subsets of a metric (or topological) space X, and let $K \subseteq X$ be a subset.

- A subset $\mathcal{U}$ of $\mathcal{T}$ is called an **open cover** of K if $K \subseteq \cup\mathcal{U}$, i.e., if every $x \in K$ is an element of some set $V \in \mathcal{U}$.
- If $\mathcal{V} \subseteq \mathcal{U}$ and if $K \subseteq \cup\mathcal{V}$ then $\mathcal{V}$ is called a **subcover** of $\mathcal{U}$.
- If moreover $\mathcal{V}$ is finite (i.e. it contains only a finite number of open sets) then it is called a **finite subcover** (sv: ändlig delövertäckning) of $\mathcal{U}$.

Now we can make the following important and fundamental definition:

Definition 4.30 (Compactness)**.** A subset K of a metric (or topological) space X is called **compact** iff every open cover of K contains a finite subcover. Similarly, the space X itself is called compact iff every open cover of X contains a finite subcover.

Theorem 4.31. *Compact sets in metric spaces are necessarily closed.*

Remark* 4.32. The same conclusion is true in any Hausdorff space: A topological space is called a **Hausdorff space iff for all $p \neq q$ in X there exist open sets V and W s.t. $V \cap W = \emptyset$, $p \in V$ and $q \in W$, i.e. "the topology separates points".

Proof. Let K be a compact subset of a metric (or Hausdorff) space X and let $p \in K^c$. To each $q \in K$ there exist open disjoint sets V_q and W_q such that $q \in V_q$ and $p \in W_q$. Since $\bigcup_{q \in K} V_q \supseteq K$ and K compact there exists a finite number of points $q_1, \ldots, q_N$ in K such that $V_{q_1} \cup V_{q_2} \cup \ldots \cup V_{q_N} \supseteq K$. Now let $W_p := W_{q_1} \cap W_{q_2} \cap \ldots \cap W_{q_N}$, then obviously $p \in W_p$,

W_p is open, and $K \cap W_p \subseteq (V_{q_1} \cup \ldots \cup V_{q_N}) \cap W_p = \emptyset$. This shows that K^c, being a union of such W_p, is open. □

Theorem 4.33 (**Heine-Borel theorem**). *A subset $K \subseteq \mathbb{R}^n$ is compact (in the sense of Definition 4.30) if and only if K is closed and bounded.*

Proof. Suppose K is closed and bounded and let $\mathcal{U}$ be an open cover of K. We now assume that K cannot be covered by a finite subcover of $\mathcal{U}$, and proceed with an iterative splitting argument, similar to the second proof of BW (Exercise 4.2), leading to a contradiction.

Namely, consider a closed cube $Q_0 := [a, b]^n \supseteq K$. Split this cube into 2^d smaller closed cubes $Q_{1,j}$ of half the side length, $(b-a)/2$. Consider the closed sets $K_{1,j} := K \cap Q_{1,j}$, $j \in \{1, \ldots, 2^n\}$. If no finite subset of $\mathcal{U}$ covers K then there is at least one of the $K_{1,j}$ that is not covered by a finite subset of $\mathcal{U}$ (otherwise K would be covered by the union over all j). Call this set $K_1 = K_{1,j}$ and consider the cube $Q_1 = Q_{1,j}$.

Iterating this argument gives us a sequence $K =: K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots$ such that for all $m \in \mathbb{N}$, K_m is closed, nonempty and is of diameter $\operatorname{diam} K_m \leq \sqrt{n}2^{-m}(b-a) \to 0$ as $m \to \infty$. Furthermore, no finite subcover of $\mathcal{U}$ covers K_m. If we pick $\mathbf{x}_m \in K_m$ then it follows by BW (Theorem 4.3) that there is a convergent subsequence $(\mathbf{x}_{m'})$ in K, converging to $\mathbf{x} \in K$ say (K is closed). But then also $\mathbf{x} \in K_m$ for each m (why?). Take some open $V \in \mathcal{U}$ such that $\mathbf{x} \in V$. But then $V \supseteq B_\varepsilon(\mathbf{x}) \supseteq K_m$ if m is big enough and $\varepsilon > 0$ small enough. Thus K_m is covered by $\{V\} \subseteq \mathcal{U}$. This is a contradiction, hence K is compact.

Conversely, any compact K is closed by Theorem 4.31, and to prove that K is bounded is left as Exercise 4.18. □

Definition 4.34. A metric space (X, d) is said to have the **Heine-Borel (HB) property** if every closed and bounded subset of X is compact.

Thus, $\mathbb{R}^n$ has the Heine-Borel property for any $n \in \mathbb{N}^+$. We also saw some similarities to Bolzano-Weierstrass in the proof above, and in fact an appropriate generalization of the approach to arbitrary metric spaces goes as follows:

Theorem 4.35. *In any metric space (or Hausdorff space), closed subsets of compact sets are also compact.*

Proof. Let K be compact and $F \subseteq K$ closed. If $\mathcal{U}$ is an open cover of F, then $\mathcal{V} = \mathcal{U} \cup \{F^c\}$ is an open cover of K. Hence it has a finite subcover $\mathcal{F} \subseteq \mathcal{V}$. But $\mathcal{F} \setminus \{F^c\} \subseteq \mathcal{U}$ is then also a finite cover of F. □

Theorem 4.36 (Compact enclosure; cf. box enclosure Theorem 4.4). *If (K_n) is a sequence of nonempty compact sets in a metric (or Hausdorff) space such that $K_n \supseteq K_{n+1}$ $\forall n \geq 1$ then $\bigcap_{n\geq 1} K_n$ is nonempty.*

Proof. $\{K_n\}_{n\geq 1}$ is a collection of compact, hence closed, subsets such that the intersection of every finite subcollection is nonempty. Note that $\{K_n^c\}_{n\geq 1}$ is a collection of open subsets. If $K_1 \cap \bigcap_{n\geq 2} K_n = \emptyset$ then (again recall (2.2))

$$K_1 \subseteq \Big(\bigcap_{n\geq 2} K_n\Big)^c = \bigcup_{n\geq 2} K_n^c \tag{4.25}$$

Since K_1 is compact there is some finite subcover, say

$$K_1 \subseteq K_{n_1}^c \cup \ldots \cup K_{n_N}^c = (K_{n_1} \cap \ldots \cap K_{n_N})^c\,, \tag{4.26}$$

but then $K_1 \cap K_{n_1} \cap \ldots \cap K_{n_N} = \emptyset$, which is a contradiction. □

Theorem 4.37 (cf. Rudin Thm 2.37 and Exc 2.26)**.** *A subset $K \subseteq X$ of a metric space is compact iff every infinite subset of K has a limit point in K.*

Remark* 4.38. A topological space K with the latter property may be called **limit point compact. Thus limit point compactness is equivalent to compactness in metric spaces.

Proof. Note that we may in fact forget about X and just consider the induced relative topology on K (cf. [Rud76, Thm 2.33]).

Assume that K is compact and consider an infinite subset $E \subseteq K$. Assume that E has no limit point in K. Then at every $p \in K$ there is some ball $B_{r_p}(p)$ which contains at most one point of E (possibly $p \in E$). Therefore any finite subcollection of $\{B_{r_p}(p) : p \in K\}$ can only cover finitely many points of E, therefore cannot cover E or K. This contradicts compactness of K.

Conversely, assume every infinite subset $E \subseteq K$ has a limit point in K. By exercises in Rudin (proving first that K is separable and thus has a countable base of open sets), every open cover of K necessarily has a *countable* subcover $\{V_n\}_{n\in\mathbb{N}^+}$. If K cannot be covered by any finite subcollection then for each n, $F_n := (V_1 \cup \ldots \cup V_n)^c \neq \emptyset$, $F_n \supseteq F_{n+1}$ closed, however $\bigcap_{n\geq 1} F_n = \emptyset$. Taking $x_n \in F_n$ we have some subsequence $x_{n'} \to y \in K$, so that by closedness also $y \in F_n$ for all $n \geq 1$. This yields a contradiction. □

Corollary 4.39. *Every compact metric space is complete.*

Proof. Take a Cauchy sequence (x_n) in a compact metric space X which is not eventually constant (converged). Then its set of points $\{x_n\}$ cannot be a finite set since if it was, the set $\{d(x_n, x_m) : n, m \in \mathbb{N}, x_n \neq x_m\} \subseteq \mathbb{R}^+$, and therefore its diameter, would be bounded from below by some $\varepsilon > 0$. Hence $\{x_n\}$ is infinite and has a limit point, say $y \in X$, by Theorem 4.37. Thus, there exists a subsequence $(x_{n'})$ s.t. $x_{n'} \to y$. Then, taking $n' \geq n$, also

$$d(x_n, y) \leq d(x_n, x_{n'}) + d(x_{n'}, y) \to 0 \text{ as } n \to \infty,$$

showing that $x_n \to y$. Hence every Cauchy sequence in X converges. □

Exercise 4.18. *Prove that a compact set is necessarily bounded.*

Exercise 4.19. *Prove that a finite set is compact.*

Exercise 4.20. *Prove that an infinite set with the discrete metric (see Example 4.12) is complete and bounded but not compact.*

Exercise 4.21. *Give an alternative proof of Theorem 4.33 (HB) using limit point compactness and BW.*

Exercise 4.22. *Prove that a compact metric space is separable.*
(Hint: consider coverings by balls $B_{1/n}(x)$.)

4.5. **Connectedness.** (cf. Rudin 2.45-47)

Definition 4.40. We call a metric space (or topological space) X **disconnected** if it can be written as the union $X = A \cup B$ of two nonempty disjoint open subsets A and B. If X is not disconnected, it is **connected** (sv: sammanhängande). Given a subset $Y \subseteq X$ we say it is connected or disconnected (relative to X) depending on whether it is connected or disconnected in the relative topology $\mathcal{T}_{Y\subseteq X}$.

Remark 4.41. Note that if $X = A \cup B$ where A and B are both open and disjoint then $A^c = B = \bar{B}$ and $B^c = A = \bar{A}$, so $A \cap \bar{B} = \emptyset = \bar{A} \cap B$, which corresponds to Rudin's definition (where A and B are called **separated**).

Example 4.42. Recall that in $\mathbb{R}$ only the sets $\emptyset$ and $\mathbb{R}$ are both open and closed (Exercise 4.14), so $\mathbb{R}$ is indeed connected. The same is true of an interval (a, b), $(a, b]$, $[a, b)$ or $[a, b]$ (why? cf. Exercise 4.17), but if e.g. $I \subseteq \mathbb{R}$ is such that $I \subseteq (a, b) \cup (b, c)$ with elements in both intervals, then $I = (I \cap (a, b)) \cup (I \cap (b, c))$ and hence it is disconnected.

4.6. **Continuity.** (cf. Rudin Chapter 4 — recall 4.4 & 4.9!)

Recall that for a function $f \colon \Omega \subseteq \mathbb{R} \to \mathbb{R}$ we can, for each inner point x in the domain Ω, consider its **left-hand** resp. **right-hand limits**,

$$f(x^-) = \lim_{y \to x^-} f(y), \qquad f(x^+) = \lim_{y \to x^+} f(y), \tag{4.27}$$

assuming that they exist, that is, if the corresponding upper and lower limits agree (compare (3.36)):

$$\limsup_{y \to x^\pm} f(y) = \liminf_{y \to x^\pm} f(y) \in \mathbb{R}. \tag{4.28}$$

In these we consider the supremum resp. infimum of all possible limits as $\Omega \ni y \to x^\pm$:

$$\{L \in \mathbb{R}_\infty : f(x_n) \to L \text{ for some sequence } x_n \to x,\ x_n > (\text{resp.} <)\, x\}. \tag{4.29}$$

We say that f is **continuous** at $x \in \Omega$ iff both these limits exist and agree with the function value,

$$f(x^-) = f(x^+) = f(x), \tag{4.30}$$

or equivalently, iff

$$\forall \varepsilon > 0\ \exists \delta > 0 \ \text{ s.t. } \forall y \in \Omega \quad \big(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \varepsilon\big). \tag{4.31}$$

If f is not continuous at $x \in \Omega$ then it is **discontinuous** at x and we may then either have a **simple discontinuity**, also called a **discontinuity of the first kind**, if $f(x^-)$ and $f(x^+)$ both exist, while otherwise it is a **discontinuity of the second kind**.

Note that in the case that $x \in \partial\Omega \subseteq \mathbb{R}$ is a boundary point, the definition (4.31) is still valid but we may in (4.30) only take the limit on the side which is in the domain Ω.

Similarly, if $f \colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{x} \in \Omega$, and there is an $\mathbf{L} \in \mathbb{R}^m$ such that

$$\forall \varepsilon > 0\ \exists \delta > 0 \ \text{ s.t. } \forall \mathbf{y} \in \Omega \quad \big(0 < |\mathbf{y} - \mathbf{x}| < \delta \Rightarrow |f(\mathbf{y}) - \mathbf{L}| < \varepsilon\big), \tag{4.32}$$

then we call this the **limit** of f at $\mathbf{x}$ and write $\mathbf{L} = \lim_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y})$. We then say that f is **continuous** at $\mathbf{x} \in \Omega$ if

$$\lim_{\mathbf{y} \to \mathbf{x}} f(\mathbf{y}) = f(\mathbf{x}), \tag{4.33}$$

or in other words that (note the differences to (4.32))

$$\forall \varepsilon > 0\ \exists \delta > 0 \ \text{ s.t. } \forall \mathbf{y} \in \Omega \quad \big(|\mathbf{y} - \mathbf{x}| < \delta \Rightarrow |f(\mathbf{y}) - f(\mathbf{x})| < \varepsilon\big). \tag{4.34}$$

Again, these notions generalize straightforwardly to metric spaces:

Definition 4.43 (Continuity)**.** A mapping $f \colon X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is said to be **continuous** at $p \in X$ iff

$$\forall \varepsilon > 0\ \exists \delta > 0 \ \text{ s.t. } \forall x \in X \quad \big(d_X(x, p) < \delta \Rightarrow d_Y(f(x), f(p)) < \varepsilon\big). \tag{4.35}$$

If f is not continuous at $p \in X$ it is called **discontinuous** at p. If f is continuous at every point of X it is simply called **continuous**. The set of all continous mappings from X to Y is denoted $C(X;Y)$ (or just $C(X)$ in the case that $Y = \mathbb{R}$).

Theorem 4.44 (Continuous maps pull back open sets to open sets)**.** *If X and Y are metric spaces, then a function $f: X \to Y$ is continuous precisely if the pulled back set*

$$f^{-1}(V) = \{x \in X : f(x) \in V\}$$

is open in X whenever V is open in Y.

Proof. Suppose f is continuous and that V is open in Y. Let $p \in f^{-1}(V)$ i.e. $f(p) \in V$ and choose $\varepsilon > 0$ such that $B_\varepsilon(f(p)) \subseteq V$. Then $\exists \delta > 0$ such that $d_X(p,x) < \delta$ implies $d_Y(f(p), f(x)) < \varepsilon$. Hence, $B_\delta(p) \subseteq f^{-1}(B_\varepsilon(f(p))) \subseteq f^{-1}(V)$, showing that $f^{-1}(V)$ is open.

Conversely, assume that $f^{-1}(V)$ is open whenever V is open, and let $p \in X$. Then for any $\varepsilon > 0$, $V = B_\varepsilon(f(p))$ is open and contains $f(p)$. Thus $f^{-1}(V)$ is open and contains p. Since p is an inner point we can find $B_\delta(p) \subseteq f^{-1}(V)$. Thus f is continuous at p according to the definition (4.35). □

This leads naturally to the following very general definition:

Definition 4.45 (Continuity in topological spaces)**.** A mapping $f: X \to Y$ between two topological spaces is called **continuous** iff $f^{-1}(V)$ is open for every open V in Y.

Exercise 4.23. *Consider for $f: \Omega \subseteq \mathbb{R} \to \mathbb{R}$ the graph* $\mathrm{graph}(f) \subseteq \mathbb{R}^2$*. Give a description of* (4.28) *and* (4.30) *in terms of limit points (x,y) of this graph in the metric space $\mathbb{R}^2$.*

Exercise 4.24. *Show that a function $f: X \to Y$ between metric (or, if you want, topological) spaces is continuous iff $f^{-1}(F)$ is closed in X for every F closed in Y.*

Exercise 4.25. *Give examples of functions $f: (a,b) \to \mathbb{R}$ with discontinuities of the second kind. Can these notions be generalized to $f: \mathbb{R}^2 \to \mathbb{R}$? (Discuss!)*

Exercise 4.26. *Consider the popcorn function $f: \mathbb{R} \to \mathbb{Q} \subseteq \mathbb{R}$ from Exercise 3.7,*

$$f(x) := \begin{cases} 1/n, & \text{if } x = m/n \in \mathbb{Q} \text{ is a fully reduced rational with } m \ \underline{\text{odd}}, \\ 0, & \text{otherwise.} \end{cases}$$

Show that f has a simple discontinuity at every odd-numerator rational, and is continuous everywhere else.

4.6.1. *Continuity and compactness.*

Theorem 4.46 (Continuous maps preserve compactness)**.** *Let $f: X \to Y$ be a continuous mapping between the metric (or topological) spaces X and Y, and let $K \subseteq X$ be a compact subset. Then $f(K) = \{f(x) \in Y : x \in K\}$ is compact in Y.*

Proof. Let $\mathcal{U} \subseteq \mathcal{P}(Y)$ be an open cover of $f(K)$. Then the set $\{f^{-1}(V) : V \in \mathcal{U}\}$ is an open cover of K. Since K is compact there is a finite subset $\mathcal{F}$ of $\mathcal{U}$ such that $\{f^{-1}(V) : V \in \mathcal{F}\}$ is a finite cover of K. But then $f(K) \subseteq \cup\mathcal{F}$ and thus $\mathcal{F}$ is a finite cover of $f(K)$. □

Corollary 4.47 (Min/max value theorem). *Any continuous map $f: [a,b] \to \mathbb{R}$ takes both a maximum and a minimum, i.e. there exists $x_{\max} \in [a,b]$ such that $f(x_{\max}) \geq f(x)$ $\forall x \in [a,b]$. and there exists $x_{\min} \in [a,b]$ such that $f(x_{\min}) \leq f(x)$ $\forall x \in [a,b]$.*

Proof. This is immediate from the fact that $f([a,b])$ is a closed and bounded set in $\mathbb{R}$ by HB, so it contains its supremum and its infimum. $\square$

Exercise 4.27. *Find various contradictions to Corollary 4.47 if $K = [a,b]$ is replaced by a non-compact set.*

4.6.2. *Continuity and connectedness.*

Theorem 4.48 (Continuous maps preserve connectedness)**.** *Let $f\colon X \to Y$ be a continuous mapping between the metric (or topological) spaces X and Y, and let $S \subseteq X$ be a connected subset. Then $f(S) = \{f(x) \in Y : x \in S\}$ is connected.*

Proof. Assume to the contrary that $f(S)$ is disconnected, i.e. there exist A and B nonempty, disjoint and open in the relative topology $\mathcal{T}_{f(S)\subseteq Y}$ such that $f(S) = A \cup B$, hence there exist $\tilde{A}$ and $\tilde{B}$ open in Y s.t. $A = \tilde{A} \cap f(S)$ and $B = \tilde{B} \cap f(S)$. Consider $f^{-1}(\tilde{A})$ and $f^{-1}(\tilde{B})$. These are open in X and $V := S \cap f^{-1}(\tilde{A})$ resp. $W := S \cap f^{-1}(\tilde{B})$ are open in S (i.e. the relative topology $\mathcal{T}_{S\subseteq X}$). Note that V and W are nonempty. Furthermore, note that they are disjoint since if $p \in V \cap W$ then $f(p) \in A \cap B = \emptyset$. Lastly, note that $V \cup W = S$ since $V \subseteq S$, $W \subseteq S$ and $S \subseteq V \cup W$ by the definitions. Thus S is disconnected, which is a contradiction. $\square$

Corollary 4.49 (Intermediate value theorem). *Any continuous map $f\colon [a,b] \to \mathbb{R}$ has the **intermediate value property**, i.e. if $y \in \mathbb{R}$ is such that $f(a) < y < f(b)$ then there exists $x \in (a,b)$ such that $f(x) = y$.*

Proof. This is immediate from the fact that $f([a,b])$ is connected (as well as bounded). Namely, if y is in the interval $(f(a), f(b))$ but not in $f([a,b])$ then we can nontrivially decompose

$$f([a,b]) = \big((-R,y) \cap f([a,b])\big) \cup \big((y,R) \cap f([a,b])\big) \tag{4.36}$$

for sufficiently large $R > 0$. The image of f in $\mathbb{R}$ is therefore disconnected (cf. Example 4.42), which is a contradiction. $\square$

4.6.3. *Monotonicity.* (cf. Rudin 4.28-31)

A function $f\colon (a,b) \subseteq \Omega \to \mathbb{R}$ is said to be **monotonically increasing** (resp. **decreasing**) on (a,b) if $a < x < y < b$ implies $f(x) \le f(y)$ (resp. $f(x) \ge f(y)$). A function is **monotonic** if it is either monotonically increasing or decreasing. A function is **strictly** increasing resp. decreasing (monotonic) if there is strict inequality. (Compare these definitions to the corresponding ones for sequences, Section 3.4.)

Interestingly, monotonic functions can have no discontinuities of the second kind, and may have at most countably many discontinuities; see Rudin 4.29-30.

Exercise 4.28 (Rudin Exercise 4.15)**.** *A map $f\colon X \to Y$ is called open if it maps open sets to open sets, i.e. $V \subseteq X$ open $\Rightarrow$ $f(V) \subseteq Y$ open. Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is open and continuous then f is strictly monotonic.*

4.7. **Baire's theorem.** (cf. Rudin Exc 1.30 and 3.21-22)

Lemma 4.50 (Ball enclosure)**.** *Let (X,d) be a complete metric space and consider sequences of points p_n, x_n, radii $r_n > 0$, and corresponding balls s.t.*

$$x_n \in \bar{B}_n := \bar{B}_{r_n}(p_n) \supseteq \bar{B}_{n+1} \neq \emptyset, \qquad r_n \to 0 \text{ as } n \to \infty. \tag{4.37}$$

Then (x_n) is Cauchy and $x_n \to x$, where $\{x\} = \bigcap_n \bar{B}_n$.

The proof of this lemma is left as Exercise 4.29.

The following theorem turns out to be extremely useful in functional analysis:

Theorem 4.51 (**Baire's category theorem**). *Let (X, d) be a complete metric space and V_n, $n \in \mathbb{N}$, a sequence of open dense subsets of X. Then $\bigcap_{n=0}^{\infty} V_n$ is dense in X (and, assuming X has no discrete/isolated points, uncountable).*

Equivalently, if $X = \bigcup_{n=0}^{\infty} F_n$ where F_n, $n \in \mathbb{N}$, are closed subsets, then at least one F_n has a nonempty interior.

Proof. It is sufficient to show that $V := \bigcap_{n=0}^{\infty} V_n$ meets every nonempty open subset. Let $W \subseteq X$ be a nonempty open subset, then for each $n \in \mathbb{N}$, $V_n \cap W$ is nonempty and open. Hence there is $p_0 \in V_0 \cap W$ and some $0 < r_0 < 1$ such that $\bar{B}_{r_0}(p_0) \subseteq B_{2r_0}(p_0) \subseteq V_0 \cap W$. Consider then the set $W_1 := B_{r_0}(p_0) \cap V_0 \cap W$. This is nonempty and open and therefore meets V_1 to an open set. Thus we may find $p_1 \in V_1 \cap W_1$ as well as some $0 < r_1 < r_0$ s.t. $\bar{B}_{r_1}(p_1) \subseteq B_{2r_1}(p_1) \subseteq V_1 \cap W_1$.

Consider then $W_2 := B_{r_1}(p_1) \cap V_1 \cap W_1$. Again this is nonempty and open and therefore meets V_2 to an open set from which we can pick p_2 and $0 < r_2 < 1/2$ s.t. $\bar{B}_{r_2}(p_2) \subseteq B_{2r_2}(p_2) \subseteq V_2 \cap W_2$.

Iterating this produces a sequence of sets $W_{n+1} := B_{r_n}(p_n) \cap V_n \cap W_n$, with $0 < r_n < 1/n$, which are nonempty and open, and furthermore $W_{n+1} \subseteq \bar{B}_{r_n}(p_n) =: \bar{B}_n$. By Lemma 4.50, the sequence (p_n) is Cauchy and converges to some $\{p\} = \bigcap_{n=0}^{\infty} \bar{B}_n$. Since $W_{n+1} \subseteq W_n$, also $p \in \bigcap_{n \geq 0} W_n \subseteq V \cap W$. Thus V is dense in X.

Finally, we claim that V is uncountable. Namely, assuming that V is countable, say $V = \{q_1, q_2, \ldots\}$, we may define $V_n' := V_n \setminus \{q_1, \ldots, q_n\}$ and redo the argument above with V_n replaced by V_n'. Assuming X has no discrete points, V_n' are still open and dense, and furthermore $p \in V' := \bigcap_{n \geq 0} V_n' \subseteq \bigcap_{n \geq 0} V_n = V$. Therefore $p = q_N$ for some $N \in \mathbb{N}$, and $q_N \in V_N' = V_N \setminus \{q_1, \ldots, q_N\}$, which is a contradiction.

The equivalent statement of the theorem follows by taking complements. □

Corollary 4.52. *The real numbers $\mathbb{R}$ (and thus also $\mathbb{R}^n$) are uncountable.*

Proof. $\mathbb{R}$ is complete by Theorem 3.22 and has no isolated points by Exercise 4.14. Thus V_n defined by removing n points from $\mathbb{R}$ will be open and dense, and their intersection is necessarily nonempty and even uncountable by Baire's theorem. □

Remark 4.53. A note concerning the title of Theorem 4.51: A space X such that the intersection of a countable collection of dense open sets is nonempty is said to be of the **second (Baire) category**. Hence a complete metric space (such as $\mathbb{R}$) is of the second category, while a space which is not of the second category is said to be of the **first category**.

Exercise 4.29. *Prove Lemma 4.50 (Hint: compare previous enclosure theorems).*

Exercise 4.30. *Show that $\mathbb{Q}$ is of the first (Baire) category.*

Exercise 4.31. *A subset $E \subseteq \mathbb{R}$ is called a* ***null set*** *if for any $\varepsilon > 0$ there exists a countable set of closed intervals $I_1, I_2, \ldots$ such that $E \subseteq \bigcup_{n>0} I_n$, and $\sum_{n>0} m(I_n) < \varepsilon$, where $m([a, b]) := b - a$ (thus, being covered by an arbitrarily narrow set, E is of 'length' zero). Show that there exists an uncountable and dense null set in $\mathbb{R}$ which contains $\mathbb{Q}$. (Hint: think of covering $\mathbb{Q}$ with intervals whose lengths tend to zero sufficiently fast.)*

4.8. **Normed and quadratic spaces.** While $\mathbb{R}^n$ could be considered our favorite example of a metric space, we frequently come across many other normed or quadratic spaces in analysis. We give here some of the most essential notions.

4.8.1. *Normed spaces.*

Definition 4.54. Let V denote a vector space over $\mathbb{R}$. A **norm** on V is a map $V \to \mathbb{R}_+$, $v \mapsto \|v\|$, such that for all $\alpha \in \mathbb{R}$, $u, v \in V$:

(i) $\|\alpha v\| = |\alpha| \, \|v\|$ (scaling linearly),
(ii) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality),
(iii) $\|v\| = 0$ if and only if $v = 0$ (positive definite / non-degenerate).

The pair $(V, \|\cdot\|)$ is called a **normed linear space**.

Example 4.55. $\|\mathbf{x}\|_2 := |\mathbf{x}|$ defined in (4.2) is our standard/euclidean norm on $\mathbb{R}^n$.

Example 4.56. It is possible to show that for any real number $p \geq 1$,

$$\|\mathbf{x}\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p} \tag{4.38}$$

defines a norm on $\mathbb{R}^n$ (the tricky part being the triangle inequality, unless $p = 1$ or $p = 2$):

Theorem 4.57 (**Minkowski's inequality**)**.** *For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $p \geq 1$ it holds*

$$\|\mathbf{x} + \mathbf{y}\|_p \leq \|\mathbf{x}\|_p + \|\mathbf{y}\|_p \, .$$

We postpone the general proof of this theorem until we discuss duality in Section 6.7 (note however that $p = 1$ is obvious and $p = 2$ also derived below from Cauchy-Schwarz).

Exercise 4.32. *Show that*

$$\|\mathbf{x}\|_\infty := \max(|x_1|, |x_2|, \ldots, |x_n|) \tag{4.39}$$

is also a norm on $\mathbb{R}^n$.

Remark 4.58. Proposition 4.2 states that

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_\infty \leq \|\mathbf{x}\|_2 \qquad \forall \mathbf{x} \in \mathbb{R}^n, \tag{4.40}$$

which means that these two norms are *equivalent* (see also Exercise 4.37).

Any normed space $(V, \|\cdot\|)$ has an induced metric

$$d(u, v) := \|u - v\| \, , \tag{4.41}$$

namely its symmetry follows from (i) with $\alpha = -1$, while the triangle inequality and non-degeneracy are directly inherited from the corresponding properties (ii) and (iii) of the norm:

$$d(u, v) = \|u - v\| = \|u - w + w - v\| \leq \|u - w\| + \|w - v\| = d(u, w) + d(w, v). \tag{4.42}$$

That two norms are equivalent, such as in Remark 4.58, implies for the metric that their families of open balls/neighborhoods are equivalent, i.e. one is always nested in the other, and thus they define the same topology on V.

4.8.2. *Quadratic spaces.*

Definition 4.59. A **bilinear form** on V is a map $V \times V \to \mathbb{R}$, $(u, v) \mapsto \langle u, v\rangle$ such that for all $\alpha, \beta \in \mathbb{R}$, $u, v, w \in V$:

(i) $\langle \alpha u + \beta v, w\rangle = \alpha \langle u, w\rangle + \beta \langle v, w\rangle$,
(ii) $\langle u, \alpha v + \beta w\rangle = \alpha \langle u, v\rangle + \beta \langle u, w\rangle$.

A **symmetric** bilinear form on V is a bilinear form satisfying, in addition:

(iii) $\langle u, v\rangle = \langle v, u\rangle$ (symmetry).

An **inner product** or **scalar product** on V is a symmetric bilinear form $\langle \cdot, \cdot\rangle$ satisfying, in addition:

(iv) $\langle v, v\rangle > 0$ for $v \neq 0$ (positive definite).

A **quadratic form** on V is a map $q \colon V \to \mathbb{R}$ such that for $\alpha \in \mathbb{R}$, $u, v \in V$:

(i') $q(\alpha v) = \alpha^2 q(v)$ (scaling quadratically),
(ii') $\langle u, v\rangle_q := \frac{1}{4}\big(q(u + v) - q(u - v)\big)$ is bilinear in (u, v).

The quadratic form q is positive definite iff $\langle \cdot, \cdot\rangle_q$ is positive definite.

The pair (V, q) is called a **quadratic space**, and the pair $(V, \langle \cdot, \cdot\rangle)$ is called an **inner product space** (or a (real) **pre-Hilbert space**).

Example 4.60. $\mathbb{R}^n$ with the standard **euclidean inner product**

$$\langle \mathbf{x}, \mathbf{y}\rangle := \mathbf{x} \cdot \mathbf{y} = \sum_{j=1}^{n} x_j y_j \tag{4.43}$$

is our typical (n-dimensional) example of an inner product space. Given a positive-definite symmetric $n \times n$ matrix A we can also consider a "deformed" inner product

$$\langle \mathbf{x}, \mathbf{y}\rangle_A := \mathbf{x} \cdot (A\mathbf{y}) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i A_{ij} y_j. \tag{4.44}$$

Recall that by diagonalizing A we can re-write this expression

$$\langle \mathbf{x}, \mathbf{y}\rangle_A = \sum_{j=1}^{n} \lambda_j x'_j y'_j \tag{4.45}$$

in the coordinates $\mathbf{x}' = R\mathbf{x}$ of a corresponding basis of eigenvectors, $A = R^T D R$, $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $R^T R = R R^T = \mathbb{1}$. Positive-definiteness of $\langle \cdot, \cdot\rangle_A$ then equivalently means that all $\lambda_j > 0$, which is how we define positive-definiteness of the matrix A. We also recall from linear algebra the notion that

$$q(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}\rangle_A = \sum_{j=1}^{n} \lambda_j x'^2_j \tag{4.46}$$

is the quadratic form associated to the matrix A (useful when studying the Hessian matrix for instance).

Example 4.61. Another important example is the space of square-summable real sequences

$$\ell^2 := \left\{ u = (u_n) \colon \mathbb{N} \to \mathbb{R} \ : \ \sum_{n=0}^{\infty} u_n^2 < \infty \right\} \tag{4.47}$$

with the inner product (we will return to this example in Section 4.9.2 after discussing series)

$$\langle u, v\rangle := \sum_{n=0}^{\infty} u_n v_n. \tag{4.48}$$

Proposition 4.62 (**Cauchy–Schwarz inequality**). *Let V be an inner product space. Then for every $u, v \in V$ we have*

$$|\langle u, v\rangle| \leq \sqrt{\langle u, u\rangle}\sqrt{\langle v, v\rangle} \tag{4.49}$$

with equality iff u and v are parallel. (See Exercise 4.33 concerning the proof.)

Any inner product space $(V, \langle\cdot,\cdot\rangle)$ induces a positive definite quadratic form

$$q(v) := \langle v, v\rangle \tag{4.50}$$

and a norm, by taking the square root,

$$\|v\| := \sqrt{\langle v, v\rangle}. \tag{4.51}$$

Namely, its linear scaling and positive definiteness follow immediately, and furthermore, because of the Cauchy-Schwarz inequality (4.49), it also satisfies the triangle inequality (also known as **Minkowski's inequality**; cf. Theorem 4.57)

$$\|u+v\|^2 = \|u\|^2 + 2\langle u, v\rangle + \|v\|^2 \leq (\|u\| + \|v\|)^2, \tag{4.52}$$

and hence becomes a norm on V.

However, it is not the case that every normed space is also an inner product space. Namely, while

$$q(v) := \|v\|^2 \tag{4.53}$$

indeed scales quadratically, and is positive definite, the condition (ii') above must also hold. It is a fact however, that the normed spaces which are inner product spaces are precisely those in which the parallelogram identity holds:

Theorem 4.63 (**Jordan–von Neumann's theorem**). *Any normed real linear space $(V, \|\cdot\|)$ satisfying the **parallelogram identity***

$$\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2 \tag{4.54}$$

for all $u, v \in V$ is a real, positive-definite, quadratic space (V, q) with quadratic form $q(v) := \|v\|^2$, and an inner product space $(V, \langle\cdot,\cdot\rangle_q)$ with inner product

$$\langle u, v\rangle_q = \frac{1}{4}\left(\|u+v\|^2 - \|u-v\|^2\right). \tag{4.55}$$

Definition 4.64. A normed vector space in which every Cauchy sequence converges is called a **complete normed space** or a **Banach space**. An inner product space which is also a complete normed space is called a **Hilbert space**.

Further, two normed vector spaces X and Y are said to be **isomorphic**, $X \cong Y$, iff there exists a linear bijection $f: X \to Y$ and an equivalence between norms, i.e. constants $C \geq c > 0$ so that

$$c\|x\|_X \leq \|f(x)\|_Y \leq C\|x\|_X \qquad \forall x \in X, \tag{4.56}$$

and they are called **isometric** iff the equivalence holds with $C = c = 1$, i.e.

$$\|f(x)\|_Y = \|x\|_X \qquad \forall x \in X. \tag{4.57}$$

Example 4.65. Since $(\mathbb{R}^n, \|\cdot\|_2)$ is complete by Theorem 4.5, it is a Hilbert space. Furthermore, by the equivalence of the norms (see Remark 4.58), also $(\mathbb{R}^n, \|\cdot\|_\infty)$ is complete, and thus a Banach space. (And in fact, by Exercise 4.37, the same is true for $(\mathbb{R}^n, \|\cdot\|)$ for any norm $\|\cdot\|$.) Further important examples will be given below.

Exercise 4.33. *Prove Proposition 4.62 for example by considering the expression* $\langle u - \alpha v, u - \alpha v\rangle$ *with* $\alpha = \langle v, u\rangle / \langle v, v\rangle$.

Exercise 4.34. *Check that* (4.54) *and* (4.55) *hold for the induced norm* (4.51) *of an inner product space.*

Exercise 4.35. *Show that* (4.54) *does not hold in general for the norm* $\|\cdot\|_\infty$ *in* (4.39)*, and therefore it does not define an inner product (despite their* <u>*topologies*</u> *being equivalent).*

***Exercise 4.36** (Difficult!). *Prove Theorem 4.63.*
(Hint: consider first linearity over -1*, then* $\mathbb{Z}$*, then* $\mathbb{Q}$ *and finally* $\mathbb{R}$*.)*

Exercise 4.37 (Difficult!). *Prove that any two norms* $\|\cdot\|_A$ *and* $\|\cdot\|_B$ *in* $\mathbb{R}^n$ *are equivalent, i.e. that there exist some constants* $C \geq c > 0$ *s.t.*

$$c\,\|\mathbf{x}\|_A \leq \|\mathbf{x}\|_B \leq C\,\|\mathbf{x}\|_A \qquad \forall \mathbf{x} \in \mathbb{R}^n \tag{4.58}$$

(compare Remark 4.58).

***Exercise 4.38.** *What is the ratio of the volume of the unit ball in* $(\mathbb{R}^n, \|\cdot\|_2)$ *to that in* $(\mathbb{R}^n, \|\cdot\|_\infty)$*? (You could interpret it as the probability of a point placed at random in one "ball" ending up in the other ball.) What happens as* $n \to \infty$*?*

4.8.3. **Metric spaces can always be embedded into Banach spaces. ...* (interesting fact)

4.9. **Series.** Recall that, given a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$, we define its **series**

$$\sum_{n=0}^{\infty} x_n \tag{4.59}$$

in terms of the **partial sums**

$$S_N := \sum_{n=0}^{N} x_n = x_0 + x_1 + \ldots + x_N, \qquad N \in \mathbb{N}, \tag{4.60}$$

and we say that the series (4.59) **converges** iff the sequence $(S_N)_{N\in\mathbb{N}}$ converges, and define its limit value to be $\lim_{N\to\infty} S_N$.

4.9.1. *Majorization of series.* The following instructs us how to deal similarly with series in arbitrary Banach spaces.

Theorem 4.66 (The **majorization theorem**). *Let* $x_0, x_1, x_2, \ldots$ *be a sequence in a Banach space* $(X, \|\cdot\|)$ *and let* $b_0, b_1, b_2, \ldots$ *be a sequence in* $\mathbb{R}_+$ *such that*

$$\|x_n\| \leq b_n \qquad \forall n \in \mathbb{N}. \tag{4.61}$$

Let $S_n := x_0 + x_1 + \ldots + x_n$ *and* $B_n := b_0 + b_1 + \ldots + b_n$*. Then the sequence* (S_n) *in* X*, and hence the series* $\sum_{n=0}^{\infty} x_n \in X$*, converges as* $n \to \infty$ *if* (B_n) *converges as* $n \to \infty$*.*

Proof. Let $m > n$. Then, by the triangle inequality for the norm, and (4.61),

$$\|S_m - S_n\| = \left\| \sum_{n<k\leq m} x_k \right\| \leq \sum_{n<k\leq m} \|x_k\| \leq \sum_{n<k\leq m} b_k = |B_m - B_n|. \tag{4.62}$$

If (B_n) converges as $n \to \infty$ it is Cauchy. Hence, if $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that

$$\|S_m - S_n\| \leq |B_m - B_n| < \varepsilon \tag{4.63}$$

whenever $m \geq n \geq N$. Hence (S_n) is also Cauchy, and since X is assumed to be complete, the series converges as $n \to \infty$. □

Theorem 4.67 (Abel)**.** *Let $x_0, x_1, x_2, \dots$ be a sequence in a Banach space $(X, \|\cdot\|)$ and let $b_0, b_1, b_2, \dots$ be a sequence in $\mathbb{R}_+$ such that*

(i) $b_n \geq b_{n+1} \to 0$ *as* $n \to \infty$,
(ii) $\exists B \in \mathbb{R}_+$ *such that* $\|x_0 + x_1 + \dots + x_n\| \leq B \ \forall n \in \mathbb{N}$.

Then the sequence (S_n) in X defined by $S_n := b_0x_0 + b_1x_1 + \dots + b_nx_n$, i.e. the series

$$\sum_{n=0}^{\infty} b_n x_n,$$

is convergent.

Proof. Let $y_n := x_0 + x_1 + \dots + x_n$ and $y_{-1} := 0$. Then if $m > n \geq 0$ we have

$$\begin{aligned}
\|S_m - S_n\| &= \left\| \sum_{n<k\leq m} b_k x_k \right\| = \left\| \sum_{n<k\leq m} b_k(y_k - y_{k-1}) \right\| = \left\| \sum_{n<k\leq m} b_k y_k - \sum_{n\leq k<m} b_{k+1} y_k \right\| \\
&\leq \|b_{n+1}y_n\| + \|b_m y_m\| + \sum_{n<k<m} \|(b_k - b_{k+1})y_k\| \\
&\leq b_{n+1}B + b_m B + B \sum_{n<k<m} (b_k - b_{k+1}) \\
&\leq 2b_n B + B(b_{n+1} - b_{n+2} + b_{n+2} - b_{n+3} + \dots + b_{m-1} - b_m) \leq 3b_n B.
\end{aligned}$$

Now let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ s.t. $3b_n B < \varepsilon$ if $n \geq N$. Hence (S_n) is Cauchy and by the completeness of X we can conclude that (S_n) converges. □

4.9.2. *Some sequential Banach spaces.* We already mentioned the space ℓ^2 in Example 4.61. Using sequences and series we can define important examples of Banach spaces which can be thought of as limiting spaces $\mathbb{R}^{n\to\infty}$, however, we may see that their topology differs in the limit depending on which norm is chosen (unlike in the finite-dimensional case; cf. Exercise 4.37).

Example 4.68. Let ℓ^∞ denote the vector space of all bounded real sequences,

$$\ell^\infty := \{u = (u_n) \colon \mathbb{N} \to \mathbb{R} \ : \exists B > 0 \text{ s.t. } |u_n| \leq B \ \forall n \in \mathbb{N}\} \tag{4.64}$$

with the norm

$$\|u\|_\infty := \sup_{n\in\mathbb{N}} |u_n| = \sup\{|u_n| : n \in \mathbb{N}\}. \tag{4.65}$$

Indeed, addition and scalar multiplication of sequences are defined component-wise, making ℓ^∞ a vector space over $\mathbb{R}$. That (4.65) defines a norm on ℓ^∞ may be verified similarly to Exercise 4.32. Furthermore, $(\ell^\infty, \|\cdot\|_\infty)$ turns out to be a Banach space. We can prove this directly from the definitions (Exercise 4.39), but it will also follow from a more general theorem in Section 6. In fact, this is also an example of a *non*-separable Banach space:

Theorem 4.69. *The space ℓ^∞ is not separable.*

Proof. Recall that the space is separable iff it contains a countable dense subset. Note that we may associate to each subset $A \subseteq \mathcal{P}(\mathbb{N})$ an element $e^A \in \ell^\infty$ (a binary sequence) of the form

$$e^A = \big((n \in A)\big)_{n\in\mathbb{N}} = \big((0 \in A), (1 \in A), (2 \in A), \ldots\big), \tag{4.66}$$

where we use the notation $(P) = 1$ if P is true and $(P) = 0$ if P is false. We then obtain, for any $A, B \in \mathcal{P}(\mathbb{N})$, $A \neq B$,

$$d(e^A, e^B) = \left\|e^A - e^B\right\|_\infty = \sup_{n\in\mathbb{N}} |e^A_n - e^B_n| = \sup_{n\in\mathbb{N}} |(n \in A) - (n \in B)| = 1. \tag{4.67}$$

Thus we can form in ℓ^∞ the set

$$\mathcal{B} := \bigcup_{A\in\mathcal{P}(\mathbb{N})} B_{1/3}(e^A) \tag{4.68}$$

of uncountably many disjoint open balls, which cannot contain a countable dense subset. □

Theorem 4.70. *For any real number $1 \le p < \infty$, let $\ell^p \subsetneq \ell^\infty$ denote the vector space of all bounded and p-summable real sequences,*

$$\ell^p := \left\{ u = (u_n) \colon \mathbb{N} \to \mathbb{R} \ : \sum_{n=0}^\infty |u_n|^p < \infty \right\} \tag{4.69}$$

with the norm

$$\|u\|_p := \left(\sum_{n=0}^\infty |u_n|^p \right)^{1/p}. \tag{4.70}$$

Then $(\ell^p, \|\cdot\|_p)$ is a separable Banach space. Furthermore, its closed unit ball $\bar{B}_1(0) \subseteq \ell^p$ is <u>not</u> compact.

Proof. Given the triangle (Minkowski) inequality of Theorem 4.57 on $\mathbb{R}^n$, any $n \in \mathbb{N}^+$, by considering the norm $\|u\|_p$ as the p:th root of the limit

$$\|u\|_p^p = \lim_{n\to\infty} \sum_{k=0}^n |u_k|^p = \sup_{n\in\mathbb{N}} \sum_{k=0}^n |u_k|^p$$

it also extends to sequences and ensures that $(\ell^p, \|\cdot\|_p)$ is a normed vector space over $\mathbb{R}$.

Let us prove that ℓ^p is complete, namely take a Cauchy sequence $(u^{(n)})_{n\in\mathbb{N}}$ in ℓ^p. Then for any $k, n, m \in \mathbb{N}$

$$\left|u_k^{(n)} - u_k^{(m)}\right|^p \le \sum_{j=0}^\infty \left|u_j^{(n)} - u_j^{(m)}\right|^p = \left\|u^{(n)} - u^{(m)}\right\|_p^p, \tag{4.71}$$

which is smaller than ε^p if $m, n \geq N(\varepsilon)$ is chosen large enough. Therefore $(u_k^{(n)})_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ and, by completeness of $\mathbb{R}$, converges to some number, say $u_k^{(n)} \to x_k \in \mathbb{R}$, as $n \to \infty$. Consider the sequence $x = (x_k)_{k\in\mathbb{N}}$ of real numbers. We need to prove that $x \in \ell^p$ and that $\left\|u^{(n)} - x\right\|_p \to 0$. By the boundedness of Cauchy sequences (Exercise 4.11), $\left\|u^{(n)}\right\|_p \leq B$ for some $B > 0$ and all n, and therefore for any $M \geq 0$

$$\sum_{k=0}^{M} |u_k^{(n)}|^p \leq \left\|u^{(n)}\right\|_p^p \leq B^p. \tag{4.72}$$

Thus the l.h.s. is bounded *uniformly* in M and n, and taking first the limit $n \to \infty$, we find

$$\sum_{k=0}^{M} |x_k|^p \leq B^p < \infty, \tag{4.73}$$

so that, after finally taking the limit $M \to \infty$, not only $x \in \ell^\infty$ (bounded sequence) but actually $x \in \ell^p$ (p-summable sequence).

We also have for any $\varepsilon > 0$, any M and $n, m \geq N(\varepsilon)$

$$\sum_{k=0}^{M} \left|u_k^{(n)} - u_k^{(m)}\right|^p \leq \left\|u^{(n)} - u^{(m)}\right\|_p^p < \varepsilon^p, \tag{4.74}$$

so that, taking first the limit $m \to \infty$ and then $M \to \infty$,

$$\sum_{k=0}^{M} \left|u_k^{(n)} - x_k\right|^p \leq \varepsilon^p \qquad \Rightarrow \qquad \left\|u^{(n)} - x\right\|_p \leq \varepsilon, \tag{4.75}$$

for any $n \geq N(\varepsilon)$. Therefore $u^{(n)} \to x$ in ℓ^p as $n \to \infty$.

It remains to prove that ℓ^p is separable. Consider the set

$$Q := \left\{u = (u_k) \in \ell^p : u_k \in \mathbb{Q}\ \forall k \text{ and } |\operatorname{supp} u| < \infty\right\},$$

where $\operatorname{supp} u := \{n \in \mathbb{N} : u_n \neq 0\}$, the support of u. Then

$$Q = \bigcup_{N>0} Q_N, \qquad Q_N := \{u \in Q : u_n = 0\ \forall n \geq N\}, \tag{4.76}$$

i.e.

$$Q_N \ni u = (u_0, u_1, \ldots, u_{N-1}, 0, 0, \ldots), \tag{4.77}$$

where each such subset $Q_N \sim \mathbb{Q}^N$ is countable, and therefore Q is countable by Theorem 2.6.

We claim that Q is dense in ℓ^p, namely take any $x \in \ell^p$ and $\varepsilon > 0$. Then by finiteness of $\|x\|_p$ there exists $N \in \mathbb{N}$ s.t. $\left(\sum_{k=N}^{\infty} |x_k|^p\right)^{1/p} < \varepsilon/2^{1/p}$. Let us write

$$x = (x_0, x_1, \ldots, x_{N-1}, 0, 0, \ldots) + (0, \ldots, 0, x_N, x_{N+1}, \ldots), \tag{4.78}$$

where the first part is identical to $\mathbb{R}^N$ and the second is bounded in ℓ^p-norm by $\varepsilon/2^{1/p}$. By density of $\mathbb{Q}$ in $\mathbb{R}$, there exists $y_k \in \mathbb{Q}$ s.t. $|y_k - x_k| < \varepsilon/(2N)^{1/p}$ for each $k \in \{0, 1, \ldots, N-1\}$. Therefore, with $y = (y_0, \ldots, y_{N-1}, 0, 0, \ldots) \in Q_N \subseteq Q$,

$$\|x - y\|^p = \sum_{k=0}^{N-1} |x_k - y_k|^p + \sum_{k=N}^{\infty} |x_k|^p < \varepsilon^p/2 + \varepsilon^p/2 = \varepsilon^p, \tag{4.79}$$

which proves our claim. Thus ℓ^p contains the dense countable subset Q.

For the last statement of the theorem, consider the sequence $e^{(n)} \in Q_{n+1} \subseteq \ell^p$, $n \in \mathbb{N}$, where

$$e_k^{(n)} = \begin{cases} 1, & k = n, \\ 0, & k \neq n. \end{cases} \tag{4.80}$$

Then $\left\|e^{(n)}\right\|_p = 1$, so $e^{(n)} \in \bar{B}_1(0)$, but no subsequence is Cauchy (why?) so cannot converge. Therefore, by Theorem 4.37, the closed bounded subset $\bar{B}_1(0)$ is not compact. □

Corollary 4.71. *The space $(\ell^2, \langle\cdot,\cdot\rangle)$ is a separable Hilbert space, and its closed unit ball $\bar{B}_1(0) \subseteq \ell^2$ is not compact.*

Proof. By the above theorem, ℓ^2 is a separable Banach space and it only remains to prove that (4.48) defines an inner product. Indeed, by $0 \leq (a \pm b)^2 = a^2 + b^2 \pm 2ab$, we have

$$|ab| \leq \frac{1}{2}(a^2 + b^2), \tag{4.81}$$

and therefore, for $x = (x_n), y = (y_n) \in \ell^2$ and all $n \in \mathbb{N}$

$$|x_n y_n| \leq \frac{1}{2}(x_n^2 + y_n^2). \tag{4.82}$$

Thus, by the majorization theorem, $\langle\cdot,\cdot\rangle$ is well defined and

$$|\langle x, y\rangle| \leq \sum_{n=0}^{\infty} |x_n y_n| \leq \frac{1}{2}(\|x\|_2^2 + \|y\|_2^2). \tag{4.83}$$

Furthermore $\langle x, x\rangle = \|x\|_2^2$, while bilinearity and symmetry follow by taking limits. □

Exercise 4.39. *Prove that ℓ^∞ with the norm* (4.65) *is complete.*

Exercise 4.40. *Construct an uncountable discrete metric space (i.e. having the discrete metric of Example 4.12) as a metric subspace of ℓ^∞.*

Exercise 4.41. *Prove that the closed unit ball $\bar{B}_1(0)$ in ℓ^∞ is bounded and closed but not compact.*

**Remark* 4.72. Side note: if curious about different notions of compactness and their equivalence depending on ZF(C), see [DlCHH+02] (advanced!).

4.10. Normed rings.

Definition 4.73. Let $(R, +, \cdot)$ be a ring. A function $R \to \mathbb{R}_+$, $x \mapsto \|x\|$, is called a **norm** on R if for all $x, y \in R$:

(i) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality / sub-additive),
(ii) $\|xy\| \leq \|x\|\,\|y\|$ (sub-multiplicative),
(iii) $\|x\| = 0$ if and only if $x = 0$ (positive definite / non-degenerate).

The pair $(R, \|\cdot\|)$ is called a **normed ring**. A normed ring is **complete** (or **Banach**) if every Cauchy sequence in R converges.

Example 4.74. Typical examples of normed rings are of course $\mathbb{R}$ and $\mathbb{C}$, but also e.g. square matrices $\mathbb{R}^{n\times n}$ with a suitable norm (will be discussed in Section 5.2.1), as well as functions with values in a normed ring and a suitable norm.

Lemma 4.75. *If $x_n \to x$ and $y_n \to y$ in a normed ring R, then $x_n y_n \to xy$ as $n \to \infty$. (In other words, multiplication is continuous.)*

Proof. By the properties of the norm,

$$\begin{aligned}\|x_n y_n - xy\| &= \|x_n y_n - xy_n + xy_n - xy\| \leq \|(x_n - x)y_n\| + \|x(y_n - y)\| \\ &\leq \|x_n - x\| \, \|y_n\| + \|x\| \, \|y_n - y\| \to 0,\end{aligned}$$

as $n \to \infty$, where we used that $\|y_n\|$ is bounded (and in fact $|\,\|y_n\| - \|y\|\,| \leq \|y_n - y\| \to 0$, so $\|y_n\| \to \|y\|$). □

4.10.1. *The exponential function on complete normed rings.*

Theorem 4.76 (The **exponential function**). *Let R be a complete normed ring with unit 1. Then there exists a function* $\exp\colon R \to R$ *such that*

$$\exp(0) = 1 \tag{4.84}$$

and

$$\exp(xy) = \exp(x)\exp(y) \tag{4.85}$$

for all $x, y \in R$ such that $xy = yx$. Moreover,

$$\exp(x) = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{x^n}{n!}. \tag{4.86}$$

Proof. Observe first that for $x \neq 0$, x^0 is, by definition, the multiplicative unit 1 in R. Define for $x \in R$ and $n \in \mathbb{N}$

$$e_n(x) := 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}. \tag{4.87}$$

If $n < m$

$$\|e_m(x) - e_n(x)\| = \left\| \sum_{n<k\leq m} \frac{x^k}{k!} \right\| \leq \sum_{n<k\leq m} \frac{\|x^k\|}{k!}, \tag{4.88}$$

where $\|x^k\| = \|xx^{k-1}\| \leq \|x\| \, \|x^{k-1}\| \leq \ldots \leq \|x\|^k$. Hence, if $r := \|x\|$,

$$\|e_m(x) - e_n(x)\| \leq \sum_{n<k\leq m} \frac{r^k}{k!} \leq \sum_{k=n+1}^{\infty} \frac{r^k}{k!}. \tag{4.89}$$

We have that $r \in \mathbb{R}_+$ and that the sequence $e_n(r) = \sum_{k=0}^{n} \frac{r^k}{k!}$ of non-negative terms converges for any r (e.g. by the ratio test, cf. Rudin 3.34). Therefore the r.h.s. of (4.89) tends to zero with $n \to \infty$. This shows that $(e_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence, which converges in R. (In fact the argument shows that $e_n(x)$ converges uniformly on the ball $\{x \in R : \|x\| \leq r\}$ for all $r \in \mathbb{R}_+$ as $n \to \infty$; we will get back to this concept later.)

Let us define

$$e^x := \exp(x) := \lim_{n\to\infty} e_n(x). \tag{4.90}$$

Obviously $e_n(0) = 1 = e^0$.

Let $x, y \in R$ s.t. $xy = yx$ and suppose $\|x\| \leq r$, $\|y\| \leq r$. Then we have

$$e_{2n}(x)e_{2n}(y) = \sum_{0 \leq j,k \leq 2n} \frac{x^j}{j!}\frac{y^k}{k!} = \sum_{\substack{0 \leq j,k \leq 2n \\ j+k \leq 2n}} \frac{x^j y^k}{j!k!} + \sum_{\substack{0 \leq j,k \leq 2n \\ j+k > 2n}} \frac{x^j y^k}{j!k!}. \tag{4.91}$$

We call the first sum on the r.h.s. A and the second B. Compute

$$A = \sum_{s=0}^{2n} \sum_{j+k=s} \frac{x^j y^k}{j!k!} = \sum_{s=0}^{2n} \frac{1}{s!} \sum_{j+k=s} \frac{s!}{j!k!} x^j y^k = \sum_{s=0}^{2n} \frac{(x+y)^s}{s!}, \tag{4.92}$$

hence $A = e_{2n}(x+y)$.

We claim that $B \to 0$ as $n \to \infty$. Indeed, (making a rough sketch of the indices in $\mathbb{N}^2$ helps here)

$$\|B\| \leq \sum_{\substack{n<j \leq 2n \\ 0 \leq k \leq 2n}} \frac{r^j r^k}{j!k!} + \sum_{\substack{0 \leq j < n \\ n < k \leq 2n}} \frac{r^j r^k}{j!k!} = (e_{2n}(r) - e_n(r))e_{2n}(r) + e_n(r)(e_{2n}(r) - e_n(r)), \tag{4.93}$$

and both these terms tend to zero as $n \to \infty$ since $(e_s(r))_{s \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$.

Thus, we have proved that

$$e_{2n}(x)e_{2n}(y) - e_{2n}(x+y) \to 0 \qquad \text{as } n \to \infty, \tag{4.94}$$

and hence, by Lemma 4.75,

$$e^x e^y = e^{x+y}, \tag{4.95}$$

which proves the theorem. □

See [Rud76, Thms. 3.31 & 8.6] for more on the exponential function.

5. Differentiation and integration [L11-15]

5.1. **Reading tip.** This Section concerns Rudin's Chapters 5 (differentiation) and 6 (integration), as well as some of the important examples in Ch 8. Whenever possible, we will here try to give general statements (i.e. for normed vector or Banach spaces), which includes parts of Chapter 9-10 (mainly 9.10-9.21, while 9.1-9.9 recalls some linear algebra, and 10.1-9) as well.

Recall the basic theorems concerning derivatives, Rudin 5.1-5.5 and 5.13 (L'Hospital), and study carefully the examples in 5.6! We will spend a little more time on the mean value theorems, 5.7-5.13, and Taylor's theorem 5.15.

Recall the definition and basic properties of the Riemann integral, and study the various examples and definitions in the suggested exercises. Note that the generalization in Rudin due to Stieltjes is *extracurricular* to this course. Therefore you may skip Rudin 6.14-19 and in the exercises and theorems in Rudin (such as Thm 6.12) choose the 'weight' function $\alpha(x) = x$ to reduce to the usual Riemann integral (which should be familiar from calculus). However, the curious student will benefit from learning and working with the Riemann-Stieltjes integral as it is not a very deep generalization. (Though note that by taking a discontinuous weight α one may extract pointwise values of the integrand.) The last chapter (11) in Rudin concerns the Lebesgue integral which is extremely useful but more technical and thus will be dealt with in another course. Similarly, integration on curves and surfaces and using differental forms, Rudin's 6.26-27 and what comes after 10.9, is the subject of other courses in vector analysis and differential geometry.

5.1.1. *Typos in Rudin.*

- in Thm 6.12(b) one also assumes $f_1, f_2 \in \mathscr{R}$
- in Eq. (16), Ch. 9 there is a "$|$" missing

5.1.2. *Exercises.* Rudin Ch. 5: 1-9,11-21; Ch. 6: (1),2,(3),4-12,15; Ch. 9: 1-11,13-15,26-31 Exam 2020-08-19: problem 7. Exam 2020-06-15: problem 7. Exam 2020-03-16: problems 4,7. Exam 2019-06-15: problems 7,8. Exam 2019-01-14: problems 4,8. Exam 2014-12-17: problem 4. Exam 2014-04-23: problem 3. Exam 2013-12-18: problem 2.

Remark 5.1. Rudin Exc 6.15 actually has a useful application in quantum mechanics, namely the proved inequality

$$\int_a^b f'(x)^2\,dx \cdot \int_a^b x^2 f(x)^2\,dx > \frac{1}{4} \tag{5.1}$$

is a form of **Heisenberg's uncertainty principle**, and says that if a particle with position x and spatial probability distribution $f(x)^2$ is contained in an interval $[a, b] \ni 0$ and has fixed, finite 'energy' $E = \int_a^b f'(x)^2\,dx$ (it turns out that f' encodes the momentum of the particle), then it cannot be localized arbitrarily close to the point $x = 0$, i.e. it is impossible to adjust the shape of f to make the variance $\int_a^b x^2 f(x)^2 dx$ arbitrarily small.

5.1.3. *Aims.* Concepts discussed in this Section:

- differentiable functions
- mean-value theorem and its consequences
- Taylor series (polynomial)
- Riemann integral

Learning outcomes: After this Section you should be able to

- explain the theoretical basis of differential and integral calculus including the formulation of central theorems and the main features of their proofs;
- apply the theory to solve mathematical problems including the construction of simple proofs.

The more abstract parts including peculiarities in infinite dimensions, such as Section 5.2.3, are recommended for students aiming for the highest grade.

5.2. **Differentiation.** Recall that differentiability is about the *local* approximation of a function by a *linear* one. In its simplest formulation, say for a function $f\colon (a,b) \subseteq \mathbb{R} \to \mathbb{R}$, we say that f is **differentiable** at $x \in (a,b)$ iff the limit

$$L = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{5.2}$$

exists, and we then denote this limit L by $f'(x)$ or $\frac{df}{dx}(x)$. Equivalently, we may formulate this statement that there exists some real number L such that

$$|f(x+h) - f(x) - Lh|/|h| \to 0 \qquad \text{as } 0 \neq h \to 0, \tag{5.3}$$

or, also equivalently, that for any $h \in \mathbb{R}$ such that $x + h \in (a,b)$ we may write

$$f(x+h) = f(x) + Lh + \varepsilon(h)h, \tag{5.4}$$

where the quantity $\varepsilon(h)$, simply defined by

$$\varepsilon(h) := \begin{cases} (f(x+h) - f(x) - Lh)/h, & h \neq 0, \\ 0, & h = 0, \end{cases} \tag{5.5}$$

is required to vanish as $h \to 0$. In other words, in the first step we approximate $f(x+h)$ as a function of h by the constant $f(x)$ and we then see that the error we are making vanishes at worst linearly in h, with the factor of linearity being L. After accounting for this behavior by subtracting the constant plus linear function $f(x) + Lh$, the remaining error $|\varepsilon(h)h|$ vanishes strictly *faster* than linearly as $h \to 0$ since we may even divide this quantity by h before taking the limit.

Similarly, if $\mathbf{f}\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, we call f differentiable at $\mathbf{x} \in \Omega$ iff there exists $\mathbf{L} \in \mathbb{R}^{m\times n}$, i.e. a linear map $\mathbb{R}^n \to \mathbb{R}^m$, such that(2)

$$\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{f}(\mathbf{x}) + \mathbf{L}[\mathbf{h}] + \varepsilon(\mathbf{h})|\mathbf{h}|, \tag{5.6}$$

where $\varepsilon(\mathbf{h}) \to 0$ as $|\mathbf{h}| \to 0$. We call the map $\mathbf{L} =: \mathbf{f}'(\mathbf{x})$ the derivative of $\mathbf{f}$ at $\mathbf{x}$ and may identify it with the **Jacobian matrix**

$$\frac{d\mathbf{f}}{d\mathbf{x}}(\mathbf{x}) = \begin{bmatrix} \frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{x}) & \frac{\partial \mathbf{f}}{\partial x_2}(\mathbf{x}) & \dots & \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}) \end{bmatrix}, \qquad \frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}) = \mathbf{L}[\mathbf{e}_j], \tag{5.7}$$

where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ denotes the standard basis in $\mathbb{R}^n$.

Note that in order to discuss linear approximation of functions in general we need at a minimum the notion of linear maps between vector spaces, since the derivative is such a map. And furthermore, in order to be able to estimate the smallness of the error in the approximation we need a metric, or a norm on these spaces. Thus, we may generalize the concept of derivatives to arbitrary normed vector spaces:

(2)We use the notation $[h]$ to stress that there is linearity in this argument.

Definition 5.2 (Derivative)**.** Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed linear spaces, and $\mathrm{Hom}(V, W)$ the space of *bounded/continuous* (clarified below) linear maps from V to W. A function $f\colon \Omega \subseteq V \to W$ is called **differentiable** at $x \in \Omega$ iff there exists a linear map $L \in \mathrm{Hom}(V, W)$ such that

$$f(x+h) = f(x) + L[h] + \varepsilon(h)\,\|h\|_V \tag{5.8}$$

where $\|\varepsilon(h)\|_W \to 0$ as $\|h\|_V \to 0$. We denote this map L by $f'(x)$ and call it the **derivative** of f at the point x.

Remark 5.3. We note that $f'(x)$ is unique, namely if there exist $L, M \in \mathrm{Hom}(V, W)$ such that

$$f(x) + L[h] + \varepsilon_L(h)\,\|h\|_V = f(x+h) = f(x) + M[h] + \varepsilon_M(h)\,\|h\|_V \tag{5.9}$$

then $(L - M)[h] = \varepsilon(h)\,\|h\|_V$ with error $\varepsilon(h) := \varepsilon_L(h) - \varepsilon_M(h) \to 0$ as $h \to 0$. Fix $h \neq 0$, then for any $t > 0$

$$(L - M)[th] = \varepsilon(th)\,\|th\|_V = \varepsilon(th)t\,\|h\|_V \quad \Rightarrow \quad (L - M)[h] = \varepsilon(th)\,\|h\|_V \to 0, \tag{5.10}$$

as $t \to 0$. Since h was arbitrary this implies $L - M = 0$ as linear maps, i.e. $L = M$.

For dealing with error terms of different order it will be tremendously convenient to introduce the following notation:

Definition 5.4 (**Ordo**)**.** Let $f, g\colon X \to Y$ be defined in a neighborhood of $0 \in X$.

We write $f(h) = O\big(g(h)\big)$ as $h \to 0$ iff there exists $\delta > 0$ and $K < \infty$ such that $\|f(h)\|_Y \le K\,\|g(h)\|_Y$ for all $h \in B_\delta(0)$, or equivalently if

$$\limsup_{h \to 0} \frac{\|f(h)\|_Y}{\|g(h)\|_Y} < \infty. \tag{5.11}$$

We then say that f is **big-O** of g as $h \to 0$.

We write $f(h) = o\big(g(h)\big)$ as $h \to 0$ iff $f(h) = O\big(g(h)\big)$ and

$$\limsup_{h \to 0} \frac{\|f(h)\|_Y}{\|g(h)\|_Y} = 0, \tag{5.12}$$

or equivalently, for any $\varepsilon > 0$ there exists $\delta > 0$ s.t. $\|f(h)\| \le \varepsilon\,\|g(h)\|$ for all $h \in B_\delta(0)$. We then say that f is **small-o** of g as $h \to 0$.

Note that $f(h) = g(h) + O\big(k(h)\big)$ iff $f(h) - g(h) = O\big(k(h)\big)$, and similarly for o. Also note that $f(h) = O(1)$ simply means that f is bounded in a neighborhood of $h = 0$, while $f(h) = o(1)$ simply means that $f(h) \to 0$ as $h \to 0$. In general we may also replace the notation $\|g(h)\|_Y = \tilde{g}(\|h\|_X)$ where $\tilde{g}\colon \mathbb{R}_+ \to \mathbb{R}_+$.

Thus $f\colon V \to W$ is differentiable at x iff there exists L linear (and bounded/continuous) s.t.

$$f(x+h) = f(x) + L[h] + o(h), \tag{5.13}$$

where $o(h) = \varepsilon(h)\,\|h\|_V$ in our earlier notation (namely $\limsup_{h\to 0} |\varepsilon(h)h|/|h| = 0$). Furthermore, we also have $f(x+h) = f(x) + O(h)$, where the error $O(h)$ is bounded by

$$\|f(x+h) - f(x)\|_W = \|L[h] + \varepsilon(h)\,\|h\|_V\|_W \le \|L[h]\|_W + \|\varepsilon(h)\|_W\,\|h\|_V \to 0, \tag{5.14}$$

as $h \to 0$, which imples that f is continuous at x (note that also continuity of L was used). Thus we have proved:

Theorem 5.5. *If f is differentiable at the point x then f is continuous at x.*

Example 5.6. Let $f\colon \mathbb{R} \to \mathbb{R}$ (with usual norm $|\cdot|$) be defined by $f(x) = x^2$. Then for any $x, h \in \mathbb{R}$

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2 = f(x) + 2xh + o(h), \tag{5.15}$$

and therefore f is differentiable on $\mathbb{R}$ with derivative $f'(x) = 2x$.

Example 5.7. Let $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $f(A) = A^2$ for 2×2-matrices A with the norm which is induced by the linear isomorphism $\mathbb{R}^{2\times 2} \cong \mathbb{R}^4$ (bijection) of vector spaces:

$$\|A\| = \left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\| := |(a_{11}, a_{12}, a_{21}, a_{22})| = \sqrt{a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2} \tag{5.16}$$

(why is this a norm?).

Let us determine if f is differentiable and, if so, compute $f'(A)$. We note that

$$f(A+H) = (A+H)^2 = A^2 + AH + HA + H^2, \tag{5.17}$$

and claim that the last term satisfies $H^2 = o(H)$, i.e. $\|H^2\| / \|H\| \to 0$ as $\|H\| \to 0$. This may be seen by either explicitly computing

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \Rightarrow \|H^2\|^2 = (h_{11}^2 + h_{12}h_{21})^2 + h_{12}^2(h_{11}+h_{22})^2 + h_{21}^2(h_{11}+h_{22})^2 + (h_{21}h_{12}+h_{22}^2)^2 \tag{5.18}$$

and estimating each term of this polynomial by the polynomial $\|H\|^4$, or for example by first using the triangle inequality

$$\begin{aligned} \left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\| &\leq \left\| \begin{bmatrix} a_{11} & 0 \\ 0 & 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \right\| + \left\| \begin{bmatrix} 0 & 0 \\ 0 & a_{22} \end{bmatrix} \right\| \\ &= |a_{11}| + |a_{12}| + |a_{21}| + |a_{22}|, \end{aligned}$$

and noting that each component of the matrix H^2 is a two-term quadratic polynomial in h_{ij} so that $\|H^2\| \leq 8\|H\|^2$. Another, much more powerful method (though quite far-fetched if one has never seen it before) is to note that (let us now use the conventional notation $A_{jk} = a_{jk}$)

$$\|A\|^2 = \sum_{j,k} A_{jk}^2 = \operatorname{Tr}(A^T A), \tag{5.19}$$

and that for another matrix B, by Cauchy-Schwarz,

$$\|AB\|^2 = \sum_{j,k} \left(\sum_l A_{jl} B_{lk} \right)^2 \leq \sum_{j,k} \left(\sum_l A_{jl}^2 \right) \left(\sum_l B_{lk}^2 \right) = \|A\|^2 \|B\|^2, \tag{5.20}$$

i.e. $\|AB\| \leq \|A\| \|B\|$. In particular we obtain $\|H^2\| \leq \|H\|^2$, and thus

$$H^2 = O(\|H\|^2) = o(\|H\|). \tag{5.21}$$

In conclusion, we may write

$$f(A+H) = f(A) + L_A[H] + o(H), \tag{5.22}$$

with the linear (check!) map $L_A\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$,

$$H \mapsto L_A[H] := AH + HA. \tag{5.23}$$

We infer from our definitions that f is differentiable at any point $A \in \mathbb{R}^{2\times 2}$ and that the derivative $f'(A)$ at that point is given by the linear map L_A in (5.23).

Definition 5.8. The above norm (5.19) on matrices, which we may denote by $\|\cdot\|_{\mathrm{F}}$, is called the **Frobenius norm** and makes $\mathbb{R}^{n\times n}$ a normed ring with unit $\mathbb{1}$. By defining

$$\langle A, B\rangle_{\mathrm{F}} := \mathrm{Tr}(A^T B) = \sum_{j,k} A_{jk}B_{jk} \tag{5.24}$$

for any $A, B \in \mathbb{R}^{m\times n}$ (!) we have an inner product associated to this norm: $\|A\|_{\mathrm{F}}^2 = \langle A, A\rangle_{\mathrm{F}}$.

Exercise 5.1. *Let $f\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be defined by $f(A) = A^3$. Show that f is differentiable at all matrices $A \in \mathbb{R}^{2\times 2}$ and compute the derivative $f'(A)$.*

Exercise 5.2. *Verify that (5.24) defines an inner product on $\mathbb{R}^{m\times n}$.*

Exercise 5.3. *Use the ordo concept to derive a simple proof of l'Hospital's rule (cf. Rudin 5.13)*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \tag{5.25}$$

if f, g are real-valued and differentiable in a neighborhood of a, $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, and the latter limit exists (and is finite).

5.2.1. *Operator norm.* Note that when we generalize the derivative to arbitrary normed vector spaces, we must speak of *bounded/continuous* linear operators.

Definition 5.9. Given normed linear spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ over $\mathbb{R}$, we call $T\colon V \to W$ an **operator** (or **linear transformation**) if it is a linear map, i.e. for all $\alpha, \beta \in \mathbb{R}$ and $u, v \in V$,

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v). \tag{5.26}$$

We call an operator T **bounded** if the image $T(\bar{B}_1(0))$ of the unit ball is bounded (note that because of the linearity, the full range $T(V)$ is always *un*bounded in W unless $T = 0$, and therefore one uses a slightly different notion of boundedness for operators).

We denote by $\mathcal{L}(V;W)$ the space of all operators $V \to W$, and by $\mathrm{Hom}(V;W)$ the space of all bounded operators (these are also called **homomorphisms** of normed vector spaces, i.e. structure-respecting maps, here w.r.t. both linearity and norm). If $S \in \mathcal{L}(U;V)$ and $T \in \mathcal{L}(V;W)$ then we may compose the operators, $TS \in \mathcal{L}(U;W)$, with $(TS)v := (T \circ S)(v) = T(Sv)$ (it is common to abbreviate $Tx = T(x) = T[x]$ for linear maps).

Define for any $T \in \mathrm{Hom}(V;W)$ the **operator norm**

$$\|T\|_{\mathrm{op}} = \|T\| := \sup_{x\in V:\|x\|_V \leq 1} \|Tx\|_W\,. \tag{5.27}$$

Indeed this is a norm because:

1. $0 \leq \|T\| < \infty$ for any $T \in \mathrm{Hom}(V,W)$ due to the boundedness condition.
2. $\|\lambda T\| = |\lambda|\,\|T\|$ for any $\lambda \in \mathbb{R}$:
$$\|\lambda T(x)\|_W = |\lambda|\,\|Tx\|_W\,. \tag{5.28}$$
3. $\|T + S\| \leq \|T\| + \|S\|$:
$$\sup\|(T+S)(x)\|_W \leq \sup\left(\|Tx\|_W + \|Sx\|_W\right) \leq \sup\|Tx\|_W + \sup\|Sx\|_W\,. \tag{5.29}$$
4. If $\|T\| = 0$ then $\sup_{\|x\|_V\leq 1}\|Tx\|_W = 0$ so $Tx = 0$ for all $x \in \bar{B}_1(0)$, and thus $T = 0$.

And, furthermore, if $S \in \mathrm{Hom}(U,V)$ and $T \in \mathrm{Hom}(V,W)$ then

5. $\|TS\| \leq \|T\|\,\|S\|$: For any $x \in U$,

$$\|TSx\|_W \leq \|T\|\,\|Sx\|_V \leq \|T\|\,\|S\|\,\|x\|_U\,. \tag{5.30}$$

We note that this in fact makes $(\mathrm{Hom}(V,V), \|\cdot\|_{\mathrm{op}})$ a normed ring with unit $\mathbb{1} = \mathrm{id}_V$. Furthermore, bounded linear operators are **continuous**, since for $T \in \mathrm{Hom}(V,W)$ one has $T(x+h) - T(x) = T(h)$, and if $h \neq 0$

$$\|T(h)\|_W = \left\|T\big(\|h\|_V \|h\|_V^{-1} h\big)\right\|_W = \|h\|_V \left\|T\big(\|h\|_V^{-1} h\big)\right\|_W \leq \|h\|_V \|T\|_{\mathrm{op}} = O(h). \tag{5.31}$$

Example 5.10. On the vector space $\mathbb{R}^{m\times n} = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)$ of matrices we now have two norms: the operator norm $\|A\|_{\mathrm{op}}$ and the Frobenius norm $\|A\|_{\mathrm{F}} = \sqrt{\mathrm{Tr}(A^T A)}$. While they are the same for $m = 1$ or $n = 1$ (reducing to $|\cdot|$), they are *not* for $m, n \geq 2$. Consider for example

$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow \quad \|\mathbb{1}\|_{\mathrm{op}} = \sup_{(x,y)\in \bar{B}_1(0)} |(x,y)| = 1, \quad \text{and} \quad \|\mathbb{1}\|_{\mathrm{F}} = \sqrt{2}, \tag{5.32}$$

but also

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \|A\|_{\mathrm{op}} = \sup_{(x,y)\in \bar{B}_1(0)} |y| = 1, \quad \text{and} \quad \|A\|_{\mathrm{F}} = 1. \tag{5.33}$$

However, recalling Exercise 4.37 (we may think of $\mathbb{R}^{m\times n}$ as $\mathbb{R}^{mn}$) these norms will anyway be necessarily equivalent. Explicitly, we have for any $A \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, by Cauchy-Schwarz,

$$|A\mathbf{x}|^2 = \sum_{j=1}^m \left(\sum_{k=1}^n A_{jk} x_k\right)^2 \leq \sum_{j=1}^m \left(\sum_{k=1}^n A_{jk}^2\right)\left(\sum_{k=1}^n x_k^2\right) = \sum_{j,k} A_{jk}^2 |\mathbf{x}|^2, \tag{5.34}$$

also

$$|A_{jk}| = |\mathbf{e}_j \cdot A\mathbf{e}_k| \leq |A\mathbf{e}_k| \leq \|A\|_{\mathrm{op}}\,, \tag{5.35}$$

and thus

$$\|A\|_{\mathrm{op}} \leq \|A\|_{\mathrm{F}} \leq \sqrt{mn}\,\|A\|_{\mathrm{op}}\,. \tag{5.36}$$

Example 5.11. Let us consider an example of a noncontinuous linear map. By Exercise 5.5 it is necessary to have infinite dimensions (unless we use a different topology). Let ℓ_c^∞ denote the subspace in $(\ell^\infty, \|\cdot\|_\infty)$ of bounded sequences with finite (compact) support, i.e.

$$\ell_c^\infty = \big\{x \in \ell^\infty : \mathrm{supp}(x) \text{ finite}\big\}, \tag{5.37}$$

where $\mathrm{supp}(x) = \{n \in \mathbb{N} : x_n \neq 0\} \subseteq \mathbb{N}$, and let $T\colon \ell_c^\infty \to \mathbb{R}$,

$$T(x) = \sum_{n=0}^\infty x_n. \tag{5.38}$$

Then, taking for $m \in \mathbb{N}^+$ the sequence $x^{(m)} = (x_n^m)_{n=0}^\infty \in \ell_c^\infty$,

$$x_n^{(m)} = \begin{cases} 1/m, & 0 \leq n \leq m-1 \\ 0, & n \geq m, \end{cases} \tag{5.39}$$

we have $x^{(m)} \to 0$ in ℓ_c^∞ as $m \to \infty$ while $Tx^{(m)} = 1 \nrightarrow 0$, and indeed

$$\|T\|_{\mathrm{op}} \geq \left\|T(mx^{(m)})\right\|_\infty \to \infty \text{ as } m \to \infty. \tag{5.40}$$

Now, using the induced metric and topology of the normed vector space $(\mathrm{Hom}(V, W), \|\cdot\|_{\mathrm{op}})$ we can also talk in general terms about *continuous* differentiability:

Definition 5.12. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces, and $\emptyset \neq \Omega \subseteq V$. A function $f\colon \Omega \to W$ is called **continuously differentiable** at $p \in \Omega$ if $f'(x) \in \mathrm{Hom}(V, W)$ exists in a neighborhood of p and the function $V \ni x \mapsto f'(x) \in \mathrm{Hom}(V, W)$ is continuous at $x = p$.

Example 5.13. For $V = W = \mathbb{R}$ we have $\mathrm{Hom}(\mathbb{R}, \mathbb{R}) = \mathbb{R}$ and all the norms reduce to $|\cdot|$. The function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) := \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{5.41}$$

is differentiable for all $x \in \mathbb{R}$, but not *continuously* differentiable at $x = 0$:

$$f'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x), & x \neq 0, \\ \lim_{h\to 0} h \sin(1/h) = 0, & x = 0. \end{cases} \tag{5.42}$$

(We have used the product and chain rules and continuous differentiability of sin and $x \mapsto 1/x$ on their respective domains; see below.) Note that f' has a discontinuity of the *second* kind (see 5.3.4).

Exercise 5.4. *Show that, if V, W are normed vector spaces, then $\mathcal{L}(V; W)$ and $\mathrm{Hom}(V; W)$ are vector spaces over $\mathbb{R}$.*

Exercise 5.5. *Show that if $V = \mathbb{R}^n$ and W is any normed vector space, then every operator $V \to W$ is bounded, i.e. $\mathcal{L}(V; W) = \mathrm{Hom}(V, W)$.*

Exercise 5.6. *Let $V \subsetneq \ell^1$ be the subspace of all absolute summable sequences with finite support (cf. Theorem 4.70), and define*

$$\begin{array}{rrcl} T\colon & V & \to & V \\ & (x_n)_{n\in\mathbb{N}} & \mapsto & \big((Tx)_n\big)_{n\in\mathbb{N}} := (nx_n)_{n\in\mathbb{N}} \end{array} \tag{5.43}$$

Show that $T \in \mathcal{L}(V, V)$ but T is <u>un</u>bounded, i.e. $\mathrm{Hom}(V, V) \subsetneq \mathcal{L}(V, V)$ in this case.

Exercise 5.7. *Show that the function*

$$f(x) := \begin{cases} 0 & x \leq 0, \\ e^{-1/x}, & x > 0, \end{cases} \tag{5.44}$$

is continuously differentiable on $\mathbb{R}$. (You may use the chain rule and continuous differentiability of exp *and $x \mapsto 1/x$; see below.) What is the error in the linear approximation at $x = 0$?*

5.2.2. *Directional derivatives.* Taking $h > 0$ resp. $h < 0$ we may also speak of **right resp. left derivatives** of $f\colon [a,b] \to \mathbb{R}$,

$$f'_+(x) := \lim_{h\to 0^+} \frac{f(x+h)-f(x)}{h}, \qquad f'_-(x) := \lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h}, \tag{5.45}$$

whenever these are defined (which includes the endpoints $f'_+(a)$ and $f'_-(b)$). For $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ or generally $f\colon \Omega \subseteq V \to W$ we may consider **directional derivatives** along the vector $v \in V$:

$$D_v^+ f(x) := \lim_{h\to 0^+} \frac{f(x+hv)-f(x)}{h}, \qquad D_v f(x) := \lim_{h\to 0} \frac{f(x+hv)-f(x)}{h}. \tag{5.46}$$

Note that, if f is differentiable at x, then

$$D_v^+ f(x) = D_v f(x) = f'(x)[v] = -D_{-v}^+ f(x), \tag{5.47}$$

however, it is not necessarily the case that if $D_v f(x)$ exists for all $v \in V$ then the derivative exists, since linearity is not guaranteed. Indeed, in the one-variable case, linearity holds iff

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{ exists} \quad \Leftrightarrow \quad f'_+(x) = f'_-(x) \text{ exist.} \tag{5.48}$$

For $\mathbf{f}\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{f} = (f_i)_{i=1}^m$, we have the **partial derivatives**

$$\frac{\partial \mathbf{f}}{\partial x_j}(\mathbf{x}) := D_{\mathbf{e}_j}\mathbf{f}(\mathbf{x}), \quad \text{i.e.} \quad \frac{\partial f_i}{\partial x_j}(\mathbf{x}) := D_{\mathbf{e}_j} f_i(\mathbf{x}), \quad i \in \{1,\ldots,m\},\ j \in \{1,\ldots,n\}, \tag{5.49}$$

and the existence of $\mathbf{f}'(\mathbf{x})$ requires continuity in $\mathbf{v} \mapsto D_{\mathbf{v}}\mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})[\mathbf{v}]$.

Example 5.14. The function $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) := \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq \mathbf{0}, \\ 0, & (x,y) = \mathbf{0}, \end{cases} \tag{5.50}$$

is not differentiable at $(x,y) = \mathbf{0}$, and not even continuous there:

$$(x,y) = (r\cos\varphi, r\sin\varphi) \quad \Rightarrow \quad f(x,y) = \sin\varphi\cos\varphi \nrightarrow 0, \text{ as } r \to 0, \tag{5.51}$$

however both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist everywhere and equal zero at $\mathbf{x} = \mathbf{0}$ (check!). Also $D_{\mathbf{v}} f(x,y)$ exists for all $\mathbf{v} \in \mathbb{R}^2$ at $\mathbf{x} \neq \mathbf{0}$, but at $\mathbf{x} = \mathbf{0}$ only for $\mathbf{v}$ aligned with the axes.

Example 5.15. If we consider instead the function $f\colon \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) := \begin{cases} \sqrt{x^2+y^2}\,\mathrm{sign}(x), & x \neq 0, \\ 0, & x = 0, \end{cases} \tag{5.52}$$

we have continuity at $(x,y) = \mathbf{0}$, and even that the directional derivatives

$$D_{(\cos\varphi,\sin\varphi)} f(0,0) = \lim_{h\to 0} \frac{|h|}{h}\,\mathrm{sign}(h\cos(\varphi)) = \begin{cases} 1, & \varphi \in (-\pi/2, \pi/2), \\ 0, & \varphi = \pm\pi/2, \\ -1, & \varphi \in (\pi/2, 3\pi/2), \end{cases} \tag{5.53}$$

exist for all angles φ, but these are not continuous in φ, and therefore $f'(0,0)$ cannot exist.

In the case of real-valued differentiable maps $f\colon \mathbb{R}^n \to \mathbb{R}$, we define the **gradient** at $\mathbf{x}$,

$$\nabla f(\mathbf{x}) := \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right)(\mathbf{x}) = [f'(\mathbf{x})]^T \tag{5.54}$$

by the transpose of the Jacobian, $\mathbb{R}^n \cong \mathbb{R}^{n\times 1} \cong \mathbb{R}^{1\times n}$. The function $\nabla f\colon \mathbb{R}^n \to \mathbb{R}^n$ defines a **vector field** on $\mathbb{R}^n$. Note that $\nabla f(\mathbf{x})$ is either zero or yields both the value $\max_{|\mathbf{h}|=1} f'(\mathbf{x})[\mathbf{h}]$ and direction $\mathbf{h} = \nabla f(\mathbf{x})/|\nabla f(\mathbf{x})|$ of steepest increase according to the linear approximation of f, i.e. the maximal directional derivative.

5.2.3. *Example: the inverse map.* We consider here a typical and interesting example of a differentiable map, namely that of the **inverse map** $x \mapsto x^{-1}$. However in order to illustrate how we may work with all these notions of operator norms and Banach spaces we take as our domain for x the invertible elements in an arbitrary complete normed ring (possibly non-commutative). In conjunction with this investigation we would also like to point out a generally useful connection between the spaces Hom and ℓ^∞, and generalize the latter from our previous definition with sequences with values in $\mathbb{R}$ to sequences or arbitrary functions with values in an arbitrary Banach space.

Theorem 5.16 (The inverse map)**.** *Let $(\mathcal{R}, \|\cdot\|)$ be a complete normed ring with unit $\mathbb{1}$, $\mathcal{R} \neq \{0\}$, and let*

$$\Omega = \{T \in \mathcal{R} : \exists S \in \mathcal{R} \ \text{ s.t. } \ ST = TS = \mathbb{1}\} = \{T \in \mathcal{R} : \exists T^{-1} \in \mathcal{R}\} \tag{5.55}$$

be the set of all invertible elements in $\mathcal{R}$. Define the map

$$\begin{array}{rccc} F\colon & \Omega & \to & \Omega \\ & T & \mapsto & T^{-1}. \end{array} \tag{5.56}$$

Then Ω is a nonempty open subset of $\mathcal{R}$ and F is continuously differentiable in Ω, with derivative $\Omega \ni T \mapsto F'(T) \in \mathrm{Hom}(\mathcal{R}, \mathcal{R})$ given by

$$F'(T)[H] = -T^{-1}HT^{-1} \quad \forall H \in \mathcal{R}. \tag{5.57}$$

Proof. Note first that $\mathbb{1} \in \Omega$, $\mathbb{1}^{-1} = \mathbb{1}$, with norm $\|\mathbb{1}\| \in \mathbb{R}^+$ (if $\|\mathbb{1}\| = 0$ then $\|T\| = \|T\mathbb{1}\| \leq \|T\|\,\|\mathbb{1}\| = 0 \ \forall T \in \mathcal{R}$, so $\mathcal{R} = \{0\}$). Assume then that $T \in \Omega$, i.e. T^{-1} exists and $\|T^{-1}\| \in \mathbb{R}_+$. We note that by $\|\mathbb{1}\| = \|TT^{-1}\| \leq \|T\|\,\|T^{-1}\|$ we must also have $\|T^{-1}\| \geq \|T\|^{-1}\,\|\mathbb{1}\| > 0$. Given $H \in \mathcal{R}$ we can write

$$T + H = T + TT^{-1}H = T(\mathbb{1} + T^{-1}H), \tag{5.58}$$

which we seek to invert. If we require $\|H\| < \|T^{-1}\|^{-1}$ then

$$\|T^{-1}H\| \leq \|T^{-1}\|\,\|H\| < 1. \tag{5.59}$$

We may thus form the inverse of $(\mathbb{1} + T^{-1}H)$. Namely, let $X = -T^{-1}H \in \mathcal{R}$, $\|X\| < 1$, and note that since the real power series

$$\|X\| + \|X\|^2 + \|X\|^3 + \ldots = \sum_{n=1}^{\infty} \|X\|^n = (1 - \|X\|)^{-1} - 1 \tag{5.60}$$

converges, by the majorization Theorem 4.66, also

$$X + X^2 + X^3 + \ldots = \sum_{n=1}^{\infty} X^n \tag{5.61}$$

converges. Furthermore, (by the same trick as in the real case)

$$(\mathbb{1} - X)\sum_{n=0}^{N} X^n = \mathbb{1} - X^{N+1} \tag{5.62}$$

so that, taking $N \to \infty$,

$$\left\|(\mathbb{1} - X)\sum_{n=0}^{N} X^n - \mathbb{1}\right\| = \left\|X^{N+1}\right\| \leq \|X\|^{N+1} \to 0, \tag{5.63}$$

and we obtain $(\mathbb{1} - X)^{-1} = \sum_{n=0}^{\infty} X^n$.

Therefore, inverting (5.58), we have an explicit expansion of $F(T+H)$ in terms of H:

$$(T+H)^{-1} = (\mathbb{1} - X)^{-1}T^{-1} = \sum_{n=0}^{\infty}(-T^{-1}H)^n T^{-1} \tag{5.64}$$

$$= T^{-1} - T^{-1}HT^{-1} + \sum_{n=2}^{\infty}(-T^{-1}H)^n T^{-1} \tag{5.65}$$

$$= F(T) - T^{-1}HT^{-1} + o(H), \tag{5.66}$$

where the last term is actually $O(\|H\|^2)$:

$$\left\|\sum_{n=2}^{\infty}(-T^{-1}H)^n T^{-1}\right\| \leq \sum_{n=2}^{\infty} \left\|T^{-1}\right\|^n \|H\|^n \left\|T^{-1}\right\| = \|H\|^2 \left\|T^{-1}\right\|^3 (1 - \left\|T^{-1}\right\| \|H\|)^{-1}. \tag{5.67}$$

This shows that F is differentiable in all of Ω with derivative $F'(T)[H] = -T^{-1}HT^{-1}$ (here any $H \in \mathcal{R}$), $\|F'(T)\|_{\text{op}} \leq \left\|T^{-1}\right\|^2 < \infty$, and also that Ω is open (if T^{-1} exists then $(T+H)^{-1}$ exists as long as $\|H\| < \left\|T^{-1}\right\|^{-1}$). Furthermore, it shows that F is continuous in Ω by Theorem 5.5. Also, since for any $H \in \mathcal{R}$

$$(F'(S) - F'(T))[H] = -S^{-1}HS^{-1} + T^{-1}HT^{-1} = (T^{-1} - S^{-1})HS^{-1} + T^{-1}H(T^{-1} - S^{-1}), \tag{5.68}$$

we have that F' is continuous in Ω (in the operator norm of $\text{Hom}(\mathcal{R}, \mathcal{R})$). □

Example 5.17. For $\mathcal{R} = \mathbb{R}$ with norm $|\cdot|$ we have $\Omega = \mathbb{R} \setminus \{0\}$ and $F\colon \Omega \to \Omega$, $F(x) = x^{-1}$. Thus we obtain $F'(x)[h] = -x^{-2}h$, i.e. the well-known result $F'(x) = -x^{-2} = -F(x)^2$.

Example 5.18. For $\mathcal{R} = \mathbb{R}^{n\times n}$ with norm $\|\cdot\|_{\text{F}}$ or $\|\cdot\|_{\text{op}}$ (finite-dimensional and therefore complete) we have

$$\Omega = \{A \in \mathbb{R}^{n\times n} : \det A \neq 0\} \tag{5.69}$$

the set of all invertible matrices and $F\colon \Omega \to \Omega$, $F(A) = A^{-1}$. Thus we obtain

$$F'(A)[H] = -A^{-1}HA^{-1}, \qquad A \in \Omega,\ H \in \mathbb{R}^{n\times n}, \tag{5.70}$$

where $F'\colon \Omega \to \text{Hom}(\mathbb{R}^{n\times n}, \mathbb{R}^{n\times n})$, $A \mapsto F'(A)$ is continuous on Ω.

As mentioned we now aim to generalize to arbitrary Banach spaces.

Theorem 5.19. *Let X be a set and $(W, |\cdot|)$ a Banach space. Let $\ell^\infty(X;W)$ denote the set of all bounded maps $f\colon X \to W$ with the norm*

$$\|f\|_\infty := \sup_{x\in X} |f(x)|, \tag{5.71}$$

then $(\ell^\infty(X;W), \|\cdot\|_\infty)$ is a Banach space.

Remark 5.20. For $X = \mathbb{N}$ and $W = \mathbb{R}$ this is exactly the space ℓ^∞ from Example 4.68.

Proof. First, note that $\ell^\infty(X;W)$ is naturally a vector space over $\mathbb{R}$, with

$$(f+g)(x) := f(x) + g(x), \quad (\lambda f)(x) := \lambda f(x). \tag{5.72}$$

Furthermore, $\|f\|_\infty < \infty$ by boundedness of f, and

$$\|f\|_\infty = 0 \quad \Leftrightarrow \quad f(x) = 0 \ \forall x \in X \quad \Leftrightarrow \quad f = 0, \tag{5.73}$$

and the triangle inequality holds exactly as before (by the triangle inequality of $|\cdot|$). Assuming that $(f_n) \subset \ell^\infty(X;W)$ is Cauchy, we have that for any $\varepsilon > 0$ $\exists N(\varepsilon)$ s.t. $m, n \geq N(\varepsilon)$ implies

$$\sup_{x\in X} |f_n(x) - f_m(x)| < \varepsilon, \tag{5.74}$$

i.e. for any fixed $x \in X$, the sequence $(f_n(x))$ in W is Cauchy and therefore converges,

$$f(x) := \lim_{n\to\infty} f_n(x). \tag{5.75}$$

Further, by boundedness of a Cauchy sequence, $\sup_{n,x} |f_n(x)| \leq B$, implying $|f(x)| \leq B$ and therefore $f \in \ell^\infty(X;W)$. Now consider

$$\|f_n - f\|_\infty = \sup_{x\in X} |f_n(x) - f(x)|. \tag{5.76}$$

By (5.74) there exists $N(\varepsilon)$ s.t. for all $x \in X$ and all $n \geq N(\varepsilon)$

$$\sup_{m\geq N(\varepsilon)} |f_n(x) - f_m(x)| < \varepsilon \quad \Rightarrow \quad |f_n(x) - f(x)| \leq \varepsilon, \tag{5.77}$$

after taking the limit $m \to \infty$. Hence $\|f_n - f\|_\infty \leq \varepsilon$ for $n \geq N(\varepsilon)$, which proves the theorem. □

The following application then clarifies the topology of the space $\mathrm{Hom}(V,W)$ for Banach spaces W (note that V does not have to be complete), and essentially states that linear maps are determined by their value on the unit sphere.

Corollary 5.21. *Let $(V, |\cdot|)$ be a normed vector space and $(W, |\cdot|)$ a Banach space. Let* $\mathrm{Hom}(V,W)$ *be the set of all bounded linear maps $T\colon V \to W$ with the norm*

$$\|T\| := \sup_{x\in \mathbb{S}_V} |T(x)|, \tag{5.78}$$

where

$$\mathbb{S}_V := \{v \in V : |v| = 1\} \tag{5.79}$$

denotes the unit 'sphere' in V. Then $\|T\| = \|T\|_{\mathrm{op}}$ and $\mathrm{Hom}(V,W)$ *is a Banach space.*

Proof. We first note that

$$\sup\{|Tx| : x \in \bar{B}_1(0)\} = \sup\{|Tx| : x \in \mathbb{S}_V\}, \tag{5.80}$$

namely the l.h.s. is greater than the r.h.s. because the supremum is over a larger set, but also the l.h.s. is smaller than the r.h.s. because every value $|Tx| = |x||T(x/|x|)| \leq |T(x/|x|)|$ with $x \in \bar{B}_1(0)$ is bounded by at least one value $|Ty|$ with $y = x/|x| \in \mathbb{S}_V$.

We then note that we may associate to any map $T \in \mathrm{Hom}(V, W)$ a corresponding map $f_T : \mathbb{S}_V \to W$, simply by restriction, $f_T := T|_{\mathbb{S}_V}$. We then have

$$\|f_T\|_\infty = \sup_{x \in \mathbb{S}_V} |T(x)| = \sup_{x \in \bar{B}_1(0)} |T(x)| = \|T\|_{\mathrm{op}} < \infty \tag{5.81}$$

and therefore $f_T \in \ell^\infty(\mathbb{S}_V; W)$. Also, we have that

$$f_{\alpha T + \beta S} = (\alpha T + \beta S)|_{\mathbb{S}_V} = \alpha T|_{\mathbb{S}_V} + \beta S|_{\mathbb{S}_V} = \alpha f_T + \beta f_S, \tag{5.82}$$

and for a Cauchy sequence (T_n) in $\mathrm{Hom}(V, W)$

$$\|f_{T_n} - f_{T_m}\|_\infty = \|f_{T_n - T_m}\|_{\mathrm{op}} = \|T_n - T_m\|_{\mathrm{op}} < \varepsilon \tag{5.83}$$

if $m, n \geq N(\varepsilon)$. By Theorem 5.19, the Cauchy sequence (f_{T_n}) converges in $\ell^\infty(\mathbb{S}_V, W)$, say $\|f_{T_n} - f_*\|_\infty \to 0$ for some $f_* \in \ell^\infty$. Let $T_* : V \to W$ be defined by

$$T_*(x) := \begin{cases} |x| f_*(x/|x|), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{5.84}$$

Then $\|T_*\|_{\mathrm{op}} = \|f_*\|_\infty < \infty$, and furthermore we claim that T_* is linear, because if $z = \alpha x + \beta y$, $x, y, z \neq 0$, then

$$T_*(z) = |z| f_*(z/|z|) = |z| \lim_{n\to\infty} f_{T_n}(z/|z|) = \lim_{n\to\infty} \big(|z| f_{T_n}(z/|z|)\big), \tag{5.85}$$

where for each n

$$|z| f_{T_n}(z/|z|) = |z| T_n(z/|z|) = T_n(z) = \alpha T_n(x) + \beta T_n(y) = \alpha |x| f_{T_n}(x/|x|) + \beta |y| f_{T_n}(y/|y|). \tag{5.86}$$

Taking the limit of the r.h.s. we obtain

$$\begin{aligned} T_*(z) &= \alpha |x| \lim_{n\to\infty} f_{T_n}(x/|x|) + \beta |y| \lim_{n\to\infty} f_{T_n}(y/|y|) = \alpha |x| f_*(x/|x|) + \beta |y| f_*(y/|y|) \\ &= \alpha T_*(x) + \beta T_*(y) \end{aligned}$$

(and analogously if any of x, y, z is zero). We conclude then that actually $T \in \mathrm{Hom}(V, W)$, $f_T = f$, and

$$\|T_n - T\|_{\mathrm{op}} = \|f_{T_n} - f\|_{\mathrm{op}} \to 0 \tag{5.87}$$

as $n \to \infty$. □

Corollary 5.22. *If V is a Banach space then the space* $(\mathrm{Hom}(V, V), \|\cdot\|_{\mathrm{op}})$ *of operators on V is a complete normed ring.*

Example 5.23. For V a Banach space we have the invertible operators on this space $\Omega \subseteq \mathrm{Hom}(V, V) = \mathcal{R}$ and $F : \Omega \to \Omega$, $F(T) = T^{-1}$. Thus we obtain by Theorem 5.16

$$F'(T)[H] = -T^{-1} H T^{-1}, \qquad T \in \Omega,\ H \in \mathrm{Hom}(V, V), \tag{5.88}$$

where $F' : \Omega \ni T \mapsto F'(T) \in \mathrm{Hom}(\mathrm{Hom}(V, V), \mathrm{Hom}(V, V))$ is continuous on Ω.

Example 5.24. The bounded operator $T\colon \ell^\infty \to \ell^\infty$, $(Tx)_n := (n+1)^{-1}x_n$, has no inverse in the ring $\mathrm{Hom}(\ell^\infty, \ell^\infty)$ because it would have to be $(T^{-1}x)_n = (n+1)x_n$ which is not a bounded operator on ℓ^∞: $\|T^{-1}\|_{\mathrm{op}} = \infty$. Thus $T \notin \Omega$.

Exercise 5.8. *Show that* $F\colon \Omega \to \mathbb{R}^{2\times 2}$, $F(A) = A^{-2} = (A^2)^{-1} = (A^{-1})^2$ *is differentiable on* $\Omega = \{A \in \mathbb{R}^{2\times 2} : \det A \neq 0\}$ *and compute the derivative* $F'(A)$ *for* $A \in \Omega$.

5.3. **Some properties of derivatives.**

5.3.1. *Generalized chain rule.* Let us consider a slightly generalized but geometrically intuitive formulation of the chain rule for derivatives (cf. Rudin 5.5 resp. 9.15 for the usual one). Note again that if $f\colon X \to Y$ is differentiable at $p \in X$ then, in a neighborhood of p,

$$f(x) = f(p) + f'(p)[x-p] + o(x-p), \tag{5.89}$$

so that if we consider the constant plus linear[(3)] function $F\colon X \to Y$,

$$F(x) := f(p) + f'(p)[x-p], \tag{5.90}$$

then

$$|f(x) - F(x)| = o(x-p). \tag{5.91}$$

Definition 5.25. Let X and Y be metric spaces and $f, F : X \to Y$. Then f and F are said to be **tangent** at $p \in X$ iff

$$d_Y(f(x), F(x)) = o(d_X(x,p)) \tag{5.92}$$

for $x \in X$ in some neighborhood of p.

A function $f\colon X \to Y$ is called **Lipschitz** at $p \in X$ iff $d_Y(f(x), f(p)) = O(d_X(x,p))$, that is

$$\exists C \in \mathbb{R}_+ \text{ s.t. } d_Y(f(x), f(p)) \le C d_X(x,p), \tag{5.93}$$

for all $x \in X$ in a neighborhood of p.

Thus a function f is differentiable at a point p iff there exists a constant plus linear (i.e. affine) map F which is tangent to f at p. Furthermore, f and F are then necessarily Lipschitz at p with a finite constant

$$\|f'(p)\|_{\mathrm{op}} \le C \le \|f'(p)\|_{\mathrm{op}} + \varepsilon, \tag{5.94}$$

where $\varepsilon > 0$ can be taken arbitrarily small (depending on how large a neighborhood of p is considered). Note that for F we may take $\varepsilon = 0$ since there are no higher-order terms.

Theorem 5.26 (**Chain rule**). *Let* X, Y, Z *be metric spaces,* $f, F : X \to Y$ *and* $g, G : Y \to Z$*. Assume that* f *and* F *are tangent at* $p \in X$*, that* g *and* G *are tangent at* $q = f(p) \in Y$*, and that* f *and* G *are Lipschitz at* p *resp.* q*. Then* $g \circ f$ *and* $G \circ F$ *are tangent at* p*.*

In particular, if X, Y, Z *are normed vector spaces, and if* f *is differentiable at* p *with tangent*

$$F(x) = f(p) + f'(p)[x-p], \tag{5.95}$$

and if g *is differentiable at* q *with tangent*

$$G(y) = g(q) + g'(q)[y-q], \tag{5.96}$$

---

(3) Constant plus linear functions are also called **affine** functions.

then $g \circ f$ is differentiable at p with tangent

$$G \circ F(x) = g(q) + g'(q)\big[f(p) + f'(p)[x-p] - q\big] = g\big(f(p)\big) + \Big(g'\big(f(p)\big) \circ f'(p)\Big)[x-p]. \quad (5.97)$$

Proof. We simply estimate (let us denote all metrics by d for simplicity)

$$\begin{aligned} d\big(g(f(x)), G(F(x))\big) &\le d\big(g(f(x)), G(f(x))\big) + d\big(G(f(x)), G(F(x))\big) \\ &\le o\big(d(f(x), f(p)\big) + C_G d\big(f(x), F(x)\big) \\ &\le o\big(C_f d(x,p)\big) + C_G o\big(d(x,p)\big) = o\big(d(x,p)\big), \end{aligned}$$

where we first used the triangle inequality and then tangency and Lipschitz with corresponding constants C_f and C_G. Therefore $d(g \circ f(x), G \circ F(x)) = o(d(x,p))$ as claimed. □

Exercise 5.9. *Let $f(x) = \sqrt{x^2 + x^4}$ and $F(x) = |x|$ for $x \in \mathbb{R}$.*

a) *Show that f and F are Lipschitz at every point of $\mathbb{R}$ and determine bounds for their respective constants.*
b) *Show that f and F are tangent in the generalized sense at $x = 0$.*
c) *Apply the generalized chain rule to f composed with $g(x) = e^x$ at $x = 0$.*

5.3.2. *Mean value inequality.* cf. Rudin 9.19

Theorem 5.27 (**Mean value inequality**). *Let X, Y be normed vector spaces with norms $|\cdot|$. Assume $f\colon X \to Y$ is differentiable on the line segment*

$$[p,q] := \{tp + (1-t)q : 0 \le t \le 1\}, \qquad p, q \in X,$$

with the bound on the derivative $\|f'(x)\|_{\mathrm{op}} \le K$ $\forall x \in [p,q]$. Then

$$|f(p) - f(q)| \le K\,|p-q|\,. \quad (5.98)$$

In particular, if $f'(x) = 0$ on $[p,q]$ then f is constant on $[p,q]$.

Remark 5.28. The last statement is also known as the **Fundamental theorem of differential calculus**.

Proof. Let us first simplify the setup a bit by considering $g\colon [0,1] \to Y$ with

$$g(t) := f(tp + (1-t)q) - f(q). \quad (5.99)$$

We then have $g(1) = f(p) - f(q)$ and $g(0) = 0$. Furthermore, by the chain rule g is differentiable with

$$g'(t)h = f'(tp + (1-t)q)[p-q]h, \qquad t \in [0,1],\ h \in \mathbb{R}, \quad (5.100)$$

so that

$$\big|g'(t)\big| = \big\|g'(t)\big\|_{\mathrm{op}} \le L := K\,|p-q|\,, \qquad \forall 0 \le t \le 1. \quad (5.101)$$

Hence it suffices to prove that $|g(1)| \le L$.

Fix an arbitrary $M > L$ and introduce the set

$$E = \big\{t \in [0,1] : |g(s)| \le Ms \ \forall s \in [0,t]\big\}. \quad (5.102)$$

It is then evident that $E = [0,t]$ for some $t \in [0,1]$ (that it is a *closed* interval follows from the closed set of conditions). Assume that $t < 1$, then we claim that also $t + k \in E$ for some $k > 0$. Namely, because of differentiability of g at t,

$$g(t+h) = g(t) + g'(t)h + o(h), \quad (5.103)$$

so there exists some $k > 0$ s.t.

$$|g(t+h) - g(t) - g'(t)h| \le (M-L)h \qquad \forall\, 0 \le h \le k. \tag{5.104}$$

Then we obtain for such h

$$|g(t+h)| \le |g(t)| + |g'(t)|h + (M-L)h \le Mt + Lh + Mh - Lh = M(t+h), \tag{5.105}$$

that is $t + k \in E$.

It follows that actually $E = [0,1]$ and thus that $|g(1)| \le M$. But $M > L$ was arbitrary, and hence also $|g(1)| \le L$. □

5.3.3. *Mean value theorems.* First we recall one of the most important applications of derivatives, as a means for finding extreme values [Rud76, Thm. 5.8]:

Theorem 5.29 (**Extreme value theorem**). *If $f\colon (a,b) \to \mathbb{R}$ has a local maximum or minimum at $x \in (a,b)$, and if $f'(x)$ exists, then $f'(x) = 0$. Similarly, if $f\colon \mathbb{R}^n \supseteq \Omega \to \mathbb{R}$ has a local maximum or minimum at $\mathbf{x} \in \Omega$, and if $f'(\mathbf{x})$ exists, then $f'(\mathbf{x}) = [\partial f/\partial x_j]_{j=1..n} = \mathbf{0}$.*

The usual mean value theorem in one variable is given in [Rud76, Thm. 5.9], and follows simply by studying extreme values of the differentiable function

$$h(t) := (f(b) - f(a))g(t) - (g(b) - g(a))f(t), \qquad t \in [a,b], \tag{5.106}$$

for which $h(a) = h(b)$.

Theorem 5.30 (**Mean value theorem**). *Assume $f, g : [a,b] \to \mathbb{R}$ are continuous on the interval $[a,b]$ and differentiable on the interval (a,b). Then there exists an $x \in (a,b)$ s.t.*

$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x). \tag{5.107}$$

In particular (taking $g(x) = x$), there exists $x \in (a,b)$ s.t.

$$f(b) - f(a) = f'(x)(b-a). \tag{5.108}$$

Remark 5.31. The simpler special case that $f(a) = f(b)$ implies $f'(x) = 0$ at some $x \in (a,b)$ is also known as **Rolle's theorem**.

This important theorem has many applications, such as concerning the growth of functions [Rud76, Thm. 5.11], l'Hospital's rule [Rud76, Thm. 5.13], and Taylor's theorem recalled below. Another important application in $\mathbb{R}^n$ is to connect full derivatives to partial derivatives:

Theorem 5.32 ([Rud76, Thm. 9.21]). *Let $\mathbf{f}\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where Ω is nonempty and open. Then $\mathbf{f}$ is continuously differentiable on Ω iff all partial derivatives $\partial f_i/\partial x_j$ exist and are continuous on Ω, $i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$.*

Proof idea. $\Rightarrow$ immediate by existence/cont. of $\mathbf{f}'$ and the definition of partial derivatives.

$\Leftarrow$ use the mean value theorem to estimate the difference

$$\begin{aligned}
&\mathbf{f}(x_1 + h_1, x_2 + h_2, \dots, x_{j-1} + h_{j-1}, x_j + h_j, x_{j+1}, \dots, x_n)\\
&- \mathbf{f}(x_1 + h_1, x_2 + h_2, \dots, x_{j-1} + h_{j-1}, x_j, x_{j+1}, \dots, x_n)\\
&= h_j \frac{\partial \mathbf{f}}{\partial x_j}(x_1 + h_1, \dots, x_{j-1} + h_{j-1}, x_j + t h_j, x_{j+1}, \dots, x_n), \quad \text{some } t \in (0,1),
\end{aligned}$$

and use continuity to estimate $\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \sum_j h_j \partial\mathbf{f}/\partial x_j(\mathbf{x})$ as $o(\mathbf{h})$. □

We also consider the following generalization for curves in normed vector spaces (Rudin 5.16-19 discusses further examples of complications arising in the higher-dimensional case).

Definition 5.33. A subset K of a vector space X is called **convex** if $\forall p, q \in K$, $[p, q] \subseteq K$. (Note: intersections of convex sets are also convex.)

To any set $A \subseteq X$ we can define the **convex hull** of A, $\operatorname{Conv} A$, as the smallest convex set in X that contains A:

$$\operatorname{Conv} A := \bigcap_{\substack{K \supseteq A \\ K \text{convex}}} K. \tag{5.109}$$

Remark 5.34. One can show that

$$\operatorname{Conv} A = \left\{ \sum_{j=1}^{n} t_j x_j : \sum_{j=1}^{n} t_j = 1, t_j \geq 0, x_j \in A \right\}. \tag{5.110}$$

Theorem 5.35 (**Mean value theorem for curves**). *Let $f \colon X \to Y$ where X and Y are normed vector spaces, and assume that $p, q \in X$ and that f is differentiable on $[p, q] \subseteq X$. Let*

$$\mathcal{C}[p, q] := \operatorname{Conv}\{f'(x) : x \in [p, q]\} \subseteq \operatorname{Hom}(X, Y) \tag{5.111}$$

be the convex hull of the derivatives of f on $[p, q]$. Then we can for every $\varepsilon > 0$ find $F \in \mathcal{C}[p, q]$ and $R \in Y$ with $|R| \leq \varepsilon$, such that

$$f(p) - f(q) = F(p - q) + |p - q|R. \tag{5.112}$$

Example 5.36. In the case that $X = \mathbb{R}$, Y is a Banach space, and f is continuously differentiable on the interval $[a, b] \subseteq \mathbb{R}$, we have that $f([a, b])$ is a (reasonably smooth) parameterized curve in Y. Note that

$$\operatorname{Hom}(\mathbb{R}, Y) = \{\mathbb{R} \ni t \mapsto ty \in Y : y \in Y\} \tag{5.113}$$

is isomorphic to Y. By the continuity of the map f' and the compactness of $[a, b]$, the set $f'([a, b]) \subseteq Y$ is compact. Furthermore, it turns out that the closure of the convex hull of a compact set in a complete metric space is also compact (see e.g. [AB06, Theorem 5.35]). Therefore, by taking in (5.112) a sequence (F_n, R_n) in $\mathcal{C}[a, b] \times Y \subseteq Y^2$ s.t. $R_n \to 0$, we may conclude by the compactness of $\overline{\mathcal{C}[a, b]}$ and Theorem 4.37 that there exists a subsequential limit $y \in \overline{\mathcal{C}[a, b]}$ s.t. $f(a) - f(b) = y(b - a)$, that is

$$y = \frac{f(b) - f(a)}{b - a} \in \overline{\mathcal{C}[a, b]}. \tag{5.114}$$

In words: *The average velocity over the curve is contained in the closure of the convex hull of all instantaneous velocities attained along the curve.*

Proof. For $\varepsilon > 0$, let's say that the interval $[p, q]$ has the ε-property if the conclusion of the theorem holds. Let $r \in [p, q]$ and assume that $[p, r]$ and $[r, q]$ has the ε-property. Thus there are $F_1 \in \mathcal{C}[p, r]$, $F_2 \in \mathcal{C}[r, q]$ and $R_1, R_2 \in Y$ s.t.

$$\begin{aligned} f(p) - f(r) &= F_1[p - r] + |p - r|R_1, \qquad |R_1| \leq \varepsilon, \\ f(r) - f(q) &= F_2[r - q] + |r - q|R_2, \qquad |R_2| \leq \varepsilon. \end{aligned}$$

Write $r = tp + (1-t)q$ for some $0 \le t \le 1$, then we have $p - r = (1-t)(p-q)$ and $r - q = t(p-q)$, and thus

$$f(p)-f(q) = f(p)-f(r)+f(r)-f(q) = F_1(1-t)(p-q)+(1-t)|p-r|R_1+F_2t(p-q)+t|p-q|R_2. \tag{5.115}$$

If we let

$$F := (1-t)F_1 + tF_2 \in \mathcal{C}[p,q] \tag{5.116}$$

and

$$R := (1-t)R' + tR'' \in Y, \qquad |R| \le \varepsilon, \tag{5.117}$$

then we obtain

$$f(p) - f(q) = F[p-q] + |p-q|R, \tag{5.118}$$

which shows that also $[p,q]$ has the ε-property.

Now take an arbitrary $r \in [p,q]$. Since f is differentiable at r we have

$$f(r+h) = f(r) + f'(r)h + E(h)|h|, \qquad E(h) \to 0,\ h \to 0, \tag{5.119}$$

hence

$$f(r+h) - f(r) = f'(r)h + |h|E(h), \tag{5.120}$$

which shows that $[r, r+h]$ has the ε-property if h is taken so small that $|E(h)| \le \varepsilon$.

Let $r_t := p + t(q-p)$ for $0 \le t \le 1$, and define

$$E = \{t \in [0,1] : [p, r_s] \text{ has the } \varepsilon\text{-property } \forall s \in [0,t]\}. \tag{5.121}$$

One may realize that either $E = [0,t)$ or $E = [0,t]$ for some $0 \le t \le 1$. But if $t < 1$ then we know that $[r_{t-\delta}, r_{t+\delta}]$ has the ε-property if δ is small enough. We may thus deduce that $E = [0,1]$. □

Exercise 5.10. *Prove Theorems 5.29 and 5.30.*

5.3.4. *Intermediate value property (Darboux' theorem).* See [Rud76, Thm. 5.12]: Derivatives, although they are not guaranteed to be continuous, do have the **intermediate value property** (compare Theorem 4.49 for continuous functions). This is also known as **Darboux' theorem**. A particularly noteworthy consequence of this property is that derivatives cannot have simple discontinuities, i.e. any point of discontinuity is of the second kind. Recall Example 5.13 whose erratic behavior around $x = 0$ solves the necessity of intermediate values.

5.3.5. *Taylor's theorem.* cf. Rudin 5.15, 8.4

By the mean value theorem (5.108), we have for $f\colon (a,b) \to \mathbb{R}$ differentiable that

$$f(x+h) = f(x) + f'(x_h)h \tag{5.122}$$

for some $x_h \in [x, x+h]$. Taylor's theorem re-iterates this argument in a clever way to extract higher orders in h from higher orders of derivatives:

Theorem 5.37 (**Taylor's theorem**). *Let $f\colon (a,b) \to \mathbb{R}$ be $n+1$ times differentiable, then*

$$f(x+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}h^k + \frac{f^{(n+1)}(x_h)}{(n+1)!}h^{n+1} \tag{5.123}$$

for some $x_h \in [x, x+h]$.

Proof. Consider the differentiable function for $t \in [x, x+h]$

$$F(t) := f(t) + f'(t)(x+h-t) + \frac{f''(t)}{2!}(x+h-t)^2 + \ldots + \frac{f^{(n)}(t)}{n!}(x+h-t)^n. \quad (5.124)$$

We compute

$$\begin{aligned} F'(t) &= f'(t) + f''(t)(x+h-t) - f'(t) + \frac{f^{(3)}(t)}{2!}(x+h-t)^2 - f''(t)(x+h-t) \\ &\quad + \ldots + \frac{f^{(n+1)}(t)}{n!}(x+h-t)^n - \frac{f^{(n)}(t)}{(n-1)!}(x+h-t)^{n-1} \\ &= \frac{f^{(n+1)}(t)}{n!}(x+h-t)^n. \end{aligned}$$

Now, let $G\colon [x, x+h] \to \mathbb{R}$ be any continuous function, differentiable with nonvanishing derivative on $(x, x+h)$. By the mean value theorem there exists $x_h \in (x, x+h)$ s.t.

$$\frac{F(x+h) - F(x)}{G(x+h) - G(x)} = \frac{F'(x_h)}{G'(x_h)} = \frac{f^{(n+1)}(x_h)(x+h-x_h)^n}{n!\, G'(x_h)}. \quad (5.125)$$

Thus, we see that by taking $G(t) := (x+h-t)^{n+1}$ we obtain in (5.125)

$$\frac{f(x+h) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k}{-h^{n+1}} = \frac{f^{(n+1)}(x_h)}{-(n+1)!}, \quad (5.126)$$

which proves the theorem. □

The multi-dimensional case is naturally more complicated:

Theorem 5.38 (Taylor's theorem in $\mathbb{R}^n$)**.** *Let $f\colon \Omega \subseteq \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives*

$$f^{(k_1,k_2,\ldots,k_n)}(\mathbf{x}) := \left(\frac{\partial}{\partial x_1}\right)^{k_1} \left(\frac{\partial}{\partial x_2}\right)^{k_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n} f(\mathbf{x}) \quad (5.127)$$

of all orders $k_j \geq 0$, $j \in \{1, \ldots, n\}$, up to the total order $k_1 + k_2 + \ldots + k_n = N+1$. Then

$$f(\mathbf{x}+\mathbf{h}) = \sum_{k=0}^{N} \sum_{\substack{k_j \geq 0 \\ \sum_{j=1}^n k_j = k}} \frac{f^{(k_1,\ldots,k_n)}(\mathbf{x})}{k_1! \ldots k_n!} h_1^{k_1} \ldots h_n^{k_n} + \sum_{\substack{k_j \geq 0 \\ \sum_{j=1}^n k_j = N+1}} R_{k_1,\ldots,k_n}(\mathbf{h}) h_1^{k_1} \ldots h_n^{k_n} \quad (5.128)$$

for some functions $R_{\mathbf{k}}\colon \Omega \to \mathbb{R}$ s.t. $R_{\mathbf{k}}(\mathbf{h}) \to 0$ as $h \to 0$.

See Rudin Exc. 9.29-30 for further discussion. The polynomials in h in (5.123) and (5.127) are the **Taylor polynomials** to f at x of order n resp. N.

5.4. **Riemann integration.**

5.4.1. *Box partitions.* If $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are points in $\mathbb{R}^n$ then we define the corresponding **rectangle/box** $R = R(\mathbf{a}, \mathbf{b})$ by

$$R = \{\mathbf{x} \in \mathbb{R}^n : \forall i = 1, \dots, n,\ a_i < x_i < b_i\}\,, \tag{5.129}$$

and its volume

$$\operatorname{vol} R = (R \neq \emptyset)(b_1 - a_1) \dots (b_n - a_n). \tag{5.130}$$

Note that R is open and $\operatorname{vol} \emptyset = 0$.

Definition 5.39 (Partition)**.** Let R be a box in $\mathbb{R}^n$ and assume that $P = \{R_1, R_2, \dots, R_N\}$ is a finite set of boxes in $\mathbb{R}^n$ such that

- $\bar{R} = \bar{R}_1 \cup \bar{R}_2 \cup \dots \cup \bar{R}_N$ (their closures cover R),
- $R_i \cap R_j = \emptyset$ if $i \neq j$ (disjoint).

Then P is called a (rectangle/box) **partition** of R.

If P and Q are partitions of R and if each box in P is a subset of some box in Q, then we say that P is a **finer partition** than Q (or a **refinement**) and write $P \leq Q$. This defines a partial order on the set of partitions of R. We leave that and the following as an exercise:

Lemma 5.40. *If P and Q are two arbitrary partitions of the box $R \subseteq \mathbb{R}^n$, then there exists a unique partition S of R such that*

$$S \leq P \qquad \textit{and} \qquad S \leq Q, \tag{5.131}$$

*and s.t. if S' is another partition which also satisfies $S' \leq P$ and $S' \leq Q$, then necessarily $S' \leq S$. This partition S is denoted $P \wedge Q$, their **common refinement**.*

The **fineness** of a partition P is defined to be the maximal diameter among the boxes in P and is denoted $d(P)$:

$$d(P) := \max\{\operatorname{diam} R_i : R_i \in P\}, \tag{5.132}$$

where we recall

$$\operatorname{diam} R_i = \sup\{|\mathbf{x} - \mathbf{y}| : \mathbf{x}, \mathbf{y} \in R_i\}. \tag{5.133}$$

Exercise 5.11. *Prove Lemma 5.40 and that $\leq$ is a partial order on partitions (take $n = 1$ if this seems easier).*

5.4.2. *Upper, lower and Riemann sums.* Let $f\colon \mathbb{R}^n \to \mathbb{R}$ and assume that

$$\operatorname{supp}(f) \subseteq R(\mathbf{a}, \mathbf{b}). \tag{5.134}$$

Let $P = \{R_1, R_2, \dots, R_N\}$ be a partition of $R = R(\mathbf{a}, \mathbf{b})$ and define

$$\begin{aligned}
m_i &:= \inf\{f(\mathbf{x}) : \mathbf{x} \in \bar{R}_i\},\\
M_i &:= \sup\{f(\mathbf{x}) : \mathbf{x} \in \bar{R}_i\},\\
L(f, P) &:= \sum_{i=1}^{N} m_i \operatorname{vol} R_i \quad \text{(lower sum / sv: undersumma)},\\
U(f, P) &:= \sum_{i=1}^{N} M_i \operatorname{vol} R_i \quad \text{(upper sum / sv: översumma)}.
\end{aligned}$$

Also, let $T = \{t_1, t_2, \ldots, t_N\}$ be such that $t_i \in R_i$ for all $i = 1, \ldots, N$ (we say that "T belongs to P"), and define

$$I(f, P, T) := \sum_{i=1}^{N} f(t_i) \operatorname{vol} R_i \quad \text{(Riemann sum / sv: Riemannsumma).}$$

We note immediately that

$$L(f, P) \leq I(f, P, T) \leq U(f, P). \tag{5.135}$$

Furthermore, if P and Q are partitions of R then

$$L(f, P) \leq L(f, P \wedge Q) \leq U(f, P \wedge Q) \leq U(f, Q), \tag{5.136}$$

in other words, given R and f, every lower sum is smaller than every upper sum.

5.4.3. *Darboux vs. Riemann vs. Stieltjes.* Again, let R be a box in $\mathbb{R}^n$.

Definition 5.41 (Darboux integral)**.** A function $f\colon R \to \mathbb{R}$ is called **Darboux integrable** iff $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$U(f, P) - L(f, P) < \varepsilon \tag{5.137}$$

for every partition P of R with $d(P) < \delta$.

We find immediately from (5.136) that if f is Darboux integrable then

$$\sup_P L(f, P) = \inf_P U(f, P) \tag{5.138}$$

(taken over all partitions P of R) both exist, are finite, and equal. This real number is denoted

$$\int f = \int_R f = \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} f(\mathbf{x})\, d\mathbf{x}, \tag{5.139}$$

the **integral** of f (over R). Also for arbitrary f we may conventionally denote the l.h.s. resp. r.h.s. of (5.138) as the **lower integral** $\underline{\int} f$ resp. **upper integral** $\overline{\int} f$, with $\underline{\int} f \leq \overline{\int} f$.

Definition 5.42 (Riemann integral)**.** A function $f\colon R \to \mathbb{R}$ is called **Riemann integrable** iff there exists a number $I(f) \in \mathbb{R}$ so that $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if T belongs to a partition P of R with $d(P) < \delta$, then

$$|I(f, P, T) - I(f)| < \varepsilon. \tag{5.140}$$

We note that actually the Darboux and Riemann integral concepts are equivalent. The **Riemann-Stieltjes integral** is extracurricular but defined in Rudin 6.2 in the one-variable setting. It uses the notion of a '**weight function**' α to associate different weights to the points along the interval. Choosing $\alpha(x) = x$ reduces it to the Riemann-Darboux integral.

Exercise 5.12. *Show that the Darboux and Riemann integral concepts coincide and that* $I(f) = \int f$.
Hint: see [Rud76, Thm. 6.7].

Exercise 5.13 (Rudin Exc 6.4)**.** *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be the '**comb function**'*

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}. \tag{5.141}$$

Show that f is not Riemann integrable on any interval $[a,b]$, $a<b$, by computing the corresponding upper and lower integrals.

Exercise 5.14. *Compute upper and lower integrals on the interval $[0,1]$ for the popcorn function f from Exercise 4.26. Is it Riemann integrable and, if so, what is $\int_0^1 f$?*

5.4.4. *Riemann-Lebesgue's theorem.* Note that the comb function (5.141) is everywhere discontinuous. We will now show that an arbtrary function f supported in some box is Riemann (Darboux) integrable, denoted $f \in \mathscr{R}$, if and only if f is bounded, and continuous at all points except on a null set.

Recall (cf. Exercise 4.31) that a subset $A \subseteq \mathbb{R}^n$ is called a **null set** if for every $\varepsilon > 0$ there exists a countable set of boxes $\{R_1, R_2, R_3, \ldots\}$ such that

- $A \subseteq \bigcup_{j=1}^{\infty} R_j$ (covering)
- $\sum_{j=1}^{\infty} \operatorname{vol} R_j < \varepsilon$ (arbitrarily small).

Lemma 5.43. *Countable unions of null sets are null sets.*

Proof. Let $A = \bigcup_{k=1}^{\infty} A_k$, where A_k is a null set in $\mathbb{R}^n$ for each $k = 1, 2, \ldots$. Take $\varepsilon > 0$. For each k we may find boxes $R_{k,1}, R_{k,2}, R_{k,3}, \ldots$ such that

$$A_k \subseteq \bigcup_{j=1}^{\infty} R_{k,j} \qquad \text{and} \qquad \sum_{j=1}^{n} \operatorname{vol} R_{k,j} < \varepsilon/2^k.$$

Hence,

$$A \subseteq \bigcup_{k,j} R_{k,j} \qquad \text{and} \qquad \sum_{k,j} \operatorname{vol} R_{k,j} < \sum_{k=1}^{\infty} \varepsilon 2^{-k} = \varepsilon.$$

□

Lemma 5.44 (**Lebesgue covering number**)**.** *Let K be a compact subset of a metric space X and assume that $K \subseteq \bigcup_{i\in I} V_i$ is an open covering with an arbitrary index set I. Then there exists a $\lambda > 0$ such that for all $x \in K$ there exists $i \in I$ s.t. $B_\lambda(x) \subseteq V_i$.*

Proof. Assume on the contrary that for every $n \geq 1$ there exists $x_n \in K$ such that $B_{1/n}(x_n) \subsetneq V_i$ for every $i \in I$. In other words, no V_i contains $B_{1/n}(x_n)$ as a subset. Since K is compact we can find a convergent subsequence $(x_{n'})_{n=1}^{\infty}$, say $\exists x^* \in K$ s.t. $x_{n'} \to x^*$ as $n \to \infty$. But then, since $\{V_i\}$ is an open covering, there is a $j \in I$ s.t. $x^* \in V_j$, and we may even find $r > 0$ with $B_{2r}(x^*) \subseteq V_j$. Now choose N large enough that $1/N < r$ and such that $d(x_{n'}, x^*) < r$ for all $n \geq N$. Then we have that

$$B_{1/n'}(x_{n'}) \subseteq B_{2r}(x^*) \subseteq V_j, \quad \text{if } n \geq N,$$

because if $x \in B_{1/n'}(x_{n'})$ then

$$d(x, x^*) \leq d(x, x_{n'}) + d(x_{n'}, x^*) < 1/n' + r < 2r.$$

This contradiction to our assumption proves the lemma. □

Theorem 5.45 (**Riemann-Lebesgue's theorem**)**.** *Let $R \subseteq \mathbb{R}^n$ be a box and $f\colon R \to \mathbb{R}$. Then $f \in \mathscr{R}$ iff f is bounded on R and continuous except on a null set.*

Proof. We assume first that $f \in \mathscr{R}$ and need to show that f is bounded and continuous except for a null set.

We start with boundedness. Assume $\operatorname{supp}(f) \subseteq R(\mathbf{a}, \mathbf{b})$, and take $\varepsilon > 0$. Then there exists $\delta > 0$ s.t. $|I(f, P, T) - \int f| < \varepsilon$ whenever $\delta(P) < \delta$ and T belongs to P. If f is not bounded then there exists some box $R_i \in P$ such that $M_i = \infty$ or $m_i = -\infty$. Assume WLOG the former. But then there exists a sequence $(t_{i,k})_{k=1}^{\infty} \subset R_i$ such that $f(t_{i,k}) \to \infty$ if $k \to \infty$. Taking

$$T_k = \{t_1, t_2, \ldots, t_{i-1}, t_{i,k}, t_{i+1}, \ldots, t_N\}$$

we find that $I(f, P, T_k) \to \infty$ as $k \to \infty$. This contradiction proves that f is bounded.

We shall now prove that f is continuous except on a null set. Let the set of discontinuities be denoted

$$D = \{x \in \mathbb{R}^n : f \text{ is not continuous at } x\}.$$

We define for any $x \in R$ the **oscillation** of f around x by

$$\operatorname{osc}(f, x) := \lim_{r \to 0} \operatorname{diam} f\big(B_r(x)\big).$$

Note that f is continuous at x iff $\operatorname{osc}(f, x) = 0$. Let us also denote for $\rho > 0$

$$D_\rho := \{x \in \mathbb{R}^n : \operatorname{osc}(f, x) > \rho\},$$

then $D = \bigcup_{k=1}^{\infty} D_{1/k}$. This means that D is a null set if and only if D_ρ is a null set for every $\rho > 0$.

Resuming our proof, let $\varepsilon > 0$ and $\rho > 0$. Fix a partition $P = \{R_1, \ldots, R_N\}$ so fine that

$$U(f, P) - L(f, P) = \sum_{i=1}^{N} (M_i - m_i) \operatorname{vol} R_i < \varepsilon \rho, \tag{5.142}$$

i.e.

$$\sum_{i: R_i \cap D_\rho \neq \emptyset} (M_i - m_i) \operatorname{vol} R_i + \sum_{i: R_i \cap D_\rho = \emptyset} (M_i - m_i) \operatorname{vol} R_i < \varepsilon \rho, \tag{5.143}$$

But if $R_i \cap D_\rho \neq \emptyset$ then $M_i - m_i > \rho$. This implies

$$\sum_{i: R_i \cap D_\rho \neq \emptyset} \rho \operatorname{vol} R_i < \varepsilon \rho \quad \Rightarrow \quad \sum_{i: R_i \cap D_\rho \neq \emptyset} \operatorname{vol} R_i < \varepsilon, \tag{5.144}$$

and further

$$D_\rho \subseteq \left(\bigcup_{i: R_i \cap D_\rho \neq \emptyset} R_i \right) \cup \left(\bigcup_{j=1}^{N} \partial R_j \right). \tag{5.145}$$

Each ∂R_j is clearly a null set, and therefore by (5.144)-(5.145), also D_ρ is a null set. Since we may choose any $\rho = 1, 1/2, 1/3, \ldots$, necessarily D is a null set.

It remains to prove the converse. Assume then that $|f(x)| \leq M$ $\forall x \in R$ and that f is continuous except on a null set $D \subseteq R$, and we need to prove that $f \in \mathscr{R}$. Take $\varepsilon > 0$ and choose a sequence of open boxes $J_1, J_2, \ldots$ such that $D \subseteq \bigcup_{m=1}^{\infty} J_m$ and $\sum_{m=1}^{\infty} \operatorname{vol} J_m < \varepsilon$. If $x \in R \backslash D$ then $\operatorname{osc}(f, x) = 0$, and we may find some open set $I_x \ni x$ such that $\operatorname{diam} f(I_x) < \varepsilon$ (we could e.g. find a ball $I_x = B_{r_x}(x)$).

Now we have an open cover of the compact set $\bar{R}$

$$\bar{R} \subseteq \left(\bigcup_{m=1}^{\infty} J_m \right) \cup \left(\bigcup_{x \in R \setminus D} I_x \right). \tag{5.146}$$

By Lemma 5.44 there exists a Lebesgue covering number $\lambda > 0$ such that for every $x \in R$, either there exists $m \in \mathbb{N}^+$ s.t. $B_\lambda(x) \subseteq J_m$, or $y \in R \setminus D$ s.t. $B_\lambda(x) \subseteq I_y$. Now choose a partition $P = \{R_1, \ldots, R_N\}$ such that $\operatorname{diam} R_k < \lambda \;\; \forall k = 1, 2, \ldots, N$. We then obtain

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{N} (M_i - m_i) \operatorname{vol} R_i \\ &= \sum_{i: R_i \subseteq \text{some } J_m} (M_i - m_i) \operatorname{vol} R_i + \sum_{i: R_i \subseteq \text{some } I_x} (M_i - m_i) \operatorname{vol} R_i \\ &\leq \sum_{m=1}^{\infty} 2M \operatorname{vol} J_m + \sum_{i=1}^{N} \varepsilon \operatorname{vol} R_i < 2M\varepsilon + \varepsilon \operatorname{vol} R. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, $f \in \mathscr{R}$. □

5.4.5. *Generalized integrability.* To define Riemann integrability for more general functions, one takes appropriate limits of boxes (**improper integrals**); see exercises 6.7-8 in Rudin:

$$\int_0^1 f(x)\, dx := \lim_{a \to 0} \int_a^1 f(x)\, dx, \tag{5.147}$$

$$\int_1^\infty f(x)\, dx := \lim_{b \to \infty} \int_1^b f(x)\, dx. \tag{5.148}$$

These may be **absolutely convergent**, i.e. with absolute values on f. However note that when they are not, there may arise some peculiarities depending on the way which the limit is taken:

Exercise 5.15.

a) *Give examples of unbounded* $f\colon [0,1] \to \mathbb{R}^+$ *such that* (5.147) *converges/diverges.*
b) *Give examples of bounded* $f\colon [1,\infty) \to \mathbb{R}^+$ *such that* (5.148) *converges/diverges.*
c) *Give an example of* $f\colon [0,1] \to \mathbb{R}$ *such that*

$$\int_0^1 f(x)\, dx := \lim_{\varepsilon \to 0} \left(\int_0^{1/2-\varepsilon} f(x)\, dx + \int_{1/2+\varepsilon}^{1} f(x)\, dx \right) \tag{5.149}$$

converges but each of the two separate integrals in the r.h.s. diverge as $\varepsilon \to 0$. *(To stress that this is one particular way of making sense of such an integral it is called the* ***Cauchy principal value*** *and is also written* p.v. $\int f$*.)*

5.4.6. *Properties and methods of computation.* Recall the basic properties of integrals, [Rud76, Thm. 6.12-13].

Recall also our foundation for integral calculus, [Rud76, Thm. 6.20,21,24]:

Theorem 5.46 (**Fundamental theorem of integral calculus**). *Let $f \in \mathscr{R}$ on $[a,b]$. If there exists a differentiable function F on $[a,b]$ such that $F' = f$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.150}$$

If, for $a \leq x \leq b$,

$$F(x) := \int_a^x f(t)\,dt \tag{5.151}$$

then F is continuous on $[a,b]$, and if f is continuous at $x \in [a,b]$ then F is differentiable at x with $F'(x) = f(x)$.

The proofs use the mean value Theorem 5.30 on sufficiently fine partitions, respectively the box estimate for integrals. For explicit computation of integrals the following is extremely useful (as well as partial integration, [Rud76, Thm. 6.22]):

Theorem 5.47 (**Change of variable**). *If $f \in \mathscr{R}$ on $[a,b]$ and if $\varphi\colon [A,B] \to [a,b]$ is differentiable, strictly increasing, and $\varphi' \in \mathscr{R}$, then the function $(f \circ \varphi)\varphi' \in \mathscr{R}$ on $[A,B]$ and*

$$\int_a^b f(x)\,dx = \int_A^B f(\varphi(y))\varphi'(y)dy. \tag{5.152}$$

Exercise 5.16.

a) *Prove* (5.152) *under the assumptions of Theorem 5.47. (Hint: Rudin 6.19)*
b) *Prove* (5.152) *under the stronger assumptions that f and $\varphi' > 0$ are continuous, by applying the chain rule to*

$$G(x) := \int_A^{\psi(x)} f(\varphi(y))\varphi'(y)\,dy, \tag{5.153}$$

where $\psi\colon [a,b] \to [A,B]$ is the inverse to φ.

Exercise 5.17. *Continuing Exercise 5.15, show that*

$$\lim_{a\to\infty} \int_{-a}^{a} \frac{2x}{x^2+1} dx = 0 \qquad \text{while} \qquad \lim_{a\to\infty} \int_{-2a}^{a} \frac{2x}{x^2+1} dx = -\ln 4. \tag{5.154}$$

(The former, symmetric limit, is defined as p.v. $\int_{-\infty}^{\infty} \frac{2x}{x^2+1} dx$*).*

Rudin's Exc. 9.28 deals with the possible non-commutativity of differentiation and integration. Later we will also discuss the following 'monsters':

- The existence of an everywhere continuous but nowhere differentiable function (e.g. **Weierstrass' function**; see Theorem 6.16).
- The existence of an everywhere differentiable function whose derivative is bounded but not Riemann integrable (e.g. **Volterra's function**).

6. Sequences and series of functions [L16-20]

6.1. **Reading tip.** This Section concerns Chapter 7 in Rudin (function spaces), the remaining parts of Chapter 4 (uniform continuity), and parts of Chapter 8 (power series).

Study carefully the examples of Rudin 7.2-6 and 7.21!

Stone-Weierstrass approximation theorem (Rudin 7.26-33) is considered extracurricular to this course, however we hope that we will find some time to discuss it anyway as a useful application of the theory.

We will discuss Taylor series, cf. Rudin 3.38-39 and 8.1, briefly 8.2-5, while some general aspects of the exponential function (cf. Rudin 8.6 and that section) have been covered in Section 4.10.1. Most of the subsequent material about the exponential, logarithmic and trigonometric functions (Rudin 8.6-7) are assumed to be known from calculus (however it may be nice to see how their properties are here proved in a rigorous way) and these parts are left for your own study. The remaining parts of Chapter 8 are extracurricular.

6.1.1. *Exercises.* Rudin Ch. 4: 8-13; Ch. 7: 1-13,(14),15-19,(20,22-26); Ch. 8: 1-7
Exam 2020-08-19: problems 3,4,8. Exam 2020-06-15: problems 3,5,8. Exam 2020-03-16: problems 3,5,8. Exam 2019-06-15: problem 3. Exam 2019-01-14: problems 3,(5). Exam 2015-12-09: problem 6. Exam 2015-03-21: problems 4,6. Exam 2014-12-17: problems 3,6. Exam 2014-04-23: problems 4,(5). Exam 2013-12-18: problems 3,5.

6.1.2. *Aims.* Concepts discussed in this Section:

- sequences and series of functions
- uniform convergence
- equicontinuous families
- Arzelà-Ascoli theorem

Learning outcomes: After this Section you should be able to

- explain the basic theory of metric spaces and its application to function spaces;
- apply the theory to solve mathematical problems including the construction of simple proofs

6.2. **Uniform continuity.** Let X and Y be metric spaces. Recall (Definition 4.43) that a map $f\colon X \to Y$ is called continuous (or perhaps **pointwise continuous**) if

$$\forall x \in X\ \forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall y \in X\ \Big(d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon\Big) \tag{6.1}$$

or equivalently, and slightly more compactly,

$$\forall x \in X\ \forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))). \tag{6.2}$$

Note here the explicit dependence of the quantifiers: the number $\delta = \delta(x, \varepsilon)$ could in principle need to depend not only on ε but also on each point $x \in X$.

We call $f\colon X \to Y$ **uniformly continuous** (sv: likformigt kontinuerlig) if

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall x, y \in X\ \Big(d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon\Big) \tag{6.3}$$

or equivalently

$$\forall \varepsilon > 0\ \exists \delta > 0 \text{ s.t. } \forall x \in X\ \ B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x))), \tag{6.4}$$

so in this case $\delta = \delta(\varepsilon)$ may be chosen independently from x.

Example 6.1. The map $f(x) = 1/x$ on the interval $(0, 1)$ is continuous but not uniformly continuous. Namely, given any $\varepsilon > 0$ and $\delta > 0$ we may find $0 < x < \min\{\delta, \varepsilon, \varepsilon^{-1}, 1\}$ and $y = x/2 \in (0, 1)$ such that

$$|x - y| = x/2 < \delta, \qquad \text{but} \qquad |f(y) - f(x)| = 1/y - 1/x = 1/x > \varepsilon,$$

which violates (6.3)-(6.4).

Theorem 6.2. *If $f : X \to Y$ is a continous map between metric spaces, and if X is compact, then f is uniformly continuous.*

See Rudin Thm 4.19 and Exc 4.10 for two different proofs. We repeat the first one for illustration:

Proof. Let $\varepsilon > 0$. For each $x \in X$ we may use (pointwise) continuity of f to find $\delta_x > 0$ such that

$$\forall y \in X \quad d(x, y) < 2\delta_x \Rightarrow d(f(x), f(y)) < \varepsilon \tag{6.5}$$

(note the factor 2 which will be used below). We have then a covering $X \subseteq \bigcup_{x\in X} B_{\delta_x}(x)$ by open sets, and thus by compactness there are finitely many points $\{x_1, x_2, \ldots, x_N\}$ in X such that

$$X = B_{\delta_{x_1}}(x_1) \cup B_{\delta_{x_2}}(x_2) \cup \ldots \cup B_{\delta_{x_N}}(x_N). \tag{6.6}$$

We define

$$\delta := \min\{\delta_{x_1}, \delta_{x_2}, \ldots, \delta_{x_N}\} > 0. \tag{6.7}$$

Now take any $x, y \in X$ such that $d(x, y) < \delta$. Then by (6.6) there is some x_k such that

$$d(x, x_k) < \delta_{x_k} < 2\delta_{x_k}, \tag{6.8}$$

and furthermore

$$d(x_k, y) \leq d(x_k, x) + d(x, y) < \delta_{x_k} + \delta < 2\delta_{x_k}. \tag{6.9}$$

Thus, by another application of the triangle inequality, and (6.5),

$$d(f(x), f(y)) \leq d(f(x), f(x_k)) + d(f(x_k), f(y)) < \varepsilon + \varepsilon, \tag{6.10}$$

which proves the theorem. □

Exercise 6.1. *Show that $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \sin(1/x)$ is not uniformly continuous.*

Exercise 6.2. *Prove that uniform continuity always implies (pointwise) continuity.*

Exercise 6.3 (Rudin Exc 4.11)**.** *Let $f : X \to Y$ be a uniformly continuous map between metric spaces. Show that if $(x_n)_{n\in\mathbb{N}}$ is Cauchy in X then $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in Y.*

6.3. **Uniform convergence and equicontinuity.**

6.3.1. *Pointwise and uniform convergence.* Consider an arbitrary set X and a metric space (Y, d_Y), and a sequence of functions $f_n : X \to Y$, $n = 0, 1, 2, \ldots$. For each $x \in X$ we then have a sequence $(f_n(x))_{n\in\mathbb{N}}$ in Y, which may or may not converge. In the case that it converges we say that (f_n) **converges pointwise** at x. If (f_n) converges pointwise at every point $x \in X$ of its domain then we may define the **pointwise limit** of the sequence as the function $f : X \to Y$,

$$f(x) := \lim_{n\to\infty} f_n(x), \qquad \forall x \in X. \tag{6.11}$$

In other words, f_n converges pointwise to f as $n \to \infty$ iff

$$\forall x \in X \ \forall \varepsilon > 0 \ \exists N = N(x, \varepsilon) \ \text{ s.t. } \ n \geq N \Rightarrow d_Y(f_n(x), f(x)) < \varepsilon. \tag{6.12}$$

We note that pointwise convergence might happen at different speeds on the domain X:

Example 6.3. Let $f_n\colon \mathbb{R}^+ \to \mathbb{R}$ for $n \in \mathbb{N}^+$ be defined

$$f_n(x) = \frac{1}{nx}. \tag{6.13}$$

Clearly, the sequence (f_n) converges pointwise to the zero function $f = 0$, however, given any error margin $\varepsilon > 0$ we need to choose $N(x, \varepsilon)$ larger the smaller $x > 0$ is:

$$N = N(x, \varepsilon) < \frac{1}{\varepsilon x} \quad \Rightarrow \quad f_N(x) = \frac{1}{Nx} > \varepsilon. \tag{6.14}$$

We say that f_n **converges uniformly** to f on X if N in (6.12) may be chosen independently of x, in other words, iff

$$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \ \text{ s.t. } \ \forall x \in X \ \ n \geq N \Rightarrow d_Y(f_n(x), f(x)) < \varepsilon. \tag{6.15}$$

Another way to phrase this property is by measuring the distance of f_n to f uniformly on X by means of the following induced (from Y) metric[(4)] on the set of all functions $X \to Y$:

$$d(f, g) := \sup_{x \in X} d_Y(f(x), g(x)). \tag{6.16}$$

Then indeed $f_n \to f$ uniformly on X iff $d(f_n, f) \to 0$:

$$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \ \text{ s.t. } \ n \geq N \Rightarrow d(f_n, f) \leq \varepsilon. \tag{6.17}$$

In the case that Y is a real vector space then also naturally the set of functions $X \to Y$ is a real vector space with the linear operations

$$(\alpha f + \beta g)(x) := \alpha f(x) + \beta g(x) \quad \forall \alpha, \beta \in \mathbb{R}, \ f, g\colon X \to Y. \tag{6.18}$$

In the case that $(Y, |\cdot|_Y)$ is a normed vector space then we may use the corresponding **supremum norm**:

$$\|f\|_\infty := \sup_{x \in X} |f(x)|_Y, \qquad d(f, g) = \|f - g\|_\infty. \tag{6.19}$$

The corresponding space of all *bounded* functions $X \to Y$ with the metric (6.16) is precisely the space $\ell^\infty(X; Y)$ introduced in Theorem 5.19, and uniform convergence on X is the same as convergence in $(\ell^\infty(X; Y), \|\cdot\|_\infty)$:

$$\|f_n - f\|_\infty \leq \varepsilon \quad \Leftrightarrow \quad |f_n(x) - f(x)|_Y \leq \varepsilon \ \forall x \in X. \tag{6.20}$$

Furthermore, we say that $(f_n\colon X \to Y)_{n \in \mathbb{N}}$ is **uniformly Cauchy** on X iff

$$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \ \text{ s.t. } \ \forall x \in X \ \ n, m \geq N \Rightarrow d_Y(f_n(x), f_m(x)) < \varepsilon. \tag{6.21}$$

i.e. iff it is Cauchy w.r.t. the metric (6.16); $d(f_n, f_m) \to 0$ as $m, n \to \infty$. Equivalently, this is iff (f_n) is Cauchy in $\ell^\infty(X; Y)$ in the case Y is a normed vector space. By Theorem 5.19, we recall that if Y is a Banach space then also $\ell^\infty(X; Y)$ with the norm $\|\cdot\|_\infty$ is a Banach space:

[(4)] Or 'pseudo-metric' as it is not necessarily finite unless we e.g. restrict to bounded functions.

Theorem 6.4. *If X is a set and Y is a Banach space (or in fact any complete metric space), and if $f_n\colon X \to Y$, $n \in \mathbb{N}$, is uniformly Cauchy, then there exists $f\colon X \to Y$ such that $f_n \to f$ uniformly as $n \to \infty$.*

We ask what properties are preserved under pointwise respectively uniform limits. Typical questions that we will address are:

- If f_n are continuous for all n, is the limit f continuous? Note that this may be equivalently phrased
$$\lim_{y\to x}\lim_{n\to\infty} f_n(y) \stackrel{?}{=} \lim_{n\to\infty}\lim_{y\to x} f_n(y) \tag{6.22}$$
- If f_n and f are differentiable, do we have that $f_n' \to f'$?
- If f_n and f are Riemann integrable, do we have that $\int f_n \to \int f$?

We shall also consider corresponding questions for **series of functions**, i.e. sequences of the form $(S_n)_{n\in\mathbb{N}}$ with $S_n = \sum_{k=0}^n f_k$ as $n \to \infty$. The series $\sum_{k=0}^\infty f_k$ converges pointwise/uniformly iff the sequence (S_n) converges pointwise/uniformly. By the majorization Theorem 4.66 for series (also known as the **Weierstrass M-test**; cf. Rudin 7.10), we may be able to easily deduce such pointwise/uniform convergence by bounding the functions f_n.

Exercise 6.4. *Prove that uniform convergence always implies pointwise convergence.*

Exercise 6.5 (Rudin Exc 7.1)**.** *Let X be a set and Y a normed vector space. A sequence $(f_n\colon X \to Y)_{n\in\mathbb{N}}$ is called **pointwise bounded** on $E \subseteq X$ iff there exists a function $g\colon E \to \mathbb{R}_+$ such that*
$$|f_n(x)| \le g(x) \;\forall x \in E, n \in \mathbb{N}, \tag{6.23}$$
*and **uniformly bounded** on E iff there exists a number $M \in \mathbb{R}_+$ such that*
$$|f_n(x)| \le M \;\forall x \in E, n \in \mathbb{N}. \tag{6.24}$$
In the case that Y is an arbitrary metric space we may replace these conditions by the existence of balls in Y containing the function values for all n either pointwise or uniformly on X. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded, and that the limit is a bounded function.

Exercise 6.6. *Prove that Theorem 6.4 holds for any complete metric space (Y, d_Y) (not necessarily vector spaces).*

6.3.2. *Uniform convergence and continuity.*

Example 6.5. Consider the sequence of functions $f_n\colon [-1,1] \to \mathbb{R}$, $n \in \mathbb{N}^+$,
$$f_n(x) = \begin{cases} -1, & -1 \le x \le -1/n, \\ nx, & -1/n < x < 1/n, \\ 1, & 1/n \le x \le 1. \end{cases}$$
Then the pointwise limit is
$$f(x) = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Thus, clearly continuity of f_n for all n is insufficient to guarantee continuity of f. We also note that

$$\|f_n - f\|_\infty = \lim_{x\to 0} |nx \pm 1| = 1 \quad \forall n \in \mathbb{N}_+, \tag{6.25}$$

so f_n does not tend to f uniformly. In fact, (f_n) is not uniformly Cauchy (exercise).

Example 6.6. Another example is Rudin 7.3 with the pointwise convergent series

$$f(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n} = \begin{cases} 0, & x = 0, \\ 1 + x^2, & x \neq 0, \end{cases} \tag{6.26}$$

which is not continuous despite all its terms being continuous functions. We note that the supremum norm of each term of the series is greater than

$$\sup_{x\in\mathbb{R}} \frac{x^2}{(1+x^2)^n} \geq \frac{(n^{-1/2})^2}{(1+(n^{-1/2})^2)^n} \geq \frac{1}{ne} \tag{6.27}$$

(see Rudin 3.31) and a sum of such terms is divergent. Therefore we cannot apply the usual majorization theorem for series (Theorem 4.66).

Definition 6.7. Given metric spaces X and Y, we denote by $C(X;Y)$ the set of all continuous functions $X \to Y$, and by $C_b(X;Y)$ the *bounded*(5) continuous functions endowed with the supremum metric (6.16). For brevity we write $C_{(b)}(X) = C_{(b)}(X;\mathbb{R})$ for the real-valued (bounded) continuous functions on X. (Note! Rudin uses the notation: $\mathscr{C} = C_b$)

In the case that X is compact then the boundedness is automatic by Theorem 4.46, i.e. $C = C_b$, and furthermore by the continuity of the metric $d_Y : Y \times Y \to \mathbb{R}_+$, implying the continuity of the map $x \mapsto d_Y(f(x), g(x))$, we may replace the supremum by the maximum,

$$d(f,g) := \sup_{x\in X} d_Y(f(x), g(x)) = \max_{x\in X} d_Y(f(x), g(x)). \tag{6.28}$$

We also note by Theorem 6.2 that $f \in C(X;Y)$ is *uniformly* continuous if X is compact.

In the case that Y is a Banach space then by Theorem 6.4, any Cauchy sequence in $C_b(X;Y) \subseteq \ell^\infty(X;Y)$ converges uniformly to some bounded function $f \in \ell^\infty$. In other words, uniform Cauchy implies uniform convergence. Note that by Examples 6.5-6.6 it is not necessarily the case that $C(X;Y)$ remains closed under pointwise limits, however the following shows that with the stronger assumptions on X and Y, $C_b(X;Y)$ is indeed closed under *uniform* limits (note that in the example, $X = [-1,1]$ is compact and $Y = \mathbb{R}$ complete, but f is not a uniform limit of f_n).

Theorem 6.8. *If X and Y are metric spaces and Y is complete, then $(C_b(X;Y), d)$ as defined above is a complete metric space. In particular, $C_b(X)$ is a Banach space.*

Proof. First let's check that we actually have a metric:

$$\begin{aligned} d(f,g) = \sup_{x\in X} d(f(x), g(x)) &\leq \sup_{x\in X} \Big(d(f(x), h(x)) + d(h(x), g(x)) \Big) \\ &\leq \sup_{x\in X} d(f(x), h(x)) + \sup_{x\in X} d(h(x), g(x)) = d(f,h) + d(h,g). \end{aligned}$$

Furthermore, obviously $d(f,g) = d(g,f)$ and $d(f,g) = 0$ iff $f = g$.

(5) Boundedness ensures the finiteness of the supremum metric.

Now assume that a sequence (f_n) in $C_b(X;Y)$ is Cauchy, that is

$$\forall \varepsilon > 0 \ \exists N \ \text{ s.t. } \ n, m \geq N \Rightarrow \sup_{x \in X} d(f_n(x), f_m(x)) < \varepsilon. \tag{6.29}$$

In other words (f_n) is uniformly Cauchy, and by Theorem 6.4 there exists $f\colon X \to Y$ such that $f_n \to f$ uniformly. We note that f is a bounded function by Exercise 6.5. It thus remains to prove that f is also continuous, $f \in C_b(X;Y)$. We note that by the triangle inequality

$$d(f(x), f(y)) \leq d(f(x), f_n(x)) + d(f_n(x), f_n(y)) + d(f_n(y), f(y)), \tag{6.30}$$

for any n. Thus, take $\varepsilon > 0$. Then there exists $N = N(\varepsilon)$ s.t. $d(f, f_n) < \varepsilon$ if $n \geq N$. Hence,

$$d(f(x), f(y)) < 2\varepsilon + d(f_N(x), f_N(y)). \tag{6.31}$$

Furthermore, by the uniform continuity of f_N, there exists $\delta = \delta(\varepsilon) > 0$ such that $d(f_N(x), f_N(y)) < \varepsilon$ if $d(x, y) < \delta$. Therefore

$$d(f(x), f(y)) < 3\varepsilon \quad \text{if} \quad d(x, y) < \delta, \tag{6.32}$$

and f is (uniformly) continuous. □

A slight generalization of the above result clarifies the possibility to make an exchange of limits (6.22):

Theorem 6.9 (Rudin Thm 7.11)**.** *Suppose $f_n\colon X \to Y$, $n \in \mathbb{N}$, is uniformly Cauchy on a subset $E \subseteq X$ of a metric space X, and Y a complete metric space. Let $x \in X$ be a limit point of E and*

$$A_n := \lim_{E \ni t \to x} f_n(t), \qquad n \in \mathbb{N}. \tag{6.33}$$

Then the sequence (A_n) converges and

$$\lim_{E \ni t \to x} f(t) = \lim_{n \to \infty} A_n, \tag{6.34}$$

where f denotes the pointwise (and uniform) limit of (f_n) on E. In other words,

$$\lim_{E \ni t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{E \ni t \to x} f_n(t). \tag{6.35}$$

Proof. Let $\varepsilon > 0$. By the assumption of uniform Cauchy, we have for $m, n \geq N$ that

$$d(f_n(t), f_m(t)) \leq \varepsilon \qquad \forall t \in E. \tag{6.36}$$

Thus, taking $t \to x$ we obtain

$$d(A_n, A_m) \leq \varepsilon. \tag{6.37}$$

It follows that $(A_n)_n$ in Y is Cauchy and therefore converges to $A := \lim_{n \to \infty} A_n \in Y$. For any $t \in E$ and $n \in \mathbb{N}$,

$$d(f(t), A) \leq d(f(t), f_n(t)) + d(f_n(t), A_n) + d(A_n, A), \tag{6.38}$$

and we can make the first and the last term smaller than $\varepsilon/3$ by just taking n large enough. After fixing such n we may also find $\delta > 0$ such that for all $t \in E \cap B_\delta(x)$ we have $d(f_n(t), A_n) < \varepsilon/3$. This proves the equality (6.34). □

Exercise 6.7. *Show that the sequence in Example 6.5 is not uniformly Cauchy.*

6.3.3. *Dini's theorem.* Dini says that we may replace pointwise limits with uniform limits if we also have a monotonicity assumption.

Theorem 6.10 (Dini's theorem). *Let X be a compact metric space and $f_n\colon X \to \mathbb{R}$ continuous for $n = 1, 2, 3, \ldots$, and pointwise monotone decreasing,*

$$f_n \geq f_{n+1} \geq \ldots . \tag{6.39}$$

Assume that for all $p \in X$, $\lim_{n\to\infty} f_n(p) = 0$, i.e. the sequence (f_n) converges pointwise to zero, then it also converges uniformly to zero, $f_n \to 0$ as $n \to \infty$.

Corollary 6.11. *With the same assumptions as above but where f_n converges pointwise to a continuous function f, then $f_n \to f$ uniformly on X.*

Proof of the corollary. Put $g_n = f_n - f$, then (g_n) is a sequence of continuous functions with $g_n \geq g_{n+1}$. Hence by Dini's theorem $g_n \to 0$ uniformly. □

The following counterexamples illustrate the necessity of the assumptions:

Example 6.12. Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}^+$, be the sequence of functions

$$f_n(x) = \begin{cases} 2nx, & 0 \leq x \leq 1/(2n), \\ 2 - 2nx, & 1/(2n) < x \leq 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

Then pointwise $f_n(x) \to 0$ for all $x \in \mathbb{R}$, however $f_n \nrightarrow 0$ because $\|f_n\|_\infty = 1$ $\forall n$. We may restrict to the compact interval $[0, 1]$ however the sequence is not monotonously decreasing.

Example 6.13. Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}^+$, be the sequence of functions

$$f_n(x) = \begin{cases} 0, & x < n - 1, \\ x - n + 1, & n - 1 < x < n, \\ 1, & x \geq n. \end{cases}$$

Then pointwise $f_n(x) \to 0$ for all $x \in \mathbb{R}$, however $f_n \nrightarrow 0$ because $\|f_n\|_\infty = 1$ $\forall n$. The sequence decreases monotonously and indeed if we restrict to any compact interval then we would eventually (say, for $n \geq N + 1$ on $[-N, N]$) get the zero function.

Proof of Dini's theorem. Let $\varepsilon > 0$. Since for each $p \in X$, $f_n(p) \geq f_{n+1}(p) \to 0$, and moreover f_n is continuous for each n, there exists $U_p \subseteq X$ open and $N_p \in \mathbb{N}$ such that

$$f_n(x) \leq f_{N_p}(x) \leq \varepsilon \quad \text{for all} \quad x \in U_p,\ n \geq N_p. \tag{6.40}$$

Note that $\{U_p : p \in X\}$ is an open covering of X. By compactness, there are finitely many $p_1, p_2 \ldots, p_m \in X$ such that $X \subseteq U_{p_1} \cup U_{p_2} \cup \ldots \cup U_{p_m}$. Let

$$N := \max\{N_{p_1}, N_{p_2}, \ldots, N_{p_m}\}, \tag{6.41}$$

then for any $x \in X$ and $n \geq N$, we have $x \in U_{p_k}$ for some k, $N \geq N_{p_k}$, and hence $f_n(x) \leq f_{N_{p_k}}(x) \leq \varepsilon$. □

Exercise 6.8. *Give an example of a bounded sequence in $C([0, 1])$ (with its standard metric) which does not have a convergent subsequence. Note that this implies that the closed unit ball $\bar{B}_1(0)$ in $C([0, 1])$ is <u>not</u> compact.*

Exercise 6.9. *Give an example of a sequence in $C_b([0, 1))$ (half-open interval) which tends to zero pointwise and monotonously but not uniformly.*

6.3.4. *Uniform limits of derivatives.* See Rudin's Example 7.5 for a case where $f_n \to f$ pointwise and all functions are differentiable, but $f_n' \nrightarrow f'$. Making a stronger assumption, of uniform convergence of the derivatives, saves the situation however:

Theorem 6.14 (Rudin Thm 7.17)**.** *Suppose $(f_n\colon [a,b] \to \mathbb{R})$ is a sequence of differentiable functions such that $(f_n(x_0))$ converges at some point $x_0 \in [a,b]$. If (f_n') converges uniformly on $[a,b]$ then also (f_n) converges uniformly on $[a,b]$, $f_n \to f$, to some differentiable function f, and furthermore*

$$f'(x) = \lim_{n\to\infty} f_n'(x), \qquad a \le x \le b. \tag{6.42}$$

In other words, we may then commute the limits

$$\lim_{h\to 0}\lim_{n\to\infty} \frac{1}{h}\big(f_n(x+h) - f_n(x)\big) = \lim_{n\to\infty}\lim_{h\to 0} \frac{1}{h}\big(f_n(x+h) - f_n(x)\big). \tag{6.43}$$

Proof. The first step is to prove that (f_n) converges uniformly. Given $\varepsilon > 0$ we may choose N such that for all $n, m \ge N$

$$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2} \tag{6.44}$$

and for all $t \in [a,b]$

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)}. \tag{6.45}$$

Then, by the triangle inequality and the mean value Theorem 5.30,

$$\begin{aligned} |(f_n - f_m)(x)| &\le |(f_n - f_m)(x) - (f_n - f_m)(x_0)| + |(f_n - f_m)(x_0)| \\ &\le |(f_n - f_m)'(t)(x - x_0)| + |(f_n - f_m)(x_0)| \\ &< \frac{\varepsilon|x - x_0|}{2(b-a)} + \frac{\varepsilon}{2} \le \varepsilon, \end{aligned}$$

for any $x \in [a,b]$ and some t between x_0 and x. Thus $f_n \to f$ uniformly by Theorem 6.4.

It remains to prove that f is differentiable and that (6.42) holds. We will apply Theorem 6.9 to the difference quotients

$$\phi_n(t) := \frac{f_n(t) - f_n(x)}{t - x} \qquad \text{and} \qquad \phi(t) := \frac{f(t) - f(x)}{t - x}, \qquad t \ne x, \tag{6.46}$$

where $x \in [a,b]$ is fixed. Note that

$$\lim_{t\to x} \phi_n(t) = f_n'(x) \qquad \text{and} \qquad \lim_{t\to x} \phi(t) = f'(x). \tag{6.47}$$

We have again by the mean value theorem, for some $\tilde{t}$ between t and x,

$$|(\phi_n - \phi_m)(t)| = \left|\frac{(f_n - f_m)(t) - (f_n - f_m)(x)}{t - x}\right| \le \frac{|(f_n - f_m)'(\tilde{t})||t - x|}{|t - x|} < \frac{\varepsilon}{2(b-a)}, \tag{6.48}$$

so that (ϕ_n) is uniformly Cauchy on $[a,b] \setminus \{x\}$, and furthermore pointwise $\lim_{n\to\infty} \phi_n(t) = \phi(t)$, $t \ne x$, by the pointwise convergence of f_n to f. It follows that $\phi_n \to \phi$ also uniformly on $[a,b] \setminus \{x\}$, and by (6.35)

$$\lim_{t\to x} \phi(t) = \lim_{n\to\infty}\lim_{t\to x} \phi_n(t), \tag{6.49}$$

which proves the theorem. □

Remark 6.15. Note that if we would not assume that the sequence (f_n) converges at some point x_0 then we could add to it an arbitrary sequence of constants without changing the sequence (f_n'), but then possibly destroying convergence of (f_n).

As an interesting counterexample we may consider Weierstrass' monster, Rudin 7.18:

Theorem 6.16 (Weierstrass' monster). *There exists a bounded, uniformly continuous function $\mathbb{R} \to \mathbb{R}$ which is nowhere differentiable.*

Proof. Weierstrass originally considered a function on the form

$$f(x) := \sum_{n=0}^{\infty} a^n \cos(b^n \pi x), \tag{6.50}$$

with suitable $a, b \in \mathbb{R}$. However we simplify things slightly by replacing cos by the uniformly continuous and periodic ('saw-tooth') function

$$f_0(x) = |x|, \ -1 \leq x \leq 1, \qquad f_0(x+2) = f_0(x) \ \forall x \in \mathbb{R}, \tag{6.51}$$

and let

$$f_N(x) := \sum_{n=0}^{N} \left(\frac{3}{4}\right)^n f_0(4^n x). \tag{6.52}$$

By the boundedness of f_0 and the majorization theorem, the series converges uniformly and furthermore $f = \lim_{N\to\infty} f_N$ is continuous by Theorem 6.8. Explicitly, by the bound

$$|f_0(x) - f_0(y)| \leq |x - y| \quad \forall x, y \in \mathbb{R}, \tag{6.53}$$

we may estimate

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq 2 \sum_{n=N+1}^{\infty} \left(\frac{3}{4}\right)^n + \sum_{n=0}^{N} \left(\frac{3}{4}\right)^n 4^n |x-y|. \end{aligned} \tag{6.54}$$

For any $\varepsilon > 0$, taking first $N = N(\varepsilon)$ large and then $0 \leq |x - y| < \delta = \delta(\varepsilon)$ small, we have uniform continuity.

Now consider differentiability of f at $x \in \mathbb{R}$. For any $m \in \mathbb{N}$, let

$$\delta_m := \pm \frac{1}{2} 4^{-m}, \tag{6.55}$$

and note that $4^m|\delta_m| = 1/2$ and that $4^n \delta_m \in 2\mathbb{Z}$ if $n > m$. The sign of δ_m is chosen so that there is no integer between $4^m x$ and $4^m(x + \delta_m)$. Let

$$\gamma_n := \frac{f_0(4^n(x+\delta_m)) - f_0(4^n x)}{\delta_m}, \tag{6.56}$$

then $\gamma_n = 0$ if $n > m$, by periodicity, while if $0 \leq n \leq m$ then $|\gamma_n| \leq 4^n$ by (6.53). Furthermore, $|\gamma_m| = 4^m$, and it follows that

$$\left| \frac{f(x+\delta_m) - f(x)}{\delta_m} \right| = \left| \sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n \right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{3^m + 1}{2}.$$

Taking $m \to \infty$ shows that f is not differentiable at x. □

Exercise 6.10 (Difficult!). *Let $f\colon \mathbb{R} \to \mathbb{R}$,*

$$f(x) = \sum_{n=0}^{\infty} \min_{m\in\mathbb{Z}} \left| x - \frac{m}{2^k} \right|. \tag{6.57}$$

Show that f is uniformly continuous but nowhere differentiable.

6.3.5. *Uniform limits of integrals.* Proving uniform convergence for integrals is easier. See Rudin's Examples 7.4 and 7.6 for cases where the pointwise limit either loses integrability or has a different value of the integral.

Theorem 6.17 (Rudin Thm 7.16). *Suppose R is a finite box in $\mathbb{R}^m$ and $(f_n\colon R \to \mathbb{R})$ a sequence of Riemann-integrable functions, $f_n \in \mathscr{R}$ on R. If $f_n \to f$ uniformly on R then $f \in \mathscr{R}$ on R, and*

$$\int_R f = \lim_{n\to\infty} \int_R f_n. \tag{6.58}$$

In other words, we may commute the limits of Riemann sums

$$\lim_{P,T:d(P)\to 0} \lim_{n\to\infty} I(f_n, P, T) = \lim_{n\to\infty} \lim_{P,T:d(P)\to 0} I(f_n, P, T). \tag{6.59}$$

Proof. Note that f_n and f are bounded by Theorem 5.45 and Exercise 6.5. Let $\varepsilon_n := \|f - f_n\|_\infty$. We may then estimate pointwise

$$f_n - \varepsilon_n \le f \le f_n + \varepsilon_n, \tag{6.60}$$

for any n, and therefore the oscillation of f on any box R_j by that of f_n:

$$M_j(f) - m_j(f) \le \sup_{\bar{R}_j}(f_n + \varepsilon_n) - \inf_{\bar{R}_j}(f_n - \varepsilon_n) \le M_j(f_n) - m_j(f_n) + 2\varepsilon_n. \tag{6.61}$$

It follows then that, for any partition P of R,

$$0 \le \overline{\int_R} f - \underline{\int_R} f \le U(f,P) - L(f,P) \le U(f_n,P) - L(f_n,P) + 2\varepsilon_n \operatorname{vol} R. \tag{6.62}$$

Since $f_n \in \mathscr{R}$, the r.h.s. can be made arbitrarily small by first taking n sufficiently large and then P sufficiently fine. Hence, $f \in \mathscr{R}$ on R.

We may then estimate using linearity and the box estimate for integrals

$$0 \le \left| \int_R f - \int_R f_n \right| = \left| \int_R (f - f_n) \right| \le \varepsilon_n \operatorname{vol} R \to 0, \tag{6.63}$$

as $n \to \infty$. □

6.3.6. *Equicontinuity.* We will need an even stronger notion of continuity, namely continuity that is *both* uniform over the choice of a function $f \in \mathcal{F} \subseteq C(X)$ in some set or sequence, and uniform over the domain X:

Definition 6.18. A subset $\mathcal{F} \subseteq C(X;Y)$ is called **equicontinuous** iff

$$\forall\varepsilon > 0\ \exists\delta = \delta(\varepsilon) \text{ s.t. } \forall f \in \mathcal{F}\ \forall x, y \in X\ \Big(d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon\Big). \tag{6.64}$$

Note that if this condition fails, then there exists some $\varepsilon > 0$ such that for any $\delta > 0$ there exist $f \in \mathcal{F}$ and $x, y \in X$ with $d(x,y) < \delta$ such that $d(f(x), f(y)) \geq \varepsilon$. In the case that X is compact we have by Theorem 6.2 uniform continuity of any $f \in \mathcal{F}$, and therefore as we take $\delta \to 0$ the necessary oscillation in the function values must come from choosing different f.

Example 6.19 (Rudin 7.21)**.** Consider the sequence of continuous functions

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2}, \qquad 0 \leq x \leq 1,\ n \in \mathbb{N}^+. \tag{6.65}$$

Then (f_n) is uniformly bounded, $\|f\|_\infty \leq 1$, and $f_n \to 0$ pointwise for all $x \in [0,1]$, however $f_n(1/n) = 1$ so that $\|f_n\|_\infty = 1\ \forall n$ and therefore no subsequence can converge uniformly on $[0,1]$. In fact $\mathcal{F} = \{f_n\}_{n \in \mathbb{N}^+}$ is not equicontinuous.

Exercise 6.11. *Show that if $\mathcal{F} \subseteq C(X;Y)$ is equicontinuous then $f \in \mathcal{F}$ is uniformly continuous.*

Exercise 6.12. *Show that if X is compact and $\mathcal{F} \subseteq C(X;Y)$ a finite set then $\mathcal{F}$ is equicontinuous.*

6.4. **Arzelà-Ascoli.** Cf. Rudin 7.23-25

The following important theorem clarifies some of the topology of the space $C(X)$ of continuous functions:

Theorem 6.20 (**Arzelà-Ascoli**)**.** *If X is a compact metric space then a subset $\mathcal{F} \subseteq C(X)$ of the real-valued functions on X (with the supremum/max norm) is compact if and only if it is closed, bounded and equicontinuous.*

In fact we can be slightly more general and consider functions in $C(X;Y)$ where Y is any complete metric space with the **Bolzano-Weierstrass (BW) property**: that every bounded sequence has a convergent subsequence. Recall also that an equivalent definition of compactness in metric spaces is that every sequence (or infinite subset) contains a convergent subsequence (or limit point), by Theorem 4.37. Thus for bounded sets the BW property is equivalent to compactness. Also recall that any compact subset is necessarily closed and bounded (Theorem 4.31 resp. Exercise 4.18), and in fact it will be shown to be sufficient for the theorem that $\mathcal{F}$ is pointwise bounded, i.e. $\{f(x) \in Y : f \in \mathcal{F}\}$ is bounded $\forall x \in X$.

The following lemma generalizes the procedure used to prove BW from $\mathbb{R}^n$ (Theorem 4.3) to countable dimensions:

Lemma 6.21. *Let (f_n) be a pointwise bounded sequence of functions $X \to Y$, where X is a countable set and Y a metric space with the BW property. Then (f_n) contains a pointwise convergent subsequence.*

Proof. Denote $X = \{x_1, x_2, \ldots\}$. The sequence $(f_n(x_1))$ is bounded and therefore contains a convergent subsequence $(f_{1,n}(x_1))$. Also $(f_{1,n}(x_2))$ is bounded and contains a convergent subsequence $(f_{2,n}(x_2))$. In this way we obtain $\forall k \in \mathbb{N}^+$ successive subsequences $(f_{k,n})_{n=1}^\infty$ such that $(f_{k,n}(x_j))_{n=1}^\infty$ converges for all $1 \leq j \leq k$ and such that $(f_{k,n})$ is a subsequence of $(f_{k-1,n})$.

Now form the 'diagonal sequence'

$$f_{n'} := f_{n,n}, \quad n = 1, 2, \ldots. \tag{6.66}$$

Then we see that $(f_{n'}(x_j))_{n\in\mathbb{N}^+}$ converges for every $j \in \mathbb{N}^+$. □

Theorem 6.22 (AA: necessity)**.** *If (f_n) is a sequence of continuous functions between metric spaces $X \to Y$, where X is compact, and which is uniformly convergent, then $\{f_n\}$ is equicontinuous. Moreover, if $\mathcal{F} \subseteq C(X;Y)$ is compact then it is equicontinuous.*

Remark 6.23. We may equally well assume that the sequence is uniformly Cauchy instead of convergent. If Y is complete then these assumptions are equivalent.

Proof. Let $\varepsilon > 0$ and choose $N = N(\varepsilon)$ such that

$$\forall x \in X \;\; d(f_n(x), f_m(x)) < \varepsilon \qquad \text{if } n, m \geq N. \tag{6.67}$$

Since X is compact we know by Theorem 6.2 that every f_n is uniformly continuous, i.e. there exists $\delta = \delta(\varepsilon) > 0$ s.t.

$$d(f_n(x), f_n(y)) < \varepsilon \qquad \text{if } 1 \leq n \leq N \text{ and } d(x,y) < \delta. \tag{6.68}$$

Thus, for $n > N$ and $d(x,y) < \delta$ we have

$$d(f_n(x), f_n(y)) \leq d(f_n(x), f_N(x)) + d(f_N(x), f_N(y)) + d(f_N(y), f_n(y)) \leq \varepsilon + \varepsilon + \varepsilon. \tag{6.69}$$

These bounds (6.68)-(6.69) prove the first part of the theorem.

Assume now that $\mathcal{F}$ is compact but not equicontinuous. Then there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there exist points $x_n, y_n \in X$ and a function $f_n \in \mathcal{F}$ such that $d(x_n, y_n) < 1/n$ but $d(f_n(x_n), f_n(y_n)) \geq \varepsilon$. By the limit point compactness of $\mathcal{F}$ this sequence (f_n) has a convergent subsequence $(f_{n'})$, $f_{n'} \to f \in \mathcal{F}$ uniformly on X. But by the first part of the theorem this subsequence is then necessarily equicontinuous, which yields a contradiction. □

Remark 6.24. Recall that a compact space X is separable (Exercise 4.22). Further, a metric space X is called **pre-compact** if $\forall \varepsilon > 0$ there exists a finite subset $A \subseteq X$ such that $X = \cup_{a\in A} B_\varepsilon(a)$. By the same proof one may conclude that any pre-compact space is separable.

Theorem 6.25 (AA: sufficiency)**.** *Let $f_1, f_2, f_3, \ldots$ be a pointwise bounded and equicontinuous sequence of functions between metric spaces $X \to Y$, where X is compact and Y is complete and has the BW property. Then the sequence is uniformly bounded and contains a uniformly convergent subsequence.*

Proof. Take $\varepsilon > 0$ and choose $\delta > 0$ such that $d(f_n(x), f_n(y)) < \varepsilon$ for all $n \in \mathbb{N}^+$ if $d(x,y) < \delta$. Since X is compact there exists a finite set $A \subseteq X$ such that $X = \bigcup_{a\in A} B_\delta(a)$. This means that

$$\forall x \in X \; \exists a \in A \;\text{ s.t. }\; d(x,a) < \delta. \tag{6.70}$$

Since (f_n) is pointwise bounded and A finite, the sequence is uniformly bounded on A, i.e.

$$\exists q \in Y \text{ and } R < \infty \;\text{ s.t. }\; d(f_n(a), q) < R \;\forall n \in \mathbb{N}^+ \;\forall a \in A \tag{6.71}$$

(we may take $R = \max_{a\in A} R_a$ where $d(f_n(a), q) < R_a \;\forall n$). Hence, if $d(x,a) < \delta$ we obtain that

$$d(f_n(x), q) \leq d(f_n(x), f_n(a)) + d(f_n(a), q) < \varepsilon + R \quad \forall n \in \mathbb{N}^+. \tag{6.72}$$

This shows that our sequence is uniformly bounded.

It remains to prove the existence of a subsequence which converges uniformly. Let T be a countable and dense subset of X and take a subsequence $(f_{n'})$ such that for every

$t \in T$, $f_{n'}(t) \to f(t)$, $n \to \infty$, where $f\colon T \to Y$, as shown possible by Lemma 6.21. Since $\bigcup_{t\in T} B_\delta(t) = X$ and X is compact, there exists a finite subset $A \subseteq T$ such that $\bigcup_{a\in A} B_\delta(a) = X$. But then $(f_{n'})$ is uniformly Cauchy on the finite set A, i.e. there exists an $N \in \mathbb{N}^+$ such that

$$d(f_{n'}(a), f_{m'}(a)) < \varepsilon \quad \text{if} \quad n, m \geq N,\ a \in A \tag{6.73}$$

(we may take $N = \max_{a\in A} N_a$ where $n, m \geq N_a \Rightarrow d(f_{n'}(a), f_{m'}(a)) < \varepsilon$).

Now, we have that if $n, m \geq N$ and $d(x, a) < \delta$ then

$$d(f_{n'}(x), f_{m'}(x)) \leq d(f_{n'}(x), f_{n'}(a)) + d(f_{n'}(a), f_{m'}(a)) + d(f_{m'}(a), f_{m'}(x)) \leq \varepsilon + \varepsilon + \varepsilon, \tag{6.74}$$

which proves that $(f_{n'})$ is uniformly Cauchy on X. Since Y is complete it follows by Theorem 6.8 that it converges uniformly on X to a (uniformly) continuous function $X \to Y$. □

Remark 6.26. Even if we relax the condition on completeness of Y the above proof still yields the existence of a subsequence which is uniformly Cauchy. Furthermore, we only need to assume that X is pre-compact.

Remark 6.27. In Rudin it is assumed that $Y = \mathbb{R}$ or $Y = \mathbb{C}$. By BW (Theorem 4.3) the AA theorem holds for $Y = \mathbb{R}^n$ and $Y = \mathbb{C}^n$ as well, or Y any compact space by Corollary 4.39.

6.4.1. *Example application of AA: integral compactness.*

Proposition 6.28. *Let (f_n) be a uniformly bounded sequence of Riemann integrable functions on $[a, b]$ and let*

$$F_n(x) := \int_a^x f_n(t)\, dt, \quad a \leq x \leq b,\ n = 1, 2, \ldots. \tag{6.75}$$

Then there exists a subsequence $(F_{n'})$ that converges uniformly on $[a, b]$.

Proof. We aim to try to apply Arzelà-Ascoli to the set $\mathcal{F} = \{F_n\}$. However to ensure $\mathcal{F} \subseteq C([a, b])$ we must first check that these functions are bounded and $\mathcal{F}$ pointwise bounded. By the uniform bound on (f_n),

$$\exists M < \infty : |f_n(t)| \leq M \quad \forall n \in \mathbb{N}^+, t \in [a, b]. \tag{6.76}$$

Then, by the box estimate for integrals,

$$|F_n(x)| \leq (b - a)M \quad \forall n \in \mathbb{N}^+, x \in [a, b]. \tag{6.77}$$

Therefore actually $\mathcal{F}$ is uniformly bounded, and by Theorem 5.46, $\mathcal{F} \subseteq C([a, b])$.

Is $\mathcal{F}$ equicontinuous? WLOG, $a \leq x \leq y \leq b$, and for any $n \in \mathbb{N}^+$

$$|F_n(x) - F_n(y)| = \left| \int_x^y f_n \right| \leq \int_x^y |f_n| \leq (y - x)M < \varepsilon, \tag{6.78}$$

if $|x - y| < \delta := \varepsilon/M$. Thus $\mathcal{F}$ is equicontinuous and by AA there exists a uniformly convergent subsequence $(F_{n'})$ in $\mathcal{F}$. □

6.5. ***Stone-Weierstrass.** Cf. Rudin 7.26

The following theorem shows that continuous functions are well approximated by polynomials even though they might lack a Taylor expansion (they may be non-differentiable):

Theorem 6.29 (Weierstrass approximation theorem). *If $f\colon [a,b] \to \mathbb{R}$ is continous and $\varepsilon > 0$ then there exists a polynomial p such that $|f(x) - p(x)| < \varepsilon$ for all $x \in [a,b]$.*

Corollary 6.30. *The space $C([a,b];\mathbb{R})$ is separable.*

Stone's generalization...

6.6. **Power series.** cf. Rudin 8.1-5

Many important functions (exp, sin, cos, ...) may be defined by means of a **power series**,

$$f(x) := \sum_{n=0}^{\infty} c_n (x-a)^n \tag{6.79}$$

for some sequence of coefficients $(c_n)_{n=0}^{\infty}$ in $\mathbb{R}$. Each term in the series is a monomial $f_n(x) := c_n(x-a)^n$, $f_n \in C(\mathbb{R})$, and the series will have a radius of convergence $R \in [0, +\infty]$,

$$\frac{1}{R} := \limsup_{n\to\infty} (|c_n|)^{1/n}, \tag{6.80}$$

such that it converges pointwise and absolutely for $|x-a| < R$ and diverges for $|x-a| > R$ (by the root test; see Rudin Thm 3.39; for $|x-a| = R$ it may or may not converge, while $f(a) = c_0$ also if $R = 0$). We note then that, on any closed interval $I_\varepsilon = [a-R+\varepsilon, a+R-\varepsilon]$, $\varepsilon > 0$ small enough (or $R - \varepsilon$ finite in the case $R = +\infty$), we have the uniform bound

$$\|f_n\|_{C(I_\varepsilon)} = \sup_{x\in I_\varepsilon} |f_n(x)| \le |c_n||R-\varepsilon|^n \quad \Rightarrow \quad \sum_{n=0}^{\infty} \|f_n\|_{C(I_\varepsilon)} \le \sum_{n=0}^{\infty} |c_n||R-\varepsilon|^n < \infty. \tag{6.81}$$

Therefore, by the completeness of $C(I_\varepsilon)$ and the majorization theorem for series, it converges uniformly on I_ε to its pointwise limit $f = \sum_{n=0}^{\infty} f_n \in C(I_\varepsilon)$. Furthermore, the term-wise derivative

$$f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \tag{6.82}$$

actually has the same radius of pointwise convergence,

$$\limsup_{n\to\infty} (n|c_n|)^{1/n} = \limsup_{n\to\infty} (|c_n|)^{1/n}, \tag{6.83}$$

so it too converges uniformly on I_ε to the r.h.s. of (6.82), and thus by Theorem 6.14, f is differentiable and the limits of difference quotients agree, justifying (6.82). Since $\varepsilon > 0$ was arbitrary, f is differentiable (and not only continuous) on the open interval $(a-R, a+R)$.

The above conclusions are summarized in Rudin Thm 8.1. By an iteration of the argument we have actually $f' \in C^1((a-R, a+R))$ (continuously differentiable), and so on. Functions with arbitrarily many derivatives are called **smooth**, $f \in C^\infty$, while functions of the form (6.79) which admit a power series expression are called (real) **analytic**, $f \in C^\omega$.

Some typical examples are the **Taylor/Maclaurin series** (recall Section 4.10.1)

$$\begin{aligned} e^x = \exp(x) = &\quad 1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \dots, && R = +\infty, \\ \sin x = \frac{e^{ix} - e^{-ix}}{2i} = &\quad x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \dots, && R = +\infty, \\ \cos x = \frac{e^{ix} + e^{-ix}}{2} = &\quad 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \dots, && R = +\infty, \\ (1-x)^{-1} = &\quad 1 + x + x^2 + x^3 + x^4 + \dots, && R = 1, \\ \ln(1+x) = &\quad x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \dots, && R = 1. \end{aligned}$$

6.7. **Duality.** Given a normed vector space $(X, |\cdot|)$ over $\mathbb{R}$, we may define the **dual** normed vector space by

$$X^* := \operatorname{Hom}(X, \mathbb{R}) = \{f : X \to \mathbb{R} \text{ linear} : \|f\| < \infty\}, \tag{6.84}$$

with its canonical linear structure and its usual, operator, norm

$$\|f\| = \|f\|_{\text{op}} = \sup_{x \in X : |x| = 1} |f(x)|. \tag{6.85}$$

These functions are also known as bounded linear **functionals** on X.

By Corollary 5.21 we have the following useful corollary:

Theorem 6.31. *If X is a normed vector space then its dual space X^* is a Banach space. (Again we get Banach for free!)*

Sometimes the dual space may be identified with some other well-known Banach space. For example, we have the canonical isomorphism (isometry) $(\mathbb{R}^n)^* \cong \mathbb{R}^{1\times n} \cong \mathbb{R}^n$ by means of the transpose

$$\mathbb{R}^{1\times n} \ni \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \leftrightarrow (x_1, x_2, \dots, x_n) \in \mathbb{R}^n. \tag{6.86}$$

Consider sequences $x = (x_n) \in \ell^p$ and $y = (y_n) \in \ell^q$, with $1 \le p, q \le +\infty$ and the relationship

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{6.87}$$

then we have **Hölder's inequality** (exercise; for $p = \infty$ or $q = \infty$ replace by the sup norm)

$$\sum_{n=0}^{\infty} |x_n y_n| \le \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{1/p} \left(\sum_{n=0}^{\infty} |y_n|^q\right)^{1/q}. \tag{6.88}$$

For any such $y \in \ell^q$ then the map $f_y : \ell^p \to \mathbb{R}$,

$$f_y(x) := \sum_{n=0}^{\infty} x_n y_n, \tag{6.89}$$

defines a bounded linear functional on ℓ^p, i.e. $f_y \in (\ell^p)^*$. For $X = \ell^p$, $1 \le p < \infty$, we have in fact that $X^* \cong \ell^q$, via the following theorem. In particular $(\ell^2)^* \cong \ell^2$ and $(\ell_1)^* \cong \ell^\infty$. However, it is only the case that $\ell^1 \subsetneq (\ell^\infty)^*$ (i.e. it turns out that arbitrary bounded functionals on ℓ^∞ are slightly more general than ℓ^1).

Theorem 6.32. *For any $x \in \ell^p$, $1 \leq p < \infty$, and q satisfying* (6.87), *we have*

$$\|x\|_p = \max_{y \in \ell^q : \|y\|_q = 1} \sum_{n=0}^{\infty} x_n y_n. \tag{6.90}$$

Proof. By Hölder, it is clear that $f_y(x) \leq \|x\|_p$ if $\|y\|_q = 1$. By rescaling x it is sufficient to assume that $\|x\|_p = 1$ and prove that $f_y(x) = 1$ for some $y \in \ell^q$ with $\|y\|_q = 1$. Let $y_n = (\operatorname{sign} x_n)|x_n|^{p/q}$, then

$$\|y\|_q = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/q} = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} x_n y_n = \sum_{n=0}^{\infty} |x_n|^p = 1. \tag{6.91}$$

Hence $f_y(x) = 1$, i.e. the maximum $\max_{y \in \ell^q : \|y\|_q = 1} \sum_{n=0}^{\infty} x_n y_n$ is assumed and equals 1. □

Theorem 6.33 (Minkowski's inequality). *For any $x, y \in \ell^p$, $p \in [1, +\infty]$,*

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p. \tag{6.92}$$

Proof. The cases $p = 1$ and $p = +\infty$ are more or less obvious; see e.g. Example 4.68. For finite $p > 1$, by the previous theorem,

$$\|x + y\|_p = \max_{z \in \ell^q : \|z\|_q = 1} \sum_{n=0}^{\infty} (x_n + y_n) z_n \leq \max_{z \in \ell^q : \|z\|_q = 1} \sum_{n=0}^{\infty} x_n z_n + \max_{z \in \ell^q : \|z\|_q = 1} \sum_{n=0}^{\infty} y_n z_n, \tag{6.93}$$

which completes the proof. □

Compare Rudin Exc 6.10-11, Young and Hölder:

Exercise 6.13 (Young's inequality). *Prove that if $p, q > 1$ are such that*

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{6.94}$$

and if $a, b \geq 0$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{6.95}$$

with equality iff $a^p = b^p$.

Exercise 6.14. *Use Young's inequality to prove Hölder's inequality* (6.88).

Exercise 6.15. *Prove the **generalized Hölder's inequality***

$$\left(\sum_{n=0}^{\infty} |x_n y_n|^r \right)^{1/r} \leq \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{1/p} \left(\sum_{n=0}^{\infty} |y_n|^q \right)^{1/q} \tag{6.96}$$

for any $1 \leq p, q, r \leq \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}. \tag{6.97}$$

In the case that any of these are ∞ we replace $(\sum |x_n|^p)^{1/p} = \|x\|_p$ by $\|x\|_\infty$.

Exercise 6.16. *Let $R \subseteq \mathbb{R}^n$ be a finite box and denote by $\mathscr{R}^p$, $1 \le p < \infty$, the real vector space of all functions $f\colon R \to \mathbb{R}$ such that $|f|^p \in \mathscr{R}$ on R (we may also consider $R = \mathbb{R}^n$ by taking limits of bigger boxes). Define for $f \in \mathscr{R}^p$*

$$\|f\|_p := \left(\int_R |f|^p \right)^{1/p}. \tag{6.98}$$

Repeat the above analysis for ℓ^p to prove ***Hölder's inequality for integrals***

$$\int_R |fg| \le \|f\|_p \, \|g\|_q \tag{6.99}$$

for $f \in \mathscr{R}^p$, $g \in \mathscr{R}^q$, $1 < p, q < \infty$ satisfying (6.87), the pairing

$$\|f\|_p = \max_{g \in \mathscr{R}^q : \|g\|_q = 1} \int_R fg, \tag{6.100}$$

and ***Minkowski's inequality for integrals***

$$\|f + g\|_p \le \|f\|_p + \|g\|_p \, , \tag{6.101}$$

for any $f, g \in \mathscr{R}^p$, $1 \le p < \infty$. Hence (6.98) defines a norm on $\mathscr{R}^p$.

7. Fixpoint, inverse and implicit function theorems [L21-24]

7.1. **Reading tip.** This final Section concerns Chapter 9 in Rudin, mainly 9.22-32. In these lecture notes we generalize some of the main theorems from $\mathbb{R}^n$ to Banach spaces. Note that, except some useful remarks on integration in many variables in Rudin 10.1-9, remaining chapters 10-11 in Rudin are extracurricular; cf. reading tips in Section 5.1.

7.1.1. *Exercises.* Rudin Ch. 9: 16-25
Exam 2020-08-19: problems 5,6. Exam 2020-06-15: problems 4,6. Exam 2020-03-16: problem 6. Exam 2019-06-15: problems 5,6. Exam 2019-01-14: problems 6,7. Exam 2015-12-09: problems 5,7. Exam 2015-03-21: problems 5,7. Exam 2014-12-17: problems 7,8. Exam 2014-04-23: problems 7,8. Exam 2013-12-18: problems 6-8.

7.1.2. *Aims.* Concepts discussed in this Section:

- Banach's fixed point theorem and applications
- Inverse and implicit function theorems

Learning outcomes: After this Section you should be able to

- apply the theory to solve mathematical problems including the construction of simple proofs

7.2. **Banach fixpoint theorem.** Cf. Rudin 9.22-23
A map $f\colon X \to Y$ between two metric spaces is called a **contraction** if

$$d_Y(f(x), f(y)) \le d_X(x, y) \qquad \forall x, y \in X. \tag{7.1}$$

and a **strict contraction** if there exists some $c < 1$ such that

$$d_Y(f(x), f(y)) \le c d_X(x, y) \qquad \forall x, y \in X. \tag{7.2}$$

Note that a contraction is continuous (and Lipschitz according to Definition 5.25).
We call $p \in X$ a **fixed point** (or fixpoint) for a map $f\colon X \to X$ if $f(p) = p$.

Theorem 7.1 (**Banach's fixed point theorem**). *If X is a complete metric space and $f\colon X \to X$ is a strict contraction, then f has a unique fixed point.*

Proof. Let us divide the proof into the following parts:
Uniqueness: Assume that $f(p) = p$ and $f(q) = q$. Then $d(p, q) = d(f(p), f(q)) \le cd(p, q)$. Since $c < 1$, necessarily $d(p, q) = 0$, and thus $p = q$.
Existence: Let $x_0 \in X$ be an arbitrary point, and define the sequence

$$x_1 := f(x_0), \quad x_2 := f(x_1), \quad \dots, \quad x_{n+1} := f(x_n), \quad n \ge 0. \tag{7.3}$$

We shall verify that $(x_n)_{n\in\mathbb{N}}$ is Cauchy and therefore converges to some $p \in X$. Then, by continuity of f,

$$\begin{array}{ccc} x_{n+1} & = & f(x_n) \\ \downarrow & & \downarrow \\ p & = & f(p) \end{array} \tag{7.4}$$

as $n \to \infty$, thus proving the existence of the fixpoint p.
Cauchy: Let $n > 0$, then

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \le cd(x_n, x_{n-1}). \tag{7.5}$$

Iterating this argument, we find for all $n \geq 0$ that

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0). \tag{7.6}$$

Now take $m < n$, then by iterating the triangle inequality and estimating by the geometric series,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{n-1}, x_n) \\ &\leq (c^m + c^{m+1} + \ldots + c^{n-1}) d(x_1, x_0) \\ &\leq c^m \frac{1}{1-c} d(x_1, x_0) \to 0, \quad m \to \infty. \end{aligned}$$

In other words, take $\varepsilon > 0$ and choose N such that $\frac{c^N}{1-c} d(x_1, x_0) < \varepsilon$. Then $d(x_m, x_n) < \varepsilon$ if $m, n \geq N$, which was to be proven. □

The following extends this result to functions or parameter families of strict contractions:

Theorem 7.2 (Stability theorem)**.** *Let $f\colon X \times Y \to X$ where X is a complete metric space and Y a metric space. Assume also that there exists $c < 1$ such that*

$$d(f(x, y), f(x', y)) \leq c d(x, x') \qquad \forall x, x' \in X,\ y \in Y. \tag{7.7}$$

and moreover that $\forall x \in X$ the function $Y \ni y \mapsto f(x, y) \in X$ is continuous. Then there exists a continuous function $Y \ni y \mapsto p(y) \in X$ such that

$$f(p(y), y) = p(y). \tag{7.8}$$

Moreover, the function p is unique.

Proof. From Banach's fixed point theorem it follows that for any $y \in Y$ there exists a unique $p(y) \in X$ such that $f(p(y), y) = p(y)$. It thus remains to prove continuity of $p\colon Y \to X$. But if $y, y' \in Y$ then we have

$$\begin{aligned} d(p(y), p(y')) &= d(f(p(y), y), f(p(y'), y')) \\ &\leq d(f(p(y), y), f(p(y), y')) + d(f(p(y), y'), f(p(y'), y')) \\ &\leq d(f(p(y), y), f(p(y), y')) + c d(p(y), p(y')), \end{aligned}$$

and thus we obtain

$$(1 - c) d(p(y), p(y')) \leq d(f(p(y), y), f(p(y), y')). \tag{7.9}$$

The r.h.s. tends to zero when $y' \to y$, proving continuity of p. □

7.3. **Application: Ordinary differential equations.** Let $F\colon X \times [-a, a] \to X$ be a continuous map, where $a > 0$ and $(X, |\cdot|)$ is a Banach space. Also assume that F is Lipschitz in the first variable and uniformly in both variables, i.e. there exists $A \in \mathbb{R}$ s.t.

$$|F(x, t) - F(y, t)| \leq A\,|x - y| \qquad \forall t \in [-a, a]\ \forall x, y \in X \tag{7.10}$$

Then we have the following:

Theorem 7.3 (**Existence and uniqueness for regular ODE**)**.** *If $a < 1/A$ then there exists for every $c \in X$ a unique differentiable function $u\colon [-a, a] \to X$ that solves the ordinary differential equation (ODE)*

$$\begin{cases} u'(t) = F(u(t), t), & |t| \leq a, \\ u(0) = c. \end{cases} \tag{7.11}$$

Proof. As conventional, we let $C([-a,a];X)$ denote the Banach space of continuous maps $[-a,a]\to X$ with the norm $\|u\| = \max_{|t|\le a}|u(t)|$.

Existence: Using the fundamental theorem of integral calculus we may write the ODE (7.11) on an integral form. Namely, if a solution u to (7.11) exists it is continuous, and thus the map $[-a,a]\ni s\mapsto F(u(s),s)$ is continuous as well and therefore Riemann integrable. Integrating both sides we obtain by Theorem 5.46

$$u(t)-u(0) = \int_0^t u'(s)\,ds = \int_0^t F(u(s),s)\,ds. \tag{7.12}$$

Thus, we proceed to try to solve the integral equation (7.12) by defining the map

$$\begin{array}{ccc} C([-a,a];X) & \xrightarrow{\varphi} & C([-a,a];X) \\ u & \mapsto & \varphi(u) \end{array} \tag{7.13}$$

by

$$\varphi(u)(t) := c + \int_0^t F(u(s),s)\,ds. \tag{7.14}$$

Then φ is a strict contraction: for any $t\in[-a,a]$

$$\begin{aligned} |\varphi(u)(t)-\varphi(v)(t)| &= \left|\int_0^t \left(F(u(s),s)-F(v(s),s)\right)\,ds\right| \\ &\le \left|\int_0^t A\,|u(s)-v(s)|\,ds\right| \le aA\,\|u-v\|\,, \end{aligned}$$

implying

$$\|\varphi(u)-\varphi(v)\| \le aA\,\|u-v\|\,. \tag{7.15}$$

Thus with our assumption $aA<1$ this is a strict contraction, and by Banach's fixpoint theorem it has a unique fixpoint, $u=\varphi(u)$. In other words, $u\colon[-a,a]\to X$ is continuous and

$$u(t) = c + \int_0^t F(u(s),s)\,ds, \qquad u(0)=c. \tag{7.16}$$

Again, by Theorem 5.46, u is then differentiable and satisfies

$$u'(t) = F(u(t),t), \qquad u(0)=c. \tag{7.17}$$

Uniqueness: If u_1 and u_2 are two solutions and $g := u_1-u_2$,

$$g'(t) = u_1'(t)-u_2'(t) = F(u_1(t),t)-F(u_2(t),t), \tag{7.18}$$

then by the mean value inequality (Theorem 5.27) and (7.10), for $0<x\le\delta\le a$

$$|g(x)| = |g(x)-g(0)| \le |x|\sup_{t\in[0,x]}|g'(t)| \le |x|A\sup_{t\in[0,x]}|g(t)| \tag{7.19}$$

that is $\|g\|_{C([0,\delta])}\le \delta A\,\|g\|_{C([0,\delta])}$, so $g=0$ on $[0,\delta]$ if $\delta<1/A$, and similarly for $[-\delta,0]$. By iterating the argument further and further away from $x=0$ we conclude that $g=0$. □

Example 7.4. If we relax the Lipschitz condition on F then there is not necessarily uniqueness. Namely, consider the ODE

$$\begin{aligned} u'(t) &= \sqrt{|u(t)|}, \quad t\in[-a,a], \\ u(0) &= 0. \end{aligned} \tag{7.20}$$

Here $F(u,t) = |u|^{1/2}$ is certainly continuous, but not Lipschitz. The ODE admits both the differentiable solution $u(t) = (\operatorname{sign} t)t^2/4$, $u'(t) = (\operatorname{sign} t)t/2 = |t|/2 = \sqrt{|u|}$, and the trivial solution $u = 0$.

If we relax the conditions on F a bit but instead strengthen those on X we may anyway obtain the following existence result:

Theorem 7.5 (Cauchy-Peano's existence theorem for ODE). *Assume that* $F\colon \mathbb{R}^d \times [0,1] \to \mathbb{R}^d$ *is bounded and continuous. There exists for every* $\mathbf{c} \in \mathbb{R}^d$ *a differentiable function* $u\colon [0,1] \to \mathbb{R}^d$ *that solves the ordinary differential equation*

$$\begin{aligned} u'(t) &= F(u(t),t), \quad t \in [0,1], \\ u(0) &= \mathbf{c}. \end{aligned} \tag{7.21}$$

Proof. We cannot use the same approach as above since potentially $A = \infty$. Instead the idea is to split the interval in smaller parts and construct approximate solutions of the corresponding integral equation (7.16) with locally constant derivatives, that is, locally *affine* (constant plus linear) functions.

Fix $n \in \mathbb{N}^+$. We will construct $f_n\colon [0,1] \to \mathbb{R}^d$ that is continuous and piecewise affine with $f_n(0) := \mathbf{c}$. Partition the interval $[0,1]$ into

$$0 = \frac{0}{n},\ \frac{1}{n},\ \frac{2}{n},\ \ldots,\ \frac{k}{n},\ \ldots,\ \frac{n}{n} = 1. \tag{7.22}$$

For $0 < t \le 1/n$ we let

$$f_n(t) := f_n(0) + (t - 0/n)F(f_n(0/n), 0/n). \tag{7.23}$$

We then have for $0 < t < 1/n$

$$f_n'(t) = F(f_n(0/n), 0/n). \tag{7.24}$$

Now assume that f_n has been defined for $0 \le t \le \frac{k}{n}$. If $\frac{k}{n} < 1$ we define for $\frac{k}{n} < t \le \frac{k+1}{n}$

$$f_n(t) := f_n(k/n) + (t - k/n)F(f_n(k/n), k/n). \tag{7.25}$$

We then have also for these t (except at the endpoint) that

$$f_n'(t) = F(f_n(k/n), k/n). \tag{7.26}$$

Let now $M \in \mathbb{R}$ be such that

$$|F(\mathbf{x},t)| \le M \quad \forall \mathbf{x} \in \mathbb{R}^d\ \forall t \in [0,1]. \tag{7.27}$$

Then we have for all $t \in [0,1]$, $t \ne k/n$, $k \in \{0,1,\ldots,n\}$ that $|f_n'(t)| \le M$. Since f_n is continuous at every point we find that $(f_n)_{n=1}^\infty$ is a pointwise bounded (actually uniformly bounded) and equicontinuous sequence in $C([0,1];\mathbb{R}^d)$. For example, if only $k/n \in [t,s]$,

$$|f_n(s) - f_n(t)| \le |f_n(s) - f_n(k/n)| + |f_n(k/n) - f_n(t)| \le M|s - k/n| + M|k/n - t| = M|s - t|, \tag{7.28}$$

etc., and in general $|f_n(s) - f_n(t)| \le M|s - t|$.

By Arzelà-Ascoli's theorem it follows then that there exists a subsequence $(f_{n'})$ which converges uniformly to some continuous $f\colon [0,1] \to \mathbb{R}^d$, which also satisfies $f(0) = \mathbf{c}$. Let us for convenience re-denote n' by n. Since every f_n is continuous and differentiable except in a finite set of points we have that $f_n \in \mathscr{R}$ on $[0,1]$, and

$$f_n(t) = c + \int_0^t f_n'(s)\,ds, \qquad t \in [0,1]. \tag{7.29}$$

Let $k_{t,n}$ be the largest integer s.t. $\frac{k_{t,n}}{n} \leq t$. Then we furthermore have

$$\int_0^t f_n' = \int_0^{1/n} + \int_{1/n}^{2/n} + \ldots + \int_{k_{t,n-1}/n}^{k_{t,n}/n} + \ldots + \int_{k_{t,n}/n}^{t} f_n' = \sum_{k=0}^{k_{t,n}} \frac{1}{n} F(f_n(k/n), k/n) + \int_{k_{t,n}/n}^{t} f_n'. \tag{7.30}$$

The last term is in absolute value smaller than M/n and thus tends to zero as $n \to \infty$. The sum however we would like to compare to

$$\sum_{k=0}^{k_{t,n}} \frac{1}{n} F(f(k/n), k/n) \to \int_0^t F(f(s), s)\,ds, \qquad \text{as } n \to \infty, \tag{7.31}$$

by definition of the Riemann integral.

We estimate the difference between (7.30) and (7.31) using that $|f_n(t)| \leq M$, $|f(t)| \leq M$, and F is uniformly continuous on the compact set $[-M,M]^d \times [0,1]$. Namely, for any $\varepsilon > 0$ $\exists \delta > 0$ s.t.

$$\forall t \in [0,1], \mathbf{x}, \mathbf{y} \in \mathbb{R}^d,\ |\mathbf{x}|, |\mathbf{y}| \leq M \ \&\ |\mathbf{x} - \mathbf{y}| < \delta \Rightarrow |F(\mathbf{x}, t) - F(\mathbf{y}, t)| < \varepsilon, \tag{7.32}$$

and by choosing N large enough,

$$|f_n(t) - f(t)| < \delta \quad \forall t \in [0,1]. \tag{7.33}$$

We therefore obtain if $n \geq N$

$$\left| \sum_{k=0}^{k_{t,n}} \frac{1}{n} \Big[F(f_n(k/n), k/n) - F(f(k/n), k/n) \Big] \right| \leq \sum_{k=0}^{k_{t,n}} \frac{1}{n} \varepsilon \leq \varepsilon. \tag{7.34}$$

In conclusion, we obtain

$$f(t) = c + \int_0^t F(f(s), s)\,ds, \tag{7.35}$$

and $u(t) = f(t)$ solves our ODE. □

7.4. **Inverse function theorem.** Let X and Y be Banach spaces. The **inverse function theorem** says that, if $f \colon X \to Y$ is continuously differentiable close to $a \in X$, $f(a) = b$, then:

1. If $f'(a)$ has a linear bounded left inverse $A \colon X \to Y$ ($A \circ f'(a) = \mathrm{id}_X$) then f has locally near b a left inverse $g \colon \Omega \to f^{-1}(\Omega)$ ($g \circ f = \mathrm{id}_{f^{-1}(\Omega)}$) that is continuously differentiable.
2. If $f'(a)$ has a linear bounded right inverse $B \colon Y \to X$ ($f'(a) \circ B = \mathrm{id}_Y$) then f has locally near b a right inverse $g \colon \Omega \to f^{-1}(\Omega)$ ($f \circ g = \mathrm{id}_\Omega$) that is continuously differentiable.
3. If both 1. and 2. then f has locally near b a continuously differentiable inverse g.

We first prove the following simplified lemma:

Lemma 7.6. *Let $(X, |\cdot|)$ be a Banach space and $f\colon X \to X$ a continuously differentiable mapping in some neighborhood of 0, such that $f(0) = 0$ and $f'(0) = \mathbb{1}$. Then there exists an open neighborhood W of 0 and a map $g\colon W \to X$ such that $f \circ g = \mathbb{1}_W$ and $g'(0) = \mathbb{1}$.*

Proof. By the continuity of f' at 0, there exists some $r > 0$ such that $\|f'(x) - \mathbb{1}\|_{\mathrm{op}} < 1/2$ if $|x| \leq r$. Let $W = B_{r/2}(0)$, and fix any $y \in \bar{B}_{r/2}(0) \supseteq W$. Define a map $\varphi\colon K \to K$, where $K = \bar{B}_{2|y|}(0)$, by

$$\varphi(x) := x + y - f(x). \tag{7.36}$$

We then see that $\varphi(x) = x$ iff $y = f(x)$, so it is sufficient to show that φ has a unique fixpoint, which will follow from the Banach fixpoint Theorem 7.1 if we can show that indeed $\operatorname{im}\varphi \subseteq K$ and that φ is a strict contraction. Indeed, we have $K \subseteq \bar{B}_r(0)$ and thus

$$\left\|\varphi'(x)\right\|_{\mathrm{op}} = \left\|\mathbb{1} - f'(x)\right\|_{\mathrm{op}} < 1/2 \qquad \forall x \in K. \tag{7.37}$$

First, note that if $x \in K$ then

$$|\varphi(x)| = |\varphi(x) - \varphi(0) + \varphi(0)| \leq \frac{1}{2}|x| + |y| \leq 2|y| \tag{7.38}$$

by the triangle and mean value inequalities. Hence, $\operatorname{Im}\varphi \subseteq K$.

Next, if $x_1, x_2 \in K$ we have similarly that

$$|\varphi(x_1) - \varphi(x_2)| \leq \frac{1}{2}|x_1 - x_2|, \tag{7.39}$$

by (7.37), showing that φ is a strict contraction from the complete metric space K into itself. Hence there exists a unique $x \in K$, denoted $g(y)$, such that $f(g(y)) = y$.

Having this construction for every $y \in W$, we conclude that there exists $g\colon W \to X$, where $0 \in W$ open, and $f \circ g = \mathbb{1}$ on W. To show that $g'(0)$ exists we note that, by differentiability of f at 0,

$$f(x) = f(0) + f'(0)x + o(x) = x + o(x), \tag{7.40}$$

so

$$y = f(g(y)) = g(y) + o(x) \quad \Rightarrow \quad g(y) = y - o(x), \tag{7.41}$$

but also $|g(y)| \leq 2|y|$, so necessarily $o(x) = o(y)$, and thus

$$g(y) = y + o(y) \quad \Rightarrow \quad g'(0) = \mathbb{1}. \tag{7.42}$$

□

Next we do a little bit better:

Theorem 7.7. *Let $f\colon X \to X$ be continuously differentiable in some neighborhood of $a \in X$ such that $f'(x)$ is invertible in that neighborhood. Then there is a neighborhood V of $f(a)$ and a continuously differentiable function $g\colon V \to X$ such that $f \circ g = \mathbb{1}_V$. Moreover, $g \circ f = \mathbb{1}_U$ in some neighborhood U of a.*

Proof. Let $A := f'(a)^{-1} \in \operatorname{Hom}(X, X)$, and put $Tx := a + x$ and $Sx := x - Af(a)$, as well as $F := S \circ A \circ f \circ T$, i.e.

$$\begin{array}{cccccccccc} F\colon & X & \xrightarrow{T} & X & \xrightarrow{f} & X & \xrightarrow{A} & X & \xrightarrow{S} & X \\ & x & \mapsto & x + a & \mapsto & f(x+a) & \mapsto & Af(x+a) & \mapsto & Af(x+a) - Af(a). \end{array} \tag{7.43}$$

Now, T, A and S are all (affine) invertible mappings. By the chain rule, $F'(x)$ exists and depends continuously on x in a neighborhood of 0, and moreover it is invertible. We also have

$$F(0) = 0 \qquad \text{and} \qquad F'(0) = \mathbb{1}. \tag{7.44}$$

Thus, by the previous lemma, there is an open neighborhood W of 0 and a function $G\colon W \to X$ such that $F \circ G = \mathbb{1}_W$ and $G'(0) = \mathbb{1}$.

Now let $V := A^{-1}S^{-1}(W)$ be an open set containing $b := f(a)$, and define on V the function $g := T \circ G \circ S \circ A$. This proves the existence of our map g since

$$f \circ g = A^{-1} \circ S^{-1} \circ S \circ A \circ f \circ T \circ G \circ S \circ A = \mathbb{1}_V, \tag{7.45}$$

and $g'(b) = f'(a)^{-1}$, by the chain rule applied to the identity (7.45).

Observe that a could have been chosen as any point in the neighborhood in which f is continuously differentiable with invertible derivative. Moreover, the maps $a \mapsto f'(a)$ and $f'(a) \mapsto (f'(a))^{-1}$ are both continuous there; recall Theorem 5.16 and Example 5.23. Hence, since $b = f(a)$ is taken freely in V we have that g' is continuous on V. This proves the first statement of the theorem.

We may now apply the theorem again with f and g interchanged (i.e. starting from g on V and using the uniqueness of inverses (2.6)), to conclude the second statement. □

Remark 7.8. In the above theorem we say that g is a **local inverse** of f. Similarly we may talk about local left/right inverses.

We can now prove the most general version:

Theorem 7.9 (**Inverse function theorem**). *Let $f \in C^1(\Omega, Y)$, where Ω is an open subset of a Banach space X, and Y is also a Banach space. Then*

a) *If $f'(x)$ has a continuous left inverse for some $x \in \Omega$ then f has a local left inverse which is C^1.*
b) *If $f'(x)$ has a continuous right inverse for some $x \in \Omega$ then f has a local right inverse which is C^1.*
c) *If $f'(x)$ has a continuous inverse for some $x \in \Omega$ then f has a local inverse which is C^1.*

Proof.

a) Assume for $a \in \Omega$ that $A = f'(a)\colon X \to Y$ is continuous linear and that $\exists B\colon Y \to X$ continuous linear s.t. $BA = \mathbb{1}_X$. Form $g(x) := B \circ f(x)$, then $g\colon \Omega \to X$ with $g'(x) = B \circ g'(x)$ continuous and $g'(a) = B \circ A = \mathbb{1}_X$. We may then apply Theorem 7.7 to g and obtain a continuously differentiable local inverse $h\colon U \to V$. Then define $k := h \circ B$, and obtain

$$k \circ f = h \circ B \circ f = h \circ g = \mathbb{1}_V, \tag{7.46}$$

locally on $V \subseteq \Omega$.
b) Let for $a \in \Omega$, $A \in \mathrm{Hom}(Y, X)$ be the right inverse of $f'(a)$, i.e. $f'(a) \circ A = \mathbb{1}_Y$. Now apply the previous theorem on $g = f \circ A$ (exercise).
c) Combine a) and b).

□

Remark 7.10. In finite dimensions we may have for example $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, with

a) $n \le m$ and the matrix $f'(x) \in \mathbb{R}^{m\times n}$ having full (column) rank n for $x \in \Omega$,
b) $n \ge m$ and the matrix $f'(x) \in \mathbb{R}^{m\times n}$ having full (row) rank m for $x \in \Omega$,
c) $n = m$ and $\det f'(x) \neq 0$ for $x \in \Omega$.

Note also that by continuity of the determinant, if $\det f'(a)$ is nonzero at a point $a \in \Omega$ then this also holds in some neighborhood of a, in which the inverse function theorem then may be applied.

Exercise 7.1. *Show that the map*

$$F\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad F(x,y) = (xe^y, \sin(x+y))$$

has locally an inverse F^{-1} in a neighborhood of $(x,y) = (\pi, 0)$ and compute its derivative at that point.

Exercise 7.2. *Show that the function*

$$F(x,y) = \begin{cases} \left(8x + x^3\cos(x^2+y^2)^{-1}, 8y + y^3\sin(x^2+y^2)^{-1}\right), & (x,y) \neq \mathbf{0}, \\ (0,0), & (x,y) = \mathbf{0} \end{cases} \tag{7.47}$$

is differentiable on $\mathbb{R}^2$ but not continuously differentiable at $\mathbf{0}$. Further, show that the principal minors of the Jacobian matrix $DF = dF/d(x,y)$ do not vanish in $B_1(0)$.

7.5. Implicit function theorem.

7.5.1. *Model problem in one variable.* Compare **explicit** and **implicit** approach to defining functions. The first is the assignment of function values, say in the one-variable case

$$f\colon \mathbb{R} \supseteq D_f \to \mathbb{R}, \qquad x \mapsto f(x) = y, \tag{7.48}$$

where we also may think of f as defined by the graph[(6)] $(x,y) \in \operatorname{graph}(f) \subseteq \mathbb{R}\times\mathbb{R}$. The other approach would be to specify the graph (at least locally) as the solution to an equation:

$$F\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \qquad F(x,y) = 0, \tag{7.49}$$

i.e. a **constraint** or relationship between x and y.

Example 7.11. Consider the set H in $\mathbb{R}^2$ defined by the relationship $xy = 1$. Here $F(x,y) := xy - 1 = 0$ exactly on H. We may indeed explicitly solve y as a function of x: $y(x) = 1/x$, $x \neq 0$, or x as a function of y: $x(y) = 1/y$, $y \neq 0$.

Example 7.12. Consider the set C in $\mathbb{R}^2$ defined by the relationship $x^2 + y^2 = 1$, i.e. the unit circle. Here $F(x,y) := x^2 + y^2 - 1 = 0$ exactly on C, but it is not a graph globally. We may only locally solve y as a function of x, with a choice of sign:

$$\begin{cases} y_1(x) := +\sqrt{1-x^2}, & -1 < x < 1, \\ y_2(x) := -\sqrt{1-x^2}, & -1 < x < 1, \end{cases} \tag{7.50}$$

however there are problems at $(x,y) = (\pm 1, 0)$ where y cannot be a function of x.

Alternatively we solve x as a function of y:

$$\begin{cases} x_1(y) := +\sqrt{1-y^2}, & -1 < y < 1, \\ x_2(y) := -\sqrt{1-y^2}, & -1 < y < 1, \end{cases} \tag{7.51}$$

with problems at $(x,y) = (0, \pm 1)$ where x cannot be a function of y.

---

(6) Recall that a graph relation xRy associates a unique y to each possible x.

There is a geometric correspondence between the tangent to the graph of the function $y = f(x)$, which lies on the level curve $F = 0$, and the normal to the curve, which indeed is given by the gradient $\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right) = [F']^T$, the direction of steepest increase of F. The places where there are potentially problems are where the tangent is vertical, or the normal horizontal, i.e. where $\frac{\partial F}{\partial y} = 0$.

Note that in the first example above we have

$$\nabla F(x, y) = \left(\frac{\partial F}{\partial x}(x, y), \frac{\partial F}{\partial y}(x, y)\right) = (y, x), \tag{7.52}$$

so that $\frac{\partial F}{\partial y} = x \neq 0$ on H, while in the second example we have

$$\nabla F(x, y) = (2x, 2y) = 2(x, y), \tag{7.53}$$

so that $\frac{\partial F}{\partial y} = 0$ exactly at $(x, y) = (\pm 1, 0)$ on C.

Next, consider the following model problem that we would like to be able to solve:

Example 7.13. *Problem:* Show that the equation

$$e^{xy} + x^2 y - x = 0 \tag{7.54}$$

implicitly defines y as a function of x near the point $(x, y) = (1, 0)$, and give the linear approximation to $y = y(x)$ at $x = 1$.

Approach: Let us define the constraint function $F \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$F(x, y) := e^{xy} + x^2 y - x. \tag{7.55}$$

First note that indeed $F(1, 0) = 0$ so we may study a possible function graph defined by the constraint $F(x, y) = 0$ locally around this point. Also note that F is smooth and that

$$\nabla F(x, y) = (ye^{xy} + 2xy - 1, xe^{xy} + x^2), \tag{7.56}$$

so $\nabla F(1, 0) = (-1, 2)$, which tells us that the normal to the curve $F = 0$ at that point is along the vector $(-1, 2)$ while the tangent to $F = 0$ is along $(1, 1/2)$. Furthermore, since the gradient varies smoothly, we may expect to be able to follow the curve locally along this tangent, and this *indicates* that we may have $y'(1) = 1/2$,

Indeed, *assuming* that $y = y(x)$ is a continuously differentiable function satisfying $F(x, y(x)) = 0$ around $x = 1$, we may differentiate this condition w.r.t. x:

$$0 = \frac{d}{dx}\left(F(x, y(x))\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} y'(x), \tag{7.57}$$

and since $\partial_y F \neq 0$ close to $x = 1$ (by continuity and $\partial_y F(1, 0) = 2 \neq 0$), we have

$$y'(x) = -\left(\frac{\partial F}{\partial y}(x, y(x))\right)^{-1} \frac{\partial F}{\partial x}(x, y(x)), \tag{7.58}$$

that is, indeed $y'(1) = 1/2$. The following theorem validates this procedure.

Theorem 7.14 (Implicit function theorem in one variable). *Let $\Omega \subseteq \mathbb{R}^2$ be a nonempty open subset, let $F : \Omega \to \mathbb{R}$ be differentiable, and assume that $\partial F/\partial y$ is nowhere vanishing on Ω and that $(a, b) \in \Omega$ is a point such that $F(a, b) = 0$. Then there exists an open rectangle $X \times Y \subseteq \Omega$ containing (a, b), and a unique function $g \colon X \to Y$ that satisfies*

- *$g(a) = b$,*
- *$F(x, g(x)) = 0$ for all $x \in X$,*

- *g is differentiable on X and satisfies*

$$g'(x) = -\frac{\frac{\partial F}{\partial x}(x, g(x))}{\frac{\partial F}{\partial y}(x, g(x))} \qquad \forall x \in X. \tag{7.59}$$

Moreover, if $\nabla F(x, y) = (\partial_x F, \partial_y F)$ is continuous at (a, b) then g' is continuous at $x = a$.

Remark 7.15. If $F \in C^1(\Omega)$ and $\partial_y F \neq 0$ at a point (a, b) then there is also some neighborhood $\Omega' \subseteq \Omega$ of (a, b) on which the above theorem applies.

Proof. The main ideas are to use Darboux' theorem and the mean value theorem to ensure a strictly increasing (or decreasing) function in the y-direction, first at $x = 0$ and then extend by continuity, and the intermediate value theorem to find a unique zero of $y \mapsto F(x, y)$ for fixed x close to 0. We break down the proof as follows:

1. WLOG we may make a transformation on the form $F(x + a, y/c + b)$, $c = \frac{\partial F}{\partial y}(a, b)$, to normalize the problem to $(a, b) = (0, 0)$ and $\frac{\partial F}{\partial y}(0, 0) = 1$.
2. (Existence and uniqueness.)
   - Consider, for $r > 0$ small enough to fit in Ω, the function $f_0(y) := F(0, y)$, $y \in [-r, r]$. Then $f_0'(y) = \frac{\partial F}{\partial y}(0, y) \neq 0$ and $f_0'(0) = 1$. By Darboux' theorem (the derivative takes intermediate values), $f_0'(y) > 0$ $\forall y \in (-r, r)$, i.e. f_0 is strictly increasing and hence $f_0(-r) < 0 < f_0(r)$.
   - By continuity of F, there exists an open interval X around 0 s.t. $X \times [-r, r] \subseteq \Omega$, and $F|_{X\times\{-r\}} < 0$ and $F|_{X\times\{r\}} > 0$.
   - Consider for fixed $x \in X$ the function $f_x(y) := F(x, y)$, $y \in [-r, r]$, then also $f_x(-r) < 0 < f_x(r)$.
   - By the mean value theorem, $\exists c \in Y := (-r, r)$ s.t. $f_x'(c) = \frac{\partial F}{\partial y}(x, c) > 0$.
   - By Darboux again, $f_x'(y) > 0$ for all $y \in Y$, i.e. f_x is strictly increasing.
   - By the intermediate value theorem $\exists! y \in Y$ s.t. $f_x(y) = 0$. Thus, for this $x \in X$ we have found our $g(x) := y$.
3. (Continuity.) Let $0 < \varepsilon < r$. Repeating the construction above, by uniqueness we find $X' \subseteq X$ s.t. $g(x) \in (-\varepsilon, \varepsilon)$ for all $x \in X'$.
4. (Differentiability.) Using differentiability of F we have for small enough (h, k)

$$F(h, k) = vh + k + \varepsilon(h, k), \qquad v = \frac{\partial F}{\partial x}(0, 0), \tag{7.60}$$

   so that with $k = g(h)$

$$0 = F(h, g(h)) = vh + g(h) + \varepsilon(h, g(h)), \tag{7.61}$$

$$\Rightarrow \quad g(h) = -vh - \varepsilon(h, g(h)) = -vh + o(h). \tag{7.62}$$

   Therefore g is differentiable at 0 with $g'(0) = -v$.
5. Finally, given $a' \in X$ put $b' = g(a')$. Then $g \colon X \to Y$ solves the h for which $F(x, h(x)) = 0$ $\forall x \in X$ and $h(a') = b'$. Thus by the initial WLOG, g is also differentiable at a' which was arbitrarily chosen.

□

7.5.2. *Generalization to* $\mathbb{R}^n$. In the multi-variable case we consider solving a system of constraints

$$\begin{cases} F_1(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0,\\ F_2(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0,\\ &\vdots\\ F_m(x_1,\ldots,x_n,y_1,\ldots,y_m) &= 0. \end{cases} \tag{7.63}$$

Note that we expect to need as many constraints as dependent variables y_k we want to solve, while the number $n \geq 1$ of independent variables x_j is arbitrary. More compactly we write for the constraint function $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$,

$$F(\mathbf{x},\mathbf{y}) = \begin{bmatrix} F_1(\mathbf{x},\mathbf{y})\\ \vdots\\ F_m(\mathbf{x},\mathbf{y}) \end{bmatrix} \tag{7.64}$$

and look for $\mathbf{y} = \mathbf{y}(\mathbf{x})$ such that $F(\mathbf{x},\mathbf{y}(\mathbf{x})) = \mathbf{0}$, at least locally around some point $(\mathbf{x},\mathbf{y})$.

Example 7.16. In the linear case

$$\begin{cases} A_{11}x_1 + \ldots + A_{1n}x_n + B_{11}y_1 + \ldots + B_{1m}y_m &= 0,\\ A_{21}x_1 + \ldots + A_{2n}x_n + B_{21}y_1 + \ldots + B_{2m}y_m &= 0,\\ &\vdots\\ A_{m1}x_1 + \ldots + A_{mn}x_n + B_{m1}y_1 + \ldots + B_{mm}y_m &= 0, \end{cases} \tag{7.65}$$

i.e. $F(\mathbf{x},\mathbf{y}) = A\mathbf{x} + B\mathbf{y}$, with matrices $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times m}$. The equation

$$A\mathbf{x} + B\mathbf{y} = \mathbf{0} \tag{7.66}$$

may, if B is invertible, be solved uniquely

$$\mathbf{y} = -B^{-1}A\mathbf{x}. \tag{7.67}$$

Recall that $B^{-1} = (\det B)^{-1}B^{\mathrm{adj}}$ where the cofactor matrix B^{adj} consists of all the minors (sub-determinants) of B. The principal minors are the elements on the diagonal of B^{adj}.

In the general case, again assuming $F \in C^1$ locally and that a suitable differentiable solution $\mathbf{y}(\mathbf{x})$ exists, we may differentiate the constraint $F = \mathbf{0}$ to obtain the condition

$$\mathbf{0} = \frac{d}{d\mathbf{x}}\Big(F(\mathbf{x},\mathbf{y}(\mathbf{x}))\Big) = \frac{\partial F}{\partial \mathbf{x}} + \frac{\partial F}{\partial \mathbf{y}}\frac{d\mathbf{y}}{d\mathbf{x}}, \tag{7.68}$$

so that if the matrix $\frac{\partial F}{\partial \mathbf{y}}(\mathbf{x},\mathbf{y})$ is invertible at $\mathbf{y} = \mathbf{y}(\mathbf{x})$, then

$$\frac{d\mathbf{y}}{d\mathbf{x}} = -\left(\frac{\partial F}{\partial \mathbf{y}}(\mathbf{x},\mathbf{y})\right)^{-1}\frac{\partial F}{\partial \mathbf{x}}(\mathbf{x},\mathbf{y}). \tag{7.69}$$

In fact we may prove the following:

Theorem 7.17 (Implicit function theorem in $\mathbb{R}^n$). *Let $F\colon \Omega \to \mathbb{R}^m$ be continuously differentiable on a nonempty open $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^m$ containing the point $(\mathbf{a},\mathbf{b})$. Suppose that $F(\mathbf{a},\mathbf{b}) = \mathbf{0}$ and that $\det[\partial_{\mathbf{y}}F(\mathbf{a},\mathbf{b})] \neq 0$. Then:*

- *There exists an open set $X \times Y \subseteq \Omega$ containing $(\mathbf{a},\mathbf{b})$, and a differentiable function $g\colon X \to Y$ that satisfies*

$$F(\mathbf{x},g(\mathbf{x})) = \mathbf{0} \quad \forall \mathbf{x} \in X \qquad \textit{and} \qquad g(\mathbf{a}) = \mathbf{b}. \tag{7.70}$$

- *We have*

$$g'(\mathbf{x}) = -\left[\frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}, g(\mathbf{x}))\right]^{-1}_{m\times m}\left[\frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}, g(\mathbf{x}))\right]_{m\times n} \quad \forall \mathbf{x} \in X. \tag{7.71}$$

- *If $h\colon X \to Y$ satisfies $F(\mathbf{x}, h(\mathbf{x})) = 0$ for all $\mathbf{x} \in X$ then $h = g$.*

Remark 7.18. Note that by continuity of the determinant, if $F \in C^1$ and $\partial_{\mathbf{y}} F(\mathbf{x}, \mathbf{y})$ invertible at some point $(\mathbf{x}, \mathbf{y}) = (\mathbf{a}, \mathbf{b})$ then it is also invertible in a neighborhood of that point.

Again one may relax some assumptions on continuous differentiability if one can ensure the existence of a well behaved local inverse by other means. For details on this more general approach we refer to e.g. [dO12, dO17]. The simpler case that $F \in C^1$ can also be obtained as a consequence of the Inverse Function Theorem as follows, cf. also Rudin 9.28.

7.5.3. *Equivalence between inverse and implicit function theorems.* Note the following useful correspondence, showing that the Inverse Function Theorem implies the Implicit Function Theorem, and vice versa:

Given $F\colon \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, $F(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, we may form

$$\begin{array}{rcl} G\colon \quad \mathbb{R}^n \times \mathbb{R}^m & \to & \mathbb{R}^n \times \mathbb{R}^m \\ (\mathbf{x}, \mathbf{y}) & \mapsto & G(\mathbf{x}, \mathbf{y}) := (\mathbf{x}, F(\mathbf{x}, \mathbf{y})) = \begin{bmatrix} \mathbf{x} \\ F(\mathbf{x}, \mathbf{y}) \end{bmatrix}, \\ (\mathbf{a}, \mathbf{b}) & \mapsto & (\mathbf{a}, F(\mathbf{a}, \mathbf{b})) = (\mathbf{a}, \mathbf{0}). \end{array} \tag{7.72}$$

Hence, if $F \in C^1$ then $G \in C^1$ and the derivative (Jacobian matrix) is

$$G'(\mathbf{x}, \mathbf{y}) = \frac{dG}{d(\mathbf{x}, \mathbf{y})} = \begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{x}} & \frac{\partial \mathbf{x}}{\partial \mathbf{y}} \\ \frac{\partial F}{\partial \mathbf{x}} & \frac{\partial F}{\partial \mathbf{y}} \end{bmatrix} = \begin{bmatrix} I_{n\times n} & 0 \\ \frac{\partial F}{\partial \mathbf{x}} & \frac{\partial F}{\partial \mathbf{y}} \end{bmatrix}. \tag{7.73}$$

Since

$$\det G'(\mathbf{x}, \mathbf{y}) = \begin{vmatrix} I_{n\times n} & 0 \\ \frac{\partial F}{\partial \mathbf{x}} & \frac{\partial F}{\partial \mathbf{y}} \end{vmatrix} = \det \frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y}) \tag{7.74}$$

we have that $G'(\mathbf{x}, \mathbf{y})$ is invertible iff the matrix $\frac{\partial F}{\partial \mathbf{y}}(\mathbf{x}, \mathbf{y})$ is. Therefore, if $\frac{\partial F}{\partial \mathbf{y}}(\mathbf{a}, \mathbf{b})$ is invertible then $G'(\mathbf{a}, \mathbf{b})$ is invertible and by the Inverse Function Therorem for G there exists locally an inverse $(\mathbf{x}, \mathbf{y}) = G^{-1}(\mathbf{x}, \mathbf{0})$, $\mathbf{x} \in U$ a neighborhood s.t. $\mathbf{a} \in U \subseteq \mathbb{R}^n$. Thus

$$g(\mathbf{x}) := \mathbf{y}(\mathbf{x}) = P_{\mathbf{y}} \circ G^{-1}(\mathbf{x}, \mathbf{0}) \tag{7.75}$$

($P_{\mathbf{y}}$ denotes projection on second set of variables in $(\mathbf{x}, \mathbf{y})$) and $g \in C^1(U)$. This proves the Implicit Function Theorem for $F(\mathbf{x}, \mathbf{y}(\mathbf{x})) = \mathbf{0}$.

Conversely, assuming that we already have the Implicit Function Theorem, given a map $G\colon \mathbb{R}^m \to \mathbb{R}^m$, $G(\mathbf{a}) = \mathbf{b}$, we may form the map

$$\begin{array}{rcl} F\colon \quad \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^m \\ (\mathbf{y}, \mathbf{x}) & \mapsto & F(\mathbf{y}, \mathbf{x}) := G(\mathbf{x}) - \mathbf{y}, \\ (\mathbf{b}, \mathbf{a}) & \mapsto & G(\mathbf{a}) - \mathbf{b} = \mathbf{0}. \end{array} \tag{7.76}$$

If $G \in C^1$ then $F \in C^1$ and

$$\frac{\partial F}{\partial \mathbf{x}}(\mathbf{y}, \mathbf{x}) = G'(\mathbf{x}), \tag{7.77}$$

so that the matrix $G'(\mathbf{x})$ is invertible iff $\frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}, \mathbf{y})$ is. Therefore, if $\frac{\partial F}{\partial \mathbf{x}}(\mathbf{b}, \mathbf{a})$ is invertible then $G'(\mathbf{a})$ is invertible and by the Implicit Function Therorem for F there exists locally an inverse

$$G^{-1}(\mathbf{y}) := \mathbf{x}(\mathbf{y}), \qquad \mathbf{0} = F(\mathbf{y}, \mathbf{x}(\mathbf{y})) = G(\mathbf{x}(\mathbf{y})) - \mathbf{y}, \tag{7.78}$$

for $\mathbf{y} \in V$ a neighborhood s.t. $(\mathbf{a}, \mathbf{b}) \in U \times V \subseteq \mathbb{R}^m \times \mathbb{R}^m$. This proves the Inverse Function Theorem for the map $G|_U : U = G^{-1}(V) \to V$.

7.5.4. *Generalization to Banach spaces.* Note that, given two normed vector spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$, we may form the vector space (also denoted $X \oplus Y$)

$$X \times Y = \{(x, y) : x \in X, y \in Y\} \tag{7.79}$$

with addition

$$(x, y) + (x', y') = (x + x', y + y') \tag{7.80}$$

and scalar multiplication

$$\lambda(x, y) = (\lambda x, \lambda y), \tag{7.81}$$

and with a norm $\|\cdot\| : X \times Y \to \mathbb{R}_+$, such as

$$\|(x, y)\| := \|x\|_X + \|y\|_Y . \tag{7.82}$$

We may verify that if X and Y are Banach spaces then so is $X \times Y$.

Theorem 7.19 (**Implicit function theorem**). *Let X, Y, Z be Banach spaces,*

$$\begin{array}{rccl} F\colon & X \times Y & \to & Z \\ & (x, y) & \mapsto & F(x, y), \\ & (a, b) & \mapsto & c, \end{array} \tag{7.83}$$

and assume F is continuously differentiable near the point (a, b) and that $\partial_y F(a, b) : Y \to Z$ is invertible. Then there exist U_a, V_b open and

$$\varphi : U_a \to V_b, \qquad a \in U_a \subseteq X, \qquad b \in V_b \subseteq Y, \tag{7.84}$$

and such that

1. *φ continuously differentiable,*
2. *$\varphi(a) = b$,*
3. *$F(x, \varphi(x)) = F(a, b)$ for all $x \in U_a$.*

Exercise 7.3. *Generalize our above correspondence with the Inverse Function Theorem to Banach spaces to prove Theorem 7.19.*

7.6. **Application: Constrained optimization.**

Definition 7.20. We say that a function $f\colon X \to Y$ between metric or topological spaces is **open** at a point $p \in X$ iff $f(p)$ is an inner point of $\operatorname{im}(f|_V) = f(V)$ for every open neighborhood V of p. A mapping f is **open** if it takes all open sets to open sets, i.e. if it is open at every point.

By the Inverse Function Theorem we now know that if $f \in C^1$, X and Y are Banach spaces and $f'(p)$ has a continuous right inverse then f has locally a right inverse at p. That implies in particular that f is open at p.

Now let $f\colon X \to \mathbb{R}$ and $g\colon X \to Y$ be continuously differentiable maps where X and Y are Banach spaces, and consider the following general **optimization problem**:

$$\boxed{\text{Optimize the value of } f(x) \text{ subject to the constraint } g(x) = 0.} \tag{7.85}$$

We introduce the **help map** $h\colon X \to \mathbb{R} \times Y$ via $h(x) := (f(x), g(x))$, and observe the following condition:

Theorem 7.21 (**Optimization in Banach spaces**). *If f has a local optimum/extremum (i.e. local maximum or local minimum) at $p \in X$ subject to the constraint $g = 0$ then h is not open at p.*

Proof. Assume on the contrary that $h(p) = (f(p), g(p)) = (f(p), 0)$ is an inner point in the image $h(V) \subseteq \mathbb{R} \times Y$ of some open neighborhood V of p. Then there are points $p', p'' \in V$ such that

$$f(p') < f(p) < f(p'') \qquad \text{and} \qquad g(p') = g(p'') = 0. \tag{7.86}$$

This contradicts the assumption that p is a local extremum. □

Finally, consider an exceptionally useful special case:

Corollary 7.22 (**Optimization in $\mathbb{R}^n$**). *Assume that $f, g_1, \ldots, g_m\colon \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable and that $\mathbf{p} \in \mathbb{R}^n$ is a local extremum for f subject to the constraints $g_1 = g_2 = \ldots = g_m = 0$, where $1 + m \leq n$. Then the vectors*

$$\nabla f(\mathbf{p}),\ \nabla g_1(\mathbf{p}),\ \nabla g_2(\mathbf{p}),\ ,\ldots,\ \nabla g_m(\mathbf{p}) \tag{7.87}$$

in $\mathbb{R}^n$ must be linearly dependent.

Proof. We form the C^1 map $h\colon \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^m$,

$$h(\mathbf{x}) = \big(f(\mathbf{x}), g_1(\mathbf{x}), \ldots, g_m(\mathbf{x})\big). \tag{7.88}$$

Its derivative is given by the Jacobian matrix

$$h'(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} (\mathbf{x}) = \begin{bmatrix} f'(\mathbf{x}) \\ g_1'(\mathbf{x}) \\ \vdots \\ g_m'(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \nabla f(\mathbf{x})^T \\ \nabla g_1(\mathbf{x})^T \\ \vdots \\ \nabla g_m(\mathbf{x})^T \end{bmatrix}. \tag{7.89}$$

If these rows would be linearly independent at $\mathbf{x} = \mathbf{p}$ then the matrix has full rank, which means that $h'(\mathbf{p})$ is surjective. But then h is an open map by the Inverse Function Theorem and therefore by the previous theorem $\mathbf{p}$ cannot be a local extremum. □

Further, by the implicit function theorem we may be able to solve the constraint $g(\mathbf{x}) = 0$ locally with e.g. $(x_1, \ldots, x_m)$ as functions of $(x_{m+1}, \ldots, x_n)$, and then study the local behavior of f in this smaller set of independent variables.

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Department of Mathematics, Uppsala University - Box 480, SE-751 06 Uppsala, Sweden
E-mail address: douglas.lundholm@math.uu.se