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Braid group statistics and exchange matrices of non-abelian anyons

with representations in Clifford algebra

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Flätningsstatistik och utbytesmatriser för icke-abelska anyoner

med representationer i Cliffordalgebra

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Abstract

When leaving classical physics and entering the realm of quantum physics, there are many new concepts being introduced. One of the most fundamental ideas in quantum mechanics is that particles no longer have exact known positions, but instead expected values and probabilities. This leads to the phenomena of truly identical particles, since they no longer can be distinguished simply by their positions. An important property differentiating different kinds of particles is how a system behaves when two such identical particles are exchanged. Historically, this divided particles into bosons and fermions, corresponding to symmetry and antisymmetry under an exchange.

However, in two dimensions a new type of particle appears. These particles are called anyons, and behave differently when particles are exchanged. Anyons can be further divided into abelian and non-abelian anyons, of which this thesis will focus on the latter. The exchanges can then be represented by the fundamental group of the configuration space of the particles, and in two dimensions this fundamental group is the braid group. Using rotors from a Clifford algebra and studying excitations of Majorana fermions, this thesis will show a way to calculate the exchange matrices of non-abelian anyons, and their corresponding eigenvalues. Furthermore, suggestions on a generalization of this framework along with areas where it can be applied are given.

Sammanfattning

När man lämnar klassisk fysik och övergår till den kvantfysikaliska världen introduceras många nya koncept. En av de mest grundläggande idéerna inom kvantmekaniken är att partiklar inte längre har exakta positioner, eftersom dessa ersatts av väntevärden och sannolikheter. Detta leder till fenomenet att partiklar kan vara verkligt identiska, eftersom de inte längre kan särskiljas med hjälp av sina positioner. En viktig egenskap som särskiljer olika typer av partiklar är hur ett system beter sig vid ett utbyte av två sådana identiska partiklar. Historiskt sett delade denna egenskap upp partiklar i bosoner och fermioner, som uppvisar symmetri respektive antisymmetri vid ett partikelutbyte.

I två dimensioner uppstår dock en ny typ av partiklar. Dessa partiklar kallas anyoner och beter sig annorlunda vid ett partikelutbyte. Vidare kan de delas upp i abelska och icke-abelska anyoner, varav denna rapport kommer fokusera på de senare. Utbytena kan representeras av den fundamentala gruppen av partiklarnas konfigurationsrum, och i två dimensioner blir denna fundamentala grupp flätgruppen. Genom att använda rotorer från en Cliffordalgebra och studera excitationer av Majoranafermioner, så visar denna rapport ett sätt att beräkna utbytesmatriserna för icke-abelska anyoner och deras tillhörande egenvärden. Vidare ges förslag på en generalisering av detta ramverk, tillsammans med områden där det kan tillämpas.

Contents

1 Introduction

One of the major differences between quantum physics and classical physics is that when leaving the classical world, identical particles are no longer fully distinguished by their position - because it is no longer possible to determine the exact position of a certain particle. Given a system of N particles, the complex multivalued wave function $|\Psi\rangle = |\Psi(x_1, ..., x_N)\rangle$ describes the system, in the sense that $|\Psi|^2$ is the probability density of finding the system in a certain state. The exact positions of classical mechanics have thus been substituted by probabilities and expected values. This gives rise to particles that are truly identical, since they can no longer be distinguished by their position - a phenomenon that does not occur in classical physics. This leads to interesting effects when it comes to these identical particles, especially concerning how the system behaves when such particles are exchanged.

Elementary particles in three (or more) dimensions can be divided into fermions and bosons, obeying Fermi-Dirac statistics and Bose-Einstein statistics, respectively. In terms of the wave functions, these two kinds of particles are distinguished by the behavior of the wave function under an exchange of two particles. For bosons the wave function is symmetric, i.e

$$|\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N)\rangle = |\Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)\rangle \ , \tag{1}$$

when the particles j and k are exchanged. Similarly, the wave function for fermions is antisymmetric, which means that

$$|\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N)\rangle = -\,|\Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)\rangle \ , \tag{2}$$

and because of this antisymmetry fermions also obey the Pauli exclusion principle. As a result, the difference between fermions and bosons can be seen as whether or not several particles are allowed to be in the same state.

For a long time, it was believed that this is all there is to it, and since we live in a three-dimensional world it is easy to understand why. However, in the special case of two dimensions there is more freedom when it comes to the choice of statistics, as a result of the topology of the resulting configuration space. Theoretically, particles called anyons appear in two dimensions, as first discovered by Leinaas and Myrheim in 1977 [1]. The name was first coined by Wilczek in 1982 [2], and is derived from *any phase* or *any statistics*. It might be hard though, to justify that a two-dimensional particle in a three-dimensional world is more than a purely mathematical construct. Planar motion is still possible nevertheless, by essentially prohibiting movement in the third dimension - allowing nature to be fooled into believing it is two-dimensional. Fermions and bosons are still the only kinds of particles having been observed today, and trying to prove the existence of particles obeying anyonic statistics is an ongoing effort. Researchers seem to be getting closer to succeeding with that in recent years [3].

1.1 Abelian anyons

Anyons can be divided into two categories; abelian and non-abelian. Instead of the symmetry relations seen earlier, the wave function for abelian anyons satisfies

$$|\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N)\rangle = e^{i\pi\alpha}\,|\Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)\rangle \ , \quad \alpha \in [0, 2)\,, \tag{3}$$

where the periodic α is called the statistics parameter. This implies that instead of the two discrete cases seen in higher dimensions, there is a continuous spectrum of statistics ranging from Bose-Einstein ($\alpha = 0$) to Fermi-Dirac ($\alpha = 1$) and back again. It is worth noting here that since $|e^{i\pi\alpha}|^2 = 1$, the probability density $|\Psi|^2$ is still unchanged, which is a requirement for any kind of particle exchange (a requirement also satisfied by bosons and fermions). From (3) we can note the peculiar property that if two abelian anyons are exchanged and then restored to their original positions by another exchange, the wave function might still be different from the original one. This type of anyon, although interesting, will not be the focus of this thesis.

1.2 Non-abelian anyons

Non-abelian anyons have similar properties, but the wave-function is altered more drastically than by just a phase change when two particles are exchanged. Instead, the wave function is rotated to another one in the same degenerate space. This rotation can be described as a transformation by an exchange matrix U, which can be seen as a matrix version of a complex phase. The wave function thus changes as

$$|\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N)\rangle = U\,|\Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)\rangle\ . \tag{4}$$

In order to preserve the probability density mentioned earlier, this exchange matrix U must be unitary. An important property here is that the order of the particle exchanges makes a difference, as opposed to when the wave function is just multiplied by complex phases. This noncommutativity is actually what makes the study of non-abelian anyons interesting, and this thesis will focus on finding the exchange matrices. These matrices have been discussed earlier by Ivanov in 2001 [4]. Here we will examine new ways to determine them, specifically through representations in Clifford algebra.

1.3 Outline of work

As mentioned earlier, our interest is focused on the non-abelian anyons. First, we will in section 2 discuss the configuration space of a two-dimensional system of identical particles, with non-coincident positions. Here, the topological properties of that space will be examined in order to find the corresponding braiding properties of the non-abelian anyons. The braid group will then be a fundamental part of the following work.

Section 3 will focus on the description of the quantum state of an arbitrary system of fermions. It is discussed in terms of the fermionic creation and annihilation operators, since these can be used to define the corresponding operators for Majorana fermions. Bound Majorana fermions can appear as a type of quasiparticle excitation that can exhibit non-abelian statistics, which is why they are discussed here. The corresponding Majorana operators will be defined through the regular fermionic operators.

The next step in finding the exchange matrices is giving a brief introduction to the Clifford algebra in section 4. Specifically, we discuss how this geometric algebra is a powerful tool when describing rotations in a vector space - using a tool called rotors. A parallel will be drawn here to the previous section, because of similarities between the Majorana operators and the basis elements in a Clifford algebra.

Interpreting the Clifford algebra basis elements as Majorana operators is the final step needed to find the matrices. In this context, the braiding properties of the rotors will be examined, and if it is a way of finding the exchange matrices. The matrices are discussed in section 5. Specifically, the case of 4 Majorana fermions will be discussed, in order to limit the size of the matrices and thus enabling more thorough examination of their properties. The corresponding eigenvalues will be determined in section 6.

Finally, in section 7, we discuss how to generalize the methods used to a system of arbitrary size. As a conclusion, a summary of our work will be given in section 8 along with suggestions of possible continuations in section 9.

2 Configuration space

We are studying N identical particles in d dimensions, where d will later be set to 2, and are interested in the corresponding configuration space M_N^d. The position of the i:th particle is described by a vector $x_i \in \mathbb{R}^d$, and N particles can therefore be seen as an element of $(\mathbb{R}^d)^N = \mathbb{R}^{dN}$. But every point in $\mathbb{R}^{dN}$ is not part of the configuration space, since we have to consider two assumptions; that the particles are indistinguishable and that two particles cannot occupy the same point. The latter assumption, which is known as the particles being hard-cored, is not as obvious as it might seem. It is however necessary in order to have a notion of how many particles there are, since if multiple particles could occupy the same point, there would be an ambiguity in the particle count. For more details, see [5].

Define the diagonal $\mathbf{\Delta}$ as

$$\mathbf{\Delta} = \{(x_1, ..., x_N) \in \mathbb{R}^{dN} \,|\, \exists\, i \neq j : x_i = x_j\}\,, \tag{5}$$

and the configuration space can be written as

$$M_N^d = \left(\mathbb{R}^{dN} \setminus \mathbf{\Delta}\right) / S_N\,, \tag{6}$$

where the quotient of the permutation group S_N symmetrizes the space and thus accounts for the indistinguishability of the particles. Before that, the hard-cored property of the particles was accounted for by removing the diagonal, since points in $\mathbb{R}^{dN}$ along the diagonal are not possible configurations of the system. It can then be shown that the fundamental group π_1 associated with this configuration space is given by

$$\pi_1(M_N^d) \cong \begin{cases} 1\,, & d = 1 \\ B_N\,, & d = 2 \\ S_N\,, & d \geq 3 \end{cases}\,, \tag{7}$$

where B_N is the braid group and S_N the permutation group [6]. The fundamental group of a topological space contains information about when loops in the space can be continuously deformed [7]. This is why the topological difference between two and three dimensions makes a big difference for the configuration space. If a loop in three dimensions encloses a particle it can still be deformed, unlike in two dimensions where the loop becomes “stuck” on anything inside it. This topological property is the very reason anyons appear in two dimensions.

2.1 Braid group

As mentioned earlier we are, in order to study anyons, interested in the special case where $d = 2$. An important part will then be to examine the properties of the braid group B_N, defined by its generators $\{\tau_1, ..., \tau_{N-1}\}$. These generators satisfy the braid relations

$$\tau_i \tau_j = \tau_j \tau_i\,, \quad |i - j| > 1\,, \tag{8}$$

$$\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}\,, \quad i = 1, 2, ..., N-2\,. \tag{9}$$

Since this construction is not purely mathematical in the sense that it has a physical meaning, it can be fitting to take a step back in order to think about what this actually means. A way to visualize the physical meaning behind the braid group is by using a representation called braid diagrams. Imagining that the Euclidean plane for $d = 2$ moves upwards along a vertical time axis, the trajectories of the particles will form world lines in this three-dimensional spacetime. If the particles remain in the same positions, these world lines will be vertical, as illustrated in figure 1. However, if the particles instead move and are exchanged, the world lines will be intertwined and form braids.

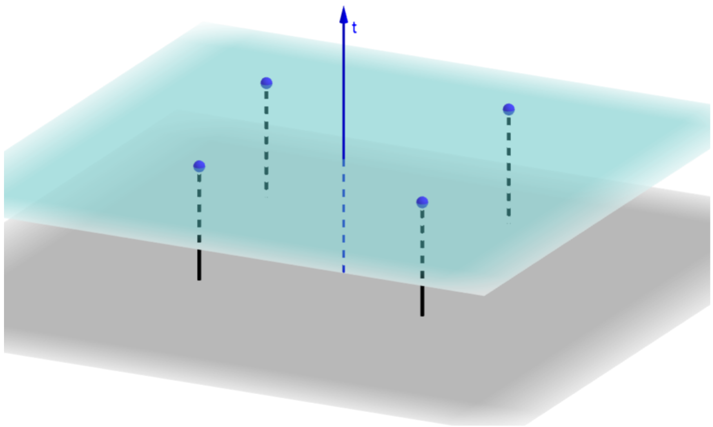

Figure 1: World lines being formed without particle exchange.

Observed from the side, the world lines formed by the particle trajectories are illustrated in these diagrams. In a braid diagram time progresses upwards, and the action of the braid group generator τ_i is interpreted as the exchange of particles i and $i+1$. Figures 2 and 3 show that the braid relations are satisfied in this representation. An equality relation between two braids simply means that the same braid has been formed in the two cases, even though different braid group generators have been applied.

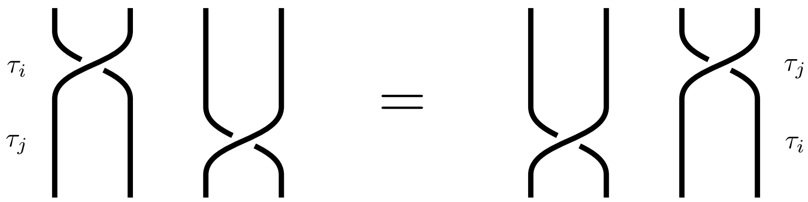

Figure 2: $\tau_i \tau_j = \tau_j \tau_i$

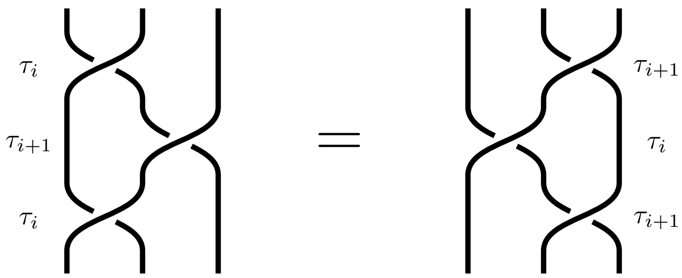

Figure 3: $\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1}$

In order to find exchange matrices, we need to examine the braiding of world lines occurring when two arbitrary particles are exchanged. However, the action of a generator of the braid group is interpreted as the exchange of two adjacent particles. If that was only type of exchange possible, it would impose a strong constraint on this model. This means that in order to use the braid group to describe an arbitrary exchange, something more is needed. Consider the braiding in figure 4, and specifically note that that all of the strands except the first and last one are unaffected by the procedure. Those two, highlighted in different colors for clarity, were exchanged.

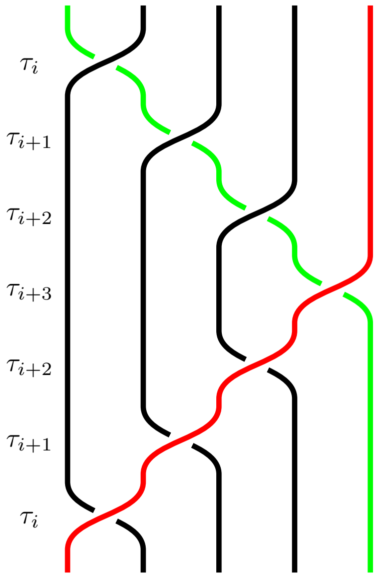

Figure 4: $\tau_i\tau_{i+1}\tau_{i+2}\tau_{i+3}\tau_{i+2}\tau_{i+1}\tau_i$

In general, the exchange between particles i and $i+1+n$ can be written as the composition

$$\tau_i\tau_{i+1}\ldots\tau_{i+n}\ldots\tau_{i+1}\tau_i\,, \tag{10}$$

which illustrates that the braid group generators are in some way fundamental. It is then sufficient to find exchange matrices for adjacent particles, since an arbitrary exchange matrix can be found by multiplying those. Actually, every combination of the braid group generators can be written using only the two elements τ_1 and $\sigma = \tau_1\tau_2...\tau_{N-1}$. For example, let $N = 4$ and we have the braid group B_4, generated by $\{\tau_1, \tau_2, \tau_3\}$. We then see, by using the braid relations (8) and (9), that

$$\begin{aligned}\sigma\tau_1\sigma^{-1} &= \tau_1\tau_2\tau_3\tau_1(\tau_1\tau_2\tau_3)^{-1}\\ &= \tau_1\tau_2\tau_1\tau_3\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}\\ &= \tau_2\tau_1\tau_2\tau_2^{-1}\tau_1^{-1} = \tau_2\,,\end{aligned} \tag{11}$$

which also gives

$$\begin{aligned}\sigma^2\tau_1\sigma^{-2} = \sigma\tau_2\sigma^{-1} &= (\tau_1\tau_2\tau_3)\tau_2(\tau_1\tau_2\tau_3)^{-1}\\ &= \tau_1\tau_2\tau_3\tau_2\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}\\ &= \tau_1\tau_3\tau_2\tau_3\tau_3^{-1}\tau_2^{-1}\tau_1^{-1}\\ &= \tau_1\tau_3\tau_1^{-1} = \tau_3\,.\end{aligned} \tag{12}$$

By using induction, it can also be shown for a braid group B_N of arbitrary size. The generator τ_{k+1}, where $1 \leq k \leq N-2$, can then be written as

$$\begin{aligned}\sigma^k\tau_1\sigma^{-k} &= \sigma^{k-1}\tau_2\sigma^{-k+1} = \sigma\tau_k\sigma^{-1}\\ &= \tau_1\tau_2\ldots\tau_{k-1}\tau_k\tau_{k+1}\ldots\tau_{N-2}\tau_{N-1}\tau_k\sigma^{-1}\\ &= \tau_1\tau_2\ldots\tau_{k-1}\tau_k\tau_{k+1}\tau_k\ldots\tau_{N-2}\tau_{N-1}\sigma^{-1}\\ &= \tau_1\tau_2\ldots\tau_{k-1}\tau_{k+1}\tau_k\tau_{k+1}\ldots\tau_{N-2}\tau_{N-1}\sigma^{-1}\\ &= \tau_{k+1}\tau_1\tau_2\ldots\tau_{k-1}\tau_k\tau_{k+1}\ldots\tau_{N-2}\tau_{N-1}\sigma^{-1} = \tau_{k+1}\sigma\sigma^{-1} = \tau_{k+1}\,.\end{aligned} \tag{13}$$

3 Fermionic operators

When working with a system of fermions, each single particle state has an annihilation operator ψ_n, and its conjugate $\psi_n^\dagger$, which is the corresponding creation operator. The general state of a system with multiple particles can be written as

$$|\Psi\rangle = |k_1 k_2 \ldots\rangle \,, \tag{14}$$

where each k_n is the occupation number for an exited state, counting how many particles there are in that state. The Pauli exclusion principle for fermions states that multiple particles cannot occupy the same state, implying that a certain state either is occupied by a single particle or not occupied at all. As a result, the occupation numbers can only take the values 0 and 1. The ground state, also known as the vacuum state, is in this notation written as

$$|0\rangle = |00\ldots\rangle \;, \tag{15}$$

i.e the state where there is no particle in any of the excited states.

The action of the creation and annihilation operators on the state of the system is then simply given by

$$\begin{aligned} \psi_n^\dagger |k_1 \ldots\rangle &= \begin{cases} 0\,, & k_n = 1 \\ (-1)^{\sum_{j<n} k_j} |k_1 \ldots k_n + 1 \ldots\rangle \;, & k_n = 0 \end{cases} \;, \\ \psi_n |k_1 \ldots\rangle &= \begin{cases} 0\,, & k_n = 0 \\ (-1)^{\sum_{j<n} k_j} |k_1 \ldots k_n - 1 \ldots\rangle \;, & k_n = 1 \end{cases} \;, \end{aligned} \tag{16}$$

which shows why these operators can be seen to be "creating" and "annihilating" states. The factor $(-1)^{\sum_{j<n} k_j}$ is a phase factor originating from the antisymmetry property of fermions, which depends on how many states below the affected one that are filled. From (16), it is then easily observed that these operators fulfill the relations

$$\begin{aligned} \{\psi_n, \psi_m\} &= 0\,, \\ \{\psi_n^\dagger, \psi_m^\dagger\} &= 0\,, \\ \{\psi_n, \psi_m^\dagger\} &= \delta_{nm}\,, \end{aligned} \tag{17}$$

where $\{A, B\} = AB + BA$ is the anticommutator and δ_{nm} is the Kronecker delta

$$\delta_{nm} = \begin{cases} 1\,, & n = m \\ 0\,, & n \neq m \end{cases} \;. \tag{18}$$

3.1 Majorana fermions

Construction of non-abelian anyon statistics can be done by observing excited states of particles called Majorana fermions. In most literature (as well as here), these quasiparticle excitations are still referred to as Majorana fermions, despite their not so fermionic behavior. From the fermionic operators we can, for each fermion, create two Majorana operators through the relations

$$\begin{aligned} \gamma_{2n-1} &= \psi_n + \psi_n^\dagger\,, \\ \gamma_{2n} &= \frac{\psi_n - \psi_n^\dagger}{i}\,. \end{aligned} \tag{19}$$

The index notation used here means that each regular fermion state gives rise to two Majorana fermions, since these operators satisfy

$$\begin{aligned} \gamma_{2n-1}^\dagger &= (\psi_n + \psi_n^\dagger)^\dagger = \psi_n^\dagger + \psi_n = \gamma_{2n-1}\,, \\ \gamma_{2n}^\dagger &= \left(\frac{\psi_n - \psi_n^\dagger}{i}\right)^\dagger = \frac{\psi_n^\dagger - \psi_n}{-i} = \gamma_{2n}\,, \end{aligned} \tag{20}$$

i.e they are hermitean (self-conjugate) and the creation and annihilation operators are identical. This is the definition of a Majorana fermion, which is its own anti-particle. By the definition (19), these Majorana operators then appear in pairs of two.

Using the relations (17), we find that all of the possible combinations of two of these operators are

$$\begin{aligned}\gamma_{2n-1}\gamma_{2m-1} &= \psi_n\psi_m + \psi_n^\dagger\psi_m^\dagger + \psi_n^\dagger\psi_m + \psi_n\psi_m^\dagger = -\gamma_{2m-1}\gamma_{2n-1}\,,\\ \gamma_{2n}\gamma_{2m} &= -(\psi_n\psi_m + \psi_n^\dagger\psi_m^\dagger - \psi_n^\dagger\psi_m - \psi_n\psi_m^\dagger) = -\gamma_{2n}\gamma_{2m}\,,\\ \gamma_{2n-1}\gamma_{2m} &= \frac{1}{i}(\psi_n\psi_m - \psi_n^\dagger\psi_m^\dagger + \psi_n^\dagger\psi_m - \psi_n\psi_m^\dagger) = -\gamma_{2m}\gamma_{2n-1}\,,\end{aligned} \tag{21}$$

which shows that

$$\gamma_k\gamma_j = -\gamma_j\gamma_k\,, \quad \forall j \neq k\,. \tag{22}$$

The indices j and k are arbitrary here, but not equal, implying that different Majorana operators always anticommute, regardless of whether they belong to the same pair or not. Furthermore, the squares of these Majorana operators become

$$\begin{aligned}(\gamma_{2n-1})^2 &= \psi_n^2 + (\psi_n^\dagger)^2 + \psi_n^\dagger\psi_n + \psi_n\psi_n^\dagger = 1\,,\\ (\gamma_{2n})^2 &= -(\psi_n^2 + (\psi_n^\dagger)^2 - \psi_n^\dagger\psi_n - \psi_n\psi_n^\dagger) = 1\,.\end{aligned} \tag{23}$$

To summarize, we have shown that the Majorana operators satisfy the relations

$$\begin{aligned}\gamma_i^2 &= 1\,,\\ \gamma_i\gamma_j &= -\gamma_j\gamma_i\,, \quad i \neq j,\end{aligned} \tag{24}$$

or, in a more compact manner,

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{25}$$

4 Clifford algebra

Consider the space $\mathbb{R}^3$ with the orthonormal basis $\{e_1, e_2, e_3\}$. We introduce the wedge product $e_1 \wedge e_2$ of e_1 and e_2, which will represent the oriented parallelogram spanned by vectors e_1 and e_2. This is a new type of vector called a 2-blade, or bivector, and we have the condition that $e_2 \wedge e_1 = -e_1 \wedge e_2$. Analogously, the wedge product $e_1 \wedge e_2 \wedge e_3$ is the oriented parallelepiped spanned by the three vectors e_1, e_2 and e_3. This is called a 3-blade, or trivector. This notion of blades can be generalized to an arbitrary dimension, where a k-blade represents a finite k-dimensional linear subspace.

Define the Clifford product of two arbitrary vectors $a, b \in \mathbb{R}^n$ as

$$ab = a \cdot b + a \wedge b\,, \tag{26}$$

where $\cdot$ denotes the usual scalar product. From (26) we then obtain the relations

$$\begin{aligned} e_i^2 &= 1 \\ e_i e_j &= -e_j e_i\,, \quad i \neq j, \end{aligned} \tag{27}$$

for the orthonormal basis vectors, which can be summarized as

$$\{e_i, e_j\} = 2\delta_{ij}. \tag{28}$$

Continuing the example in $\mathbb{R}^3$, we can generate a space using this Clifford product, where the basis for this new space will be $\{1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, e_1e_2e_3\}$. This space is the Clifford algebra of $\mathbb{R}^3$, denoted by $\mathrm{Cl}(\mathbb{R}^3)$ [8]. The Clifford algebra of $\mathbb{R}^n$ can be written in a more compact way if we introduce $\wedge^k(\mathbb{R}^n)$ as the set of all k-blades in $\mathbb{R}^n$ (i.e $\wedge^0(\mathbb{R}^n)$ is the scalars, $\wedge^1(\mathbb{R}^n)$ is the vectors, $\wedge^2(\mathbb{R}^n)$ is the bivectors, etc.), because we can then express $\mathrm{Cl}(\mathbb{R}^n)$ as

$$\mathrm{Cl}(\mathbb{R}^n) = \wedge^0(\mathbb{R}^n) \oplus \wedge^1(\mathbb{R}^n) \oplus \wedge^2(\mathbb{R}^n) \oplus ... \oplus \wedge^n(\mathbb{R}^n) = \bigoplus_{k=0}^{n} \wedge^k(\mathbb{R}^n)\,. \tag{29}$$

From this way of expressing the Clifford algebra, the dimension of the space can be calculated via combinatorics. We can easily see that

$$\dim\, \wedge^k(\mathbb{R}^n) = \binom{n}{k}, \tag{30}$$

just by the number of ways the basis vectors in the blade can be rearranged. This gives

$$\dim \mathrm{Cl}(\mathbb{R}^n) = \dim \bigoplus_{k=0}^{n} \wedge^k(\mathbb{R}^n) = \sum_{k=0}^{n} \binom{n}{k} = 2^n\,, \tag{31}$$

where the last sum is a well known result.

4.1 Rotors

Now consider the Clifford algebra created from an ordinary N-dimensional vector space $\mathbb{R}^N$, with a set of orthonormal basis vectors $\{e_1, \ldots, e_N\}$. The notation e_ie_j then, as mentioned earlier, represents the 2-blade corresponding to the coordinate plane spanned by e_i and e_j. The relation (28), however, is strikingly similar to the property (25) of the Majorana operators defined in the previous section - implying that if we rename the basis vectors as γ_i, they can be interpreted as representations of the Majorana fermions.

A Clifford algebra, being a geometric algebra, is very efficient when it comes to describing rotations. Noting that, by using the rules (27),

$$(\gamma_i\gamma_{i+1})^2 = \gamma_i\gamma_{i+1}\gamma_i\gamma_{i+1} = -\gamma_i\gamma_i\gamma_{i+1}\gamma_{i+1} = -1\,, \tag{32}$$

we see that this 2-blade has the same property as an imaginary unit. By the definition of the exponential function e^x through its Taylor series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,, \tag{33}$$

the 2-blade $\gamma_i\gamma_{i+1}$ can actually be interpreted as an imaginary unit - enabling use of Euler's formula, where the imaginary unit is replaced by $\gamma_i\gamma_{i+1}$.

The action of a rotation Φ_i by an angle θ in the plane spanned by γ_i and γ_{i+1} on the basis vectors that span the plane can then be written as

$$\begin{aligned}\Phi_i(\gamma_i) =&\gamma_i \cos\theta + \gamma_{i+1}\sin\theta = \gamma_i\cos\theta + \gamma_i\gamma_i\gamma_{i+1}\sin\theta = \gamma_i(\cos\theta + \gamma_i\gamma_{i+1}\sin\theta) = \gamma_i e^{\theta\gamma_i\gamma_{i+1}}\,,\\ \Phi_i(\gamma_{i+1}) =& -\gamma_i\sin\theta + \gamma_{i+1}\cos\theta = -\gamma_{i+1}\gamma_{i+1}\gamma_i\sin\theta + \gamma_{i+1}\cos\theta\\ =&\gamma_{i+1}(\gamma_i\gamma_{i+1}\sin\theta + \cos\theta) = \gamma_{i+1}e^{\theta\gamma_i\gamma_{i+1}}\,.\end{aligned} \tag{34}$$

A more convenient way to rewrite those two expressions can be found through

$$\begin{aligned}\Phi_i(\gamma_i) =&\gamma_i e^{\theta\gamma_i\gamma_{i+1}} = \gamma_i\left(\cos\left(\frac{\theta}{2}\right) + \gamma_i\gamma_{i+1}\sin\left(\frac{\theta}{2}\right)\right)e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}}\\ =&\left(\cos\left(\frac{\theta}{2}\right) - \gamma_i\gamma_{i+1}\sin\left(\frac{\theta}{2}\right)\right)\gamma_i e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}} = e^{-\frac{\theta}{2}\gamma_i\gamma_{i+1}}\gamma_i e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}}\,,\end{aligned} \tag{35}$$

and

$$\begin{aligned}\Phi_i(\gamma_{i+1}) =&\gamma_{i+1} e^{\theta\gamma_i\gamma_{i+1}} = \gamma_{i+1}\left(\cos\left(\frac{\theta}{2}\right) + \gamma_i\gamma_{i+1}\sin\left(\frac{\theta}{2}\right)\right)e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}}\\ =&\left(\cos\left(\frac{\theta}{2}\right) - \gamma_i\gamma_{i+1}\sin\left(\frac{\theta}{2}\right)\right)\gamma_{i+1} e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}} = e^{-\frac{\theta}{2}\gamma_i\gamma_{i+1}}\gamma_{i+1} e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}}\,.\end{aligned} \tag{36}$$

This allows the expression to be used even for vectors γ_j not in the plane of rotation (i.e $j \neq i$ and $j \neq i+1$), since

$$\begin{aligned}\Phi_i(\gamma_j) =&e^{-\frac{\theta}{2}\gamma_i\gamma_{i+1}}\gamma_j e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}} = \left(\cos\left(\frac{\theta}{2}\right) - \gamma_i\gamma_{i+1}\sin\left(\frac{\theta}{2}\right)\right)\gamma_j\left(\cos\left(\frac{\theta}{2}\right) + \gamma_i\gamma_{i+1}\sin\left(\frac{\theta}{2}\right)\right)\\ =&\gamma_j e^{-\frac{\theta}{2}\gamma_i\gamma_{i+1}} e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}} = \gamma_j\,,\end{aligned} \tag{37}$$

where the fact that $\gamma_i\gamma_{i+1}\gamma_j = \gamma_j\gamma_i\gamma_{i+1}$ was used. This is the expected result, since a rotation in a certain plane should leave vectors perpendicular to that plane unaffected. Since an arbitrary vector $x \in \mathbb{R}^N$ can be written as

$$x = \sum_{i=1}^{N} c_i\gamma_i\,, \tag{38}$$

the linearity allows the rotation of this vector an angle θ in the $\gamma_i\gamma_{i+1}$-plane to be written as

$$\Phi_i(x) = e^{-\frac{\theta}{2}\gamma_i\gamma_{i+1}} x e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}}\,. \tag{39}$$

The object $R_i = e^{\frac{\theta}{2}\gamma_i\gamma_{i+1}}$ appearing here is called a rotor, and in a more compact manner we can now write

$$\Phi_i(x) = R_i^{-1} x R_i\,. \tag{40}$$

Now define an operator T_i, as suggested by Ivanov in 2001 [4], through

$$T_i(\gamma_j) = \begin{cases}\gamma_{i+1}\,, & j = i\\ -\gamma_i\,, & j = i+1\\ \gamma_j\,, & j \neq i\,, j \neq i+1\end{cases}\,. \tag{41}$$

Originally, this was just a transformation of the Majorana operators. However, with the interpretation of γ_i as a basis element in a Clifford algebra, the operator T_i can be seen as acting on $\mathbb{R}^N$, where it is a rotation of an angle $\theta = \frac{\pi}{2}$ counterclockwise in the $\gamma_i\gamma_{i+1}$-plane. Using the newfound rotor expression of a rotation, with the rotation angle now being $\theta = \frac{\pi}{2}$, the operator T_i can be expressed using the rotors

$$R_i = e^{\frac{\pi}{4}\gamma_i\gamma_{i+1}} = \left(\cos\left(\frac{\pi}{4}\right) + \gamma_i\gamma_{i+1}\sin\left(\frac{\pi}{4}\right)\right) = \frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_{i+1})\,. \tag{42}$$

In the following sections, this is the expression referred to when mentioning the rotor R_i. This specific rotor also satisfies

$$R_i^\dagger = \left(\frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_{i+1})\right)^\dagger = \frac{1}{\sqrt{2}}(1 - \gamma_i\gamma_{i+1}) = R_i^{-1}\,, \tag{43}$$

and we can rewrite (40) as

$$T_i(x) = R_i^\dagger x R_i\,. \tag{44}$$

In the following subsection, it will be shown that these rotors corresponding to the operator T_i satisfy the braid relations. This result can in turn be used to show that the rotors are similar, implying that they share the same eigenvalues. With the choice of $S = R_i R_{i+1}$, we see that

$$SR_iS^\dagger = R_iR_{i+1}R_i(R_iR_{i+1})^\dagger = R_iR_{i+1}R_iR_{i+1}^\dagger R_i^\dagger = R_{i+1}R_iR_{i+1}R_{i+1}^\dagger R_i^\dagger = R_{i+1}\,, \tag{45}$$

which is the condition that proves R_i and R_{i+1} are similar. Since this holds for an arbitrary value of $i = 1, 2, \ldots N-2$, by induction all of the rotors are similar.

4.1.1 Braid relations

There are several ways of verifying that these rotors satisfy the braid relations, by using the different expressions. We can see that, by using the last expression in equation (42),

$$\begin{aligned}2\sqrt{2}R_iR_{i+1}R_i &= (1+\gamma_i\gamma_{i+1})(1+\gamma_{i+1}\gamma_{i+2})(1+\gamma_i\gamma_{i+1})\\ &= (1+\gamma_i\gamma_{i+1})(1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_{i+1}\gamma_{i+2}\gamma_i\gamma_{i+1})\\ &= (1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_{i+2}\gamma_i)+(\gamma_i\gamma_{i+1}-1+\gamma_i\gamma_{i+2}+\gamma_{i+1}\gamma_{i+2})\\ &= 2(\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2})\,,\end{aligned} \tag{46}$$

and

$$\begin{aligned}2\sqrt{2}R_{i+1}R_iR_{i+1} &= (1+\gamma_{i+1}\gamma_{i+2})(1+\gamma_i\gamma_{i+1})(1+\gamma_{i+1}\gamma_{i+2})\\ &= (1+\gamma_{i+1}\gamma_{i+2})(1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_i\gamma_{i+1}\gamma_{i+1}\gamma_{i+2})\\ &= (1+\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2}+\gamma_i\gamma_{i+2})+(\gamma_{i+1}\gamma_{i+2}+\gamma_{i+2}\gamma_i-1+\gamma_i\gamma_{i+1})\\ &= 2(\gamma_i\gamma_{i+1}+\gamma_{i+1}\gamma_{i+2})\,,\end{aligned} \tag{47}$$

which means that the first braid relation, i.e

$$R_iR_{i+1}R_i = R_{i+1}R_iR_{i+1}\,, \quad i = 1, 2, ..., N-2\,, \tag{48}$$

is satisfied for the rotors.

An interesting note here that might be misinterpreted as an inconsistency is that if the expression

$$R_i = e^{\frac{\pi}{4}\gamma_i\gamma_{i+1}} \tag{49}$$

is used along with the ordinary exponential rule $e^Ae^B = e^{A+B}$, one will soon find that the first braid relation is, in fact, not satisfied. This is because for noncommutative A and B, the Baker-Campbell-Hausdorff formula

$$\begin{aligned}\ln\left(e^Ae^B\right) =& A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])\\ &-\frac{1}{24}[B,[A,[A,B]]]-\frac{1}{720}([B,[B,[B,[B,A]]]]+[A,[A,[A,[A,B]]]])\\ &+\frac{1}{360}([A,[B,[B,[B,A]]]]+[B,[A,[A,[A,B]]]])\\ &+\frac{1}{120}([B,[A,[B,[A,B]]]]+[A,[B,[A,[B,A]]]])+\ldots\end{aligned} \tag{50}$$

must be used [9], where $[A, B] = AB - BA$ is the commutator. In the case of the rotors,

$$A = \gamma_i\gamma_{i+1}\,, \quad B = \gamma_{i+1}\gamma_{i+2}\,, \tag{51}$$

and thus, using (24),

$$\begin{aligned}[A,B] &= \gamma_i\gamma_{i+1}\gamma_{i+1}\gamma_{i+2}-\gamma_{i+1}\gamma_{i+2}\gamma_i\gamma_{i+1}\\ &= \gamma_i\gamma_{i+2}-\gamma_{i+2}\gamma_i = 2\gamma_i\gamma_{i+2}\neq 0\,,\end{aligned} \tag{52}$$

which shows why the extra terms in (50) needs to be accounted for in order to use (49) to show that the rotors satisfy the braid relations.

Let us now examine the commutator of different rotors. For future reference, we see that

$$R_iR_j = \frac{1}{2}(1+\gamma_i\gamma_{i+1})(1+\gamma_j\gamma_{j+1}) = \frac{1}{2}(1+\gamma_i\gamma_{i+1}+\gamma_j\gamma_{j+1}+\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1})\,, \tag{53}$$

which gives

$$\begin{aligned}[R_i,R_j] &= R_iR_j - R_jR_i \\ &= \frac{1}{2}(1+\gamma_i\gamma_{i+1}+\gamma_j\gamma_{j+1}+\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}) - \frac{1}{2}(1+\gamma_j\gamma_{j+1}+\gamma_i\gamma_{i+1}+\gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1}) \\ &= \frac{1}{2}(\gamma_i\gamma_{i+1}+\gamma_j\gamma_{j+1}+\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}-\gamma_j\gamma_{j+1}-\gamma_i\gamma_{i+1}-\gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1}) \\ &= \frac{1}{2}(\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}-\gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1})\,.\end{aligned} \tag{54}$$

Assuming that $|i-j|>1$, and with the properties (24),

$$\begin{aligned}[R_i,R_j] =& \frac{1}{2}(\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}-\gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1}) \\ =& \frac{1}{2}(\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}-\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}) = 0\,,\end{aligned} \tag{55}$$

which confirms that the rotors satisfy the second braid relation, i.e

$$R_iR_j = R_jR_i\,, \quad |i-j|>1\,. \tag{56}$$

Equations (48) and (56) show that rotors satisfy the braid relations and actually are a representation of the braid group generators. If the basis vector γ_i, as mentioned earlier, is interpreted as a representation of the i:th particle, the rotors R_i are a representation of the generators of the braid group, which describes the exchange of particles. For the operators T_i, this gives

$$\begin{aligned}T_i(T_{i+1}(T_i(x))) &= R_i^\dagger R_{i+1}^\dagger R_i^\dagger xR_iR_{i+1}R_i = (R_iR_{i+1}R_i)^\dagger xR_iR_{i+1}R_i \\ &= (R_{i+1}R_iR_{i+1})^\dagger xR_{i+1}R_iR_{i+1} = R_{i+1}^\dagger R_i^\dagger R_{i+1}^\dagger xR_{i+1}R_iR_{i+1} \\ &= T_{i+1}(T_i(T_{i+1}(x)))\,,\end{aligned} \tag{57}$$

and in the same manner

$$\begin{aligned}T_i(T_j(x)) &= R_i^\dagger R_j^\dagger xR_jR_i = (R_jR_i)^\dagger xR_jR_i \\ &= (R_iR_j)^\dagger xR_iR_j = R_j^\dagger R_i^\dagger xR_iR_j = T_j(T_i(x))\,, \quad |i-j|>1\,.\end{aligned} \tag{58}$$

This result, that the operators T_i are another representation of the braid group generators, could also have been shown directly from the definition (41). Doing it in this order highlights the more fundamental rotors as the cause of these relations, and finding this property also gives a second way of creating the exchange matrices.

4.1.2 Non-abelian properties

For the case when $|i-j|=1$, we can relabel $j=i+1$, and (54) gives

$$\begin{aligned}[R_i,R_j] =& \frac{1}{2}(\gamma_i\gamma_{i+1}\gamma_j\gamma_{j+1}-\gamma_j\gamma_{j+1}\gamma_i\gamma_{i+1}) \\ =& \frac{1}{2}(\gamma_i\gamma_{i+1}\gamma_{i+1}\gamma_{i+2}-\gamma_{i+1}\gamma_{i+2}\gamma_i\gamma_{i+1}) \\ =& \frac{1}{2}(\gamma_i\gamma_{i+2}-\gamma_{i+2}\gamma_i) = \gamma_i\gamma_{i+2}\,.\end{aligned} \tag{59}$$

Since the commutator $[R_i,R_{i+1}]\neq 0$, R_i and R_{i+1} do not commute and it is shown that $R_iR_{i+1}\neq R_{i+1}R_i$. This is an important result, since it shows that this representation describes non-abelian anyons, for which the order of exchange makes a difference. This separates them from abelian anyons, where a particle exchange of adjacent particles, independent of which particles are exchanged, contributes with a phase $e^{i\pi\alpha}$. The corresponding operators should then commute, since the exchange between abelian anyons does not depend on the order of exchange.

5 Exchange matrices

Everything discussed in the previous sections now enables us to find the exchange matrices. Let us here, in order to be able to perform all of the calculations thoroughly, restrict ourselves to the case of four Majorana operators. The results can be generalized to larger systems, which will be discussed in a later section. We now have $N = 4$ in the previous sections. Exchange matrices can then be created both from the rotors R_i and from the operators T_i, since both were shown to be representations of the braid group generators τ_i.

5.1 Rotors

With $N = 4$ there are three different rotors: R_1, R_2 and R_3. Viewing the states

$$\begin{aligned} |00\rangle &= |0\rangle \ , \\ |10\rangle &= \psi_1^\dagger |0\rangle \ , \\ |01\rangle &= \psi_2^\dagger |0\rangle \ , \\ |11\rangle &= \psi_1^\dagger \psi_2^\dagger |0\rangle \ , \end{aligned} \tag{60}$$

as a basis for the state space, we can through linear algebra find a matrix representation for each rotor by observing its action on each element in the basis. We find that

$$\begin{aligned} R_1 |00\rangle =& \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2) |00\rangle = \frac{1}{\sqrt{2}}(1 + \frac{1}{i}(\psi_1^2 - (\psi_1^\dagger)^2 + \psi_1^\dagger\psi_1 - \psi_1\psi_1^\dagger)) |00\rangle \\ =& \frac{1}{\sqrt{2}}(1 + i) |00\rangle = e^{i\frac{\pi}{4}} |00\rangle \ , \\ R_1 |10\rangle =& \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2) |10\rangle = \frac{1}{\sqrt{2}}(1 + \frac{1}{i}(\psi_1^2 - (\psi_1^\dagger)^2 + \psi_1^\dagger\psi_1 - \psi_1\psi_1^\dagger)) |10\rangle \\ =& \frac{1}{\sqrt{2}}(1 - i) |10\rangle = e^{-i\frac{\pi}{4}} |10\rangle \ , \\ R_1 |01\rangle =& \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2) |01\rangle = \frac{1}{\sqrt{2}}(1 + \frac{1}{i}(\psi_1^2 - (\psi_1^\dagger)^2 + \psi_1^\dagger\psi_1 - \psi_1\psi_1^\dagger)) |01\rangle \\ =& \frac{1}{\sqrt{2}}(1 + i) |01\rangle = e^{i\frac{\pi}{4}} |01\rangle \ , \\ R_1 |11\rangle =& \frac{1}{\sqrt{2}}(1 + \gamma_1\gamma_2) |11\rangle = \frac{1}{\sqrt{2}}(1 + \frac{1}{i}(\psi_1^2 - (\psi_1^\dagger)^2 + \psi_1^\dagger\psi_1 - \psi_1\psi_1^\dagger)) |11\rangle \\ =& \frac{1}{\sqrt{2}}(1 - i) |11\rangle = e^{-i\frac{\pi}{4}} |11\rangle \ . \end{aligned} \tag{61}$$

The matrix representation of R_1 in this basis can thus be written as

$$[R_1] = \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} . \tag{62}$$

These calculations were relatively simply since the Majorana operators in the expression for R_1 belong to the same pair. For R_3 this is also the case, where the same calculations instead yield

$$\begin{aligned} R_3 |00\rangle =& \frac{1}{\sqrt{2}}(1 + \gamma_3\gamma_4) |00\rangle = \frac{1}{\sqrt{2}}(1 + \frac{1}{i}(\psi_2^2 - (\psi_2^\dagger)^2 + \psi_2^\dagger\psi_2 - \psi_2\psi_2^\dagger)) |00\rangle \\ =& \frac{1}{\sqrt{2}}(1 + i) |00\rangle = e^{i\frac{\pi}{4}} |00\rangle \ , \\ R_3 |10\rangle =& \frac{1}{\sqrt{2}}(1 + \gamma_3\gamma_4) |10\rangle = \frac{1}{\sqrt{2}}(1 + \frac{1}{i}(\psi_2^2 - (\psi_2^\dagger)^2 + \psi_2^\dagger\psi_2 - \psi_2\psi_2^\dagger)) |10\rangle \\ =& \frac{1}{\sqrt{2}}(1 + i) |10\rangle = e^{i\frac{\pi}{4}} |10\rangle \ , \end{aligned} \tag{63}$$

$$
\begin{aligned}
R_3 |01\rangle =& \frac{1}{\sqrt{2}}(1+\gamma_3\gamma_4)|01\rangle = \frac{1}{\sqrt{2}}(1+\frac{1}{i}(\psi_2^2 - (\psi_2^\dagger)^2 + \psi_2^\dagger\psi_2 - \psi_2\psi_2^\dagger))|01\rangle \\
=& \frac{1}{\sqrt{2}}(1-i)|01\rangle = e^{-i\frac{\pi}{4}}|01\rangle \ , \\
R_3 |11\rangle =& \frac{1}{\sqrt{2}}(1+\gamma_3\gamma_4)|11\rangle = \frac{1}{\sqrt{2}}(1+\frac{1}{i}(\psi_2^2 - (\psi_2^\dagger)^2 + \psi_2^\dagger\psi_2 - \psi_2\psi_2^\dagger))|11\rangle \\
=& \frac{1}{\sqrt{2}}(1-i)|11\rangle = e^{-i\frac{\pi}{4}}|11\rangle \ ,
\end{aligned}
$$

and the matrix representation becomes

$$
[R_3] = \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} . \tag{64}
$$

For R_2 it becomes more complicated though, as $R_2 = \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3)$ contains Majorana operators from two different pairs. From the definition (19) we can see that

$$
\begin{aligned}
R_2 &= \frac{1}{\sqrt{2}}(1+\gamma_2\gamma_3) = \frac{1}{\sqrt{2}}(1+\frac{1}{i}((\psi_1-\psi_1^\dagger)(\psi_2+\psi_2^\dagger))) \\
&= \frac{1}{\sqrt{2}}(1-i(\psi_1\psi_2+\psi_1\psi_2^\dagger-\psi_1^\dagger\psi_2-\psi_1^\dagger\psi_2^\dagger)) ,
\end{aligned} \tag{65}
$$

and examining how this acts on the basis elements, we find that

$$
\begin{aligned}
R_2 |00\rangle =& \frac{1}{\sqrt{2}}(1-i(\psi_1\psi_2+\psi_1\psi_2^\dagger-\psi_1^\dagger\psi_2-\psi_1^\dagger\psi_2^\dagger))|00\rangle \\
=& \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle) , \\
R_2 |10\rangle =& \frac{1}{\sqrt{2}}(1-i(\psi_1\psi_2+\psi_1\psi_2^\dagger-\psi_1^\dagger\psi_2-\psi_1^\dagger\psi_2^\dagger))|10\rangle \\
=& \frac{1}{\sqrt{2}}(|10\rangle + i|01\rangle) , \\
R_2 |01\rangle =& \frac{1}{\sqrt{2}}(1-i(\psi_1\psi_2+\psi_1\psi_2^\dagger-\psi_1^\dagger\psi_2-\psi_1^\dagger\psi_2^\dagger))|01\rangle \\
=& \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle) , \\
R_2 |11\rangle =& \frac{1}{\sqrt{2}}(1-i(\psi_1\psi_2+\psi_1\psi_2^\dagger-\psi_1^\dagger\psi_2-\psi_1^\dagger\psi_2^\dagger))|11\rangle \\
=& \frac{1}{\sqrt{2}}(|11\rangle + i|00\rangle) ,
\end{aligned} \tag{66}
$$

which gives the matrix representation

$$
[R_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} . \tag{67}
$$

Having calculated the matrix representations for all of the rotors, we can calculate the exchange matrices for an exchange between any of the four particles, using the compositions of the braid group generators seen in the end of section 2. Define the matrix

$$
U_{p,i} = [R_i][R_{i+1}]\dots[R_{i+p}]\dots[R_{i+1}][R_i] \tag{68}
$$

as the matrix interchanging particles i and $i+1+p$. These matrices are the ones mentioned in the introduction, describing how the wave function changes as a result of a particle exchange. The possibilities where $p = 0$ in this limited case are

$$U_{0,1} = [R_1] = \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix}, \tag{69}$$

$$U_{0,2} = [R_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}, \tag{70}$$

and

$$U_{0,3} = [R_3] = \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix}. \tag{71}$$

With $p = 1$, i.e when there is one particle between the ones being exchanged, we get

$$\begin{aligned} U_{1,1} = [R_1][R_2][R_1] &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & e^{i\frac{\pi}{4}} \\ 0 & e^{-i\frac{\pi}{4}} & e^{i\frac{3\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & 0 \\ e^{i\frac{3\pi}{4}} & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & i \\ 0 & -i & i & 0 \\ 0 & i & i & 0 \\ i & 0 & 0 & -i \end{bmatrix}, \end{aligned} \tag{72}$$

and

$$\begin{aligned} U_{1,2} = [R_2][R_3][R_2] &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & e^{i\frac{3\pi}{4}} \\ 0 & e^{i\frac{\pi}{4}} & e^{i\frac{3\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & 0 \\ e^{i\frac{\pi}{4}} & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & i \\ 0 & i & i & 0 \\ 0 & i & -i & 0 \\ i & 0 & 0 & -i \end{bmatrix} \end{aligned} \tag{73}$$

as the possible exchange matrices. In this case, there is only one matrix with $p = 2$. Using the result from (73), we find

$$U_{2,1} = [R_1][R_2][R_3][R_2][R_1] = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} i & 0 & 0 & i \\ 0 & i & i & 0 \\ 0 & i & -i & 0 \\ i & 0 & 0 & -i \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} e^{i\frac{3\pi}{4}} & 0 & 0 & e^{i\frac{\pi}{4}} \\ 0 & e^{i\frac{\pi}{4}} & e^{i\frac{3\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} & e^{-i\frac{\pi}{4}} & 0 \\ e^{i\frac{3\pi}{4}} & 0 & 0 & e^{-i\frac{3\pi}{4}} \end{bmatrix}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & -1 \end{bmatrix}, \tag{74}$$

as the final exchange matrix.

5.1.1 Verifying relations

After all these calculations, we can verify them by checking that these matrices still have the right properties. The matrix representations of the rotors should satisfy the braid relations, and all of the exchange matrices should be unitary. The last property is important, since it is required in order to preserve the probability density of the wave function under the particle exchange. The diagonal matrices $U_{0,1} = [R_1]$ and $U_{0,3} = [R_3]$ can trivially be seen to be unitary, since their conjugate transpose equals their inverse. We also see that

$$U_{0,2}U_{0,2}^{\dagger} = [R_2][R_2]^{\dagger} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = I\,, \tag{75}$$

where I denotes the 4×4 identity matrix. This confirms that $[R_2]$ is unitary as well. In order to check the other exchange matrices we can note that given two unitary matrices A and B, we have

$$A^{\dagger} = A^{-1}\,, \quad B^{\dagger} = B^{-1}\,, \tag{76}$$

which gives

$$(AB)^{\dagger} = B^{\dagger}A^{\dagger} = B^{-1}A^{-1} = (AB)^{-1}\,. \tag{77}$$

Equation (77) shows that the product of two unitary matrices also is unitary - implying that all of the other exchange matrices are unitary since they are products of the unitary rotor matrices.

Furthermore, using (72), we see that

$$[R_2][R_1][R_2] = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix}$$
$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & e^{i\frac{3\pi}{4}} \\ 0 & e^{-i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{3\pi}{4}} & e^{i\frac{\pi}{4}} & 0 \\ e^{i\frac{\pi}{4}} & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \tag{78}$$
$$= \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & i \\ 0 & -i & i & 0 \\ 0 & i & i & 0 \\ i & 0 & 0 & -i \end{bmatrix} = [R_1][R_2][R_1]\,.$$

This shows that the matrices satisfy the first of the braid relations when $i = 1$, i.e $[R_1][R_2][R_1] =$

$[R_2][R_1][R_2]$. For the other possible case, similar calculations show that, using (73),

$$
\begin{aligned}
[R_3][R_2][R_3] &= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & i \\ 0 & 1 & i & 0 \\ 0 & i & 1 & 0 \\ i & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & e^{i\frac{\pi}{4}} \\ 0 & e^{i\frac{\pi}{4}} & e^{i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{3\pi}{4}} & e^{-i\frac{\pi}{4}} & 0 \\ e^{i\frac{3\pi}{4}} & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} i & 0 & 0 & i \\ 0 & i & i & 0 \\ 0 & i & -i & 0 \\ i & 0 & 0 & -i \end{bmatrix} = [R_2][R_3][R_2]\,.
\end{aligned}
\tag{79}
$$

Thus the first braid relation is satisfied also in that case, i.e $[R_2][R_3][R_2] = [R_3][R_2][R_3]$. Checking the second braid relation is easier, we see that

$$
\begin{aligned}
[R_1][R_3] &= \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\
&= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix},
\end{aligned}
\tag{80}
$$

while

$$
\begin{aligned}
[R_3][R_1] &= \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{-i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \begin{bmatrix} e^{i\frac{\pi}{4}} & 0 & 0 & 0 \\ 0 & e^{-i\frac{\pi}{4}} & 0 & 0 \\ 0 & 0 & e^{i\frac{\pi}{4}} & 0 \\ 0 & 0 & 0 & e^{-i\frac{\pi}{4}} \end{bmatrix} \\
&= \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix},
\end{aligned}
\tag{81}
$$

which shows that $[R_1][R_3] = [R_3][R_1]$. In this limited scenario, that is the only case that needs to be verified for the second relation.

5.2 T operators

The operators T_i introduced in section 4.1 can also be expressed as matrices, since they are ordinary rotations as known from linear algebra. Remembering that they also satisfy the braid relations, just like the rotors, exchange matrices can be created through them as well. The operator was defined as

$$
T_i(\gamma_j) = \begin{cases} \gamma_{i+1}\,, & j = i \\ -\gamma_i\,, & j = i+1 \\ \gamma_j\,, & j \neq i\,, j \neq i+1 \end{cases}\,,
\tag{82}
$$

and observing its action on the basis vectors γ_i, the matrix $[T_i]$ has non-zero elements

$$
[T_i]_{ab} = \begin{cases} 1\,, & a = b \neq i, i+1 \\ 1\,, & (a,b) = (i+1, i) \\ -1\,, & (a,b) = (i, i+1) \end{cases}\,.
\tag{83}
$$

In the case considered in this section, where $N = 4$, this would give the three matrices

$$
[T_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},
\tag{84}
$$

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{85}$$

and

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{86}$$

which, just like the rotors, are matrix representations of the braid group generators $\{\tau_1, \tau_2, \tau_3\}$.

The compositions of these create exchange matrices

$$\tilde{U}_{p,i} = [T_i][T_{i+1}] \dots [T_{i+p}] \dots [T_{i+1}][T_i]\,, \tag{87}$$

in analogy with how the exchange matrices $U_{p,i}$ were created. The notation with the tilde is used to distinguish them from the exchange matrices $U_{p,i}$ calculated earlier. We find

$$\tilde{U}_{0,1} = [T_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{88}$$

$$\tilde{U}_{0,2} = [T_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{89}$$

and

$$\tilde{U}_{0,3} = [T_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{90}$$

as the three exchange matrices with $p = 0$. With $p = 1$, there are the matrices

$$\begin{aligned} \tilde{U}_{1,1} = [T_1][T_2][T_1] &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned} \tag{91}$$

and

$$\begin{aligned} \tilde{U}_{1,2} = [T_2][T_3][T_2] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{92}$$

For the final exchange matrix, with $p = 2$, equation (92) gives

$$
\begin{aligned}
\tilde{U}_{2,1} = [T_1][T_2][T_3][T_2][T_1] &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .
\end{aligned} \tag{93}
$$

5.2.1 Verifying relations

Just like earlier, it is now a good idea to verify that these exchange matrices satisfy the same relations. Since the braid properties of the rotors were used to show that the operators T_i satisfied the braid relations, the matrix representations should do the same. All of the exchange matrices should also be unitary, as explained earlier. We find that

$$
\tilde{U}_{0,1}\tilde{U}_{0,1}^{\dagger} = [T_1][T_1]^{\dagger} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \,, \tag{94}
$$

$$
\tilde{U}_{0,2}\tilde{U}_{0,2}^{\dagger} = [T_2][T_2]^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \,, \tag{95}
$$

as well as

$$
\tilde{U}_{0,3}\tilde{U}_{0,3}^{\dagger} = [T_3][T_3]^{\dagger} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \,, \tag{96}
$$

which shows that the first three exchange matrices are unitary. Using the same reasoning as in the previous subsection, equation (77) implies that the rest of them also are unitary, since they themselves are products of unitary matrices.

Furthermore, using equation (91), we see that

$$
\begin{aligned}
[T_2][T_1][T_2] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [T_1][T_2][T_1] .
\end{aligned} \tag{97}
$$

This shows that the $[T_i]$-matrices satisfy the first of the braid relations when $i = 1$, i.e $[T_1][T_2][T_1] =$

$[T_2][T_1][T_2]$. With 4 matrices, there is one more possible case for this relation. Using equation (92),

$$\begin{aligned}
[T_3][T_2][T_3] &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = [T_2][T_3][T_2] \,,
\end{aligned} \tag{98}$$

and thus $[T_2][T_3][T_2] = [T_3][T_2][T_3]$, which means that the first braid relation is satisfied in all of the cases. Just as before, checking the second braid relation only requires the single case with

$$[T_1][T_3] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \tag{99}$$

and

$$[T_3][T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} , \tag{100}$$

which shows that $[T_1][T_3] = [T_3][T_1]$.

6 Eigenvalues

Given the exchange matrices calculated in the previous section, one property that specifically is of interest is the eigenvalues of the matrices. They contain information about the repelling forces between particles, and how the exchange statistics differ from ordinary fermions and bosons [10]. The eigenvalues are also invariant under a change of basis, which makes them more of an intrinsic property of the system than a result of our choice of basis.

6.1 Rotors

For the exchange matrices $U_{p,i}$ calculated through the rotor representation of the braid group generators, we find the characteristic polynomials

$$\begin{aligned}
\det(U_{0,1}-\lambda I) &= (e^{i\frac{\pi}{4}}-\lambda)^2(e^{-i\frac{\pi}{4}}-\lambda)^2 = \left(\frac{1+i}{\sqrt{2}}-\lambda\right)^2\left(\frac{1-i}{\sqrt{2}}-\lambda\right)^2 ,\\
\det(U_{0,2}-\lambda I) &= \lambda^4-2\sqrt{2}\lambda^3+4\lambda^2-2\sqrt{2}\lambda+1 = (\lambda^2-\sqrt{2}\lambda+1)^2\\
&= \left(\frac{1+i}{\sqrt{2}}-\lambda\right)^2\left(\frac{1-i}{\sqrt{2}}-\lambda\right)^2 ,\\
\det(U_{0,3}-\lambda I) &= (e^{i\frac{\pi}{4}}-\lambda)^2(e^{-i\frac{\pi}{4}}-\lambda)^2 = \left(\frac{1+i}{\sqrt{2}}-\lambda\right)^2\left(\frac{1-i}{\sqrt{2}}-\lambda\right)^2 ,\\
\det(U_{1,1}-\lambda I) &= \lambda^4+2\lambda^2+1 = (i-\lambda)^2(-i-\lambda)^2 ,\\
\det(U_{1,2}-\lambda I) &= \lambda^4+2\lambda^2+1 = (i-\lambda)^2(-i-\lambda)^2 ,\\
\det(U_{2,1}-\lambda I) &= \lambda^4+1\\
&= \left(\frac{1+i}{\sqrt{2}}-\lambda\right)\left(\frac{1-i}{\sqrt{2}}-\lambda\right)\left(\frac{-1+i}{\sqrt{2}}-\lambda\right)\left(\frac{-1-i}{\sqrt{2}}-\lambda\right) .
\end{aligned} \tag{101}$$

If we denote the set of eigenvalues of a matrix A as $\mathrm{Eigen}(A)$, the roots to these polynomials give

$$\mathrm{Eigen}(U_{0,1}) = \mathrm{Eigen}(U_{0,2}) = \mathrm{Eigen}(U_{0,3}) = \left\{\frac{1\pm i}{\sqrt{2}}\right\} , \tag{102}$$

where both values have the algebraic multiplicity 2. This also verifies the similarity of the rotors, since they share the same eigenvalues. For the next exchange matrices, we instead get the eigenvalues

$$\mathrm{Eigen}(U_{1,1}) = \mathrm{Eigen}(U_{1,2}) = \{\pm i\} , \tag{103}$$

again both with multiplicity 2. Finally, for the last matrix,

$$\mathrm{Eigen}(U_{2,1}) = \left\{\pm\frac{1+i}{\sqrt{2}}, \pm\frac{1-i}{\sqrt{2}}\right\} . \tag{104}$$

An illustration of how these eigenvalues differ from the three-dimensional case (± 1) can be done by mapping the eigenvalues onto the unit circle in the complex plane, as seen in figure 5.

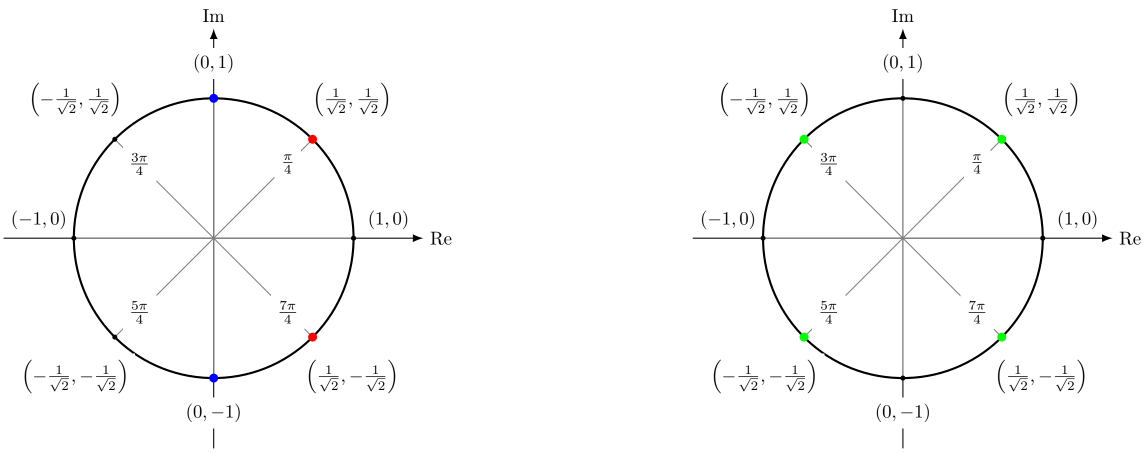

Figure 5: Eigenvalues of exchange matrices $U_{p,i}$, with $p = 0$ (red), $p = 1$ (blue) and $p = 2$ (green).

6.2 T operators

For the exchange matrices obtained through the matrix representations of the operators T_i, we find the characteristic polynomials

$$\begin{aligned}
\det(\tilde{U}_{0,1} - \lambda I) &= \lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2(\lambda^2 + 1)\,, \\
\det(\tilde{U}_{0,2} - \lambda I) &= \lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2(\lambda^2 + 1)\,, \\
\det(\tilde{U}_{0,3} - \lambda I) &= \lambda^4 - 2\lambda^3 + 2\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2(\lambda^2 + 1)\,, \\
\det(\tilde{U}_{1,1} - \lambda I) &= \lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2\,, \\
\det(\tilde{U}_{1,2} - \lambda I) &= \lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2\,, \\
\det(\tilde{U}_{2,1} - \lambda I) &= \lambda^4 + 2\lambda^3 + 2\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2(\lambda^2 + 1)\,,
\end{aligned} \tag{105}$$

which gives the eigenvalues

$$\mathrm{Eigen}(\tilde{U}_{0,1}) = \mathrm{Eigen}(\tilde{U}_{0,2}) = \mathrm{Eigen}(\tilde{U}_{0,3}) = \{1, \pm i\}\;, \tag{106}$$

where the eigenvalue $\lambda = 1$ has the algebraic multiplicity 2. We also see that

$$\mathrm{Eigen}(\tilde{U}_{1,1}) = \mathrm{Eigen}(\tilde{U}_{1,2}) = \{\pm 1\}\,, \tag{107}$$

where both values has multiplicity 2. For the last matrix,

$$\mathrm{Eigen}(\tilde{U}_{2,1}) = \{-1, \pm i\}\,, \tag{108}$$

where, similarly to the first ones, the eigenvalue $\lambda = -1$ has multiplicity 2. In the same way as in the previous section, these eigenvalues can also be illustrated on the unit circle in the complex plane. This is shown in figure 6.

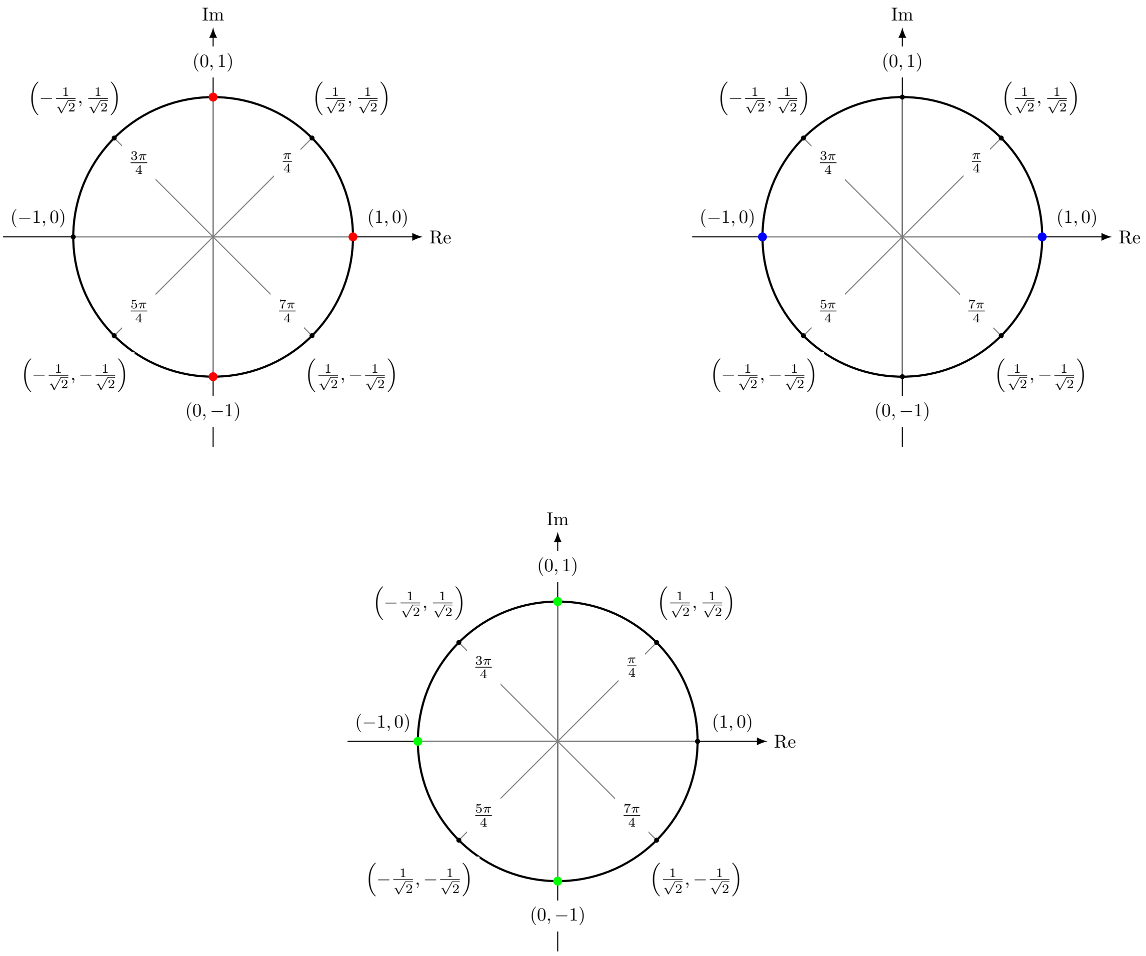

Figure 6: Eigenvalues of exchange matrices $\tilde{U}_{p,i}$, with $p = 0$ (red), $p = 1$ (blue) and $p = 2$ (green).

7 Generalization

We have concluded that the exchange of non-abelian anyons can be represented by the braid group, where the degree of the braid group is determined by the number of particles. In this study we thoroughly examined the case of four Majorana fermions, and the next step would naturally be to look at a system with more particles. The same methods that have been used to acquire the 4×4 exchange matrices can be generalized to larger systems of arbitrary size. The expression (83), i.e

$$[T_i]_{ab} = \begin{cases} 1\,, & a = b \neq i, i+1 \\ 1\,, & (a,b) = (i+1, i) \\ -1\,, & (a,b) = (i, i+1) \end{cases}\,, \tag{109}$$

which determines the components of the matrix representations of the operators T_i, is independent of the size of the system and can thus easily be generalized, along with the corresponding exchange matrices

$$\tilde{U}_{p,i} = [T_i][T_{i+1}]\dots[T_{i+p}]\dots[T_{i+1}][T_i]\,. \tag{110}$$

These matrices, just like the matrices $[T_i]$, are of size $N \times N$, since they are operators acting on the vector space $\mathbb{R}^N$.

The matrix representations of the rotors will require some more work. The basis now becomes

$$\begin{aligned} |00\dots0\rangle &= |0\rangle\;, \\ |10\dots0\rangle &= \psi_1^\dagger\,|0\rangle\;, \\ |01\dots0\rangle &= \psi_2^\dagger\,|0\rangle\;, \\ &\;\;\vdots \\ |11\dots1\rangle &= \psi_1^\dagger\psi_2^\dagger\dots\psi_{\frac{N}{2}}^\dagger\;|0\rangle\;, \end{aligned} \tag{111}$$

which contains exactly $2^{\frac{N}{2}}$ elements. The factor $\frac{1}{2}$ is a result of the pairing between Majorana fermions. This number of basis elements is the reason these matrices grow exponentially larger when increasing the number of particles, since the corresponding matrices will be of size $2^{\frac{N}{2}} \times 2^{\frac{N}{2}}$. It is thus also the reason why such a relatively small case was examined in depth earlier. The matrices can be determined in the same way as earlier, by observing how the rotor

$$R_i = \frac{1}{\sqrt{2}}(1 + \gamma_i\gamma_{i+1}) \tag{112}$$

acts on the basis elements. When expanding the Majorana operators γ_i in terms of the fermion operators according to equation (19), one will also encounter the difference between the matrices corresponding to rotors R_i with even and odd i. If i is odd, both γ_i and γ_{i+1} belong to the same pair, while if i is even they do not - resulting in a more complicated matrix, as seen earlier. When these matrices are known, the exchange matrices can again be determined through

$$U_{p,i} = [R_i][R_{i+1}]\dots[R_{i+p}]\dots[R_{i+1}][R_i]\,. \tag{113}$$

We can here also briefly discuss the differences between the exchange matrices $\tilde{U}_{p,i}$ and $U_{p,i}$. They are both representations of the braid group generators, but differ since they act on different spaces. We have seen that T_i operate on the Majorana operators γ_i, while the rotors R_i instead operate on the state $|\Psi\rangle$ of the system. This also gives rise to the difference in size between the two kinds of exchange matrices; $\tilde{U}_{p,i}$ is of size $N \times N$ while $U_{p,i}$ is of size $2^{\frac{N}{2}} \times 2^{\frac{N}{2}}$. These sizes show that it was only a coincidence that they happened to be the same size in the case discussed in section 5, where $N = 4$. An interesting note here is the similarity between these sizes and the dimensions of $\mathbb{R}^N$ (which obviously is N) and the corresponding Clifford algebra. The latter is given by (31) as

$$\dim \mathrm{Cl}(\mathbb{R}^N) = 2^N\,. \tag{114}$$

8 Summary

In this thesis we discussed non-abelian anyons and their properties. The configuration space M_N^d was created, which lead to the conclusion that the fundamental group of the configuration space in two dimensions is the braid group, i.e $\pi_1(M_N^2) \cong B_N$. We then examined the braid relations, and their connection to the physical system of particles. We realized that in order to find exchange matrices, we would need to look for matrix representations of the braid group generators τ_i.

With the purpose of reaching this main goal, we discussed the fermionic creation and annihilation operators along with their effect on quantum states. From these operators we also showed how to create Majorana fermions, since excitations of these can exhibit non-abelian statistics. We then introduced the Clifford algebra through the definitions of k-blades and the Clifford product. It became clear that it was a very powerful tool in this setting, both for the ability to cooperate with the Majorana operators and for producing representations of the braid group generators. The latter was shown to be the rotors R_i of the Clifford algebra.

These rotors, and the corresponding operators T_i, were shown to satisfy the braid relations and were then used to create exchange matrices for the non-abelian anyons. The case of 4 Majorana fermions was studied thoroughly, and it was verified that the matrices then satisfied all of the required conditions. In that limited case, we also calculated all of the eigenvalues of the exchange matrices. In conclusion, we found that a Clifford algebra has a natural connection to the framework of non-abelian anyons, and can be of great help when calculating the corresponding exchange matrices.

9 Future work

As mentioned earlier, one obvious continuation of our work is to use the same methods to produce exchange matrices, but for larger systems. The procedure for this was summarized in section 7. An interesting thing would be to examine if it is possible to verify the properties of the matrices in a more general setting, i.e without confining the system to a specific value of N. Furthermore, there are two other areas where our results could be used for continued work.

9.1 Lower limit of the kinetic energy

One possible continuation that would require some more study is to more thoroughly examine the exchange matrix eigenvalues. As mentioned in section 6, they contain information about the repelling force between particles. More specifically, they could be used to find a lower limit of this force in this kind of system. It would then also be possible to give a lower limit of the kinetic energy of the system [6]. Given a system of N particles, a purely kinetic Hamiltonian is given by

$$\hat{H} = \hat{T} = -\frac{\hbar^2}{2m}\sum_{j=1}^{N} \nabla_j^2 \,, \tag{115}$$

where ∇_j is the gradient acting on the j:th coordinate. In a system with anyons, Hardy's inequality could be used to give a lower limit of the kinetic energy

$$T = \langle \Psi | \hat{T} | \Psi \rangle \ . \tag{116}$$

This limit would then also depend on the eigenvalues of the exchange operators [10]. It could then be of interest to generalize this limit to the case of non-abelian anyons, and examine whether the eigenvalues calculated in our work could be used to draw any conclusions.

9.2 Topological quantum computers

Theoretically, one of the more relevant applications of anyons and the theory of braid group statistics is found in the field of quantum computers. In a regular computer, the memory consists of bits, which can take the values 0 or 1. In a quantum computer these are replaced by quantum bits (qubits), which are quantum states given by

$$|\psi\rangle = ae^{i\alpha}|0\rangle + be^{i\beta}|1\rangle \ , \quad a, b \in \mathbb{R} \,, \tag{117}$$

where $|0\rangle$ and $|1\rangle$ are the two classical states of a bit. One of the major problems of this setup is how sensitive these quantum mechanical systems are to interference from the outside world, which is a phenomenon known as quantum decoherence [11]. To avoid this, one would need to completely isolate the system from its environment.

However, another approach would be to use the braids of non-abelian anyons to form the logic gates of the computer, which would then be called a topological quantum computer. The advantage of using these braids as opposed to trapped "particles in a box" is that it would greatly improve the stability of the computer [12]. The reasoning behind this is quite intuitive; a small perturbation of trapped particles can cause decoherence, but will not change the topological properties of the world line braids. Further study could then show whether the exchange matrices and methods from our work could be applied in this area, when performing calculations involving these topological qubits.

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