KTH Engineering Sciences

# Families of cycles and the Chow scheme

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Doctoral Thesis
Stockholm, Sweden 2008

TRITA-MAT-08-MA-06
ISSN 1401-2278
ISRN KTH/MAT/DA 08/05-SE
ISBN 978-91-7178-999-0

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SE-100 44 Stockholm
SWEDEN

Akademisk avhandling som med tillstånd av Kungl Tekniska högskolan framlägges till offentlig granskning för avläggande av teknologie doktorsexamen i matematik måndagen den 11 augusti 2008 klockan 13.00 i Nya kollegiesalen, F3, Kungl Tekniska högskolan, Lindstedtsvägen 26, Stockholm.



Tryck: Universitetsservice US AB

## Abstract

The objects studied in this thesis are *families of cycles* on schemes. A space — the *Chow variety* — parameterizing effective equidimensional cycles was constructed by Chow and van der Waerden in the first half of the twentieth century. Even though cycles are simple objects, the Chow variety is a rather intractable object. In particular, a good functorial description of this space is missing. Consequently, descriptions of the corresponding families and the infinitesimal structure are incomplete. Moreover, the Chow variety is not intrinsic but has the unpleasant property that it depends on a given projective embedding. A main objective of this thesis is to construct a closely related space which has a good functorial description. This is partly accomplished in the last paper.

The first three papers are concerned with families of *zero-cycles*. In the first paper, a functor parameterizing zero-cycles is defined and it is shown that this functor is represented by a scheme — *the scheme of divided powers*. This scheme is closely related to the symmetric product. In fact, the scheme of divided powers and the symmetric product coincide in many situations.

In the second paper, several aspects of the scheme of divided powers are discussed. In particular, a universal family is constructed. A different description of the families as *multi-morphisms* is also given. Finally, the set of $k$-points of the scheme of divided powers is described. Somewhat surprisingly, cycles with certain rational coefficients are included in this description in positive characteristic.

The third paper explains the relation between the Hilbert scheme, the Chow scheme, the symmetric product and the scheme of divided powers. It is shown that the last three schemes coincide as topological spaces and that all four schemes are isomorphic outside the degeneracy locus.

The last paper gives a definition of families of cycles of arbitrary dimension and a corresponding Chow functor. In characteristic zero, this functor agrees with the functors of Barlet, Guerra, Kollár and Suslin-Voevodsky when these are defined. There is also a monomorphism from Angéniol's functor to the Chow functor which is an isomorphism in many instances. It is also confirmed that the morphism from the Hilbert functor to the Chow functor is an isomorphism over the locus parameterizing normal subschemes and a local immersion over the locus parameterizing reduced subschemes — at least in characteristic zero.

# Contents

# Included papers

**Paper I:** **Families of zero-cycles and divided powers: I. Representability**, submitted, arXiv:0803.0618v1.

**Paper II:** **Families of zero-cycles and divided powers: II. The universal family**.

**Paper III:** **Hilbert and Chow schemes of points, symmetric products and divided powers**.

**Paper IV:** **Families of cycles**.

# Related papers

**A minimal set of generators for the ring of multisymmetric functions**, Ann. Inst. Fourier (Grenoble) **57** (2007), no. 6, 1741–1769, arXiv:0710.0470.

**Submersions and effective descent of étale morphisms**, To appear in *Bull. Sci. Math. France*, arXiv:0710.2488v3.

**Existence of quotients by finite groups and coarse moduli spaces**, Preprint, Aug 2007, arXiv:0708.3333v1.

**Representability of Hilbert schemes and Hilbert stacks of points**, Preprint, Feb 2008, arXiv:0802.3807v1.

(with Roy Skjelnes) **The space of generically étale families**, Preprint, Mar 2007, arXiv:math.AG/0703329.

# Introduction

## Families of cycles

Let $X$ be a scheme. The cycles on $X$ is the free abelian group $C(X)$ generated by the set of reduced and irreducible closed subschemes of $X$. A cycle on $X$ is thus a formal sum $\sum_i m_i[Z_i]$ where the $m_i$'s are integers and the $Z_i$'s are closed subvarieties of $X$. A cycle is effective if all the $m_i$'s are positive. We let $C_r(X)$ be the subgroup of cycles which are equidimensional of dimension $r$. Given a projective embedding $X \hookrightarrow \mathbb{P}^n$, every $r$-dimensional cycle comes with a *degree*. Geometrically the degree can be interpreted as the number of points, counted with multiplicity, after intersecting the cycle with $r$ general hyperplanes.

Let $X$ be a quasi-projective variety with a given embedding $X \hookrightarrow \mathbb{P}^n$. A fundamental result [CW37] in classical algebraic geometry, due to Chow and van der Waerden, is the existence of a quasi-projective variety $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ — the *Chow variety* — parameterizing cycles of dimension $r$ and degree $d$ on $X$. The main goal of this thesis is to obtain a better understanding of this variety.

A family of cycles, parameterized by a variety $S$, is roughly a collection of cycles $\{\mathcal{Z}_s\}_{s\in S}$ on $X$ which is "continuous". A natural interpretation of continuity is that we require $\{\mathcal{Z}_s\}$ to be induced by a morphism $S \to \mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$. As expected, such a family is then represented by a single cycle $\mathcal{Z}$ on $X \times S$. There are, however, several serious problems with this approach.

- It can be shown that $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ depends on the chosen projective embedding in positive characteristic. Thus, we do not have a good notion of "continuous" in this case.

- The cycle $\mathcal{Z}$ on $X \times S$ representing the family $\{\mathcal{Z}_s\}$ is not "flat", as the objects usually are in other similar problems. This has several drawbacks, for example, $\mathcal{Z}_s$ is not simply the fiber of $\mathcal{Z}$ over $s$.

- It is desirable to have a notion of families of cycles also over non-reduced *schemes*. In particular, it is important to have an infinitesimal theory to be able to study deformations of cycles. The classical construction of the Chow variety comes without any infinitesimal structure, that is, it is a variety and not a scheme. It is therefore not at all clear what a family parameterized by a scheme is.

## Methods

The first problem is due to a deficiency in the classical construction. After choosing a sufficiently ample projective embedding, the classical construction gives the "correct" Chow variety, at least for zero-cycles. Also, families of cycles can be defined without referring to the Chow variety so this is not a serious problem.

The second problem is mainly technical. It also indicates that representing $\{\mathscr{Z}_s\}$ as a cycle on $X \times S$, although appealing, is not an ideal approach.

The third problem is paramount. A possible solution, closely related to the construction of the Chow variety, is to represent families of cycles as certain families of divisors on a grassmannian parameterizing linear subspaces of complementary dimension. This seems to be somewhat cumbersome and has not been systematically studied. The method introduced by Barlet [Bar75] is of a dual nature. Instead of intersecting with subspaces, he studies *projections* onto affine spaces of the same dimension as the cycle. A family is then an object, in his case a cycle $\mathscr{Z}$ on $X \times S$, which induces a family of zero-cycles over every projection.

When the parameter scheme $S$ is not reduced, then in general there is not such a simple object as a cycle on $X \times S$ which induces a family of zero-cycles over each projection. The method advocated in this thesis is to not require the existence of such an object a priori. Instead, a family $\alpha$ is defined to be a collection of zero-cycles, indexed by all projections, satisfying natural compatibility conditions. Under appropriate conditions on $S$ and the family $\alpha$, one can then find simpler geometric objects inducing this family. For example, if $S$ is reduced then $\alpha$ is represented by a cycle $\mathscr{Z}$ on $X \times S$ as above. If the cycles of the family $\alpha$ are without multiplicities or are divisors, then there is a subscheme $Z$ of $X \times S$ inducing $\alpha$. If $S$ is of characteristic zero, then $\alpha$ is represented by its relative fundamental class.

Angéniol [Ang80], working exclusively in characteristic zero, starts from the opposite end and defines a family as a class, the relative fundamental class, and imposes conditions on this class ensuring that it induces a zero-cycle over each projection. It is reasonable to believe that Angéniol's definition of a family agrees with the definition outlined above but this is not at all clear.

Angéniol's approach, applying duality and residue theory, requires deeper theory and is arguably more complicated. On the other hand, he is able to give a deformation theory for families of cycles and to show representability. My method, although also technical and sometimes cumbersome, has the advantage of giving a definition in great generality without assuming projectivity, smoothness, characteristic zero etc. It is also more geometric and easier to relate with other definitions, such as those of Kollár [Kol96] and Suslin-Voevodsky [SV00].

## Overview of the thesis and results

In Paper I, a functor $\underline{\Gamma}^d_{X/S}$ parameterizing families of zero-cycles on $X/S$ is defined and shown to be represented by an algebraic space $\Gamma^d(X/S)$. This space — the space of divided powers — is closely related to the divided powers algebra and can be viewed as a functorially well-behaved version of the symmetric product $\operatorname{Sym}^d(X/S)$. The algebraicity of $\Gamma^d(X/S)$, for an arbitrary algebraic space $X/S$, is obtained via an explicit étale covering.

With similar methods, the existence of geometric quotients [Ryd07b] and the algebraicity of Hilbert stacks [Ryd08] can be shown.

In Paper II, several aspects of the space of divided powers are discussed. A universal family of zero-cycles is constructed and a description of the $k$-points of $\Gamma^d(X/S)$ is given. Also, a different description of the functor $\underline{\Gamma}^d_{X/S}$ in terms of multi-morphisms is given.

In Paper III, the relation between the Hilbert scheme of points, the symmetric product, the space of divided powers and the Chow variety of zero-cycles is studied. It is shown that all four of these schemes coincide over the locus parameterizing non-degenerate families and it is shown that the last three schemes coincide as topological spaces. The dependence of the Chow variety on a given projective embedding is explained using weighted projective spaces. This is related to the fact that the ring of multisymmetric functions is not generated by elementary multisymmetric functions in positive characteristic [Ryd07a]. The morphism from the Hilbert scheme of points to the Chow variety, is essentially a blow-up [Hai98, ES04, RS07, Ran08] and has been used to study the Hilbert scheme of points [Fog68, Göt94].

In Paper IV, families of cycles of arbitrary dimension are defined. This definition generalizes previous definitions by Barlet [Bar75], Guerra [Gue96], Kollár [Kol96] and Suslin-Voevodsky [SV00]. Conjecturally, this definition also coincides with Angéniol's definition [Ang80] in characteristic zero. Indeed, this is so in many cases. The Chow functor, parameterizing families of proper and equidimensional cycles, is representable in similar situations.

There are natural morphisms from the Hilbert scheme [FGA], the Hilbert stack [Art74, App.], the space of Cohen-Macaulay curves [Høn05], the stack of branchvarieties [AK06] and the Kontsevich space of stable maps [Kon95] into the Chow functor. It is shown that all these morphisms are isomorphisms over the subset parameterizing normal subschemes, at least in characteristic zero. It is also shown that the Hilbert-Chow morphism is a local immersion over the subset parameterizing reduced subschemes.

The interdependence between the papers is as follows. Paper II presupposes Paper I and Papers III and IV depend on both Paper I and II. The fourth paper is to a large extent work in progress.

## Lawson homology

There is a natural equivalence relation on $C_r(X)$ called *rational equivalence* and the quotient by this relation is the *Chow group* $A_r(X)$. If $X$ is a smooth and projective scheme of dimension $n$, then $A^\bullet(X) = \bigoplus_{i=0}^n A_{n-i}(X)$ is a graded ring — the *Chow ring* — under the intersection product. The Chow ring is a central object of study in algebraic geometry and an alternative to usual cohomology theories. In fact, for any Weil cohomology $H^\bullet$ on $X$, such as Betti cohomology, $l$-adic cohomology or algebraic singular cohomology, there is a ring homomorphism $A^\bullet(X) \to H^{2\bullet}(X)$.

The cycle map $C_r(X) \to A_r(X) \to H_{2r}(X)$ factors through *algebraic equivalence.* Two cycles $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are algebraically equivalent if there exists an effective cycle $\mathcal{W}$ such that $\mathcal{Z}_1 + \mathcal{W}$ and $\mathcal{Z}_2 + \mathcal{W}$ corresponds to two points in the same connected component of the Chow variety $\mathrm{ChowVar}_r(X)$. The quotient of $C_{n-1}(X)$ by algebraic equivalence is the Néron-Severi group of $X$.

One of the more spectacular applications of Chow varieties is Lawson (co)homology. The Lawson homology groups of $X$ are defined as $L_rH_k(X) = \pi_{k-2r}(\mathrm{Chow}_r(X)^+)$ where

$+$ is a topological group completion [Law89, Fri89]. In particular, it immediately follows that $L_rH_{2r}(X)$ is the group of $r$-dimensional cycles up to algebraic equivalence. Dold-Thom's theorem is that the singular homology group $H_k(X,\mathbb{Z})$ is naturally isomorphic to $\pi_k\left(\left(\coprod_d \operatorname{Sym}^d(X)\right)^+\right) = \pi_k\left(\operatorname{Chow}_0(X)^+\right)$. Thus $L_0H_k(X) = H_k(X,\mathbb{Z})$ and Lawson homology interpolates between topological homology groups and algebraic groups.

When studying Lawson homology, operations such as proper push-forward, flat pull-back and proper intersections are used extensively. These are commonly only defined as algebraic maps between Chow varieties, i.e., as continuous maps induced by algebraic correspondences. This is equivalent with giving morphisms on the *semi-normalizations*. For topological purposes, it is enough to define these operations as algebraic maps. Nevertheless, it is expected that these operations exist as morphisms. In Paper IV, it is shown that the push-forward and the pull-back are defined as morphisms under certain assumptions.

## An example

To illustrate the difference between the Chow scheme and other parameter spaces we study curves of degree two in $\mathbb{P}^3$. Recall that such a curve, if reduced, is either a conic contained in a plane ($g = 0$) or two skew lines ($g = -1$). The main distinction between the parameter spaces is the various descriptions of curves with multiplicities, that is, double lines in our example. Note that all these parameter schemes are isomorphic over the locus parameterizing smooth curves. The schemes are illustrated with figures where ovals indicate closed subsets and the numbers are the dimensions of the corresponding closed subsets.

**The Chow scheme.** The Chow scheme parameterizes one-dimensional cycles $\mathcal{Z}$ of degree two on $\mathbb{P}^3$. It is connected and has two irreducible components. One of these parameterizes conics contained in a plane and lines of multiplicity two. The other component parameterizes pairs of skew lines, singular conics and lines of multiplicity two. In characteristic zero, the Chow scheme is *non-reduced* over the locus parameterizing lines of multiplicity two [Ang80, Rem. 6.4.3].

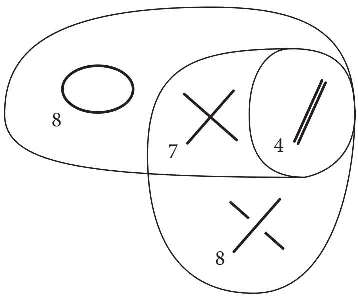


Figure 1: The Chow scheme of degree two curves on $\mathbb{P}^3$.

**The Hilbert scheme.** The Hilbert scheme parameterizes one-dimensional subschemes of $\mathbb{P}^3$ of degree two. The Hilbert scheme has an infinite number of connected components indexed by the (arithmetic) genus $g$ of the curve and is non-empty for any integer $g \leq 0$. In fact, there are even subschemes of arbitrary negative genus which are Cohen-Macaulay, that is, without components of dimension zero [Har04]. When $g = 0$, the Hilbert scheme is smooth and irreducible and parameterizes conics. When $g = -1$, the Hilbert scheme consists of several irreducible components. The main component, with generic member a pair of skew lines, is generically smooth.

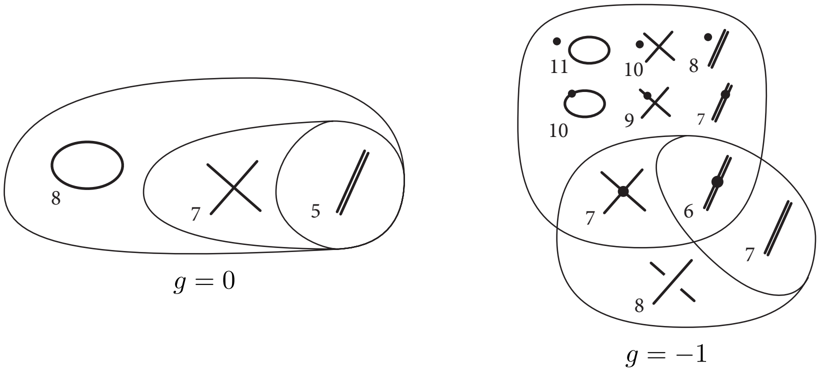


Figure 2: Part of the Hilbert scheme of degree two curves on $\mathbb{P}^3$.

**The stack of Branch-varieties.** The stack of Branch varieties parameterizes reduced curves $C$ together with a finite morphism to $\mathbb{P}^3$. It has an infinite number of connected components indexed by the genus $g$ of the curve $C$. It is non-empty for every integer $g \geq -1$. For positive $g$, it parameterizes reduced, possibly singular, genus $g$ curves $C$ equipped with a ramified degree two covering $C \to \mathbb{P}^1$ of a line in $\mathbb{P}^3$. In particular, $C$ is hyperelliptic.

When $g = 0$ then $C$ is either a smooth rational curve or two secant lines. The map $C \to \mathbb{P}^3$ either embedds $C$ as a conic in a plane or is a ramified cover of degree two over a line in $\mathbb{P}^3$.

When $g = -1$ then $C$ is a pair of skew lines which either sits inside $\mathbb{P}^3$, maps onto two secant lines of $\mathbb{P}^3$ or maps onto a single line of $\mathbb{P}^3$. This component is the stack quotient of a product of grassmannians $[\mathrm{Gr}(2,4)^2/\mathfrak{S}_2]$ and hence smooth.

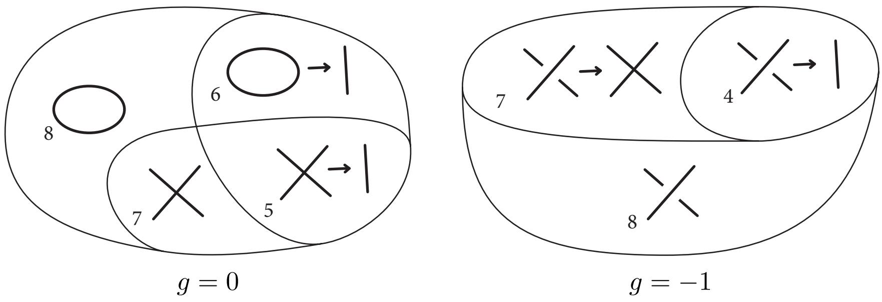


Figure 3: Part of the stack of Branch-varieties of degree two curves mapping to $\mathbb{P}^3$.

## Open questions

- An explicit description of the deformation theory of multiplicity-free cycles is yet missing. This should be a far more amenable problem than a description of the deformation theory of general cycles.
- Representability of the Chow functor is not shown except in the cases when the functor is shown to coincide with other descriptions. To show the representability using Artin's criteria, knowledge of the deformation theory is crucial. In the projective case, a projective embedding is conjecturally given by the classical Chow construction.
- Many of the operations on families of cycles, such as push-forward, pull-back and intersections, have only been defined for families satisfying certain properties. It should be possible to define these operations in general.
- Given a sheaf $\mathcal{F}$ on $X$, there is an induced sheaf on $\mathrm{Chow}_{0,d}(X)$. Are there similar Chow sheaves in higher dimension?
- Even though there is a good functorial description of the Chow-scheme, there are some features similar to that of a coarse functor to a stack. In characteristic zero $\mathrm{Chow}_{0,d}(X) = \mathrm{Sym}^d(X)$ is the coarse moduli space of the symmetric stack. Is there a similar stack in higher dimension? This stack would probably be without automorphisms over the locus parameterizing multiplicity-free cycles.

## Acknowledgments

First of all I am indebted to my advisor, Dan Laksov, for all help and support. I also thank Roy Skjelnes for countless numbers of discussions and for carefully reading several manuscripts and Bill Fulton for his hospitality during my stay at U. of Michigan. I am also grateful to my brother Jonatan for rendering the figures at short notice. A special thanks goes to my current and former officemates: M. Better, M. Blomgren, D. Eklund, A. Enblom, L. Halle and J. Söderberg. Last, but not least, I thank my wife for her everlasting patience, unconditional love and support, our two wonderful sons and our gracious Lord.

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# Paper I

# FAMILIES OF ZERO-CYCLES AND DIVIDED POWERS: I. REPRESENTABILITY

DAVID RYDH

ABSTRACT. Let $X/S$ be a separated algebraic space. We construct an algebraic space $\Gamma^d(X/S)$, *the space of divided powers*, which parameterizes zero cycles of degree $d$ on $X$. When $X/S$ is affine, this space is affine and given by the spectrum of the ring of divided powers. In characteristic zero or when $X/S$ is flat, the constructed space coincides with the symmetric product $\mathrm{Sym}^d(X/S)$. We also prove several fundamental results on the kernels of multiplicative polynomial laws necessary for the construction of $\Gamma^d(X/S)$.

## INTRODUCTION

Chow varieties, parameterizing families of cycles of a certain dimension and degree, are classically constructed using explicit projective methods [CW37, Sam55]. Moreover, Chow varieties are defined as *reduced* schemes and in positive characteristic the classical construction has the unpleasant property that it depends on a given projective embedding [Nag55].

Many attempts to give a nice functorial description of Chow varieties have been made and some successful steps towards this goal have been taken. For families parameterized by *seminormal* schemes, Kollár, Suslin and Voevodsky, have given a functorial description [Kol96, SV00]. In characteristic zero, Barlet [Bar75] has given an analytic description over *reduced* $\mathbb{C}$-schemes and Angéniol [Ang80] has given an algebraic description over, not necessarily reduced, $\mathbb{Q}$-schemes. The situation in characteristic zero is simplified by the fact that for a finite extension $A \hookrightarrow B$ such that the determinant $B \to A$ is defined, the determinant is determined by the trace.

In this article we will restrict our attention to Chow varieties of zero cycles, that is, families of cycles of relative dimension zero. We will construct an algebraic space $\Gamma^d(X/S)$, parameterizing zero-cycles, which coincides with Angéniol's Chow space in characteristic zero. As with Angéniol's Chow space, the algebraic space $\Gamma^d(X/S)$ is not always reduced but its reduction coincides with the classical Chow variety if we use a *sufficiently good* projective embedding. The relation with the Chow variety will be discussed in a subsequent article [**III**]. A good understanding of families of zero-cycles is crucial for the understanding of families of higher-dimensional cycles.

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2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 14C25, 14L30.

*Key words and phrases.* Families of cycles, zero-cycles, divided powers, symmetric tensors, symmetric product, Chow scheme, Hilbert scheme.

In fact, a family of higher-dimensional cycles is defined by giving zero-dimensional families on "smooth projections" [Bar75, **IV**].

A natural space parameterizing zero-cycles is the symmetric product $\mathrm{Sym}^d(X/S)$. This is the correct choice, in the sense that it coincides with $\Gamma^d(X/S)$, when $X$ is of characteristic zero or when $X/S$ is flat. In general, however, $\mathrm{Sym}^d(X/S)$ is not functorially well-behaved and should be replaced with the "scheme of divided powers". In the affine case, this is the spectrum of the algebra of divided power $\Gamma^d_A(B)$ and it coincides with the symmetric product when $d!$ is invertible in $A$ or when $B$ is a flat $A$-algebra.

Although the ring of divided powers $\Gamma^d_A(B)$ and multiplicative polynomial laws have been studied by many authors [Rob63, Rob80, Ber65, Zip86, Fer98], there are some important results missing. We provide these missing parts, giving a full treatment of the *kernel* of a multiplicative law. Somewhat surprisingly, the kernel does not commute with flat base change, except in characteristic zero. We will show that the kernel does commute with étale base change.

After this preliminary study of $\Gamma^d_A(B)$ we define, for any separated algebraic space $X/S$, a functor $\underline{\Gamma}^d_{X/S}$ which parameterizes families of zero-cycles. From the definition of $\underline{\Gamma}^d_{X/S}$ and the results on the kernel of a multiplicative law, it will be obvious that $\underline{\Gamma}^d_{X/S}$ is represented by $\mathrm{Spec}(\Gamma^d_A(B))$ in the affine case. If $X/S$ is a scheme such that for every $s \in S$, every finite subset of the fiber $X_s$ is contained in an *affine* open subset of $X$, then we say that $X/S$ is an AF-scheme, cf. Appendix A.1. In particular, this is the case if $X/S$ is quasi-projective. For an AF-scheme $X/S$ it is easy to show that $\underline{\Gamma}^d_{X/S}$ is representable by a scheme.

To treat the general case — when $X/S$ is any separated scheme or separated algebraic space — we use the fact that $\underline{\Gamma}^d_{X/S}$ is functorial in $X$: For any morphism $f : U \to X$ there is an induced *push-forward* $f_* : \underline{\Gamma}^d_{U/S} \to \underline{\Gamma}^d_{X/S}$. We show that when $f$ is étale, then $f_*$ is étale over a certain open subset corresponding to families of cycles which are *regular* with respect to $f$. We then show that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$ giving an explicit étale covering.

In the last part of the article we introduce "addition of cycles" and investigate the relation between the symmetric product $\mathrm{Sym}^d(X/S)$ and the algebraic space $\Gamma^d(X/S)$. Intuitively, the universal family of $\Gamma^d(X/S)$ should be related to the addition of cycles morphism $\Phi_{X/S} : \Gamma^{d-1}(X/S) \times_S X \to \Gamma^d(X/S)$. In the special case when $\Phi_{X/S}$ is *flat*, e.g., when $X/S$ is a smooth curve, Iversen has shown that the universal family is given by the *norm* of $\Phi_{X/S}$ [Ive70]. In general, there is a similar but more subtle description. The universal family and some other properties of $\Gamma^d(X/S)$ are treated in [**II**].

We now discuss the results and methods in more detail:

**Multiplicative polynomial laws.** In §1 we recall the basic properties of the algebra of divided powers $\Gamma_A(B)$ and the algebra $\Gamma^d_A(B)$. We also mention the *universal multiplication* of laws which later on will be described geometrically as *addition of cycles*.

**Kernel of a multiplicative polynomial law.** Let $B$ be an $A$-algebra. In §2 the basic properties of the *kernel* $\ker(F)$ of a multiplicative law $F : B \to A$ is established. First we show that $B/\ker(F)$ is integral over $A$ using Cayley-Hamilton's theorem. We then show that the kernel commutes with limits, localization and smooth base change. As mentioned above, the kernel does not commute with flat base change in general and showing that the kernel commutes with smooth base change takes some effort. Finally, we show some topological properties of the kernel: The radical of the kernel commutes with arbitrary base change, the fibers of $\operatorname{Spec}(B/\ker(F)) \to \operatorname{Spec}(A)$ are finite sets, and $\operatorname{Spec}(B/\ker(F)) \to \operatorname{Spec}(A)$ is universally open.

**The functor $\underline{\Gamma}^d_{X/S}$.** Guided by the knowledge that $\Gamma^d_A(B)$ is what we want in the affine case, we define in §3.1 a well-behaved functor $\underline{\Gamma}^d_{X/S}$ parameterizing families of zero-cycles of degree $d$ as follows. A family over an affine $S$-scheme $T = \operatorname{Spec}(A)$ is given by the following data

(i) A closed subspace $Z \hookrightarrow X \times_S T$ such that $Z \to T$ is *integral.* In particular $Z = \operatorname{Spec}(B)$ is *affine.*
(ii) A family $\alpha$ on $Z$, i.e., a morphism $T \to \Gamma^d(Z/T) := \operatorname{Spec}(\Gamma^d_A(B))$.

Moreover, two families are equivalent if they are both induced by a family for some common smaller subspace $Z$. We often suppress the subspace $Z$ and talk about the family $\alpha$. The smallest subspace $Z \hookrightarrow X \times_S T$ in the equivalence class containing $\alpha$ is the *image* of the family $\alpha$ and the reduction $Z_{\mathrm{red}}$ of the image is the *support* of the family. The image of $\alpha$ is given by the kernel of the multiplicative law corresponding to $\alpha$. Since the kernel commutes with étale base change, as shown in §2, so does the image of a family. This is the key result needed to show that $\underline{\Gamma}^d_{X/S}$ is a sheaf in the étale topology.

In contrast to the Hilbert functor, for which families over $T$ are determined by a subspace $Z \hookrightarrow X \times_S T$, a family of zero-cycles is not determined by its image $Z$. If $T$ is *reduced*, then the image $Z$ of a family parameterized by $T$ is reduced and the family is determined by an effective cycle supported on $Z$. In positive characteristic, over non-perfect fields, this cycle may have rational coefficients. This is discussed in [**II**].

**Push-forward of cycles.** A morphism $f : X \to Y$ of separated algebraic spaces induces a natural transformation $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ which we call the *push-forward.* When $Y/S$ is locally of finite type, the existence of $f_*$ follows from standard results. In general, we need a technical result on integral morphisms given in Appendix A.2.

We say that a family $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is *regular* if the restriction of $f_T$ to the image of $\alpha$ is an isomorphism. If $f : X \to Y$ is étale then the regular locus is an open subfunctor of $\underline{\Gamma}^d_{X/S}$. A main result is that under certain regularity constraints, push-forward commutes with products, cf. Proposition (3.3.10). Using this fact we show that the push-forward along an étale morphism is representable and étale over the regular locus. This is Proposition (3.3.15).

**Representability.** The representability of $\underline{\Gamma}^d_{X/S}$ when $X/S$ is affine or AF is, as already mentioned, not difficult and given in 3.1. When $X/S$ is any separated algebraic space, the representability is proven in Theorem (3.4.1) using the results on the push-forward.

**Addition of cycles.** Using the push-forward we define in §4.1 a morphism

$$\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$$

which on points is addition of cycles. This induces a morphism $(X/S)^d \to \Gamma^d(X/S)$ which has the topological properties of a quotient of $(X/S)^d$ by the symmetric group.

**Relation with the symmetric product.** The morphism $(X/S)^d \to \Gamma^d(X/S)$ factors through the quotient map $(X/S)^d \to \mathrm{Sym}^d(X/S)$ and it is easily proven that $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions, cf. Corollary (4.2.5). It is further easy to show that $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over the non-degeneracy locus, cf. Proposition (4.2.6).

**Comparison of representability techniques.** Consider the following inclusions of categories:

$$\begin{tikzcd}
 & \begin{array}{l} X/S \text{ quasi-projective} \\ \text{of finite presentation} \end{array} \arrow[r, hook] \arrow[d, hook] & \begin{array}{l} X/S \text{ separated algebraic space,} \\ \text{locally of finite presentation} \end{array} \arrow[d, hook] \\
X/S \text{ affine} \arrow[r, hook] & X/S \text{ AF-scheme} \arrow[r, hook] & X/S \text{ separated algebraic space.}
\end{tikzcd}$$

When $X/S$ is affine, it is fairly easy to show the existence of the quotient $\mathrm{Sym}^d(X/S)$ [Bou64, Ch. V, §2, No. 2, Thm. 2], the representability of $\underline{\Gamma}^d_{X/S}$ and the representability of the Hilbert functor of points $\mathcal{H}ilb^d_{X/S}$ [Nor78, GLS07]. The existence of $\mathrm{Sym}^d(X/S)$ and the representability of $\underline{\Gamma}^d_{X/S}$ and $\mathcal{H}ilb^d_{X/S}$ in the category of AF-schemes is then a simple consequence.

When $X/S$ is (quasi-)projective and $S$ is noetherian, one can also show the existence and (quasi-)projectivity of $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Hilb}^d(X/S)$ with projective methods, cf. [**III**] and [FGA, No. 221]. The representability of the Hilbert scheme in the category of separated algebraic spaces locally of finite presentation can be established using Artin's algebraization theorem [Art69, Cor. 6.2]. We could likewise have used Artin's algebraization theorem to prove the representability of $\underline{\Gamma}^d_{X/S}$ when $X/S$ is locally of finite presentation. The crucial criterion, that $\underline{\Gamma}^d_{X/S}$ is effectively pro-representable, is shown in §3.2.

Finally, the methods that we have used in this article to show that $\underline{\Gamma}^d_{X/S}$ is representable in the category of all separated algebraic spaces can be applied, mutatis mutandis, to the Hilbert functor of points. The proofs become significantly simpler as the difficulties encountered for $\underline{\Gamma}^d_{X/S}$ are almost trivial for the Hilbert functor.

More generally, these methods apply to the Hilbert stack of points [Ryd08b]. The existence of $\mathrm{Sym}^d(X/S)$ can also be proven in the same vein and this is done in [Ryd07].

**Notation and conventions.** We denote a *closed* immersion of schemes or algebraic spaces with $X \hookrightarrow Y$. When $A$ and $B$ are rings or modules we use $A \hookrightarrow B$ for an injective homomorphism. We let $\mathbb{N}$ denote the set of non-negative integers $0, 1, 2, \dots$ and use the notation $((a, b)) = \binom{a+b}{a}$ for binomial coefficients.

## Contents



## 1. The algebra of divided powers

We begin this section by briefly recalling the definition of polynomial laws in §1.1, the algebra of divided powers $\Gamma_A(M)$ in §1.2 and the multiplicative structure of $\Gamma^d_A(B)$ in §1.3.

1.1. **Polynomial laws and symmetric tensors.** We recall the definition of a polynomial law [Rob63, Rob80].

**Definition (1.1.1).** Let $M$ and $N$ be $A$-modules. We denote by $\mathcal{F}_M$ the functor

$$\mathcal{F}_M\ :\ A\text{–}\mathbf{Alg}\to\mathbf{Sets},\qquad A'\mapsto M\otimes_A A'$$

A *polynomial law* from $M$ to $N$ is a natural transformation $F:\mathcal{F}_M\to\mathcal{F}_N$. More concretely, a polynomial law is a set of *maps* $F_{A'}\ :\ M\otimes_A A'\to N\otimes_A A'$ for every $A$-algebra $A'$ such that for any homomorphism of $A$-algebras $g:A'\to A''$ the diagram

$$\begin{tikzcd}
M\otimes_A A' \arrow[r, "F_{A'}"] \arrow[d, "\mathrm{id}_M\otimes g"'] \arrow[dr, phantom, "\circ"] & N\otimes_A A' \arrow[d, "\mathrm{id}_N\otimes g"] \\
M\otimes_A A'' \arrow[r, "F_{A''}"] & N\otimes_A A''
\end{tikzcd}$$

commutes. The polynomial law $F$ is *homogeneous of degree* $d$ if for any $A$-algebra $A'$, the corresponding map $F_{A'}\ :\ M\otimes_A A'\to N\otimes_A A'$ is such that $F_{A'}(ax)=a^dF_{A'}(x)$ for any $a\in A'$ and $x\in M\otimes_A A'$. If $B$ and $C$ are $A$-algebras then a polynomial law from $B$ to $C$ is *multiplicative* if for any $A$-algebra $A'$, the corresponding map $F_{A'}\ :\ B\otimes_A A'\to C\otimes_A A'$ is such that $F_{A'}(1)=1$ and $F_{A'}(xy)=F_{A'}(x)F_{A'}(y)$ for any $x,y\in B\otimes_A A'$.

**Notation (1.1.2).** Let $A$ be a ring and $M$ and $N$ be $A$-modules (resp. $A$-algebras). We let $\mathrm{Pol}^d(M,N)$ (resp. $\mathrm{Pol}^d_{\mathrm{mult}}(M,N)$) denote the polynomial laws (resp. multiplicative polynomial laws) $M\to N$ which are homogeneous of degree $d$.

**Notation (1.1.3).** Let $A$ be a ring and $M$ an $A$-algebra. We denote the $d^{\mathrm{th}}$ tensor product of $M$ over $A$ by $\mathrm{T}^d_A(M)$. We have an action of the symmetric group $\mathfrak{S}_d$ on $\mathrm{T}^d_A(M)$ permuting the factors. The invariant ring of this action is the symmetric tensors and is denoted $\mathrm{TS}^d_A(M)$. By $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ we denote the graded $A$-modules $\bigoplus_{d\geq 0}\mathrm{T}^d_A(M)$ and $\bigoplus_{d\geq 0}\mathrm{TS}^d_A(M)$ respectively.

**(1.1.4)** The covariant functor $\mathrm{TS}^d_A(\cdot)$ commutes with filtered direct limits. In fact, denoting the group ring of $\mathfrak{S}_d$ by $\mathbb{Z}[\mathfrak{S}_d]$ we have that

$$\mathrm{TS}^d_A(\cdot)=\mathrm{T}^d_A(\cdot)^{\mathfrak{S}_d}=\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_d]}\big(\mathbb{Z},\mathrm{T}^d_A(\cdot)\big)$$

where $\mathfrak{S}_d$ acts trivially on $\mathbb{Z}$. As tensor products, being left adjoints, commute with any (small) direct limit so does $\mathrm{T}^d$. Reasoning as in [$\mathrm{EGA_I}$, Prop. 0.6.3.2] it follows that $\mathrm{Hom}_{\mathbb{Z}[\mathfrak{S}_d]}(\mathbb{Z},\cdot)$ commutes with filtered direct limits. In fact, $\mathbb{Z}$ is a $\mathbb{Z}[\mathfrak{S}_d]$-module of finite presentation and that $\mathbb{Z}[\mathfrak{S}_d]$ is non-commutative is not a problem here.

**(1.1.5)** *Shuffle product* — When $B$ is an $A$-algebra, then $\mathrm{TS}^d_A(B)$ has a natural $A$-algebra structure induced from the $A$-algebra structure of $\mathrm{T}^d_A(B)$. The multiplication on $\mathrm{TS}^d_A(B)$ will be written as juxtaposition. For any $A$-module $M$, we can equip $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ with $A$-algebra structures. The multiplication on $\mathrm{T}_A(M)$ is the ordinary tensor product and the multiplication on $\mathrm{TS}_A(M)$ is called the *shuffle product* and is denoted by $\times$. If $x \in \mathrm{TS}^d_A(M)$ and $y \in \mathrm{TS}^e_A(M)$ then

$$x \times y = \sum_{\sigma \in \mathfrak{S}_{d,e}} \sigma\,(x \otimes_A y)$$

where $\mathfrak{S}_{d,e}$ is the subset of $\mathfrak{S}_{d+e}$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(d)$ and $\sigma(d+1) < \sigma(d+2) < \ldots \sigma(d+e)$.

1.2. **Divided powers.** Most of the material in this section can be found in [Rob63] and [Fer98].

**(1.2.1)** Let $A$ be a ring and $M$ an $A$-module. Then there exists a graded $A$-algebra, the algebra of divided powers, denoted $\Gamma_A(M) = \bigoplus_{d \geq 0} \Gamma^d_A(M)$ equipped with maps $\gamma^d : M \to \Gamma^d_A(M)$ such that, denoting the multiplication with $\times$ as in [Fer98], we have that for every $x, y \in M$, $a \in A$ and $d, e \in \mathbb{N}$

$$\Gamma^0_A(M) = A, \quad \text{and} \quad \gamma^0(x) = 1 \tag{1.2.1.1}$$

$$\Gamma^1_A(M) = M, \quad \text{and} \quad \gamma^1(x) = x \tag{1.2.1.2}$$

$$\gamma^d(ax) = a^d \gamma^d(x) \tag{1.2.1.3}$$

$$\gamma^d(x+y) = \textstyle\sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y) \tag{1.2.1.4}$$

$$\gamma^d(x) \times \gamma^e(x) = ((d,e))\gamma^{d+e}(x) \tag{1.2.1.5}$$

Using (1.2.1.1) and (1.2.1.2) we will identify $A$ with $\Gamma^0_A(M)$ and $M$ with $\Gamma^1_A(M)$. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a family of elements of $M$ and $\nu \in \mathbb{N}^{(\mathcal{I})}$ then we let

$$\gamma^\nu(x) = \mathop{\times}_{\alpha \in \mathcal{I}} \gamma^{\nu_\alpha}(x_\alpha)$$

which is an element of $\Gamma^d_A(M)$ with $d = |\nu| = \sum_{\alpha \in \mathcal{I}} \nu_\alpha$.

**(1.2.2)** *Functoriality* — $\Gamma_A(\cdot)$ is a covariant functor from the category of $A$-modules to the category of graded $A$-algebras [Rob63, Ch. III §4, p. 251].

**(1.2.3)** *Base change* — If $A'$ is an $A$-algebra then there is a natural isomorphism $\Gamma_A(M) \otimes_A A' \to \Gamma_{A'}(M \otimes_A A')$ mapping $\gamma^d(x) \otimes_A 1$ to $\gamma^d(x \otimes_A 1)$ [Rob63, Thm. III.3, p. 262]. This shows that $\gamma^d$ is a homogeneous polynomial law of degree $d$.

**(1.2.4)** *Universal property* — The map $\mathrm{Hom}_A\bigl(\Gamma^d_A(M), N\bigr) \to \mathrm{Pol}^d(M, N)$ given by $f \to f \circ \gamma^d$ is a bijection [Rob63, Thm. IV.1, p. 266].

**(1.2.5)** *Basis and generators* — If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a set of generators of $M$, then $\bigl(\gamma^\nu(x)\bigr)_{\nu \in \mathbb{N}^{(\mathcal{I})}}$ is a set of generators of $\Gamma_A(M)$ as an $A$-module. If $(x_\alpha)_{\alpha \in \mathcal{I}}$ is a basis of $M$ then $\bigl(\gamma^\nu(x)\bigr)_{\nu \in \mathbb{N}^{(\mathcal{I})}}$ is a basis of $\Gamma_A(M)$ [Rob63, Thm. IV.2, p. 272]. Furthermore, if $A$ is an algebra over an infinite field or $A$ is an algebra over $\Lambda_d =$

$\mathbb{Z}[T]/P_d(T)$ where $P_d$ is the unitary polynomial $P_d(T) = \prod_{0\leq i<j\leq d}(T^i - T^j) - 1$, then $\gamma^d(M)$ generates $\Gamma^d_A(M)$ [Fer98, Lemme 2.3.1]. In particular, there is always a finite *faithfully flat* base change $A \to A'$ such that $\Gamma^d_{A'}(M')$ is generated by $\gamma^d(M')$. More generally $\gamma^d(M)$ generates $\Gamma^d_A(M)$ if and only if every residue field of $A$ has at least $d$ elements [**III**].

**(1.2.6)** *Exactness* — The functor $\Gamma_A(\cdot)$ is a left adjoint [Rob63, Thm. III.1, p. 257] and thus commutes with any (small) direct limit. It is thus *right exact* [GV72, Def. 2.4.1] but note that $\Gamma_A(\cdot)$ is a functor from $A$–**Mod** to $A$–**Alg** and that the latter category is not abelian. By [GV72, Rem. 2.4.2] a functor is right exact if and only if it takes the initial object onto the initial object and commutes with finite coproducts and coequalizers. Thus $\Gamma_A(0) = A$ and given an exact diagram of $A$-modules

$$M' \underset{g}{\overset{f}{\rightrightarrows}} M \xrightarrow{h} M''$$

the diagram

$$\Gamma_A(M') \underset{\Gamma g}{\overset{\Gamma f}{\rightrightarrows}} \Gamma_A(M) \xrightarrow{\Gamma h} \Gamma_A(M'')$$

is exact in the category of $A$-algebras and

$$\Gamma_A(M \oplus M') = \Gamma_A(M) \otimes_A \Gamma_A(M').$$

The latter identification can be made explicit [Rob63, Thm. III.4, p. 262] as

$$\begin{aligned}\Gamma^d_A(M \oplus M') &= \bigoplus_{a+b=d} \left(\Gamma^a_A(M) \otimes_A \Gamma^b_A(M')\right) \\ \gamma^d(x+y) &= \sum_{a+b=d} \gamma^a(x) \otimes \gamma^b(y).\end{aligned} \tag{1.2.6.1}$$

This makes $\Gamma_A(M \oplus M') = \bigoplus_{a,b\geq 0} \Gamma^{a,b}(M \oplus M')$ into a bigraded algebra where $\Gamma^{a,b}(M \oplus M') = \Gamma^a_A(M) \otimes_A \Gamma^b_A(M')$.

**(1.2.7)** *Surjectivity* — If $M \twoheadrightarrow N$ is a surjection then it is easily seen from the explicit generators of $\Gamma(N)$ in (1.2.5) that $\Gamma_A(M) \twoheadrightarrow \Gamma_A(N)$ is surjective. This also follows from the right-exactness of $\Gamma_A(\cdot)$ as any right-exact functor from modules to rings takes surjections onto surjections, cf. (1.2.8)

**(1.2.8)** *Presentation* — Let $M = G/R$ be a presentation of the $A$-module $M$. Then $\Gamma_A(M) = \Gamma_A(G)/I$ where $I$ is the ideal of $\Gamma_A(G)$ generated by the images in $\Gamma_A(G)$ of $\gamma^d(x)$ for every $x \in R$ and $d \geq 1$ [Rob63, Prop. IV.8, p. 284]. In fact, denoting the inclusion of $R$ in $G$ by $i$, we can write $M$ as a coequalizer of $A$-modules

$$R \oplus G \underset{0\oplus \mathrm{id}_G}{\overset{i\oplus \mathrm{id}_G}{\rightrightarrows}} G \xrightarrow{h} M$$

which by (1.2.6) gives the exact sequence

$$\Gamma_A(R\oplus G) \overset{\Gamma(i\oplus \mathrm{id}_G)}{\underset{\Gamma(0\oplus \mathrm{id}_G)}{\rightrightarrows}} \Gamma_A(G) \xrightarrow{\Gamma(h)} \Gamma_A(M)$$

of $A$-algebras. Since $\Gamma^0_A(0)=\Gamma^0_A(i)=\mathrm{id}_A$ and $\Gamma^d_A(0)=0$ for $d>0$ it follows that $\Gamma_A(M)$ is the quotient of $\Gamma_A(G)$ by $\bigoplus_{d\geq 1,e\geq 0}\Gamma^{d,e}(R\oplus G)$, i.e., the quotient of $\Gamma_A(G)$ by the ideal generated by $\Gamma(i)\big(\bigoplus_{d\geq 1}\Gamma^d(R)\big)$.

**(1.2.9)** *Exactness of* $\Gamma^d_A(\cdot)$ — If $M\twoheadrightarrow N$ is a surjection then $\Gamma^d_A(M)\twoheadrightarrow\Gamma^d_A(N)$ is surjective since $\Gamma_A(M)\twoheadrightarrow\Gamma_A(N)$ is surjective. This does, however, not imply that $\Gamma^d_A(\cdot)$ is right exact. In fact, in general it is not since we have that $\Gamma^d_A(M\oplus M')\neq\Gamma^d_A(M)\oplus\Gamma^d_A(M')$.

**(1.2.10)** *Presentation of* $\Gamma^d_A(\cdot)$ — If $M=G/R$ is a quotient of $A$-modules then $\Gamma^d_A(M)=\Gamma^d_A(G)/I$ where $I$ is the $A$-submodule generated by the elements $\gamma^k(x)\times y$ for $1\leq k\leq d$, $x\in R$ and $y\in\Gamma^{d-k}_A(G)$. This follows immediately from (1.2.8).

**(1.2.11)** *Filtered direct limits* — The functor $\Gamma^d_A(\cdot)$ commutes with *filtered* direct limits. In fact, if $(M_\alpha)$ is a directed filtered system of $A$-modules then

$$\begin{aligned}\bigoplus_{d\geq 0}\Gamma^d_A(\varinjlim_{A\text{–}\mathbf{Mod}} M_\alpha) &= \varinjlim_{A\text{–}\mathbf{Alg}}\bigoplus_{d\geq 0}\Gamma^d_A(M_\alpha)=\\ &=\varinjlim_{A\text{–}\mathbf{Mod}}\bigoplus_{d\geq 0}\Gamma^d_A(M_\alpha)=\bigoplus_{d\geq 0}\varinjlim_{A\text{–}\mathbf{Mod}}\Gamma^d_A(M_\alpha).\end{aligned}$$

The first equality follows from (1.2.6) and the second from the fact that a filtered direct limit in the category of $A$-algebras coincides with the corresponding filtered direct limit in the category of $A$-modules [GV72, Cor. 2.9].

**(1.2.12)** If $M$ is a free (resp. flat) $A$-module then $\Gamma^d_A(M)$ is a free (resp. flat) $A$-module. This follows from (1.2.5) and (1.2.11) as any flat module is a filtered direct limit of free modules [Laz69, Thm. 1.2].

**(1.2.13)** $\Gamma$ *and* TS — The homogeneous polynomial law $M\to\mathrm{TS}^d_A(M)$ of degree $d$ given by $x\mapsto x^{\otimes_A d}=x\otimes_A\cdots\otimes_A x$ corresponds by the universal property (1.2.4) to an $A$-module homomorphism $\varphi:\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$. This extends to an $A$-algebra homomorphism $\Gamma_A(M)\to\mathrm{TS}_A(M)$, where the multiplication in $\mathrm{TS}_A(M)$ is the shuffle product (1.1.5), cf. [Rob63, Prop. III.1, p. 254].

When $M$ is a free $A$-module the homomorphisms $\Gamma^d_A(M)\to\mathrm{TS}^d_A(M)$ and $\Gamma_A(M)\to\mathrm{TS}_A(M)$ are isomorphisms of $A$-modules respectively $A$-algebras [Rob63, Prop. IV.5, p. 272]. The functors $\mathrm{TS}^d_A$ and $\Gamma^d_A$ commute with filtered direct limits by (1.1.4) and (1.2.11). Since any flat $A$-module is the filtered direct limit of free $A$-modules [Laz69, Thm. 1.2], it thus follows that $\Gamma_A(M)\to\mathrm{TS}_A(M)$ is an isomorphism of graded $A$-algebras for any flat $A$-module $M$.

Moreover by [Rob63, Prop. III.3, p. 256], there are natural $A$-module homomorphisms $\mathrm{TS}^d_A(M) \hookrightarrow \mathrm{T}^d_A(M) \twoheadrightarrow \mathrm{S}^d_A(M) \to \Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ such that going around one turn in the diagram

$$\begin{tikzcd}
& \mathrm{S}^d_A(M) \arrow[dl] & \\
\Gamma^d_A(M) \arrow[rr] & & \mathrm{TS}^d_A(M) \arrow[ul]
\end{tikzcd}$$

is multiplication by $d!$. Here $\mathrm{S}^d_A(M)$ denotes the degree $d$ part of the symmetric algebra. Thus if $d!$ is invertible then $\Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ is an isomorphism. In particular, this is the case when $A$ is purely of characteristic zero, i.e., contains the field of rationals.

**(1.2.14)** *Universal multiplication of laws* — Let $d, e \in \mathbb{N}$. There is a canonical homomorphism

$$\rho_{d,e} \,:\, \Gamma^{d+e}_A(M) \to \Gamma^d_A(M) \otimes_A \Gamma^e_A(M)$$

given by the homogeneous polynomial law $x \mapsto \gamma^d(x) \otimes \gamma^e(x)$ of degree $d+e$ and the universal property (1.2.4). In particular

$$\rho_{d,e}\big(\gamma^\nu(x)\big) = \sum_{\substack{\nu'+\nu''=\nu \\ |\nu'|=d,\ |\nu''|=e}} \gamma^{\nu'}(x) \otimes \gamma^{\nu''}(x). \tag{1.2.14.1}$$

We can factor $\rho_{d,e}$ as $\pi_{d,e} \circ \Gamma^{d+e}(p)$ where $p \,:\, M \to M \oplus M$ is the diagonal map $x \mapsto x \oplus x$ and $\pi_{d,e}$ is the projection on the factor of bidegree $(d,e)$ of $\Gamma^{d+e}(M \oplus M)$, cf. Equation (1.2.6.1).

If $F_1 \,:\, M \to N_1$ and $F_2 \,:\, M \to N_2$ are polynomial laws homogeneous of degrees $d$ and $e$ respectively we can form the polynomial law $F_1 \otimes F_2 \,:\, M \to N_1 \otimes_A N_2$ given by $(F_1 \otimes F_2)(x) = F_1(x) \otimes F_2(x)$. The law $F_1 \otimes F_2$ is homogeneous of degree $d+e$. If $f_1 \,:\, \Gamma^d(M) \to N_1$, $f_2 \,:\, \Gamma^e(M) \to N_2$ and $f_{1,2} \,:\, \Gamma^{d+e}(M) \to N_1 \otimes_A N_2$ are the corresponding homomorphisms then $f_{1,2} = (f_1 \otimes f_2) \circ \rho_{d,e}$.

1.3. **Multiplicative structure.** Let $M, N$ be $A$-modules and $d$ a positive integer. There is a unique homomorphism

$$\mu \,:\, \Gamma^d_A(M) \otimes_A \Gamma^d_A(N) \to \Gamma^d(M \otimes_A N)$$

sending $\mu(\gamma^d(x) \otimes \gamma^d(y))$ to $\gamma^d(x \otimes y)$ [Rob80]. When $B$ is an $A$-algebra, the composition of $\mu$ and the multiplication homomorphism $B \otimes_A B \to B$ induces a multiplication on $\Gamma^d_A(B)$ which we will denote by juxtaposition. The multiplication is such that $\gamma^d(x)\gamma^d(y) = \gamma^d(xy)$ and this makes $\gamma^d$ into a multiplicative polynomial law homogeneous of degree $d$. The unit in $\Gamma^d_A(B)$ is $\gamma^d(1)$.

If $B$ is an $A$-algebra and $M$ is a $B$-module, then $\mu$ together with the module structure $B \otimes_A M \to M$ induces a $\Gamma^d_A(B)$-module structure on $\Gamma^d_A(M)$.

**(1.3.1)** *Universal property* — Let $B$ and $C$ be $A$-algebras. Then the map

$$\operatorname{Hom}_{A\text{-}\mathbf{Alg}}\bigl(\Gamma^d_A(B), C\bigr) \to \operatorname{Pol}^d_{\mathrm{mult}}(B, C)$$

given by $f \to f \circ \gamma^d$ is a bijection [Rob80]. Also see [Fer98, Prop. 2.5.1].

**(1.3.2)** $\Gamma$ *and* TS — The homogeneous polynomial law $M \to \mathrm{TS}^d_A(M)$ of degree $d$ given by $x \mapsto x^{\otimes_A d} = x \otimes_A \cdots \otimes_A x$ is multiplicative. The homomorphism $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ in (1.2.13) is thus an $A$-algebra homomorphism. It is an isomorphism when $B$ is a flat over $A$ or when $A$ is of pure characteristic zero (1.2.13). The morphism $\operatorname{Spec}\bigl(\mathrm{TS}^d_A(B)\bigr) \to \operatorname{Spec}\bigl(\Gamma^d_A(B)\bigr)$ is a universal homeomorphism with trivial residue field extensions, see Corollary (4.2.5). Further results about this morphism is found in [**III**].

**(1.3.3)** *Filtered direct limits* — The functor $B \mapsto \Gamma^d_A(B)$ commutes with filtered direct limits. This follows from (1.2.11) and the fact that a filtered direct limit in the category of $A$-algebras coincides with the corresponding filtered direct limit in the category of $A$-modules [GV72, Cor. 2.9].

**(1.3.4)** The isomorphism of $A$-modules given by equation (1.2.6.1) gives an isomorphism of $A$-algebras

$$\begin{aligned}\Gamma^d_A(B \times C) &= \prod_{a+b=d} \bigl(\Gamma^a_A(B) \otimes_A \Gamma^b_A(C)\bigr)\\ \gamma^d\bigl((x,y)\bigr) &= \bigl(\gamma^a(x) \otimes \gamma^b(y)\bigr)_{a+b=d}.\end{aligned}$$

**(1.3.5)** *Universal multiplication of laws* — If $M$ is an $A$-algebra in (1.2.14), then the polynomial law defining the homomorphism $\rho_{d,e}$ is multiplicative. The homomorphism $\rho_{d,e}$ is thus an $A$-algebra homomorphism. For a geometrical interpretation of $\rho_{d,e}$ as "addition of cycles" see section §4.1.

**Formula (1.3.6)** (Multiplication formula [Fer98, Form. 2.4.2]). *Let $(x_\alpha)_{\alpha\in\mathcal{I}}$ be a set of elements in $B$ and let $\mu, \nu \in \mathbb{N}^{(\mathcal{I})}$ with $d = |\mu| = |\nu|$. Then we have the following identity in $\Gamma^d_A(B)$*

$$\gamma^\mu(x)\gamma^\nu(x) = \sum_{\xi\in N_{\mu,\nu}} \gamma^\xi(x_{(1)}x_{(2)}) = \sum_{\xi\in N_{\mu,\nu}} \mathop{\times}_{(\alpha,\beta)\in\mathcal{I}\times\mathcal{I}} \gamma^{\xi_{\alpha,\beta}}(x_\alpha x_\beta)$$

*where $N_{\mu,\nu}$ is the set of multi-indices $\xi \in \mathbb{N}^{(\mathcal{I}\times\mathcal{I})}$ such that $\sum_{\beta\in\mathcal{I}} \xi_{\alpha,\beta} = \mu_\alpha$ for every $\alpha \in \mathcal{I}$ and $\sum_{\alpha\in\mathcal{I}} \xi_{\alpha,\beta} = \nu_\beta$ for every $\beta \in \mathcal{I}$.*

**Proposition (1.3.7).** *If $B$ is an $A$-algebra of finite type (resp. of finite presentation, resp. finite over $A$, resp. integral over $A$) then $\Gamma^d_A(B)$ is an $A$-algebra of finite type (resp. of finite presentation, resp. finite, resp. integral).*

*Proof.* If $B$ is an $A$-algebra of finite type then $B$ is a quotient of a polynomial ring $A[x_1, x_2, \dots, x_n]$. The induced homomorphism $\Gamma^d(A[x_1, x_2, \dots, x_n]) \to \Gamma^d_A(B)$ is surjective, and thus it is enough to show that $\Gamma^d(A[x_1, x_2, \dots, x_n])$ is an $A$-algebra of finite type. As $\Gamma^d$ commutes with base change it is further enough to show that $\Gamma^d_{\mathbb{Z}}(\mathbb{Z}[x_1,x_2,\dots,x_n]) = \mathrm{TS}^d_{\mathbb{Z}}(\mathbb{Z}[x_1,x_2,\dots,x_n])$ is a $\mathbb{Z}$-algebra of finite type. This is well-known, cf. [Bou64, Ch. V, §1, No. 9, Thm. 2].

If $B$ is an $A$-algebra of finite presentation then there is a noetherian ring $A_0$ and a $A_0$-algebra of finite type $B_0$ such that $B = B_0 \otimes_{A_0} A$. The first part of the proposition shows that $\Gamma^d_{A_0}(B_0)$ is an $A_0$-algebra of finite type and thus also of finite presentation as $A_0$ is noetherian. As $\Gamma^d$ commutes with base change this shows that $\Gamma^d_A(B)$ is an $A$-algebra of finite presentation.

If $B$ is a finite $A$-algebra then $\Gamma^d_A(B)$ is a finite $A$-algebra by (1.2.5). If $B$ is an integral $A$-algebra then $B$ is a filtered direct limit of finite $A$-algebras. As $\Gamma^d$ commutes with filtered direct limits this shows that $\Gamma^d_A(B)$ is an integral $A$-algebra. □

1.4. **The scheme $\Gamma^d(X/S)$ for $X/S$ affine.** Let $S$ be any scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. As the construction of $\Gamma^d_A(B)$ commutes with localization with respect to multiplicatively closed subsets of $A$ we may define a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\Gamma^d_{\mathcal{O}_S}(\mathcal{A})$. This extends the definition of the covariant functor $\Gamma^d$ to the category of quasi-coherent algebras on $S$. If $f : X \to S$ is an affine morphism we let $\Gamma^d(X/S) = \mathrm{Spec}\big(\Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_X)\big)$. This defines a covariant functor

$$\Gamma^d : \mathbf{Aff}_{/S} \to \mathbf{Aff}_{/S}, \qquad X/S \mapsto \Gamma^d(X/S)$$

where $\mathbf{Aff}_{/S}$ is the category of schemes affine over $S$. When it is not likely to cause confusion, we will sometimes abbreviate $\Gamma^d(X/S)$ with $\Gamma^d(X)$.

A polynomial law in this setting is a natural transformation of functors from quasi-coherent $\mathcal{O}_S$-algebras to sheaves of sets on $S$. We obtain an isomorphism $\mathrm{Hom}_S\big(S', \Gamma^d(X/S)\big) \to \mathrm{Pol}^d_{\mathrm{mult},\mathcal{O}_S}(\mathcal{O}_X, \mathcal{O}_{S'})$ for any affine $S$-scheme $S'$. Also observe that

$$\begin{aligned}\mathrm{Hom}_S\big(S', \Gamma^d(X/S)\big) &\cong \mathrm{Hom}_{S'}\big(S', \Gamma^d(X/S)\times_S S'\big)\\ &\cong \mathrm{Hom}_{S'}\big(S', \Gamma^d(X'/S')\big).\end{aligned}$$

More generally, if $S$ is an algebraic space and $X \to S$ is affine we define $\Gamma^d(X/S)$ by étale descent.

Defining $\Gamma^d(X/S)$ for any $S$-scheme $X$ is non-trivial. In the following sections we will give a functorial description of $\Gamma^d(X/S)$ and then show that this functor is represented by a scheme or algebraic space $\Gamma^d(X/S)$.

A very useful fact that will repeatedly be used in the sequel is the following rephrasing of paragraph (1.3.4):

**Proposition (1.4.1).** *Let $S$ be an algebraic space and let $X_1, X_2, \dots, X_n$ be algebraic spaces affine over $S$. Then*

$$\Gamma^d\left(\coprod_{i=1}^n X_i\right) = \coprod_{\substack{d_i\in\mathbb{N}\\ \sum_i d_i = d}} \Gamma^{d_1}(X_1)\times_S \Gamma^{d_2}(X_2)\times_S\cdots\times_S \Gamma^{d_n}(X_n).$$

Similarly, the following Proposition is a translation of paragraph (1.2.9):

**Proposition (1.4.2).** *If $Y$ is an algebraic space affine over $S$ and $X \hookrightarrow Y$ a closed subspace, then $\Gamma^d(X/S)$ is a closed subspace of $\Gamma^d(Y/S)$.*

## 2. Support and image of a family of zero-cycles

Let $X/S$ be a scheme or an algebraic space, affine over $S$. In this section we will show that a "family of zero-cycles" $\alpha$ on $X$ parameterized by $S$, that is, a morphism $\alpha : S \to \Gamma^d(X/S)$, has a unique minimal closed subspace $Z = \mathrm{Image}(\alpha) \hookrightarrow X$, the *image of* $\alpha$, such that $\alpha$ factors through the closed subspace $\Gamma^d(Z/S) \hookrightarrow \Gamma^d(X/S)$. The reduction $Z_{\mathrm{red}}$ will be denoted the *support of* $\alpha$ and written as $\mathrm{Supp}(\alpha)$.

For general $X/S$ a family of zero-cycles $\alpha$, parameterized by a $S$-scheme $T$, should be thought of as one of the following

(i) A morphism $T \to \Gamma^d(X/S)$.
(ii) An "object" living over $\mathrm{Image}(\alpha) \hookrightarrow X \times_S T$.
(iii) A "multi-section" $T \to X \times_S T$ with image $\mathrm{Image}(\alpha)$.

Note that in contrast to ordinary sections and families of closed subschemes, a family of zero-cycles is *not* uniquely determined by its image. If $\alpha$ is a family over a reduced scheme $T$, then $\mathrm{Supp}(\alpha) = \mathrm{Image}(\alpha)$ is reduced, cf. Proposition (2.1.4). In this case, the "object" in (ii) can be interpreted as a cycle in the ordinary sense.

We will show the following results about the image and the support:

(i) The image is integral over $S$. (§2.1)
(ii) The image commutes with essentially smooth base change $S' \to S$ and projective limits. In particular it commutes with étale base change and henselization. (§2.2)
(iii) The support commutes with any base change. (§2.3)
(iv) The support has *universally topologically finite fibers*, i.e., each fiber over $S$ consists of a finite number of points and the separable degrees of the corresponding field extensions are finite. (§2.4)
(v) The support is universally open over $S$. (§2.5)

Many of the results require rather technical but standard demonstrations. In particular we will often need to reduce from the integral to the finite case by the standard limit techniques of [$\mathrm{EGA}_{\mathrm{IV}}$, §8]. The fact that the support is universally open over $S$ will not be needed in the following sections but this result, as well as the fact that the support has universally topologically finite fibers, shows that topologically the support behaves as if it was of finite presentation over $S$.

2.1. **Kernel of a multiplicative law.** We will first define the *kernel* of a multiplicative polynomial law $F : B \to C$ of $A$-algebras. If $F$ is of degree 1, i.e., a ring homomorphism, then the kernel is the usual kernel. In general, the kernel of $F$ is the largest ideal $I$ such that $F$ factors through $B \twoheadrightarrow B/I$. We will focus our attention on the case when $C = A$. Then $B/\ker(F)$ is integral over $A$ as shown in

Proposition (2.1.6) and there is a canonical filtration of $\ker(F)$ which degenerates in characteristic zero.

**Definition (2.1.1).** Let $B$ and $C$ be $A$-algebras. Given a multiplicative law $F : B \to C$ we define its *kernel* $\ker(F)$ as the largest ideal $I$ such that $F$ factors as $B \twoheadrightarrow B/I \to C$. This is a well-defined ideal since if $F$ factors through $B \twoheadrightarrow B/I$ and $B \twoheadrightarrow B/J$ then $F$ factors through $B/(I+J)$.

Note that $F$ factors through $B \twoheadrightarrow B/I$ if and only if $F_{A'}(b' + IB') = F_{A'}(b')$ for any $A$-algebra $A'$ and $b' \in B' = B \otimes_A A'$. Also note that the kernel $\ker(F_{A'})$ contains $\ker(F)B'$ but this inclusion is often strict.

**Notation (2.1.2).** We will in the following denote homogeneous laws by upper-case Latin letters and the corresponding homomorphisms by lower-case letters. For example, if $F : B \to C$ is a homogeneous multiplicative polynomial law of degree $d$ we let $f : \Gamma^d_A(B) \to C$ be the corresponding homomorphism. If $A'$ is an $A$-algebra we denote by $F' : B' \to C'$ the multiplicative law given by $F'_R = F_R$ for every $A'$-algebra $R$. The corresponding homomorphism $f' : \Gamma^d_{A'}(B') \to C'$ is then the base change of $f$ along $A \to A'$.

**Lemma (2.1.3).** *Let $A$ be a ring and let $B$ and $C$ be $A$-algebras. Given a multiplicative law $F : B \to C$ homogeneous of degree $d$, or equivalently given a morphism $f : \Gamma^d_A(B) \to C$, define the following subsets of $B$*

$$L_1 = \bigl\{ b \in B \;:\; f\bigl(\gamma^k(b) \times y\bigr) = 0,\ \forall k, y \bigr\}$$
$$L_2 = \bigl\{ b \in B \;:\; f\bigl(\gamma^k(bx) \times y\bigr) = 0,\ \forall k, x, y \bigr\}$$
$$L_3 = \bigl\{ b \in B \;:\; f'\bigl(\gamma^k(bx') \times y'\bigr) = 0,\ \forall k, A', x', y' \bigr\}$$

*where $1 \leq k \leq d$, $x \in B$, $y \in \Gamma^{d-k}_A(B)$, $x' \in B'$, $y' \in \Gamma^{d-k}_{A'}(B')$ and $A \to A'$ is a ring homomorphism. Then $\ker(F) = L_1 = L_2 = L_3$. In particular, these sets are ideals.*

*Proof.* Clearly $L_3 \subseteq L_2 \subseteq L_1$. Let $b \in L_1$ and let $x \in B$. The multiplication formula (1.3.6) shows that for any $y \in \Gamma^{d-k}_A(B)$

$$\gamma^k(bx) \times y = \bigl(\gamma^k(b) \times y\bigr)\bigl(\gamma^k(x) \times \gamma^{d-k}(1)\bigr) + \sum_{i=1}^{k} \gamma^i(b) \times y_i$$

for some $y_i \in \Gamma^{d-i}_A(B)$. Thus $b \in L_2$ and hence $L_1 = L_2$. From Equations (1.2.1.3) and (1.2.1.4) it follows that $L_2 = L_3$ and that this set is an ideal.

If $I$ is an ideal in $B$ then $\Gamma^d_A(B/I) = \Gamma^d_A(B)/J$ where $J$ is the ideal generated by $\gamma^k(b) \times y$ where $b \in I$, $1 \leq k \leq d$ and $y \in \Gamma^{d-k}_A(B)$, cf. (1.2.10). Thus $\ker(F)$ is contained in $L_2$. On the other hand, if $b$ is contained in $L_3$ then for any $A$-algebra $A'$ and $b', x' \in B' = B \otimes_A A'$ we have that

$$F_{A'}(b' + bx') = \sum_{k=0}^{d} f'\bigl(\gamma^k(bx') \times \gamma^{d-k}(b')\bigr) = f'\bigl(\gamma^d(b')\bigr) = F_{A'}(b')$$

and thus $b \in \ker(F)$. □

**Proposition (2.1.4)** ([Zip88, Lem. 7.6])**.** *Let $A$ be a ring and $B, C$ be $A$-algebras together with a multiplicative law $F : B \to C$ homogeneous of degree $d$. If $C$ is reduced then $B/\ker(F)$ is reduced.*

*Proof.* Let $f : \Gamma^d_A(B) \to C$ be the homomorphism corresponding to $F$. Let $b \in B$ such that $b^n \in \ker(F)$ for some $n \in \mathbb{N}$. Then by Lemma (2.1.3) we have that $f\big(\gamma^k(b^n x) \times y\big) = 0$ for every $1 \leq k \leq d$, $x \in B$ and $y \in \Gamma^{d-k}_A(B)$. An easy calculation using the multiplication formula (1.3.6) shows that the element $\big(\gamma^k(b) \times y\big)^{\lceil dn/k \rceil}$ is in the kernel of $f$ for every $1 \leq k \leq d$ and $y \in \Gamma^{d-k}_A(B)$. As $C$ is reduced this implies that $\gamma^k(b) \times y$ is in the kernel of $f$ and thus $b \in \ker(F)$. □

**Definition (2.1.5).** Let $F : B \to A$ be a multiplicative law homogeneous of degree $d$. For any $b \in B$ we define its characteristic polynomial as

$$\chi_{F,b}(t) = F_{A[t]}(b - t) = \sum_{k=0}^{d} (-1)^k f\big(\gamma^{d-k}(b) \times \gamma^k(1)\big) t^k \in A[t].$$

We let

$$I_{\mathrm{CH}}(F) = \big(\chi_{F,b}(b)\big)_{b \in B} \subseteq B$$

be the *Cayley-Hamilton ideal* of $F$. Here $\chi_{F,b}(b)$ is the evaluation of $\chi_{F,b}(t)$ at $b \in B$, i.e., the image of $\chi_{F,b}(t)$ along $A[t] \to B[t] \to B[t]/(t - b) = B$.

**Proposition (2.1.6)** ([Ber65, Satz 4])**.** *Let $F : B \to A$ be a multiplicative law. Then $I_{\mathrm{CH}}(F) \subseteq \ker(F) \subseteq \sqrt{I_{\mathrm{CH}}(F)}$. In particular it follows that $B/\ker(F)$ is integral over $A$.*

*Proof.* Let $P \twoheadrightarrow B$ be a surjection from a flat $A$-algebra $P$ and let $F' : P \to A$ be the multiplicative law given as the composition of $F$ with $P \twoheadrightarrow B$. As the images of $I_{\mathrm{CH}}(F')$ and $\ker(F')$ in $B$ are $I_{\mathrm{CH}}(F)$ and $\ker(F)$ respectively, we can, replacing $B$ with $P$ and $F$ with $F'$, assume that $B$ is flat over $A$. Then $\Gamma^d_A(B) = \mathrm{TS}^d_A(B)$.

We will first show the inclusion $I_{\mathrm{CH}}(F) \subseteq \ker(F)$. By definition this is equivalent with the following: For every base change $A \to A'$, every $b \in B$ and every $b', x' \in B' = B \otimes_A A'$, the identity $F_{A'}\big(\chi_{F,b}(b)x' + b'\big) = F_{A'}(b')$ holds.

For any ring $R$ we let $\mathrm{Diag}_d(R) = R^d$ denote the diagonal $d \times d$-matrices with coefficients in $R$. Let $\Psi : B \to \mathrm{Diag}_d\big(\mathrm{T}^d_A(B)\big)$ be the ring homomorphism such that $\Psi(b) = \mathrm{diag}\,(b_1, b_2, \dots, b_d)$ where $b_k = 1^{\otimes k-1} \otimes b \otimes 1^{\otimes d-k} \in \mathrm{T}^d_A(B)$. The determinant gives a multiplicative law

$$\det : \mathrm{Diag}_d\big(\mathrm{T}^d_A(B)\big) \to \mathrm{T}^d_A(B)$$

which is homogeneous of degree $d$. Let $E = \mathrm{TS}^d_A(A[t]) = A[e_1, e_2, \dots, e_d]$ be the polynomial ring over $A$ in $d$ variables. Here $e_k$ denotes the elementary symmetric function $t^{\otimes k} \times 1^{\otimes d-k}$. Let $b \in B$ be any element. We have a homomorphism $\rho_b : E \hookrightarrow \mathrm{TS}^d_A(B)$ induced by the morphism $A[t] \to B$ mapping $t$ on $b$. More explicitly $\rho_b(e_k) = b^{\otimes k} \times 1^{\otimes d-k}$.

Let $A \to A'$ be any ring homomorphism and let $B' = B \otimes_A A'$, $E' = E \otimes_A A'$. We have a commutative diagram

$$\begin{tikzcd}
B' \arrow[r, "\gamma^d"] & \mathrm{TS}^d_{A'}(B') \arrow[r, "f'"] & A' \\
B' \otimes_{A'} E' \arrow[u, two heads, "(\mathrm{id}, f' \circ \rho'_b)"'] \arrow[r, "\gamma^d"] \arrow[d, "\Psi"] \arrow[ur, phantom, "\circ"] \arrow[dr, phantom, "\circ"] & \mathrm{TS}^d_{A'}(B') \otimes_{A'} E' \arrow[u, two heads, "(\mathrm{id}, f' \circ \rho'_b)"'] \arrow[r, two heads, "(\mathrm{id}, \rho'_b)"] \arrow[d, hook] \arrow[ur, phantom, "\circ"] \arrow[dr, phantom, "\circ"] & \mathrm{TS}^d_{A'}(B') \arrow[u, "f'"'] \arrow[dd, hook] \\
\mathrm{Diag}_d\big(\mathrm{T}^d_{A'}(B') \otimes_{A'} E'\big) \arrow[r, "\det"] \arrow[d, two heads, "\mathrm{Diag}(\mathrm{id}, \rho'_b)"] \arrow[dr, phantom, "\circ"] & \mathrm{T}^d_{A'}(B') \otimes_{A'} E' \arrow[dr, two heads, "(\mathrm{id}, \rho'_b)"] & \\
\mathrm{Diag}_d\big(\mathrm{T}^d_{A'}(B')\big) \arrow[rr, "\det"] & & \mathrm{T}^d_{A'}(B').
\end{tikzcd}$$

Let $\chi(t) = \sum_{k=0}^{d} (-1)^k e_{d-k} t^k \in E[t]$ where we let $e_0 = 1$. Let

$$\chi_b(t) = \rho_b \circ \chi(t) = \sum_{k=0}^{d} (-1)^k \gamma^{d-k}(b) \times \gamma^k(1) t^k \in \mathrm{TS}^d_A(B)[t].$$

Then $f(\chi_b(t)) = \chi_{F,b}(t) \in B[t]$. Let $b', x' \in B'$ be any elements. We begin with the elements $\chi_{F,b}(b)x' + b'$ and $b'$ in the upper-left corner $B'$ of the diagram and want to show that their images by $F_{A'} = f' \circ \gamma^d$ in the upper-right corner $A'$ coincide. As $\chi_{F,b}(b)x' + b'$ lifts to $\chi(b)x' + b' \in B' \otimes_{A'} E'$ it is enough to show that images of $b', \chi(b)x' + b' \in B' \otimes_{A'} E'$ in the lower-left corner $\mathrm{Diag}_d\big(\mathrm{T}^d_{A'}(B')\big)$ are equal.

For any ring $R$ and diagonal matrix $D \in \mathrm{Diag}_d(R)$ let $P_D(t) \in R[t]$ be the characteristic polynomial of $D$. Then by Cayley-Hamilton's theorem $P_D(D) = 0$ in $\mathrm{Diag}_d(R)$. Note that the determinant and the characteristic polynomial commute with arbitrary base change $R \to R'$. Now, the image of $\chi(b)$ by $\mathrm{Diag}(\mathrm{id}, \rho_b) \circ \Psi$ is easily seen to be $\chi_b(\Psi(b)) = P_{\Psi(b)}\big(\Psi(b)\big) = 0$. Thus the images of $\chi(b)x' + b'$ and $b'$ in the lower-left corner are equal. This concludes the proof of the inclusion $I_{\mathrm{CH}}(F) \subseteq \ker(F)$.

If $b \in \ker(F)$ then by Lemma (2.1.3) $f\big(\gamma^k(b) \times \gamma^{d-k}(1)\big) = 0$ for every $k = 1, 2, \dots, d$. Thus $\chi_{F,b}(t) = t^d$ and hence $b^d \in I_{\mathrm{CH}}(F)$ which shows the second inclusion. Finally $B/I_{\mathrm{CH}}(F)$ is clearly integral over $A$ and thus also $B/\ker(F)$. $\square$

*Remark* **(2.1.7).** Ziplies defines the *radical* of a not necessarily homogeneous polynomial law in [Zip88, Def. 6.7]. When the polynomial law is homogeneous the radical coincides with the kernel as defined in (2.1.5). Ziplies further proves in [Zip88, Lem. 7.4] that if $I_{\mathrm{CH}}(F)$ is zero in $B$ then $\ker(F)$ is contained in the Jacobson radical of $B$. Proposition (2.1.6) shows more generally that under this assumption $\ker(F)$ is contained in the nilradical of $B$. Note that both inclusions $I_{\mathrm{CH}}(F) \subseteq \ker(F) \subseteq \sqrt{I_{\mathrm{CH}}(F)}$ can be strict[1].

In [Zip86, 3.4] Ziplies also shows that $I_{\mathrm{CH}}(F)$ is contained in the ideal

$$\begin{aligned} I_F^{(1)} &= \bigl\{ b \in B \;:\; f\bigl(bx \times \gamma^{d-1}(1)\bigr) = 0,\ \forall x \in B \bigr\} \\ &= \bigl\{ b \in B \;:\; f(b \times y) = 0,\ \forall y \in \Gamma_A^{d-1}(B) \bigr\}. \end{aligned}$$

As this ideal by Lemma (2.1.3) clearly contains $\ker(F)$, the first inclusion of Proposition (2.1.6) is a generalization of this result.

## 2.2. Kernel and base change.

**Definition (2.2.1).** Let $A$ be a ring and let $B$ and $C$ be $A$-algebras. Given a multiplicative law $F : B \to C$ homogeneous of degree $d$, or equivalently given a morphism $f : \Gamma_A^d(B) \to C$, we let

$$I_F^{(k)} = \bigl\{ b \in B \;:\; f\bigl(\gamma^i(b) \times y\bigr) = 0,\ \forall 1 \leq i \leq k,\ y \in \Gamma_A^{d-i}(B) \bigr\}.$$

for $k = 0, 1, 2, \dots, d$.

**Proposition (2.2.2).** *Let $B$ and $C$ be $A$-algebras and let $F : B \to C$ be a multiplicative law homogeneous of degree $d$. Then the sets $I_F^{(k)}$ are ideals of $B$ and we have a filtration*

$$B = I_F^{(0)} \supseteq I_F^{(1)} \supseteq \dots \supseteq I_F^{(d)} = \ker(F).$$

*If $A'$ is an $A$-algebra and $B' = B \otimes_A A'$ then $I_{F_{A'}}^{(k)} \supseteq I_F^{(k)} B'$. In particular* $\ker(F_{A'}) \supseteq \ker(F)B'$.

*Proof.* That $I_F^{(k)}$ are ideals follows exactly as in the proof of Lemma (2.1.3). That $I_F^{(d)} = \ker(F)$ is Lemma (2.1.3) and the other assertions are trivial. □

The main application for the filtration $I_F^{(0)} \supseteq I_F^{(1)} \supseteq \dots \supseteq I_F^{(d)}$ is that the elements in $I_F^{(k-1)}$ behave "quasi-linear" modulo $I_F^{(k)}$ with respect to $\gamma^k$ in a certain sense. This will be utilized in Lemma (2.2.10).

**Lemma (2.2.3).** *Let $n \in \mathbb{N}$ and $p$ be a prime. Then $p \mid \binom{n}{k}$ for every $1 \leq k \leq n-1$ if and only if $n = p^s$.*

*Proof.* Assume that $p \mid \binom{n}{k}$ for $1 \leq k \leq n-1$. It easily follows that $a^n = a$ in $\mathbb{F}_p$ for every $a \in \mathbb{F}_p$. Thus $x^p - x$ divides $x^n - x$ in $\mathbb{F}_p[x]$ which shows that $p \mid n$. We obtain that $a^{n/p} = a$ for every $a \in \mathbb{F}_p$ and by induction on $s$ that $n = p^s$. The converse is easy. □

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[1]There is a misprint in [Zip88, Lem. 7.4]. "equals" should be replaced with "is contained in". Also $A$ should be a $B$-algebra as well as an $R$-algebra in his notation.

**Proposition (2.2.4).** *Let $A$ be either a $\mathbb{Z}_{(p)}$-algebra with $p$ a prime or a $\mathbb{Q}$-algebra in which case we let $p=1$. Let $k\geq 1$ be an integer. Then $I_F^{(k)}=I_F^{(k-1)}$ if $k\neq p^s$. In particular, if $A$ is a $\mathbb{Q}$-algebra then* $\ker(F)=I_F^{(1)}$.

*Proof.* Let $A'=A[t]$ and $b'_1,b'_2\in I_{F'}^{(k-1)}$. Then for any $y'\in\Gamma_{A'}^{d-k}(B')$

$$f'\bigl(\gamma^k(b'_1+b'_2)\times y'\bigr)=f'\bigl(\gamma^k(b'_1)\times y'\bigr)+f'\bigl(\gamma^k(b'_2)\times y'\bigr).$$

In particular for any $b\in I_F^{(k-1)}$ and $y\in\Gamma_A^{d-k}(B)$

$$(1+t)^k f'\bigl(\gamma^k(b)\times y\bigr)=f'\bigl(\gamma^k((1+t)b)\times y\bigr)=(1+t^k)f\bigl(\gamma^k(b)\times y\bigr)$$

which shows that $\binom{k}{i}$ annihilates $f\bigl(\gamma^k(b)\times y\bigr)$ for any $1\leq i\leq k-1$. If $k\neq p^s$ then $f\bigl(\gamma^k(b)\times y\bigr)=0$ by Lemma (2.2.3) and thus $b\in I_F^{(k)}$. □

**Lemma (2.2.5).** *Let $A$ be a ring and $B=\varinjlim B_\lambda$ be a filtered direct limit of $A$-algebras with induced homomorphisms $\varphi_\lambda : B_\lambda\to B$. Let $f:\Gamma_A^d(B)\to C$ and denote by $f_\lambda$ the composition of $\Gamma_A^d(\varphi_\lambda):\Gamma_A^d(B_\lambda)\to\Gamma_A^d(B)$ and $f$. Then $I_F^{(k)}=\varinjlim I_{F_\lambda}^{(k)}$ for every $k=0,1,\dots,d$. In particular* $\ker(F)=\varinjlim\ker(F_\lambda)$.

*Proof.* As $f_\lambda$ factors as $\Gamma_A^d(B_\lambda)\to\Gamma_A^d(B)\to C$ it follows that $\varphi_\lambda^{-1}\bigl(I_F^{(k)}\bigr)\subseteq I_{F_\lambda}^{(k)}$. Thus $I_F^{(k)}\subseteq\varinjlim I_{F_\lambda}^{(k)}$. Conversely, for any $b\in B\setminus I_F^{(k)}$ there is an $i\leq k$ and $y\in\Gamma_A^{d-i}(B)$ such that $f\bigl(\gamma^i(b)\times y\bigr)\neq 0$. If we let $\alpha$ be such that ${\varphi_\alpha}^{-1}(b)\neq\emptyset$ and $\Gamma^{d-i}(\varphi_\alpha)^{-1}(y)\neq\emptyset$ then for any $\lambda\geq\alpha$ and $b_\lambda\in B_\lambda$ such that $\varphi_\lambda(b_\lambda)=b$ we have that $b_\lambda\notin I_{F_\lambda}^{(k)}$. Thus $\varinjlim I_{F_\lambda}^{(k)}\subseteq I_F^{(k)}$. □

**Proposition (2.2.6).** *Let $A$ be a ring and $S$ a multiplicative closed subset. Let $F:B\to A$ be a multiplicative homogeneous law of degree $d$ and denote by $S^{-1}F:S^{-1}B\to S^{-1}A$ the map corresponding to the $A$-algebra $S^{-1}A$. Then $S^{-1}I_F^{(k)}=I_{S^{-1}F}^{(k)}$. In particular $S^{-1}\ker(F)=\ker(S^{-1}F)$, i.e., the kernel commutes with localization.*

*Proof.* By Proposition (2.1.6) the quotient $B/\ker(F)$ is integral over $A$. Replacing $B$ by $B/\ker(F)$ we can thus assume that $B$ is integral over $A$. As $B$ is the filtered direct limit of its finite sub-$A$-algebras and both the kernel of a multiplicative law, Lemma (2.2.5), and tensor products commute with filtered direct limits we can assume that $B$ is a finite $A$-algebra. Then $\Gamma_A^i(B)$ is a finite $A$-algebra for all $i=0,1,\dots,d$ by Proposition (1.3.7).

Let $x/s\in I_{S^{-1}F}^{(k)}$, i.e., by definition $x/s\in S^{-1}B$ such that $S^{-1}f\bigl(\gamma^i(x/s)\times y\bigr)=0$ for all $1\leq i\leq k$ and $y\in\Gamma_A^{d-i}(B)$. For any $y\in\Gamma_A^{d-i}(B)$ there is then a $t\in S$ such that $tf\bigl(\gamma^i(x)\times y\bigr)=0$ in $A$. As $\Gamma_A^{d-i}(B)$ is a finite $A$-algebra we can find a common $t$ that works for all $i\leq k$ and $y$. Then $f\bigl(\gamma^i(tx)\times y\bigr)=t^if\bigl(\gamma^i(x)\times y\bigr)=0$ for all $i\leq k$ and $y$. As $x/s=tx/st$, this shows that $I_{S^{-1}F}^{(k)}=S^{-1}I_F^{(k)}$. □

**Proposition (2.2.7).** *Let $A$ be a ring and $B$ an $A$-algebra. Let $A' = \varinjlim A'_\lambda$ be a filtered direct limit of $A$-algebras with induced homomorphisms $\varphi_\lambda : A'_\lambda \to A'$. Let $F : B \to A$ be a multiplicative polynomial law of degree $d$. Then $I^{(k)}_{F_{A'}} = \varinjlim I^{(k)}_{F_{A'_\lambda}}$ for every $k = 0, 1, \dots, d$. In particular* $\ker(F_{A'}) = \varinjlim \ker(F_{A'_\lambda})$.

*Proof.* As in the proof of Proposition (2.2.6) we can assume that $B$ is finite over $A$ and hence that $\Gamma^i_A(B)$ is a finite $A$-module. Choose generators $y_{i1}, y_{i2}, \dots, y_{in_i}$ of $\Gamma^{d-i}_A(B)$ as an $A$-module for $i = 1, 2, \dots, d$. Let $B' = B \otimes_A A'$ and $B'_\lambda = B \otimes_A A'_\lambda$. Let $b' \in I^{(k)}_{F_{A'}}$. Then there exists an $\alpha$ and $b'_\alpha \in B'_\alpha$ such that $b'$ is the image of $b'_\alpha$ by $B'_\alpha \to B$. As the image of $f_{A'_\alpha}(\gamma^i(b'_\alpha) \times y_{ij})$ in $A'$ is $f_{A'}(\gamma^i(b') \times y_{ij})$ and hence zero for $i = 1, 2, \dots, k$, there is a $\beta \geq \alpha$ such that $b'_\alpha \in I^{(k)}_{F_{A'_\lambda}}$ for all $\lambda \geq \beta$. Thus $b' \in \varinjlim_\lambda I^{(k)}_{F_{A'_\lambda}}$ and $I^{(k)}_{F_{A'}} \subseteq \varinjlim_\lambda I^{(k)}_{F_{A'_\lambda}}$. The reverse inclusion is obvious. □

We will now show that the kernel, always commutes with smooth base change and that it commutes with flat base change in characteristic zero.

**Proposition (2.2.8).** *Let $A$ be a ring and let $F : B \to A$ a multiplicative homogeneous law of degree $d$. Let $A'$ be a flat $A$-algebra and denote by $F'$ the multiplicative law corresponding to $A'$. Then $I^{(1)}_F B' = I^{(1)}_{F'}$. In particular, if $A$ is a $\mathbb{Q}$-algebra then the kernel commutes with flat base change.*

*Proof.* We reduce to $B$ a finite $A$-algebra as in the proof of Proposition (2.2.6). For any $y \in \Gamma^{d-1}_A(B)$ let $\varphi_y$ be the $A$-module homomorphism $B \to \Gamma^d_A(B)$ given by $b \mapsto b \times y$. Then $I^{(1)}_f = \bigcap_{y \in \Gamma^{d-1}_A(B)} \ker(f \circ \varphi_y)$. As $\Gamma^{d-1}_A(B)$ is a finitely generated $A$-module and $\varphi_y$ is linear in $y$, this intersection coincides with an intersection over a finite number of $y$'s. As both finite intersections and kernels commute with flat base change the first statement of the proposition follows. The last statement follows from Proposition (2.2.4). □

Recall that a monic polynomial $g \in A[t]$ is *separable* if $(g, g') = A[t]$, where $g'$ is the formal derivative of $g$. Further recall that $A \hookrightarrow A[t]/g$ is *étale* if and only if $g$ is separable. We will need the following basic lemma to which we, for a lack of suitable reference, include a proof.

**Lemma (2.2.9).** *Let $A \hookrightarrow A' = A[t]/g$ be an étale homomorphism, i.e., such that $g$ is a separable polynomial. If $A$ is a local ring of residue characteristic $p > 0$ then for any prime power $q = p^s$, $s \in \mathbb{N}$, the elements $1, t^q, t^{2q}, \dots, t^{(n-1)q}$ form an $A$-module basis of $A'$ where $n = \deg(g)$.*

*Proof.* Let $k = A/\mathfrak{m}_A$. By Nakayama's lemma it is enough to show that a basis of $A'/\mathfrak{m}_A A' = k[t]/\overline{g}$ over $k$ is given by $1, t^q, t^{2q}, \dots, t^{(n-1)q}$. Replacing $A$, $A'$ and $g$ with $k$, $A'/\mathfrak{m}_A A'$ and $\overline{g}$ respectively, we can thus assume that $A = k$ is a field of characteristic $p$.

Let $g = g_1 g_2 \dots g_m$ be a factorization of $g$ into irreducible polynomials. We have that $A' = k[t]/g = k'_1 \times k'_2 \times \dots \times k'_m$ where $k \hookrightarrow k'_i = k[t]/g_i$ are separable field extensions. The subring generated by $t^q$ is the image of $k[t^q]/g^q = \prod k[t^q]/g_i^q$ in $\prod_i k'_i$. To show that $t^q$ generates $k[t]/g$ it is thus enough to show that its image in $k'_i$ generates $k'_i$ for every $i$. Thus, we can assume that $g$ is irreducible such that $A' = k[t]/g = k'$ is a field.

The field extension $k \hookrightarrow k(t^q) \hookrightarrow k(t) = k'$ is separable which shows that so is $k(t^q) \hookrightarrow k(t)$. Thus $k(t^q) = k(t)$ and $t^q$ generates $k'$. □

**Lemma (2.2.10).** *Let $F : B \to A$ be a multiplicative polynomial law of degree $d$. Let $A' = A[t]/g$ where either $g = 0$ or $g$ is separable. Then $I_F^{(k)}$ and $\ker(F)$ commute with the base change $A \hookrightarrow A'$.*

*Proof.* If $g = 0$ we let $n = \infty$ and otherwise we let $n = \deg(g)$. A basis of $A'$ as an $A$-module is then given by $1, t, t^2, \dots, t^{n-1}$. By Proposition (2.2.6) we can assume that $A$ is a local ring. Let $p$ be the exponential characteristic of the residue field $A/\mathfrak{m}_A$, i.e., $p$ equals the characteristic if it is positive and 1 if the characteristic is zero.

We will proceed by induction on $k$ to show that $I_F^{(k)} B' = I_{F'}^{(k)}$. As $I_F^{(0)} = B$ and $I_{F'}^{(0)} = B'$ the case $k = 0$ is obvious. Proposition (2.2.4) shows that $I_F^{(k)} = I_F^{(k-1)}$ if $k \neq p^s$ and we can thus assume that $k = p^s$.

Let $x' \in I_{F'}^{(p^s)} \subseteq I_{F'}^{(p^s-1)}$. By induction $x' \in I_F^{(p^s-1)} B'$ and we can thus write uniquely $x' = \sum_{i=0}^{n-1} x_i t^i$ where $x_i \in I_F^{(p^s-1)}$ are almost all zero. Let $y \in \Gamma_A^{d-p^s}(B)$. Then

$$f'\big(\gamma^{p^s}(x') \times y\big) = \sum_{i=0}^{n-1} t^{p^s i} f\big(\gamma^{p^s}(x_i) \times y\big).$$

If $g = 0$ then $1, t^{p^s}, t^{2p^s}, \dots$ are linearly independent in $A' = A[t]$. If $g$ is separable then $1, t^{p^s}, t^{2p^s}, t^{(n-1)p^s}$ are linearly independent by Lemma (2.2.9). This shows that $f\big(\gamma^{p^s}(x_i) \times y\big) = 0$ for every $y$ and thus $x_i \in I_F^{(p^s)}$ as $x_i \in I_F^{(p^s-1)}$. Hence $x' \in I_F^{(p^s)} B'$ which shows that $I_F^{(p^s)} B' = I_{F'}^{(p^s)}$. □

## 2.3. **Image and base change.**

As the kernel of a multiplicative law commutes with localization by Proposition (2.2.6) it is possible to define the kernel for a multiplicative law for schemes:

**Definition (2.3.1).** Let $S$ be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras and $F : \mathcal{A} \to \mathcal{O}_S$ a multiplicative polynomial law, cf. §1.4. We let $\ker(F) \subseteq \mathcal{A}$ be the quasi-coherent ideal sheaf given by $\ker(F)|_U = \ker\big(F|_U\big)$ for any affine open subset $U \subseteq S$. If $f : X \to S$ is an affine morphism of schemes and $\alpha : S \to \Gamma^d(X/S)$ is a morphism then we let the *image* of $\alpha$, denoted $\mathrm{Image}(\alpha)$, be the closed subscheme of $X$ corresponding to the ideal sheaf $\ker(F_\alpha)$ where $F_\alpha : f_*\mathcal{O}_X \to \mathcal{O}_S$ is the polynomial law corresponding to $\alpha$.

We say that a morphism $S' \to S$ is *essentially smooth* if every local ring of $S'$ is a local ring of a scheme which is smooth over $S$. The results of the previous section are summarized in the following proposition.

**Theorem (2.3.2).** *Let $f : X \to S$ be an affine morphism of schemes and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. If $S' \to S$ is an* essentially smooth *morphism then* $\mathrm{Image}(\alpha) \times_S S' = \mathrm{Image}(\alpha \times_S S')$*, i.e., the image commutes with essentially smooth base change.*

*Proof.* As $\mathrm{Image}(\alpha)$ commutes with localization we can assume that $S = \mathrm{Spec}(A)$ is local and that $S' \to S$ is smooth. Further it is enough that for any $x \in S'$ there is an affine neighborhood $S'' \subseteq S'$ such that the image commutes with the base change $S'' \to S$. By [$\mathrm{EGA_{IV}}$, Cor. 17.11.4] we can choose $S''$ such that $S'' \to S$ is the composition of an étale morphism followed by a morphism $\mathbb{A}^n_S = \mathrm{Spec}(A[t_1, t_2, \dots, t_n]) \to S = \mathrm{Spec}(A)$. We can thus assume that either $S' \to S$ is étale or $S' = \mathbb{A}^1_S$.

If $S' \to S$ is étale and $S = \mathrm{Spec}(A)$ is local, then for any $s' \in S'$ we have that $\mathcal{O}_{S',s'} = A[t]/g$ where $g \in A[t]$ is a separable polynomial [$\mathrm{EGA_{IV}}$, Thm. 18.4.6 (ii)] and it is thus enough to consider base changes $S' \to S$ of the form $A \to A[t]/g$. The result now follows from Lemma (2.2.10). □

**Corollary (2.3.3).** *Let* $S = \mathrm{Spec}(A)$ *and* $S' = \mathrm{Spec}(A')$ *such that $A'$ is a direct limit of essentially smooth $A$-algebras. Let $f : X \to S$ be an affine morphism and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then* $\mathrm{Image}(\alpha') = \mathrm{Image}(\alpha) \times_S S'$*. In particular this holds if $S'$ is the henselization or the strict henselization of a local ring of $S$.*

*Proof.* Follows from Proposition (2.2.7) and Theorem (2.3.2). □

*Remark* **(2.3.4).** If $S$ and $S'$ are locally noetherian and $S' \to S$ is a flat morphism with geometrically regular fibers, then $S'$ is a filtered direct limit of smooth morphisms by Popescu's theorem [Swa98, Spi99]. Thus the image of a family $\alpha : S \to \Gamma^d(X/S)$ commutes with the base change $S' \to S$ under this hypothesis. In particular we can apply this with $S' = \mathrm{Spec}(\widehat{\mathcal{O}_{S,s}})$ for $s \in S$ if $S$ is an excellent scheme [$\mathrm{EGA_{IV}}$, Def. 7.8.2].

**Definition (2.3.5).** Let $f : X \to S$ be an affine morphism of *algebraic spaces* and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. We let $\mathrm{Image}(\alpha)$ be the closed subspace of $X$ such that for any *scheme* $S'$ and étale morphism $S' \to S$ we have that $\mathrm{Image}(\alpha) \times_S S' = \mathrm{Image}(\alpha \times_S S')$. As étale morphisms descend closed subspaces and the image commutes with étale base change, this is a unique and well-defined closed subspace. When $S$ is a scheme, this definition of $\mathrm{Image}(\alpha)$ and the one in Definition (2.3.1) agree. We let $\mathrm{Supp}(\alpha) = \mathrm{Image}(\alpha)_{\mathrm{red}}$ and call this subscheme the *support* of $\alpha$.

**Theorem (2.3.6).** *Let $S$ and $X$ be algebraic spaces such that $X$ is affine over $S$. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism and let $S' \to S$ be any morphism. Then* $\big(\mathrm{Supp}(\alpha) \times_S S'\big)_{\mathrm{red}} = \mathrm{Supp}(\alpha \times_S S')$*, i.e., the support commutes with arbitrary base change.*

*Proof.* We can assume that $S = \operatorname{Spec}(A)$ and $S' = \operatorname{Spec}(A')$ are affine. Let $P$ be a, possibly infinite-dimensional, polynomial algebra over $A$ such that there is a surjection $P \to A'$. Then as $\operatorname{Spec}(P)$ is a limit of smooth $S$-schemes we can by Theorem (2.3.2) replace $A$ with $P$ and assume that $A \to A'$ is surjective.

Let $X = \operatorname{Spec}(B)$, let $f : \Gamma^d_A(B) \to A$ correspond to $\alpha$ and let $F : B \to A$ be the corresponding multiplicative law. Pick an element $b' \in \ker(F_{A'}) \subseteq B \otimes_A A'$ and choose a lifting $b \in B$ of $b'$. Then by Lemma (2.1.3), the elements $f(\gamma^{d-k}(b) \times \gamma^k(1))$, $k = 0, 1, 2, \dots, d-1$ lie in the kernel of $A \to A'$. In particular, the image of $\chi_{F,b}(b)$ in $B$ is $b'^d$. Thus $\ker(F_{A'}) \subseteq \sqrt{I_{\mathrm{CH}}(F)}(B \otimes_A A')$. As $\sqrt{I_{\mathrm{CH}}(F)} = \sqrt{\ker F}$ by Proposition (2.1.6) the theorem follows. $\square$

**Examples (2.3.7).** We give two examples. The first shows that $\ker(F)$ does not commute with arbitrary base change even in characteristic zero. The second shows that $\ker(F)$ does not commute with flat base change in positive characteristic.

(i) Let $A = k[x]$ and $B = k[x, y]/(x^2 - y^2)$. Then $B$ is a free $A$-module of rank 2. The norm $N : B \to A$ is a multiplicative law of degree 2. It can further be seen that $\ker(N) = 0$. Let $A' = k[x]/x$. Then $B' = B \otimes_A A' = k[y]/y^2$ is not reduced and by Proposition (2.1.4) the kernel of $N'$ cannot be trivial. In fact, we have that $\ker(N') = (y)$.

(ii) Let $k$ be a field of characteristic $p$ and $A = B = k$. We let $F : B \to A$ be the polynomial law given by $x \mapsto x^p$, i.e., the Frobenius. Clearly $\ker(F) = 0$. Let $A' = A[t]/t^p$ which is a flat $A$-algebra. Then $\ker(F') = (t)$ as $(b'' + tx'')^p = b''^p + t^p x''^p = b''^p$ for any $A' \to A''$ and $b'', x'' \in B'' = A''$.

It is further easily seen that $\ker(F)$ does not commute with any base change such that $A'$ is not reduced. In fact, if $t \in A'$ is such that $t^p = 0$ then $t \in \ker(F')$.

2.4. **Various properties of the image and support.** A morphism $\alpha : S \to \Gamma^d(X/S)$ is, as we will see later on, a "family of zero-cycles of degree $d$ on $X$ parameterized by $S$". The subscheme $\operatorname{Supp}(\alpha) \hookrightarrow X$ is the support of this family of cycles. In particular it should, topologically at least, have finite fibers over $S$.

**Proposition (2.4.1).** *Let $S$ be a* connected *algebraic space and $X$ a space affine over $S$. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. If $X = \coprod_{i=1}^n X_i$, then there are uniquely defined integers $d_1, d_2, \dots, d_n \in \mathbb{N}$ such that $d = d_1 + d_2 + \dots + d_n$ and such that $\alpha$ factors through the closed subspace $\Gamma^{d_1}(X_1) \times_S \Gamma^{d_2}(X_2) \times_S \dots \times_S \Gamma^{d_n}(X_n) \hookrightarrow \Gamma^d(X/S)$. The support* $\operatorname{Supp}(\alpha)$ *is contained in the union of the $X_i$'s with $d_i > 0$. In particular* $\operatorname{Supp}(\alpha)$ *has at most $d$ connected components.*

*Proof.* By Proposition (1.4.1) there is a decomposition

$$\Gamma^d(X/S) = \coprod_{\substack{d_i \in \mathbb{N} \\ \sum_i d_i = d}} \Gamma^{d_1}(X_1) \times_S \Gamma^{d_2}(X_2) \times_S \dots \times_S \Gamma^{d_n}(X_n).$$

As $S$ is connected $\alpha$ factors uniquely through one of the spaces in this decomposition. It is further clear that $X_i \cap \operatorname{Supp}(\alpha) \neq \emptyset$ if and only if $d_i > 0$. The last observation follows after replacing $X$ with $\operatorname{Image}(\alpha)$ as then $n$ is at most $d$ in any decomposition. □

**Definition (2.4.2).** Let $S$ and $X$ be as in Proposition (2.4.1). The *multiplicity* of $\alpha$ on $X_i$ is the integer $d_i$.

**Proposition (2.4.3).** *Let $S = \operatorname{Spec}(k)$ where $k$ is a field and let $X/S$ be an affine scheme. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then $\operatorname{Image}(\alpha) = \operatorname{Supp}(\alpha) = \coprod_{i=1}^n \operatorname{Spec}(k_i)$ is a disjoint union of a at most $d$ points such that the separable degree of each $k_i/k$ is finite.*

*Proof.* Propositions (2.1.4) and (2.1.6) shows that $\operatorname{Image}(\alpha)$ is reduced and affine of dimension zero, hence totally disconnected. By Proposition (2.4.1) it is thus a disjoint union of at most $d$ reduced points. As the support commutes with arbitrary base change by Theorem (2.3.6), it follows after considering the base change $k \hookrightarrow \overline{k}$ that the separable degree of $k_i/k$ is finite. □

**Corollary (2.4.4).** *Let $X$, $Y$ and $S$ be algebraic spaces with affine morphisms $f : X \to Y$ and $g : Y \to S$. Let $\alpha : S \to \Gamma^d(X/S)$ be a morphism and denote by $f_*\alpha$ the composition of $\alpha$ and the morphism $\Gamma^d(f) : \Gamma^d(X/S) \to \Gamma^d(Y/S)$. Then $\operatorname{Supp}(f_*\alpha) = f\bigl(\operatorname{Supp}(\alpha)\bigr)$.*

*Proof.* As the support and the set-theoretic image commute with any base change, we can assume that $S = \operatorname{Spec}(k)$ where $k$ is a field. Then

$$\operatorname{Image}(\alpha) = \coprod_{i=1}^{n} \operatorname{Spec}(k_i) = \{x_1, x_2, \dots, x_n\}$$

by Proposition (2.4.3). Further, by Proposition (1.4.1) there are positive integers $d_1, d_2, \dots, d_n$ such that $\alpha$ factors through $\prod_{i=1}^n \Gamma^{d_i}\bigl(\operatorname{Spec}(k_i)\bigr) \hookrightarrow \Gamma^d(X/S)$. Let

$$f\bigl(\operatorname{Image}(\alpha)\bigr) = \coprod_{j=1}^{m} \operatorname{Spec}(k'_j) = \{y_1, y_2, \dots, y_m\}$$

where $m \leq n$. It is then immediately seen that $f_*\alpha$ factors through the closed subspace $\prod_{j=1}^m \Gamma^{e_j}\bigl(\operatorname{Spec}(k'_j)\bigr) \hookrightarrow \Gamma^d(Y/S)$ where $e_j = \sum_{f(x_i)=y_j} d_i$. As $d_i$ is positive so is $e_j$ and thus $y_j \in \operatorname{Supp}(f_*\alpha)$. This shows that $\operatorname{Supp}(f_*\alpha) = f\bigl(\operatorname{Supp}(\alpha)\bigr)$. □

**Proposition (2.4.5).** *Let $X$ be an algebraic space affine over $S$ and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then every irreducible component of $\operatorname{Supp}(\alpha)$ maps onto an irreducible component of $S$.*

*Proof.* As the support commutes with any base change it is enough to consider the case where $S = \operatorname{Spec}(A)$ is irreducible, reduced and affine. Let $\operatorname{Image}(\alpha) = \operatorname{Spec}(B)$ and $F : B \to A$ be the multiplicative polynomial law corresponding to $\alpha$. We have a commutative diagram

$$\begin{tikzcd} \Gamma^d_A(B) \arrow[r, two heads] \arrow[d, "f"'] & \Gamma^d_A(B/I) \arrow[r] \arrow[d, phantom, "\circ"] & \Gamma^d_{K(A)}\big(B \otimes_A K(A)\big) \arrow[d, "f"'] \\ A \arrow[rr, hook] & & K(A) \end{tikzcd}$$

where $I = \ker\big(B \to B \otimes_A K(A)\big)$. This shows that $I \subseteq \ker(F) = 0$. As $V(I)$ is the union of the irreducible components of $\mathrm{Supp}(\alpha)$ which dominate $S$ this shows that every component surjects onto $S$. □

In the following theorem we restate the main properties of the image and support of a family of cycles:

**Theorem (2.4.6).** *Let $X$ be an algebraic space affine over $S$ and let $\alpha : S \to \Gamma^d(X/S)$ be a morphism. Then*

(i) *If $S$ is reduced then* $\mathrm{Image}(\alpha)$ *is reduced.*
(ii) $\mathrm{Image}(\alpha) \to S$ *is integral.*
(iii) *If $S$ is connected then* $\mathrm{Supp}(\alpha)$ *has at most $d$ connected components.*
(iv) *If $S = \mathrm{Spec}(k)$ where $k$ is a field then* $\mathrm{Image}(\alpha) = \coprod_{i=1}^n \mathrm{Spec}(k_i)$ *is a disjoint union of a finite number of points, at most $d$, such that the separable degree of each $k_i/k$ is finite.*
(v) *Each geometric fiber of* $\mathrm{Supp}(\alpha) \to S$ *has at most $d$ points. In particular,* $\mathrm{Supp}(\alpha) \to S$ *has universally topologically finite fibers, cf. Definition* (A.2.1).
(vi) *If $S$ is a semi-local scheme, i.e., the spectrum of a semi-local ring, then* $\mathrm{Supp}(\alpha)$ *is semi-local.*
(vii) *Every irreducible component of* $\mathrm{Supp}(\alpha)$ *maps onto an irreducible component of $S$.*

*Proof.* Properties (i) and (ii) follows from Propositions (2.1.4) and (2.1.6) respectively. Properties (iii) and (iv) are Propositions (2.4.1) and (2.4.3) respectively. Property (v) follows from (iv) as the support commutes with any base change and property (vi) follows immediately from (ii) and (v). Property (vii) is Proposition (2.4.5). □

The following examples show that the support is not always finite.

**Example (2.4.7).** Let $k = \mathbb{F}_p(t_1, t_2, \dots)$ and $K = \mathbb{F}_p(t_1^{1/p}, t_2^{1/p}, \dots)$. We have a polynomial law $F : K \to k$ given by $a \mapsto a^p$. The support of the corresponding family $\alpha : \mathrm{Spec}(k) \to \Gamma^d(\mathrm{Spec}(K))$ is $\mathrm{Spec}(K)$ and $k \hookrightarrow K$ is not finite.

The following example shows that even if $X \to S$ is of finite presentation then the image of a family $\alpha : S \to \Gamma^d(X/S)$ need not be of finite presentation.

**Example (2.4.8).** Let $X = S = \mathrm{Spec}(A)$ where $A = k[t_1, t_2, \dots]/(t_1^p, t_2^p, \dots)$ and $k$ is a field of characteristic $p$. Let $\alpha$ correspond to the multiplicative polynomial law

$F : A \to A$, $x \mapsto x^p$. Then, as in Examples (2.3.7) the kernel of $F$ is $(t_1, t_2, \dots)$ which is not finitely generated. Hence $\mathrm{Image}(\alpha) = \mathrm{Spec}(k) \hookrightarrow X$ is not finitely presented over $S$.

## 2.5. Topological properties of the support.

**Definition (2.5.1)** ([EGA$_\mathrm{I}$, Def. 3.9.2])**.** We say that a morphism of algebraic spaces $f : X \to Y$ is *generizing* if for any $x \in X$ and generization $y' \in Y$ of $y = f(x)$ there exists a generization $x'$ of $x$ such that $f(x') = y'$. Equivalently, if $X$ and $Y$ are schemes, the image of $\mathrm{Spec}(\mathcal{O}_{X,x})$ by $f$ is $\mathrm{Spec}(\mathcal{O}_{Y,y})$. We say that $f$ is *component-wise dominating* if every irreducible component of $X$ dominates an irreducible component of $Y$. We say that $f$ is *universally generizing* (resp. *universally component-wise dominating*) if $f' : X' \to Y'$ is generizing (resp. dominating) for any morphism $g : Y' \to Y$ where $X' = X \times_Y Y'$.

*Remark* **(2.5.2).** A morphism $f : X \to Y$ is generizing (resp. universally generizing) if and only if $f_{\mathrm{red}}$ is generizing (resp. universally generizing). If $g : Y' \to Y$ is a generizing surjective morphism, we have that $f$ is generizing if $f'$ is generizing. If $g : Y' \to Y$ is a universally generizing surjective morphism, then $f$ is generizing (resp. universally generizing) if and only if $f'$ is generizing (resp. universally generizing). Any flat morphism $Y' \to Y$ of algebraic spaces is universally generizing.

**Lemma (2.5.3).** *Let $f : X \to Y$ be a morphism of algebraic spaces. Then $f$ is universally generizing if and only if it is universally component-wise dominating.*

*Proof.* A generizing morphism is component-wise dominating so the condition is necessary. For sufficiency, assume that $f$ is universally component-wise dominating. Let $x \in X$, $y = f(x)$ and choose a generization $y' \in Y$. Let $Y' = \overline{\{y'\}}$ with the reduced structure and consider the base change $Y' \hookrightarrow Y$. As $f'$ is component-wise dominating, there is a generization $x'$ of $x$ above $y'$. □

**Proposition (2.5.4).** *Let $f : X \to S$ be an affine morphism of algebraic spaces. Let $\alpha : S \to \Gamma^d(X/S)$ be a family with support $Z = \mathrm{Supp}(\alpha) \hookrightarrow X$. Then $f|_Z$ is universally generizing.*

*Proof.* Follows immediately from Lemma (2.5.3) as the support of a family of cycles is universally component-wise dominating by Theorems (2.4.6) (vii) and (2.3.6). □

*Remark* **(2.5.5).** If $Z \to S$ is of finite presentation, e.g., if $S$ is locally noetherian and $X \to S$ is locally of finite type, then it immediately follows that $f|_Z$ is *universally open* from [EGA$_\mathrm{I}$, Prop. 7.3.10]. We will show that $f|_Z$ is universally open without any hypothesis on $f$. The following lemma settles the case when $X \to S$ is locally of finite type.

**Lemma (2.5.6).** *Let $S$ and $X$ be* affine *schemes and $f : X \to S$ a morphism of* finite type*. Let $\alpha : S \to \Gamma^d(X/S)$ be a family of cycles and $Z = \mathrm{Supp}(\alpha)$ its* *support. There is then a bijective closed immersion $Z \hookrightarrow Z'$ such that $Z'$ is of finite presentation over $S$.*

*Proof.* Let $S = \operatorname{Spec}(A)$, $Z = \operatorname{Spec}(B)$ and let $F : B \to A$ be the multiplicative law corresponding to $\alpha$ restricted to its image. Let $C = A[t_1, t_2, \dots, t_n] \to B$ be a surjection. The multiplicative law $F$ induces a multiplicative law $G : C \twoheadrightarrow B \to A$. Note that $B = C/\ker(G)$. Corresponding to $G$ is a homomorphism $g : \Gamma^d_A(C) \twoheadrightarrow \Gamma^d_A(B) \to A$. As $\Gamma^d_A(C)$ is a finitely presented $A$-algebra, cf. Proposition (1.3.7), this homomorphism descends to a homomorphism $g_0 : \Gamma^d_{A_0}(C_0) \to A_0$ with $A_0$ noetherian such that $C = C_0 \otimes_{A_0} A$ and $g = g_0 \otimes_{A_0} \mathrm{id}_A$. As $A_0$ is noetherian $C_0/\ker(G_0)$ is a finite $A_0$-algebra of finite presentation.

Let $Z_0 = \operatorname{Spec}\bigl(C_0/\ker(G_0)\bigr)$ and $Z' = Z_0 \times_{\operatorname{Spec}(A_0)} \operatorname{Spec}(A)$. As the support commutes with base change by Theorem (2.3.6) we have that $Z \hookrightarrow Z'$ is a bijective closed immersion. □

**Proposition (2.5.7).** *Let $S$ and $X$ be algebraic spaces and $f : X \to S$ an affine morphism. Let $Z$ be the support of a family $\alpha : S \to \Gamma^d(X/S)$. Then the restriction of $f$ to $Z$ is* universally open*.*

*Proof.* The statement is étale-local so we can assume that $S = \operatorname{Spec}(A)$ and $Z = \operatorname{Spec}(B)$. Further as the support commutes with any base change, cf. Theorem (2.3.6), it is enough to show that $f|_Z : Z \to S$ is open.

We can write $B$ as a filtered direct limit of finite $A$-subalgebras $B_\lambda \hookrightarrow B$. Let $Z_\lambda = \operatorname{Spec}(B_\lambda)$. As $B_\lambda \hookrightarrow B$ is integral and injective it follows that $Z \to Z_\lambda$ is closed and dominating and thus surjective. Let $\alpha : S \to \Gamma^d(Z/S)$ be a family with support $Z$ and let $\alpha_\lambda : S \to \Gamma^d(Z_\lambda/S)$ be the family given by push-forward along $\varphi_\lambda : Z \to Z_\lambda$.

By Corollary (2.4.4) we have that $\operatorname{Supp}(\alpha_\lambda) = \varphi_\lambda(Z_{\mathrm{red}}) = (Z_\lambda)_{\mathrm{red}}$. Further by Lemma (2.5.6) there is a scheme $Z'_\lambda$ of finite presentation over $S$ such that $\operatorname{Supp}(\alpha_\lambda)$ and $Z_\lambda$ are homeomorphic to $Z'_\lambda$. As $\operatorname{Supp}(\alpha_\lambda) \to S$ is generizing by Proposition (2.5.4) so is $Z'_\lambda \to S$. As $Z'_\lambda \to S$ is also of finite presentation it is *open* by [$\mathrm{EGA_I}$, Prop. 7.3.10] and hence so is $Z_\lambda \to S$.

To show that $f|_Z : Z \to S$ is open it is enough to show that the image of any quasi-compact open subset of $Z$ is open. Let $U \subseteq Z$ be a quasi-compact open subset. Then according to [$\mathrm{EGA_{IV}}$, Cor. 8.2.11] there is a $\lambda$ and $U_\lambda \subseteq Z_\lambda$ such that $U = {\varphi_\lambda}^{-1}(U_\lambda)$. As $\varphi_\lambda$ is surjective and $Z_\lambda \to S$ is open this shows that $f|_Z(U)$ is open. □

## 3. Definition and representability of $\underline{\Gamma}^d_{X/S}$

We will define a functor $\underline{\Gamma}^d_{X/S}$ and show that when $X/S$ is affine it is represented by $\Gamma^d(X/S)$. It is then easy to prove that $\underline{\Gamma}^d_{X/S}$ is represented by a scheme for any AF-scheme $X/S$. To prove representability in general, i.e., when $X/S$ is any separated algebraic space, is more difficult. For *any* morphism $f : X \to Y$ there is a natural transformation $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ which is "push-forward of cycles". If

$f$ is étale, then $f_*$ is étale over a certain open subset of $\Gamma^d(X/S)$. We will use this result to show representability of $\underline{\Gamma}^d_{X/S}$ giving an explicit étale covering.

3.1. **The functor $\underline{\Gamma}^d_{X/S}$.** Recall that a morphism of algebraic spaces $f : X \to S$ is said to be *integral* if it is affine and the corresponding homomorphism $\mathcal{O}_S \to f_*\mathcal{O}_X$ is integral. Equivalently, for any affine scheme $T = \operatorname{Spec}(A)$ and morphism $T \to S$ the space $X \times_S T = \operatorname{Spec}(B)$ is affine and $A \to B$ is integral. Further recall, Proposition (1.4.2), that if $X/S$ is affine and $Z$ is a closed subspace of $X$, then $\Gamma^d(Z/S)$ is a closed subspace of $\Gamma^d(X/S)$.

**Definition (3.1.1).** Let $S$ be an algebraic space and $X/S$ an algebraic space *separated* over $S$. A family of zero-cycles of degree $d$ consists of a closed subscheme $Z \hookrightarrow X$ such that $Z \hookrightarrow X \to S$ is *integral* together with a morphism $\alpha : S \to \Gamma^d(Z/S)$. Two families $(Z_1, \alpha_1)$ and $(Z_2, \alpha_2)$ are equivalent if there is a closed subscheme $Z$ of both $Z_1$ and $Z_2$ and a morphism $\alpha : S \to \Gamma^d(Z/S)$ such that $\alpha_i$ is the composition of $\alpha$ and the morphism $\Gamma^d(Z/S) \hookrightarrow \Gamma^d(Z_i/S)$ for $i = 1, 2$.

If $g : S' \to S$ is a morphism of spaces and $(Z, \alpha)$ a family of cycles on $X/S$, we let $g^*(Z, \alpha) = \big(g^*(Z), g^*\alpha\big)$ be the pull-back along $g$. The image and support of a family of cycles $(Z, \alpha)$ is the image and support of $\alpha$, cf. Definitions cf. (2.3.1) and (2.3.5).

*Remark* **(3.1.2).** It is clear that the pull-backs of equivalent families are equivalent and that the image and support of equivalent families coincide. If $(Z, \alpha)$ is a family then the family $(\operatorname{Image}(\alpha), \alpha')$ is a minimal representative in the same equivalence class. Here $\alpha'$ is the restriction of $\alpha$ to its image, i.e., the morphism $S \to \Gamma^d(\operatorname{Image}(\alpha)/S)$ which composed with $\Gamma^d(\operatorname{Image}(\alpha)/S) \hookrightarrow \Gamma^d(Z/S)$ is $\alpha$.

The pull-back $g^*\alpha$ of a minimal representative $\alpha$ will not in general be a minimal representative. However note that by Theorem (2.3.6) we have a canonical bijective closed immersion $\operatorname{Image}(g^*\alpha) \hookrightarrow g^*\operatorname{Image}(\alpha)$.

**Definition (3.1.3).** We let $\underline{\Gamma}^d_{X/S}$ be the contravariant functor from $S$-schemes to sets defined as follows. For any $S$-scheme $T$ we let $\underline{\Gamma}^d_{X/S}(T)$ be the set of equivalence classes of families of zero-cycles $(Z, \alpha)$ of degree $d$ of $X \times_S T/T$. For any morphism $g : T' \to T$ of $S$-schemes, the map $\underline{\Gamma}^d_{X/S}(g)$ is the pull-back of families of cycles as defined above.

In the sequel we will suppress the space of definition $Z$ and write $\alpha \in \underline{\Gamma}^d_{X/S}(T)$. We will not make explicit use of $Z$. Instead, we will use the subspace $\operatorname{Image}(\alpha) \hookrightarrow X \times_S T$ which is independent on the choice of $Z$ by Remark (3.1.2).

**Proposition (3.1.4).** *If $X$ is* affine *over $S$ then the functor $\underline{\Gamma}^d_{X/S}$ is represented by the algebraic space $\Gamma^d(X/S)$, defined in §1.4, which is affine over $S$.*

*Proof.* There is a natural transformation from $\underline{\Gamma}^d_{X/S}$ to $\operatorname{Hom}_S(-, \Gamma^d(X/S))$ given by composing a family $\alpha : T \to \Gamma^d(Z/T)$ with $\Gamma^d(Z/T) \hookrightarrow \Gamma^d(X \times_S T/T) =$

$\Gamma^d(X/S) \times_S T \to \Gamma^d(X/S)$. If $\alpha : T \to \Gamma^d(X/S)$ is any morphism then $\alpha \times_S \mathrm{id}_T$ factors through $\Gamma^d(Z/T) \hookrightarrow \Gamma^d(X \times_S T/T)$ where $Z \hookrightarrow X \times_S T$ is the image of $\alpha \times_S \mathrm{id}_T$. As $Z$ is integral over $S$ by Theorem (2.4.6) (ii), we have that the morphism $\alpha$ corresponds to a unique equivalence class of families. It is thus clear that $\Gamma^d(X/S)$ represents $\underline{\Gamma}^d_{X/S}$. □

*Remark* **(3.1.5).** For an affine morphism of algebraic spaces $X \to S$, we have that $\Gamma^1(X/S) = X$ and that the $T$-points of $\Gamma^1(X/S)$ parameterizes sections of $X \times_S T \to T$. Thus, for any separated algebraic space $X/S$ it follows that $\underline{\Gamma}^1_{X/S}$ parameterizes sections of $X \to S$ and that $\underline{\Gamma}^1_{X/S}$ is represented by $X$.

**Proposition (3.1.6).** *The functor $\underline{\Gamma}^d_{X/S}$ is a sheaf in the étale topology.*

*Proof.* Let $T$ be an $S$-scheme and $f : T' \to T$ an étale surjective morphism. Let $T'' = T' \times_T T'$ with projections $\pi_1$ and $\pi_2$. Given an element $\alpha' \in \underline{\Gamma}^d_{X/S}(T')$ such that $\pi_1^*\alpha' = \pi_2^*\alpha'$ we have to show that there is a unique $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ such that $f^*\alpha = \alpha'$. Let $Z' \hookrightarrow X \times_S T'$ be the image of $\alpha'$. As the image commutes with étale base change, cf. Theorem (2.3.2), the image of $\alpha''$ is $Z'' = \pi_1^{-1}(Z') = \pi_2^{-1}(Z')$. As closed immersions satisfy effective descent with respect to étale morphisms [$\mathrm{SGA}_1$, Exp. VIII, Cor. 1.9], there is a closed subspace $Z \hookrightarrow X \times_S T$ such that $Z' = Z \times_T T'$. Moreover $Z$ is affine over $T$. Any $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ such that $f^*\alpha = \alpha'$ is then in the subset $\underline{\Gamma}^d_{Z/T}(T) \subseteq \underline{\Gamma}^d_{X/S}(T)$. It is thus enough to show that $\underline{\Gamma}^d_{Z/S}$ is a sheaf in the étale topology. But $\underline{\Gamma}^d_{Z/S}$ is represented by the space $\Gamma^d(Z/S)$ which is affine over $S$. As the étale topology is sub-canonical, it follows that $\underline{\Gamma}^d_{Z/S}$ is a sheaf. □

**Proposition (3.1.7).** *Let $X/S$ and $Y/S$ be separated algebraic spaces. If $f : X \to Y$ is an immersion (resp. a closed immersion, resp. an open immersion) then $\underline{\Gamma}^d_{X/S}$ is a locally closed subfunctor (resp. a closed subfunctor, resp. an open subfunctor) of $\underline{\Gamma}^d_{Y/S}$.*

*Proof.* Let $T$ be an $S$-scheme and let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ be a family with $Z = \mathrm{Image}(\alpha) \hookrightarrow X_T$. Then $Z \hookrightarrow X_T$ is a closed subscheme such that $Z \to T$ is integral and hence universally closed. As $Y \to S$ is separated it thus follows that $Z \hookrightarrow X_T \hookrightarrow Y_T$ is a closed subscheme. It follows that $\underline{\Gamma}^d_{X/S}$ is a subfunctor of $\underline{\Gamma}^d_{Y/S}$.

Let $\alpha : T \to \underline{\Gamma}^d_{Y/S}$ be a family of cycles. We have to show that if $f$ is a closed (resp. open) immersion then there is a closed (resp. open) subscheme $U \hookrightarrow T$ such that if $g : T' \to T$ and $\alpha' = g^*\alpha \in \underline{\Gamma}^d_{X/S}(T')$ then $g$ factors through $U$. Let $X_T = X \times_S T$, $Y_T = Y \times_S T$, $Z = \mathrm{Image}(\alpha) \subset Y_T$ and $W = Z \cap X_T = Z \times_{Y_T} X_T \hookrightarrow X_T$.

If $f$ is an open immersion we let $V$ be the closed subset $Y_T \setminus X_T$ and $U$ be the complement of the image of $V \cap Z = Z \setminus W$ by $Z \to T$. Thus $U$ is the open subset of $T$ such that $t \in U$ if and only if the fiber $Z_t$ does not meet $V$ or equivalently is contained in $W$. As the support commutes with arbitrary base change, see Theorem (2.3.6), it is easily seen that $Z \times_T T'$ factors through $X_{T'}$ if and only if

$T' \to T$ factors through $U$. Hence $T \times_{\underline{\Gamma}^d_{Y/S}} \underline{\Gamma}^d_{X/S} = T|_U$ which shows that $\underline{\Gamma}^d_{X/S}$ is an open subfunctor.

If $f$ is a closed immersion we consider the cartesian diagram

$$\begin{tikzcd} T \times_{\underline{\Gamma}^d_{Z/T}} \underline{\Gamma}^d_{W/T} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{W/T} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X_T/T} \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X/S} \arrow[d, hook] \\ T \arrow[r] & \underline{\Gamma}^d_{Z/T} \arrow[r, hook] & \underline{\Gamma}^d_{Y_T/T} \arrow[r] & \underline{\Gamma}^d_{Y/S}. \end{tikzcd}$$

As $W$ and $Z$ are affine over $S$, the functors $\underline{\Gamma}^d_{W/T}$ and $\underline{\Gamma}^d_{Z/T}$ are represented by $\Gamma^d(W/T)$ and $\Gamma^d(Z/T)$ respectively. As $\Gamma^d(W/T) \hookrightarrow \Gamma^d(Z/T)$ is a closed immersion by Proposition (1.4.2) it follows that $\underline{\Gamma}^d_{X/S}$ is a closed subfunctor of $\underline{\Gamma}^d_{Y/S}$. □

**Proposition (3.1.8).** *Let $S$ be an algebraic space and let $X_1, X_2, \dots, X_n$ be algebraic spaces separated over $S$. Then*

$$\underline{\Gamma}^d_{\coprod_{i=1}^n X_i} = \coprod_{\substack{d_i \in \mathbb{N} \\ \sum_i d_i = d}} \underline{\Gamma}^{d_1}_{X_1} \times_S \underline{\Gamma}^{d_2}_{X_2} \times_S \cdots \times_S \underline{\Gamma}^{d_n}_{X_n}.$$

*Proof.* Follows from Proposition (1.4.1). □

**Corollary (3.1.9).** *Let $X/S$ be a separated algebraic space. Let $k$ be an algebraically closed field and $s$ : $\mathrm{Spec}(k) \to S$ a geometric point of $S$. There is a one-to-one correspondence between $k$-points of $\underline{\Gamma}^d_{X/S}$ and effective zero-cycles of degree $d$ on $X_s$. In this correspondence, a zero-cycle $\sum_{i=1}^n d_i[x_i]$ on $X_s$ corresponds to the family $(Z, \alpha)$ where $Z = \{x_1, x_2, \dots, x_n\} \subseteq X_s$ and $\alpha$ is the morphism*

$$\alpha : \mathrm{Spec}(k) \cong \Gamma^{d_1}(x_1/k) \times_k \Gamma^{d_2}(x_2/k) \times_k \cdots \times_k \Gamma^{d_n}(x_n/k) \hookrightarrow \Gamma^d(Z/k)$$

*Proof.* Let $\alpha \in \underline{\Gamma}^d_{X/S}(k)$ be a $k$-point. By Theorem (2.4.6) (iv) we have that $Z = \mathrm{Image}(\alpha) \hookrightarrow X_s$ is a finite disjoint union of points $x_1, x_2, \dots, x_n$, all with residue field $k$ as $k$ is algebraically closed. According to Proposition (3.1.8), there are positive integers $d_1, d_2, \dots, d_n$ such that $d = d_1 + d_2 + \cdots + d_n$ and such that $\alpha : k \to \Gamma^d(Z/k)$ factors through the open and closed subscheme $\Gamma^{d_1}(x_1/k) \times_k \Gamma^{d_2}(x_2/k) \times_k \cdots \times_k \Gamma^{d_n}(x_n/k)$. As $k(x_i) = k$, we have that $\Gamma^{d_i}(x_i/k) \cong k$. The point $\alpha$ corresponds to $\sum_{i=1}^n d_i[x_i]$. □

**Proposition (3.1.10).** *Let $X/S$ be a separated algebraic space. Let $\{U_\beta\}$ be an open covering of $X$ such that any set of $d$ points in $X$ above the same point in $S$ lies in one of the $U_\beta$'s. Then $\coprod_\beta \underline{\Gamma}^d_{U_\beta/S} \to \underline{\Gamma}^d_{X/S}$ is an open covering. If $X/S$ is an* AF*-scheme then such a covering with the $U_\beta$'s* affine *exists.*

*Proof.* Let $k$ be a field and $\alpha \in \underline{\Gamma}^d_{X/S}(k)$. Then by Theorem (2.4.6) (iv) there is a $\beta$ such that $\alpha \in \underline{\Gamma}^d_{U_\beta/S}(k) \subseteq \underline{\Gamma}^d_{X/S}(k)$. Thus $\coprod_\beta \underline{\Gamma}^d_{U_\beta/S} \to \underline{\Gamma}^d_{X/S}$ is an open covering by Proposition (3.1.7). □

**Theorem (3.1.11).** *Let $S$ be a scheme and $X/S$ an* AF*-scheme. The functor $\underline{\Gamma}^d_{X/S}$ is then represented by an* AF*-scheme $\Gamma^d(X/S)$.*

*Proof.* As $\underline{\Gamma}^d_{X/S}$ is a sheaf in the Zariski topology, we can assume that $S$ is affine. Let $\{U_\beta\}$ be an open covering of $X$ by affines such that any set of $d$ points in $X$ above the same point in $S$ lies in one of the $U_\beta$'s. As $\underline{\Gamma}^d_{U_\beta/S}$ is represented by an affine scheme, Proposition (3.1.10) shows that $\underline{\Gamma}^d_{X/S}$ is represented by a scheme $\Gamma^d(X/S)$.

If $\alpha_1, \alpha_2, \dots, \alpha_m$ are points of $\Gamma^d(X/S)$ above the same point of $S$, then the union of their supports consists of at most $dm$ points and there is thus an affine subset $U \subseteq X$ such that $\alpha_1, \alpha_2, \dots, \alpha_m \in \Gamma^d(U/S)$. This shows that $\Gamma^d(X/S)/S$ is an AF-scheme. □

3.2. **Effective pro-representability of $\underline{\Gamma}^d_{X/S}$.** Let $A$ be a henselian local ring and $T = \operatorname{Spec}(A)$ together with a morphism $T \to S$. The image of a family of cycles $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ over $T$ is then a semi-local scheme $Z$, integral over $T$ by Theorem (2.4.6) (ii), (vi). Furthermore, Proposition (A.2.7) implies that $Z$ is a finite disjoint union of local henselian schemes.

Let $z_1, z_2, \dots, z_n$ be the closed points of $Z \hookrightarrow X_T$ and $\{x_1, x_2, \dots, x_m\}$ their images in $X$ where the $x_i$'s are chosen to be distinct. As $z_i$ lies over the closed point of $T$, all $x_i$ lies over a common point $s \in S$. Let ${}^{\mathrm{h}}X_{x_i} = \operatorname{Spec}({}^{\mathrm{h}}\mathcal{O}_{X,x_i})$, ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m} = \coprod_{i=1}^m {}^{\mathrm{h}}X_{x_i}$ and ${}^{\mathrm{h}}S_s = \operatorname{Spec}({}^{\mathrm{h}}\mathcal{O}_{S,s})$ be the henselizations of $X$ and $S$ at the $x_i$'s and $s$. As $\mathcal{O}_{Z,z_i}$ is henselian it follows that $Z \hookrightarrow X_T \to X$ factors uniquely through ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m} \to X$. Thus $Z \hookrightarrow X_T$ factors uniquely through ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m} \times_{{}^{\mathrm{h}}S_s} T \to X_T$ and $\alpha$ corresponds to a unique element of $\underline{\Gamma}^d_{{}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s}(T)$. As ${}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}$ is affine, we have a unique morphism $T \to \Gamma^d({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s)$.

Further, by Proposition (1.4.1)

$$\Gamma^d\left({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s\right) = \coprod_{\substack{d_i \in \mathbb{N} \\ \sum_i d_i = d}} \prod_{i=1}^m \Gamma^{d_i}\left({}^{\mathrm{h}}X_{x_i}/{}^{\mathrm{h}}S_s\right).$$

and as $T$ is connected $T \to \Gamma^d({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s)$ factors through one of these components.

To conclude, there are uniquely determined points $x_1, x_2, \dots, x_m \in X$, unique positive integers $d_i$ and a unique morphism

$$\varphi : T \to \prod_{i=1}^m \Gamma^{d_i}\left({}^{\mathrm{h}}X_{x_i}/{}^{\mathrm{h}}S_s\right) \hookrightarrow \Gamma^d\left({}^{\mathrm{h}}X_{x_1,x_2,\dots,x_m}/{}^{\mathrm{h}}S_s\right)$$

such that $\alpha$ is equivalent to $\varphi \times_{{}^{\mathrm{h}}S_s} \mathrm{id}_T$. This implies the following:

**Proposition (3.2.1).** *Let $X/S$ be a separated algebraic space and assume that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$. Let $\beta \in \Gamma^d(X/S)$ be a point with residue field $k$ and $s$ its image in $S$. The point $\beta$ corresponds uniquely to points*

*$x_1, x_2, \dots, x_m \in X$, positive integers $d_1, d_2, \dots, d_m$ with sum $d$ and morphisms $\varphi_i : k \to \Gamma^d\big(k(x_i)/k(s)\big)$. The local henselian ring (resp. strictly local ring) at $\beta$ is the local henselian ring (resp. strictly local ring) of $\prod_{i=1}^m \Gamma^{d_i}({}^{\mathrm{h}}X_{x_i}/{}^{\mathrm{h}}S_s)$ at the point corresponding to the morphisms $\varphi_i$.*

**(3.2.2)** If $X/S$ is *locally of finite type* and $A$ is a complete local *noetherian* ring, then the support of any family of cycles $\alpha$ on $X$ parameterized by $T = \mathrm{Spec}(A)$ is finite over $T$. Thus $\mathrm{Image}(\alpha)$ is a disjoint union of a finite number of complete local rings. Let $s \in S$ and $x_i \in X$ be defined as above and let $\widehat{X}_{x_i} = \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x_i})$, $\widehat{S}_s = \mathrm{Spec}(\widehat{\mathcal{O}}_{S,s})$ and $\widehat{X}_{x_1,x_2,\dots,x_m} = \coprod_{i=1}^m \widehat{X}_{x_i}$ be the completions of $X$ and $S$ at the corresponding points. Repeating the reasoning above we conclude that there is a unique morphism

$$\varphi : T \to \prod_{i=1}^m \Gamma^{d_i}(\widehat{X}_{x_i}/\widehat{S}_s) \hookrightarrow \Gamma^d(\widehat{X}_{x_1,x_2,\dots,x_m}/\widehat{S}_s)$$

such that $\alpha$ is equivalent to $\varphi \times_{\widehat{S}_s} \mathrm{id}_T$. Thus we obtain:

**Proposition (3.2.3).** *Let $S$ be locally noetherian and $X$ an algebraic space separated and locally of finite type over $S$ and assume that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$. Let $\beta \in \Gamma^d(X/S)$ be a point with residue field $k$ and $s$ its image in $S$. The point $\beta$ corresponds uniquely to points $x_1, x_2, \dots, x_m \in X$, positive integers $d_1, d_2, \dots, d_m$ with sum $d$ and morphisms $\varphi_i : k \to \Gamma^d\big(k(x_i)/k(s)\big)$. The formal local ring at $\beta$ is the formal local ring of $\prod_{i=1}^m \Gamma^{d_i}(\widehat{X}_{x_i}/\widehat{S}_s)$ at the point corresponding to the morphisms $\varphi_i$.*

**Corollary (3.2.4).** *Let $S$ be locally noetherian and $X$ an algebraic space separated and locally of finite type over $S$. The functor $\underline{\Gamma}^d_{X/S}$ is* effectively pro-representable *by which we mean the following: Let $k$ be any field and $\beta_0 \in \underline{\Gamma}^d_{X/S}(k)$. There is then a complete local noetherian ring $\widehat{A}$ and an object $\widehat{\beta} \in \underline{\Gamma}^d_{X/S}(\mathrm{Spec}(A))$ such that for any local artinian scheme $T$ and family $\alpha \in \underline{\Gamma}^d_{X/S}(T)$, coinciding with $\beta_0$ at the closed point of $T$, there is a unique morphism $f : T \to \mathrm{Spec}(\widehat{A})$ such that $\alpha = f^*\widehat{\beta}$.*

*Remark* **(3.2.5).** Assume that $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space $\Gamma^d(X/S)$. Questions about properties of $\Gamma^d(X/S)$ which only depend on the strictly local rings, such as being flat or reduced, can be reduced to the case where $X$ is affine using Proposition (3.2.1). As some properties cannot be read from the strictly local rings we will need the stronger result of Proposition (3.4.2) which shows that any point in $\Gamma^d(X/S)$ has an étale neighborhood which is an open subset of $\Gamma^d(U/S)$ for some affine scheme $U$.

## 3.3. **Push-forward of families of cycles.**

**Definition (3.3.1).** Let $f : X \to Y$ be a morphism of algebraic spaces separated over $S$. If $(Z,\alpha) \in \underline{\Gamma}^d_{X/S}(T)$ is a family of cycles over $T$ we let $f_*(Z,\alpha) = (f_T(Z), f_*\alpha)$ where $f_T(Z)$ is the schematic image of $Z$ along $X \times_S T \to Y \times_S T$ and $f_*\alpha$ is the composition of $\alpha : T \to \Gamma^d(Z/T)$ and $\Gamma^d(Z/T) \to \Gamma^d(f_T(Z)/T)$. This induces a natural transformation of functors $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ denoted the push-forward.

*Remark* **(3.3.2).** If $g : Y \to Z$ is another morphism of $S$-spaces then clearly $g_* \circ f_* = (g \circ f)_*$. If $X$ and $Y$ are affine over $S$, the push-forward $f_* : \underline{\Gamma}^d_{X/S} \to \underline{\Gamma}^d_{Y/S}$ coincides with the morphism $\Gamma^d(X/S) \to \Gamma^d(Y/S)$ given by the covariance of the functor $\Gamma^d$.

Definition (3.3.1) only makes sense after we have checked that $f_T(Z)$ is integral over $T$. If $Y/S$ is locally of finite type then $f_T(Z)$ is quasi-finite and proper and hence finite, cf. Proposition (A.2.3). More generally, as $Z \to T$ is integral with topological finite fibers by Theorem (2.4.6) (v), it follows from Theorem (A.2.2) that $f_T(Z)$ is integral without any hypothesis on $Y/S$ except the separatedness.

**Definition (3.3.3).** Let $X/S$ and $Y/S$ be separated algebraic spaces and let $f : X \to Y$ be any morphism of $S$-spaces. We say that $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is *regular* (resp. *quasi-regular*) with respect to $f$ if $f_T|_{\mathrm{Image}(\alpha)}$ is a closed immersion (resp. universally injective) or equivalently if $f_T|_{\mathrm{Image}(\alpha)} : \mathrm{Image}(\alpha) \to f_T\bigl(\mathrm{Image}(\alpha)\bigr)$ is an isomorphism (resp. a universal bijection). We let $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f}(T)$ (resp. $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}(T)$) be the elements which are regular (resp. quasi-regular) with respect to $f$.

**Definition (3.3.4).** Let $\mathcal{F}$ and $\mathcal{G}$ be contravariant functors from $S$-schemes to sets. We say that a morphism of functors $f : \mathcal{F} \to \mathcal{G}$ is *topologically surjective* if for any field $k$ and element $y \in \mathcal{G}\bigl(\mathrm{Spec}(k)\bigr)$ there is a field extension $g : \mathrm{Spec}(k') \to \mathrm{Spec}(k)$ and an element $x \in \mathcal{F}\bigl(\mathrm{Spec}(k')\bigr)$ such that $f(x) = g^*y$ in $\mathcal{G}\bigl(\mathrm{Spec}(k')\bigr)$. If $\mathcal{F}$ and $\mathcal{G}$ are represented by algebraic spaces, we have that $f$ is topologically surjective if and only if the corresponding morphism of spaces is surjective.

**Definition (3.3.5).** A morphism $f : X \to Y$ is *unramified* if it is formally unramified and locally of finite type.

In [$\mathrm{EGA_{IV}}$] unramified morphisms are locally of finite presentation but the above definition is more useful and also commonly used.

**Proposition (3.3.6).** *Let $X/S$ and $Y/S$ be separated algebraic spaces and let $f : X \to Y$ be a morphism of $S$-spaces. Let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$. If $f$ is unramified then $\alpha$ is quasi-regular if and only if $\alpha$ is regular.*

*Proof.* If $\alpha$ is quasi-regular and $f$ unramified then $\mathrm{Image}(\alpha) \hookrightarrow X \times_S T \to Y \times_S T$ is unramified and universally injective. By [$\mathrm{EGA_{IV}}$, Prop. 17.2.6] this implies that $f_T|_{\mathrm{Image}(\alpha)} : \mathrm{Image}(\alpha) \to Y \times_S T$ is a monomorphism. As $\mathrm{Image}(\alpha) \to T$ is universally closed and $Y_T \to T$ is separated it follows that $f_T|_{\mathrm{Image}(\alpha)}$ is a proper monomorphism and hence a closed immersion [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 18.12.6]. □

**Proposition (3.3.7).** *Let $X/S$ and $Y/S$ be separated algebraic spaces and let $f : X \to Y$ be a morphism of $S$-spaces. Let $T$ be an $S$-scheme and $f_T : X \times_S T \to Y \times_S T$ the base change of $f$ along $T \to S$. Let $\alpha \in \underline{\Gamma}^d_{X/S}(T)$. Then*

- (i) $\mathrm{Image}(f_*\alpha) \hookrightarrow f_T\bigl(\mathrm{Image}(\alpha)\bigr)$.
- (ii) $\mathrm{Supp}(f_*\alpha) = f_T\bigl(\mathrm{Supp}(\alpha)\bigr)$.
- (iii) $\mathrm{Supp}(\alpha) \to f_T\bigl(\mathrm{Supp}(\alpha)\bigr) = \mathrm{Supp}(f_*\alpha)$ *is a bijection if $\alpha$ is quasi-regular with respect to $f_*$.*
- (iv) $\mathrm{Image}(\alpha) \cong f_T\bigl(\mathrm{Image}(\alpha)\bigr) = \mathrm{Image}(f_*\alpha)$ *if $\alpha$ is regular with respect to $f_*$.*

*Proof.* (i) follows immediately by the definition of $f_*$ and (ii) follows from Corollary (2.4.4). (iii) follows from the definition of a quasi-regular family and (ii). (iv) follows by the definition of regular as $\mathrm{Image}(\alpha) \cong f_T\bigl(\mathrm{Image}(\alpha)\bigr)$ easily implies that $f_T\bigl(\mathrm{Image}(\alpha)\bigr) = \mathrm{Image}(f_*\alpha)$. □

**Examples (3.3.8).** We give two examples on bad behavior of the image with respect to push-forward. In the first example $f$ is étale, $\alpha$ not (quasi-)regular and $\mathrm{Image}(f_*\alpha) \hookrightarrow f_T\bigl(\mathrm{Image}(\alpha)\bigr)$ is not an isomorphism. In the second example $f$ is universally injective and $\alpha$ quasi-regular but not regular.

- (i) Let $S = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $X = Y \amalg Y = \mathrm{Spec}(B \times B)$ where $A = k[\epsilon]/\epsilon^2$ and $B = k[\epsilon,\delta]/(\epsilon^2,\delta^2,\epsilon\delta)$. We let $f : X \to Y$ be the étale map given by the identity on the two components. Finally we let $\alpha \in \underline{\Gamma}^2_{X/S}(S)$ be the family of cycles corresponding to the multiplicative polynomial law $F : B \times B \to B/(\delta-\epsilon) \times B/(\delta+\epsilon) \cong A \times A \to A \otimes_A A \cong A$ which is homogeneous of degree 2. The support of $\alpha$ corresponds to $\ker(F) = \bigl((\delta-\epsilon),(\delta+\epsilon)\bigr) \subset B \times B$. It is easily seen that $f\bigl(\mathrm{Image}(\alpha)\bigr) = V(0)$. On the other hand an easy calculation shows that $\mathrm{Image}(f_*\alpha) = V(\delta)$.
- (ii) Let $k$ be a field of characteristic different from 2. Let $S = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(C)$ where $A = k[\epsilon]/\epsilon^2$, $B = k[\epsilon,\delta]/(\epsilon,\delta)^2$ and $C = k[\epsilon,\delta,\tau]/\bigl(\epsilon^2,\epsilon\delta,\epsilon\tau,\delta^2,\tau^2,\delta\tau-\epsilon\bigr)$. Let $f : X \to Y$ be the natural morphism. An easy calculation shows that $\Gamma^2_A(C)$ is generated by $\gamma^2(\delta)$, $\gamma^2(\tau)$, $\delta \times 1$, $\tau \times 1$ and $\delta \times \tau$. After finding explicit relations for these generators in $\Gamma^2_A(C)$, it can also be shown that $\gamma^2(\delta), \gamma^2(\tau), \delta \times 1, \tau \times 1 \mapsto 0$ and $\delta \times \tau \mapsto -2\epsilon$ defines a family $\alpha : S \to \Gamma^2(X/S)$. It is easy to check that $\mathrm{Image}(\alpha) = X$, $f(\mathrm{Image}(\alpha)) = Y$ but $\mathrm{Image}(f_*\alpha) = V(\delta)$.

**Proposition (3.3.9).** *Let $f : X \to Y$ be a morphism between algebraic spaces separated over $S$. Then:*

- (i) $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f}$ *and* $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}$ *are subfunctors of* $\underline{\Gamma}^d_{X/S}$.
- (ii) *If $f : X \to Y$ is unramified then $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} = \underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}$ is an open subfunctor of $\underline{\Gamma}^d_{X/S}$.*

(iii) *If $f$ is an immersion then* $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} = \underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} = \underline{\Gamma}^d_{X/S}$.
(iv) *If $f$ is surjective then* $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ *is topologically surjective.*

*Proof.* (i) As the support commutes with arbitrary base change it follows that the requirement for $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ to be quasi-regular is stable under arbitrary base change. Thus the pull-back $\underline{\Gamma}^d_{X/S}(T) \to \underline{\Gamma}^d_{X/S}(T')$ induced by $T' \to T$ restricts to $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f}$. If $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is regular then by definition $\mathrm{Image}(\alpha) \cong f_T\big(\mathrm{Image}(\alpha)\big) = \mathrm{Image}(f_*\alpha)$. If $g : T' \to T$ is any morphism then clearly $\mathrm{Image}(g^*\alpha) \cong \mathrm{Image}(g^*f_*\alpha) = \mathrm{Image}(f_*g^*\alpha)$ and thus $g^*\alpha \in \underline{\Gamma}^d_{X/S,\mathrm{reg}/f}(T')$.

(ii) Proposition (3.3.6) shows that $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} = \underline{\Gamma}^d_{X/S,\mathrm{reg}/f}$. To show that $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \subseteq \underline{\Gamma}^d_{X/S}$ is open we let $\alpha : T \to \underline{\Gamma}^d_{X/S}$ be a morphism. This factors through $T \to \Gamma^d(Z/T)$ where $Z = \mathrm{Image}(\alpha) \hookrightarrow X_T$ and $X_T = X \times_S T$. As $f$ is unramified $(f_T)|_Z : Z \hookrightarrow X_T \to Y_T$ is unramified. In particular $(f_T)|_Z : Z \to f_T(Z)$ is finite and unramified. By Nakayama's lemma, the rank of the fibers of a finite morphism is upper semicontinuous. Thus, the subset $W$ of $f_T(Z)$ over which the geometric fibers of $(f_T)|_Z$ contain more than one point is closed. Let $U = T \setminus g_T(W)$, where $g : Y \to S$ is the structure morphism. Then $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} \times_{\underline{\Gamma}^d_{X/S}} T = U$ which shows that $\underline{\Gamma}^d_{X/S,\mathrm{qreg}/f} \subseteq \underline{\Gamma}^d_{X/S}$ is an open subfunctor.

(iii) Obvious from the definitions.

(iv) Let $\beta \in \underline{\Gamma}^d_{Y/S}(k)$ where $k = \overline{k}$ is an algebraically closed field. Then by Theorem (2.4.6) (iv) the image $W := \mathrm{Image}(\beta) \hookrightarrow Y_k$ is a finite disjoint union of reduced points, each with residue field $k$. As $f$ is surjective we can then find a field extension $k \hookrightarrow k'$ and a closed subspace $Z \hookrightarrow X_{k'}$ such that $f_{k'}(Z) = W_{k'}$ and $f_{k'}|_Z : Z \to W_{k'}$ is an isomorphism. This gives an element $\alpha \in \underline{\Gamma}^d_{X/S}(k')$ such that $f_*\alpha = \beta$. $\square$

**Proposition (3.3.10).** *Let*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

*be a cartesian square of algebraic spaces separated over $S$. Let*

$$\begin{aligned} \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} &= \underline{\Gamma}^d_{X'/S} \times_{\underline{\Gamma}^d_{Y'/S}} \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g} \\ &= \Big\{ \alpha \in \underline{\Gamma}^d_{X'/S} \; : \; f'_*\alpha \text{ is regular with respect to } g \Big\}. \end{aligned}$$

*Then*

(i) *If $g$ is unramified or $f$ is an immersion then*

$$\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \subseteq \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}.$$

(ii) *If $g$ is étale or $f$ is an immersion then we have a cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} & \xrightarrow{g'_*} & \underline{\Gamma}^d_{X/S} \\ \Big\downarrow{\scriptstyle f'_*} & \square & \Big\downarrow{\scriptstyle f_*} \\ \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g} & \xrightarrow{g_*} & \underline{\Gamma}^d_{Y/S}. \end{array}$$

(iii) *For arbitrary $g$ the results of* (i) *and* (ii) *are true over reduced $S$-schemes, i.e., for any reduced $S$-scheme $T$ we have that*

$$\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g}(T) \subseteq \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}(T)$$

*and the diagram in* (ii) *is cartesian in the subcategory of functors from reduced $S$-schemes.*

*Proof.* (i) Let $\alpha' \in \underline{\Gamma}^d_{X'/S}(T)$. If $f$ is an immersion then $\mathrm{Image}(f'_*\alpha') = \mathrm{Image}(\alpha')$ and $\mathrm{Image}(f_*g'_*\alpha') = \mathrm{Image}(g'_*\alpha')$. It is thus obvious that $\alpha'$ is regular if and only if $f'_*\alpha'$ is regular, i.e., $\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} = \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}$.

Assume instead that $f$ is arbitrary but $g$ is unramified. Let $Z' = \mathrm{Image}(\alpha')$ and $W' = f'_T(Z')$. If $\alpha' \in \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g}(T)$, i.e., if $f'_*\alpha'$ is regular with respect to $g$, we have that $\mathrm{Image}(f'_*\alpha') \hookrightarrow W' \hookrightarrow Y'_T \to Y_T$ is a closed immersion. But $\mathrm{Image}(f'_*\alpha') \hookrightarrow W'$ is universally bijective and thus $W' \to Y_T$ is universally injective and unramified. By [$\mathrm{EGA_{IV}}$, Prop. 17.2.6] this implies that $W' \to Y_T$ is a monomorphism and hence a closed immersion. Thus $Z' \hookrightarrow W' \times_{Y'_T} X'_T = W' \times_{Y_T} X_T \hookrightarrow X_T$ is a closed immersion which shows that $\alpha'$ is regular with respect to $g'$.

(ii) The commutativity of the diagrams is obvious. This gives us a canonical morphism

$$\Lambda : \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \to \underline{\Gamma}^d_{X/S} \times_{\underline{\Gamma}^d_{Y/S}} \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g}.$$

We construct an inverse $\Lambda^{-1}$ of this morphism as follows: Let $T$ be an $S$-scheme, $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ and $\beta' \in \underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g}(T)$ such that $\beta = g_*\beta' = f_*\alpha \in \underline{\Gamma}^d_{Y/S}(T)$. As $\beta'$ is regular with respect to $g$ we have that $\mathrm{Image}(\beta') \hookrightarrow Y'_T$ is isomorphic to $\mathrm{Image}(\beta) \hookrightarrow Y_T$. Let $Z = \mathrm{Image}(\alpha) \hookrightarrow X_T$. If $f$ is an immersion then $\alpha$ is regular with respect to $f$ and $Z \hookrightarrow X_T$ is isomorphic to $\mathrm{Image}(\beta)$ and we let $Z' = \mathrm{Image}(\beta') \times_{\mathrm{Image}(\beta)} \mathrm{Image}(\alpha) \cong Z$.

For arbitrary $f$ but étale $g$, let $W = f_T(Z)$. Then $\mathrm{Image}(\beta) \hookrightarrow W$ is a bijective closed immersion. By the regularity of $\beta'$, we have that $\mathrm{Image}(\beta')$ is a section of $g_T^{-1}\big(\mathrm{Image}(\beta)\big) \to \mathrm{Image}(\beta)$. As $g$ is unramified it thus follows that $\mathrm{Image}(\beta')$ is open and closed in $g_T^{-1}\big(\mathrm{Image}(\beta)\big) \hookrightarrow g_T^{-1}W$. Let $W'$ be the corresponding open and closed subscheme of $g_T^{-1}W$. As $g$ is étale $W' \cong W$ and we let $Z' = W' \times_W Z$.

In both cases we have obtained a canonical closed subscheme $Z' \hookrightarrow X'_T$ such that $Z' \cong Z$. This gives a unique lifting of the family $\alpha \in \underline{\Gamma}^d_Z(T)$ to a family $\alpha' \in \underline{\Gamma}^d_{Z'}(T) \subseteq \underline{\Gamma}^d_{X'/S}(T)$. By the construction of $Z'$ and the regularity of $\beta'$, it is clear that $f'_*\alpha' = \beta'$. We let $\Lambda^{-1}(T)(\alpha, \beta') = \alpha'$ and it is obvious that $\Lambda$ is a morphism since the construction is functorial. By construction $\Lambda \circ \Lambda^{-1}$ is the identity and as $\underline{\Gamma}^d_{X'/S,\mathrm{reg}/g} \subseteq \underline{\Gamma}^d_{X'/S,\mathrm{reg}/g'}$ it follows that $\Lambda^{-1}\circ\Lambda$ is the identity as well.

(iii) Over reduced schemes, all images involved are reduced by Theorem (2.4.6) (i) and the support of the push-forward coincides with the image. The arguments of (i) and (ii) then simplify and go through without any hypotheses on $f$ and $g$. □

**Corollary (3.3.11).** *Let $f : X \to Y$ and $g : Y' \to Y$ be morphism of algebraic spaces, separated over $S$. Assume that for every involved space $Z$, the functor $\underline{\Gamma}^d_{Z/S}$ is represented by a space which we denote by $\Gamma^d(Z/S)$.*

(i) *If $g$ is unramified, then $\underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g}$ is represented by an open subspace $U = \mathrm{reg}(g)$ of $\Gamma^d(Y'/S)$.*

(ii) *If $g$ is étale, then we have a cartesian diagram*

$$\begin{array}{ccc} \Gamma^d(X'/S)|_{f'^{-1}_*(U)} & \xrightarrow{g'_*} & \Gamma^d(X/S) \\ \Big\downarrow{\scriptstyle f'_*} & \square & \Big\downarrow{\scriptstyle f_*} \\ \Gamma^d(Y'/S)|_U & \xrightarrow{g_*} & \Gamma^d(Y/S). \end{array}$$

(iii) *If $g$ is unramified, the canonical morphism*

$$\Lambda : \Gamma^d(X'/S)|_{f'^{-1}_*(U)} \to \Gamma^d(Y'/S)|_U \times_{\Gamma^d(Y/S)} \Gamma^d(X/S)$$

*is a universal homeomorphism such that $\Lambda_{\mathrm{red}}$ is an isomorphism.*

*Proof.* Follows immediately from Propositions (3.3.9) and (3.3.10). □

**Corollary (3.3.12).** *Let $f_i : X_i \to Y$, $i = 1, 2$ be morphism of algebraic spaces, separated over $S$. Let $\pi_i : X_1 \times_Y X_2 \to X_i$ be the projections. Assume that for every involved space $Z$, the functor $\underline{\Gamma}^d_{Z/S}$ is represented by a space which we denote by $\Gamma^d(Z/S)$. Assume that $f_1$ and $f_2$ are both étale and let $U_i = \mathrm{reg}(f_i)$ and $U_{12} = \mathrm{reg}(f_1 \circ \pi_1) = \mathrm{reg}(f_2 \circ \pi_2)$. Then*

(i) *$U_{12} = ((\pi_1)_*)^{-1}(U_1) \cap ((\pi_2)_*)^{-1}(U_2)$.*

(ii) *The diagram*

$$\begin{array}{ccc} \Gamma^d(X_1 \times_Y X_2/S)|_{U_{12}} & \xrightarrow{(\pi_2)_*} & \Gamma^d(X_2/S)|_{U_2} \\ \Big\downarrow{\scriptstyle (\pi_1)_*} & \square & \Big\downarrow{\scriptstyle (f_2)_*} \\ \Gamma^d(X_1/S)|_{U_1} & \xrightarrow{(f_1)_*} & \Gamma^d(Y/S) \end{array}$$

*is cartesian.*

*Proof.* It follows from (i) of Proposition (3.3.10) that

$$((\pi_1)_*)^{-1}(U_1) \cap ((\pi_2)_*)^{-1}(U_2) \subseteq U_{12}$$

and the reverse inclusion is obvious. That the diagram is cartesian now follows from Corollary (3.3.11). □

*Remark* **(3.3.13).** The diagrams in Proposition (3.3.10) and Corollary (3.3.11) are not always cartesian if $g$ is unramified but not étale. In fact, by Examples (3.3.8) there is a morphism $f : X \to Y$ and a family $\alpha \in \underline{\Gamma}^d_{X/S}(S)$ such that $\mathrm{Image}(\alpha) = X$, $f(\mathrm{Image}(\alpha)) = Y$ and such that $\mathrm{Image}(f_*\alpha) \hookrightarrow Y$ is not an isomorphism. If we let $Y' = \mathrm{Image}(f_*\alpha)$ and $\beta' = f_*\alpha \in \underline{\Gamma}^d_{Y'/S}(S)$, then we cannot lift $(\alpha, \beta')$ to a family $\alpha' \in \underline{\Gamma}^d_{X'/S}(S)$. On the other hand, it is easily seen that Corollary (3.3.12) remains valid if we replace étale with unramified.

*Remark* **(3.3.14).** Let $X$, $Y$, $U$, $f$ and $g$ as in Corollary (3.3.11) and let $U'$ be the open subscheme of $\Gamma^d(X'/S)$ which represents $\underline{\Gamma}^d_{Y'/S,\mathrm{reg}/g'}$. Then ${f'_*}^{-1}(U) \subseteq U'$ by Proposition (3.3.10) (i), i.e., the points of $\Gamma^d(X'/S)|_{{f'_*}^{-1}(U)}$ are regular with respect to $g'$. On the other hand, a point which is regular with respect to $g'$ need not be regular with respect to $g$, i.e., the inclusion ${f'_*}^{-1}(U) \subseteq U'$ is strict in general.

**Proposition (3.3.15).** *If $f : X/S \to Y/S$ is an étale (resp. étale and surjective) morphism of algebraic spaces separated over $S$, then the push-forward $f_* : \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ is representable and étale (resp. étale and surjective).*

*Proof.* If $f$ is surjective then $f_* : \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ is topologically surjective by Proposition (3.3.9) (iv).

**I)** *Reduction to $X \to S$ quasi-compact.* Let $\{U_\beta\}$ be an open cover of $X$ such that $U_\beta$ is quasi-compact and any set of $d$ points in $X$ over the same point in $S$ lies in some $U_\beta$. Then $\{\underline{\Gamma}^d_{U_\beta,\mathrm{reg}/f|_{U_\beta}} \to \underline{\Gamma}^d_{X,\mathrm{reg}/f}\}$ is an open cover by Proposition (3.1.10). Replacing $X$ with $U_\beta$ we can thus assume that $X$ is quasi-compact.

**II)** *Reduction to $X, Y$ and $S$ affine and $Y$ integral over $S$.* Let $T$ be an affine scheme and $T \to \underline{\Gamma}^d_{Y/S}$ a morphism. Then it factors as $T \to \Gamma^d(W/T)$ where $W \hookrightarrow Y_T = Y \times_S T$ is a closed subspace such that $W \to T$ is integral. Let $Z = f_T^{-1}(W)$. Note that $f$ is separated and quasi-compact as $X \to S$ is separated and quasi-compact. Hence $f$ is quasi-affine as well as $Z \to W \to T$ which is the composition of two quasi-affine morphisms. Thus $\underline{\Gamma}_{Z/T}$ and $\underline{\Gamma}_{W/T}$ are both representable by Theorem (3.1.11). As $W \hookrightarrow Y_T$ is a closed immersion it follows from Proposition (3.3.10) (ii) that we have a cartesian diagram

$$\begin{tikzcd}
\Gamma^d(Z/T)|_{\mathrm{reg}(f_T|_Z)} \arrow[r, hook] \arrow[d, "(f_T|_Z)_*"] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X_T/T,\mathrm{reg}/f_T} \arrow[r] \arrow[d, "(f_T)_*"] \arrow[dr, phantom, "\square"] & \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \arrow[d, "f_*"] \\
\Gamma^d(W/T) \arrow[r, hook] & \underline{\Gamma}^d_{Y_T/T} \arrow[r] & \underline{\Gamma}^d_{Y/S}.
\end{tikzcd}$$

This shows that $f_*$ is representable. To show that $f_* : \underline{\Gamma}^d_{X/S,\mathrm{reg}/f} \to \underline{\Gamma}^d_{Y/S}$ is étale it is thus enough to show that $\Gamma^d(Z/T) \to \Gamma^d(W/T)$ is étale over the open subset $\mathrm{reg}\,(f_T|_Z)$. Further, as $\Gamma^d(Z/T)$ is covered by open affine subsets of the form $\Gamma^d(U/T)$ where $U \subseteq Z$ is an affine open subset by Proposition (3.1.10), we can assume that $Z/T$ is affine. Replacing $X$, $Y$ and $S$ with $Z$, $W$ and $T$ we can then assume that $X$ and $S$ are affine and $Y$ is integral over $S$.

**III)** *Reduction to $X$ and $Y$ quasi-finite and finitely presented over $S$.* Let $S = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(C)$. We can write $B$ as a filtered direct limit of finite and finitely presented $A$-algebras $B_\lambda$. As $B \to C$ is of finite presentation, we can find an $\mu$ and a $B_\mu$-algebra $C_\mu$ such that $C = C_\mu \otimes_{B_\mu} B$. Let $C_\lambda = C_\mu \otimes_{B_\mu} B_\lambda$, $X_\lambda = \operatorname{Spec}(C_\lambda)$ and $Y_\lambda = \operatorname{Spec}(B_\lambda)$ for every $\lambda \geq \mu$. As $\Gamma^d$ commutes with filtered direct limits, cf. paragraph (1.3.3), we have that $\Gamma^d_A(B) = \varinjlim_\lambda \Gamma^d_A(B_\lambda)$ and $\Gamma^d_A(C) = \varinjlim_\lambda \Gamma^d_A(C_\lambda)$.

Let $U = \operatorname{reg}(f) \subseteq \Gamma^d(X/S)$ and let $u \in U$ be a point with residue field $k$ and let $\alpha \in \underline{\Gamma}_{X/S}(k)$ be the corresponding family of cycles with image $Z \hookrightarrow X_k$. Let $\beta = f_*\alpha$ and $W = \operatorname{Image}(\beta)$. As $\alpha$ is regular $Z \to W$ is an isomorphism. Now as $W$ consists of a finite number of points each with a residue field of finite separable degree over $k$, it is easily seen that there is a $\lambda \geq \mu$ such that $(Y \times_S k)|_W \to Y_\lambda \times_S k$ is universally injective. Thus the push-forward of $\alpha$ along $\psi_\lambda : X \to X_\lambda$ is quasi-regular with respect to $f_\lambda$ and thus regular as $f_\lambda$ is étale. Corollary (3.3.11) gives the cartesian diagram

$$\begin{tikzcd}
\Gamma^d(X/S)|_{\psi_\lambda^{-1}(V)} \arrow[r, "f_*"] \arrow[d, "\psi_{\lambda *}"'] \arrow[dr, phantom, "\square"] & \Gamma^d(Y/S) \arrow[d] \\
\Gamma^d(X_\lambda)|_V \arrow[r, "(f_\lambda)_*"] & \Gamma^d(Y_\lambda/S)
\end{tikzcd}$$

where $V = \operatorname{reg}(f_\lambda)$ and $u \in \psi_\lambda^{-1}(V)$ as $(\psi_\lambda)_*\alpha$ is regular.

Replacing $X$ and $Y$ with $X_\lambda$ and $Y_\lambda$ we can thus assume that $X$ and $Y$ are of finite presentation over $S$. Further as $f$ is quasi-finite and of finite presentation and $Y \to S$ is finite and of finite presentation it follows that $X \to S$ is quasi-finite and of finite presentation. Proposition (1.3.7) then shows that $\Gamma^d(X/S)$ and $\Gamma^d(Y/S)$ are of finite presentation over $S$. Thus $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ is also of finite presentation.

**IV)** *Reduction to $S$ strictly local.* Let $\alpha \in \Gamma^d(X/S)$ and let $\beta = f_*(\alpha)$ and $s \in S$ be its images. Let $S' \to S$ be a flat morphism such that $s$ is in its image. Then, as $f_*$ is of finite presentation, $f_*$ is étale at a point $\alpha \in \Gamma^d(X/S)$ if the morphism $\Gamma^d(X'/S') \to \Gamma^d(Y'/S')$ is étale at a point $\alpha' \in \Gamma^d(X'/S')$ above $\alpha$ [EGA$_{\mathrm{IV}}$, Prop. 17.7.1]. We take $S'$ as the strict henselization of $\mathcal{O}_{S,s}$. As $\underline{\Gamma}^d_{X/S,\mathrm{reg}/f}$ is an open subfunctor of $\underline{\Gamma}^d_{X/S}$ we have that $\operatorname{reg}(f) \times_S S' = \operatorname{reg}(f')$. We can thus replace $X$, $Y$ and $S$ with $X'$, $Y'$ and $S'$ and assume that $S$ is strictly local.

**V)** *Conclusion* We have now reduced the proposition to the following situation: $S$ is strictly local, $X \to S$ is quasi-finite and finitely presented and $Y \to S$ is finite and finitely presented. The support of $\alpha \in \Gamma^d(X/S)$ consists of a finite number of points $x_1, x_2, \dots, x_m \in X$ lying above the closed point $s \in S$. As $X \to S$ is quasi-finite and $S$ is henselian it follows that $X = \left(\coprod_{i=1}^m X_i\right) \amalg X'$ where $X_i$ are strictly local schemes, finite over $S$, such that $x_i \in X_i$. Then $\alpha \in \Gamma^d\bigl(\coprod_{i=1}^m X_i\bigr) \hookrightarrow \Gamma^d(X/S)$ and we can thus assume that $X = \coprod_{i=1}^m X_i$ is finite over $S$.

As $S$ is strictly local and $Y \to S$ is finite it follows that $Y = \coprod_{j=1}^n Y_j$ is a finite disjoint union of strictly local schemes. For every $i = 1, 2, \dots, m$ there is a $j(i)$ such that $f(X_i) \hookrightarrow Y_{j(i)}$ and $f|_{X_i} : X_i \to Y_{j(i)}$ is an isomorphism as $f$ is étale. We have further by Proposition (1.4.1) that

$$\Gamma^d(X/S) = \coprod_{\sum_i d_i = d} \prod_{i=1}^m \Gamma^{d_i}(X_i), \quad \Gamma^d(Y/S) = \coprod_{\sum_j e_j = d} \prod_{j=1}^n \Gamma^{e_j}(Y_j).$$

It is obvious that the regular subset $U \subseteq \Gamma^d(X/S)$ is given by the connected components with $d_1, d_2, \dots, d_m$ such that for every $j = 1, 2, \dots, n$ there is at most one $i$ with $d_i > 0$ such that $j(i) = j$. As

$$\prod_{i=1}^m \Gamma^{d_i}(X_i) \to \prod_{i=1}^m \Gamma^{d_i}\left(Y_{j(i)}\right)$$

is an isomorphism this completes the demonstration. □

**Corollary (3.3.16).** *Let $X/S$ be a separated algebraic space and $\{f_\alpha : U_\alpha \to X\}_\alpha$ an étale separated cover. Assume that for every involved space $Z$, the functor $\underline{\Gamma}^d_{Z/S}$ is represented by a space which we denote by $\Gamma^d(Z/S)$. Then*

$$\text{(3.3.16.1)} \qquad \coprod_{\alpha,\beta} \Gamma^d(U_\alpha \times_X U_\beta/S)|_{\mathrm{reg}} \rightrightarrows \coprod_{\alpha} \Gamma^d(U_\alpha/S)|_{\mathrm{reg}} \longrightarrow \Gamma^d(X/S)$$

*is an étale equivalence relation. Here* reg *denotes the regular locus with respect to the push-forward to $X$.*

*Proof.* This follows from Corollary (3.3.12) and Proposition (3.3.15). □

3.4. **Representability of $\underline{\Gamma}^d_{X/S}$ by an algebraic space.** In this subsection, it will be shown that for *any* algebraic space $X$ separated over $S$, the functor $\underline{\Gamma}^d_{X/S}$ is represented by an algebraic space, separated over $S$.

**Theorem (3.4.1).** *Let $S$ be an algebraic space and $X/S$ a separated algebraic space. Then the functor $\underline{\Gamma}^d_{X/S}$ is represented by a separated algebraic space $\Gamma^d(X/S)$.*

*Proof.* Let $f : X' \to X$ be an étale cover such that $X'$ is a disjoint union of affine schemes. Then $X'$ is an AF-scheme and $\underline{\Gamma}^d_{X'/S}$ is represented by the scheme $\Gamma^d(X'/S)$, cf. Theorem (3.1.11). By Propositions (3.1.6) and (3.3.15), the functor $\underline{\Gamma}^d_{X/S}$ is a sheaf in the étale topology and the push-forward $f_* : \Gamma^d(X'/S)|_{\mathrm{reg}(f)} \to \underline{\Gamma}^d_{X/S}$ is an étale presentation.

To show that $\underline{\Gamma}^d_{X/S}$ is a separated algebraic space, it is thus sufficient to show that the diagonal is represented by closed immersions. Let $T$ be an $S$-scheme and $\alpha, \beta \in \underline{\Gamma}^d_{X/S}(T)$. Let $Z_\alpha, Z_\beta \hookrightarrow X \times_S T$ be the images of $\alpha$ and $\beta$. Let $Z_0 = Z_\alpha \cap Z_\beta = Z_\alpha \times_{X_T} Z_\beta$. We then let $T_0 = \alpha^{-1}(\Gamma^d(Z_0/S)) \cap \beta^{-1}(\Gamma^d(Z_0/S))$ where we have considered $\alpha$ and $\beta$ as morphisms $T \to \Gamma^d(Z_\alpha/T)$ and $T \to \Gamma^d(Z_\beta/T)$ respectively. Then $T_0 \hookrightarrow T$ is a closed subscheme and

$$\begin{aligned}(\alpha,\beta)^*\Delta_{\underline{\Gamma}^d_{X/S}/S} &= \underline{\Gamma}^d_{X/S} \times_{\underline{\Gamma}^d_{X/S}\times_S \underline{\Gamma}^d_{X/S}} T \\ &= \underline{\Gamma}^d(Z_0/T) \times_{\underline{\Gamma}^d(Z_0/T)\times_S \underline{\Gamma}^d(Z_0/T)} T_0 \\ &= (\alpha|_{T_0}, \beta|_{T_0})^*\Delta_{\Gamma^d(Z_0/T)/T}\end{aligned}$$

which is a closed subscheme of $T_0$ as $\Gamma^d(Z_0/T) \to T$ is affine. □

**Proposition (3.4.2).** *Let $X/S$ be a separated algebraic space. Let $s \in S$ and let $\alpha \in \Gamma^d(X/S)$ be a point over $s \in S$. There is then a finite number of points $x_1, x_2, \dots, x_n \in X$ with $n \leq d$ such that the following condition holds:*

- (*) *Choose an étale neighborhood $S' \to S$ of $s$ and étale neighborhoods $\{U_i \to X\}$ of $\{x_i\}$ such that the $U_i$'s are algebraic $S'$-spaces. There is then an open subset $V$ of $\Gamma^d\big(\coprod_{i=1}^n U_i/S'\big)$ such that $V \to \Gamma^d(X/S)$ is an étale neighborhood of $\alpha$.*

*Furthermore, if we choose the $U_i$'s such that there is a point above $x_i$ with trivial residue field extension, then there is a point in $V$ above $\alpha$ with trivial residue field extension.*

*In particular, $\Gamma^d(X/S)$ has an étale covering of the form $\coprod_i \Gamma^d(X_i/S_i)|_{V_i}$ where $S_i$ and $X_i$ are affine and $S_i \to S$ and $X_i \to X$ étale.*

*Proof.* The point $\alpha$ corresponds to a family $\operatorname{Spec}(k(\alpha)) \to \Gamma^d(X/S)$ where $k(\alpha)$ is the residue field. Let $Z \hookrightarrow X \times_S \operatorname{Spec}(k(\alpha))$ be the image of this family. Then $Z$ is reduced and consists of a finite number of points $z_1, z_2, \dots, z_m$ such that $m \leq d$. Let $W = \{x_1, x_2, \dots, x_n\}$ be the projection of $Z$ on $X$. Then $\alpha$ lies in the closed subset $\Gamma^d(W/S) \hookrightarrow \Gamma^d(X/S)$.

If $f : U \to X$ is an étale neighborhood of $W$ then it is obvious that there is a lifting of $\alpha$ to $V = \Gamma^d(U/S)|_{\mathrm{reg}(f)}$. Furthermore, if $f$ has trivial residue field extensions over $W$, then we can choose a lifting with the residue field $k(\alpha)$. That $V \to \Gamma^d(X/S)$ is étale is Proposition (3.3.15). □

## 4. Further properties of $\Gamma^d(X/S)$

4.1. **Addition of cycles and non-degenerate families.** In paragraphs (1.2.14) and (1.3.5) we defined the universal multiplication of laws $\rho_{d,e} : \Gamma_A^{d+e}(B) \to \Gamma_A^d(B) \otimes_A \Gamma_A^e(B)$. We will give a corresponding morphism $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ for arbitrary $X/S$.

**Definition-Proposition (4.1.1).** *Let $X/S$ be a separated algebraic space and let $d, e$ be positive integers. Then there exists a morphism*

$$+ : \Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$$

*which on points is addition of cycles. When $X/S$ is affine, this morphism corresponds to the homomorphism $\rho_{d,e}$. The operation $+$ makes the space $\Gamma(X/S) = \coprod_{d\geq 0} \Gamma^d(X/S)$ into a graded commutative monoid.*

*Proof.* The morphism $+$ is the composition of the open and closed immersion $\Gamma^d(X/S)\times\Gamma^e(X/S)\hookrightarrow\Gamma^{d+e}(X\amalg X/S)$ of Proposition (3.1.8) and the push-forward along $X\amalg X\to X$. It is clear that this is an associative and commutative operation as push-forward is functorial. When $X/S$ is affine, it is clear from (1.2.14) that the addition of cycles corresponds to the homomorphism $\rho_{d,e}$. □

**Proposition (4.1.2).** *Let $X/S$ be a separated algebraic space and $T$ an $S$-scheme. Let $\alpha\in\underline{\Gamma}^d_{X/S}(T)$ and $\beta\in\underline{\Gamma}^e_{X/S}(T)$.*

(i) *If $T$ is connected and* $\mathrm{Image}(\alpha)=\coprod_{i=1}^n Z_i$ *then there are integers $d_i\geq 1$ and families of cycles $\alpha_i\in\underline{\Gamma}^{d_i}_{Z_i/S}(T)$ such that $d=d_1+d_2+\dots+d_n$ and $\alpha=\alpha_1+\alpha_2+\dots+\alpha_n$.*

(ii) $\mathrm{Supp}(\alpha+\beta)=\mathrm{Supp}(\alpha)\cup\mathrm{Supp}(\beta)$.

(iii) *Let $f:X\to Y$ be a morphism of separated algebraic spaces. Then $f_*(\alpha+\beta)=f_*\alpha+f_*\beta$.*

*Proof.* (i) and (ii) follows from Propositions (3.1.8) and (3.3.7) (ii) respectively. (iii) follows easily from the definitions and the functoriality of the push-forward. □

**Proposition (4.1.3).** *The morphism $\Gamma^d(X/S)\times_S\Gamma^e(X/S)\to\Gamma^{d+e}(X/S)$ is étale over the open subset $U\subseteq\Gamma^d(X/S)\times_S\Gamma^e(X/S)$ where $(\alpha,\beta)\in U$ if* $\mathrm{Supp}(\alpha)$ *and* $\mathrm{Supp}(\beta)$ *are disjoint.*

*Proof.* The morphism $X\amalg X\to X$ is étale. By Propositions (3.1.8) and (3.3.15) we have that $\Gamma^d(X/S)\times_S\Gamma^e(X/S)\to\Gamma^{d+e}(X/S)$ is étale at $(\alpha,\beta)$ if $\alpha\amalg\beta$ is regular with respect to $X\amalg X\to X$. This is fulfilled if and only if $\mathrm{Supp}(\alpha)$ and $\mathrm{Supp}(\beta)$ are disjoint. □

**Notation (4.1.4).** We let $(X/S)^d$ denote the fiber product $X\times_S X\times_S\cdots\times_S X$ of $d$ copies of $X$ over $S$.

**Proposition (4.1.5).** *Let $X/S$ be a separated algebraic space. The symmetric group on $d$ letters $\mathfrak{S}_d$ acts on $(X/S)^d$ by permutation of factors. We equip $\Gamma^d(X/S)$ with the trivial $\mathfrak{S}_d$-action. Then:*

(i) *There is a canonical $\mathfrak{S}_d$-equivariant morphism $\Psi_X:(X/S)^d\to\Gamma^d(X/S)$.*

(ii) *$\Psi_X$ is integral and universally open. Its fibers are the orbits of $(X/S)^d$ and this also holds after base change.*

(iii) *$\Psi_X$ is étale outside the diagonals of $(X/S)^d$.*

(iv) *If $f:X\to Y$ is a morphism of separated algebraic spaces we have a commutative diagram*

$$\begin{tikzcd}
(X/S)^d \arrow[r, "f^d"] \arrow[d, "\Psi_X"'] \arrow[dr, phantom, "\circ"] & (Y/S)^d \arrow[d, "\Psi_Y"] \\
\Gamma^d(X/S) \arrow[r, "f_*"] & \Gamma^d(Y/S).
\end{tikzcd}$$

*If $f$ is unramified (resp. étale) and $U = \operatorname{reg}(f)$ then the canonical morphism*

$$\Lambda : (X/S)^d|_{\Psi_X^{-1}(U)} \to \Gamma^d(X/S)|_U \times_{\Gamma^d(Y/S)} (Y/S)^d$$

*is a universal homeomorphism (resp. an isomorphism).*

*Proof.* (i) As $\operatorname{Hom}_S\bigl(T,(X/S)^d\bigr) = \operatorname{Hom}_S(T,X)^d = \underline{\Gamma}^1_{X/S}(T)^d$ by Remark (3.1.5) we obtain by addition of cycles the morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ and this is clearly an $\mathfrak{S}_d$-equivariant morphism as addition of cycles is commutative.

(iii) Follows immediately from Proposition (4.1.3).

(iv) Follows from the definition of $\Psi$ and Corollary (3.3.11) since

$$\begin{tikzcd} (X/S)^d \arrow[r, "f^d"] \arrow[d] \arrow[dr, phantom, "\square"] & (Y/S)^d \arrow[d] \\ \Gamma^d(\coprod_{i=1}^d X) \arrow[r] & \Gamma^d(\coprod_{i=1}^d Y) \end{tikzcd}$$

is cartesian.

(ii) We first show that the fibers of $\Psi$ are the $\mathfrak{S}_d$-orbits and that this holds after any base change. Let $f : \operatorname{Spec}(k) \to \Gamma^d(X/S)$ be a morphism. Then $f$ factors through $\Gamma^d(Z/k) \to \Gamma^d(X/S)$ where $Z \hookrightarrow X \times_S \operatorname{Spec}(k)$ is a closed subspace integral over $k$.

As $\Gamma^d$ commutes with base change, we can replace $S$ with $\operatorname{Spec}(k)$. Furthermore, using the unramified part of (iv), we can replace $X$ with $Z$. We can thus assume that $S = \operatorname{Spec}(k)$ and that $X = Z = \operatorname{Spec}(B)$. Then $(X/k)^d = \operatorname{Spec}\bigl(\mathrm{T}^d_k(B)\bigr)$ and $\Gamma^d(X/k) = \operatorname{Spec}\bigl(\mathrm{TS}^d_k(B)\bigr) = \operatorname{Sym}^d(X/k)$. As the fibers of $(X/k)^d \to \operatorname{Sym}^d(X/k)$ are the $\mathfrak{S}_d$-orbits it follows that the same holds for $\Psi$.

If $U \hookrightarrow (X/S)^d$ is an open (resp. closed subset) then $\Psi^{-1}\bigl(\Psi(U)\bigr) = \bigcup_{\sigma\in\mathfrak{S}_d} \sigma U$. As this also holds after any base change $T \to \Gamma^d(X/S)$ it follows that $\Psi$ is universally closed and universally open.

We will now show that $\Psi_X$ is affine. As $\Psi_X$ is universally closed it then follows that $\Psi_X$ is integral by [$\mathrm{EGA_{IV}}$, Prop. 18.12.8]. As affineness is local in the étale topology we can assume that $S$ is affine. Let $f : X' \to X$ be an étale covering such that $X'$ is a disjoint union of affine schemes and in particular an AF-scheme. By Proposition (3.3.15) the push-forward morphism $f_* : \Gamma^d(X'/S)|_{\operatorname{reg}(f)} \to \Gamma^d(X/S)$ is an étalecover. Using (iv) and replacing $X$ with $X'$ we can thus assume that $X$ is AF. Proposition (3.1.10) then shows that $\Gamma^d(X/S)$ is covered by open subsets $\Gamma^d(U/S)$ where $U$ is affine. Finally $\Psi_U$ is affine as $(U/S)^d$ is affine. $\square$

**Definition (4.1.6).** Let $X/S$ be a separated algebraic space, $T$ an $S$-space and $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ a family of cycles. Let $t \in T$ be a point and let $\overline{k}$ be an algebraic closure of its residue field $k$. We say that $\alpha$ is *non-degenerated* in a point $t \in T$ if the support of the cycle $\alpha_t \times_k \overline{k}$ consists of $d$ distinct points. Here $\alpha_t \times_k \overline{k}$ denotes the family given by the composition of $\operatorname{Spec}(\overline{k}) \to \operatorname{Spec}(k) \to T$ and $\alpha$.

The *non-degeneracy locus* is the set of points $t \in T$ such that $\alpha$ is non-degenerate in $t$.

**Definition (4.1.7).** We let $\Gamma^d(X/S)_{\mathrm{nondeg}} \subseteq \Gamma^d(X/S)$ denote the subset of non-degenerate families.

**Proposition (4.1.8).** *The subset $\Gamma^d(X/S)_{\mathrm{nondeg}} \subseteq \Gamma^d(X/S)$ is open. The morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ is étale of rank $d!$ over $\Gamma^d(X/S)_{\mathrm{nondeg}}$ and the addition morphism $+ : \Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ is étale of rank $((d,e))$ over $\Gamma^{d+e}(X/S)_{\mathrm{nondeg}}$.*

*Proof.* Let $U$ be the complement of the diagonals of $(X/S)^d$, which is an open subset. Then $\Gamma^d(X/S)_{\mathrm{nondeg}} = \Psi_X(U)$ which is an open subset as $\Psi_X$ is open. The last two statements follow from Proposition (4.1.3). $\square$

## 4.2. The $\mathrm{Sym}^d \to \Gamma^d$ morphism.

**Definition (4.2.1)** ([Kol97, Ryd07])**.** If $G$ is a group and $f : X \to Y$ a $G$-equivariant morphism, then we say that $f$ is fixed-point reflecting, or fpr, at $x \in X$ if the stabilizer of $x$ coincides with the stabilizer of $f(x)$. The subset of $X$ where $G$ is fixed-point reflecting is $G$-stable and denoted $\mathrm{fpr}(f)$.

*Remark* **(4.2.2).** Let $X/S$ be a separated algebraic space. There is then a uniform geometric and categorical quotient $\mathrm{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$, cf. [Ryd07]. Furthermore we have that $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ is integral, universally open and a topological quotient, i.e., it satisfies (ii) of Proposition (4.1.5). Moreover (iii) and the étale part of (iv) also holds for $q$ instead of $\Psi$ if we replace $\mathrm{reg}(f)$ with $\mathrm{fpr}(f)$, cf. [Ryd07].

As $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ is $\mathfrak{S}_d$-equivariant and $\mathrm{Sym}^d(X/S)$ is a *categorical* quotient, we obtain a canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ such that $\Psi_X = \mathrm{SG}_X \circ q$.

**Lemma (4.2.3).** *Let $f : X \to Y$ be a morphism of algebraic spaces and let $\alpha \in \Gamma^d(X/S)$ be a point. Then $\alpha$ is quasi-regular with respect to $f$ if and only if $f^d$ is fixed-point reflecting at $\Psi_X^{-1}(\alpha)$ with respect to the action of $\mathfrak{S}_d$.*

*Proof.* Let $k$ be the algebraic closure of the residue field $k(\alpha)$. The supports of $\alpha$ and $f_*\alpha$ are finite disjoint unions of points. Thus $\alpha : \mathrm{Spec}(k) \to \Gamma^d(X/S)$ and $f_*\alpha : \mathrm{Spec}(k) \to \Gamma^d(Y/S)$ factors as

$$\mathrm{Spec}(k) \to \prod_{i=1}^{n} \Gamma^{d_i}(x_i/k) \to \Gamma^d(X/S)$$

and

$$\mathrm{Spec}(k) \to \prod_{j=1}^{m} \Gamma^{e_i}(y_j/k) \to \Gamma^d(Y/S)$$

where $x_i$ and $y_j$ are points of $X \times_S \operatorname{Spec}(k)$ and $Y \times_S \operatorname{Spec}(k)$ respectively and $k(x_i) = k(y_j) = k$. Every point of $(X/S)^d$ (resp. $(Y/S)^d$) above $\alpha$ (resp. $f_*\alpha$) is thus such that, after a permutation, the first $d_1$ (resp. $e_1$) projections agree, the next $d_2$ (resp. $e_2$) projections agree, etc, but no other two projections are equal. Thus the stabilizers of the points of $\Psi_X^{-1}(\alpha)$ (resp. $\Psi_Y^{-1}(f_*\alpha)$) are $\mathfrak{S}_{d_1} \times \mathfrak{S}_{d_2} \times \cdots \times \mathfrak{S}_{d_n}$ (resp. $\mathfrak{S}_{e_1} \times \mathfrak{S}_{e_2} \times \cdots \times \mathfrak{S}_{e_m}$). Equality holds if and only if $f$ is quasi-regular. $\square$

**Proposition (4.2.4).** *Let $f : X \to Y$ be an étale morphism of algebraic spaces. Then $\Psi_X^{-1}\big(\operatorname{reg}(f)\big) = \operatorname{fpr}(f^d)$, and we have a cartesian diagram*

$$\begin{tikzcd}
(X/S)^d|_{\operatorname{fpr}(f^d)} \arrow[r, "q"] \arrow[d, "f^d"] \arrow[dr, phantom, "\square"] & \operatorname{Sym}^d(X/S)|_{\operatorname{fpr}(f^d)} \arrow[r, "\operatorname{SG}_X"] \arrow[d, "f^d/\mathfrak{S}_d"] \arrow[dr, phantom, "\square"] & \Gamma^d(X/S)|_{\operatorname{reg}(f)} \arrow[d, "f_*"] \\
(Y/S)^d \arrow[r, "q"] & \operatorname{Sym}^d(Y/S) \arrow[r, "\operatorname{SG}_Y"] & \Gamma^d(Y/S)
\end{tikzcd}$$

*In particular $f^d/\mathfrak{S}_d$ is étale over the open subset $q\big(\operatorname{fpr}(f^d)\big) = \operatorname{SG}_X^{-1}\big(\operatorname{reg}(f)\big)$.*

*Proof.* As $f$ is unramified $\operatorname{reg}(f) = \operatorname{qreg}(f)$ by Proposition (3.3.6), the first statement follows from Lemma (4.2.3). The outer square is cartesian by Proposition (4.1.5) (iv) and as $q$ is a uniform quotient the formation of the quotient commutes with étale base change which shows that the right square is cartesian. It follows that the left square is cartesian too. $\square$

**Corollary (4.2.5).** *Let $X/S$ be a separated algebraic space. The canonical morphism $\operatorname{SG}_X : \operatorname{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions. If $S$ has pure characteristic zero or $X/S$ is flat, then $\operatorname{SG}_X$ is an isomorphism.*

*Proof.* Using Proposition (4.2.4) and the covering in Proposition (3.4.2) we can assume that $X = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(A)$ are affine. Then $(X/S)^d = \operatorname{Spec}\big(\mathrm{T}_A^d(B)\big)$, $\Gamma^d(X/S) = \operatorname{Spec}\big(\Gamma_A^d(B)\big)$ and $\operatorname{Sym}^d(X/S) = \operatorname{Spec}\big(\mathrm{TS}_A^d(B)\big)$ are all affine. As $\Gamma_A^d(B) \to \mathrm{TS}_A^d(B) \hookrightarrow \mathrm{T}_A^d(B)$ is integral by Proposition (4.1.5) (ii), we have that $\operatorname{SG}_X : \operatorname{Spec}\big(\mathrm{TS}_A^d(B)\big) \to \operatorname{Spec}\big(\Gamma_A^d(B)\big)$ is integral.

The geometric fibers of both $\Psi_X$ and $q : (X/S)^d \to \operatorname{Sym}^d(X/S)$ are the geometric orbits of $(X/S)^d$. Thus $\operatorname{SG}_X$ is universally bijective and hence a universal homeomorphism. That $\operatorname{SG}_X$ is an isomorphism when $S$ is purely of characteristic zero or $X/S$ is flat follows from paragraph (1.3.2) as $X$ and $S$ are affine.

Let $a \in \operatorname{Sym}^d(X/S)$ be any point, $b = \operatorname{SG}_X(a) \in \Gamma^d(X/S)$ and $s$ its image in $S$. We have a commutative diagram

$$\begin{tikzcd}
\operatorname{Sym}^d\big(X_s/k(s)\big) \arrow[r, "\operatorname{SG}_{X_s}", "\cong"'] \arrow[d] \arrow[dr, phantom, "\circ"] & \Gamma^d\big(X_s/k(s)\big) \arrow[d, "\cong"] \\
\operatorname{Sym}^d(X/S) \times_S k(s) \arrow[r, "\operatorname{SG}_X \times_S \operatorname{id}_{k(s)}"] & \Gamma^d(X/S) \times_S k(s)
\end{tikzcd}$$

which gives a commutative diagram of residue fields

$$\begin{tikzcd} k(\overline{a}) & k(\overline{b}) \arrow[l, hook', "\cong"'] \\ k(a) \arrow[u, hook] \arrow[ur, phantom, "\circ"] & k(b). \arrow[l, hook'] \arrow[u, hook, "\cong"'] \end{tikzcd}$$

and thus $k(a) = k(b)$. □

**Proposition (4.2.6).** *Let $X/S$ be a separated algebraic space. The canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over $\Gamma^d(X/S)_{\mathrm{nondeg}}$.*

*Proof.* Let $U$ be the complement of the diagonals in $(X/S)^d$. Then $\Psi_X(U) = \Gamma^d(X/S)_{\mathrm{nondeg}}$ and $\mathfrak{S}_d$ acts freely on $U$. By Proposition (4.1.8) the morphism $\Psi_X$ is étale of rank $d!$ over $\Gamma^d(X/S)_{\mathrm{nondeg}}$. It is further well-known that $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ is étale of rank $d!$ over $q(U)$. In fact, $\mathrm{Sym}^d(X/S)|_{q(U)}$ is the quotient sheaf in the étale topology of the étale equivalence relation $\mathfrak{S}_d \times U \rightrightarrows U$. □

## 4.3. Properties of $\Gamma^d(X/S)$ and the push-forward.

**Proposition (4.3.1).** *Let $S$ be an algebraic space and let $X$ be an algebraic space separated over $S$. Consider for a morphism the property of being*

(i) *quasi-compact*
(ii) *finite type*
(iii) *finite presentation*
(iv) *locally of finite type*
(v) *locally of finite presentation*
(vi) *flat*
(vii) *finite*
(viii) *integral*
(ix) *affine*
(x) *quasi-affine*

*If $X \to S$ has one of these properties then so does $\Gamma^d(X/S) \to S$.*

*Proof.* If $X \to S$ is quasi-compact then $(X/S)^d \to S$ is quasi-compact. As there is a surjective morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$ it follows that $\Gamma^d(X/S)$ is quasi-compact over $S$. This shows (i). As $\Gamma^d(X/S)$ is separated, (ii) and (iii) follows from (i), (iv) and (v). It is thus enough to show (iv)–(x).

As the question is local over $S$ we can assume that $S$ is affine. If $X \to S$ is affine (resp. quasi-affine) then $\Gamma^d(X/S)$ is affine (resp. quasi-affine) by Propositions (3.1.4) and (3.1.7). If $X \to S$ is finite (resp. integral), then $X \to S$ is affine and $\Gamma^d(X/S) \to S$ is finite (resp. integral) by Proposition (1.3.7).

By Proposition (3.4.2) any point of $\Gamma^d(X/S)$ has an étale neighborhood $V$ such that $V$ is an open subset of $\Gamma^d(U/S)$ where $U$ is an affine scheme and $U \to X$ étale. If $V \to S$ is locally of finite type (resp. locally of finite presentation, resp. flat) for any such neighborhood $V$ then it follows by [$\mathrm{EGA_{IV}}$, Lem. 17.7.5] and

[$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 2.2.11 (iv)] that $\Gamma^d(X/S)$ is locally of finite type (resp. locally of finite presentation, resp. flat) over $S$. Replacing $X$ with $U$ we can thus assume that $X$ is affine. The proposition now follows from Proposition (1.3.7) and paragraph (1.2.12). □

**Corollary (4.3.2).** *Let $S$ and $X$ be algebraic spaces. If $f : X \to S$ is flat with geometric reduced fibers then $\Gamma^d(X/S) \to S$ is flat with geometric reduced fibers. In particular, if in addition $S$ is reduced then $\Gamma^d(X/S)$ is reduced.*

*Proof.* Proposition (4.3.1) shows that $\Gamma^d(X/S) \to S$ is flat. It is thus enough to show that $\Gamma^d(X_k/k)$ reduced for any algebraic closed field $k$ and morphism $\mathrm{Spec}(k) \to S$. As $X_k$ is reduced by hypothesis and hence also $(X_k/k)^d$ it follows that $\mathrm{Sym}^d(X_k/k)$ is reduced and $\Gamma^d(X_k/k) = \mathrm{Sym}^d(X_k/k)$ by Corollary (4.2.5). The last statement follows by [Pic98, Prop. 5.17]. □

**Proposition (4.3.3).** *Let $S$ and $X$ be algebraic spaces. If $f : X \to S$ is smooth of relative dimension* $0$ *(resp.* $1$*, resp. at most* $1$*) then $\Gamma^d(X/S) \to S$ is smooth of relative dimension* $0$ *(resp. $d$, resp. at most $d$).*

*Proof.* As $\Gamma^d(X/S) \to S$ is flat and locally of finite presentation by Proposition (4.3.1), it is enough to show that its geometric fibers are regular [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 17.5.1]. Thus we can assume that $S = \mathrm{Spec}(k)$ where $k$ is algebraically closed. Let $y \in \Gamma^d(X/k)$. Then by Proposition (3.2.3), the formal local ring $\widehat{\mathcal{O}}_{\Gamma^d(X/k),y}$ is the completion at a point of the scheme $\prod_{i=1}^n \Gamma^{d_i}(\widehat{X}_{x_i}/k)$ where $x_1, x_2, \dots, x_n$ are points of $X$ and $d = d_1 + d_2 + \dots + d_n$. If $f$ has relative dimension 0 at $x_i$ then $\mathcal{O}_{X,x_i} = k$ and if $f$ has relative dimension 1 at $x_i$ then $\widehat{\mathcal{O}}_{X,x_i} = k[[t]]$, cf. [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 17.5.3]. The proposition now easily follows if we can show that $\Gamma^e\big(\mathrm{Spec}(k[t])/\mathrm{Spec}(k)\big)$ is smooth of relative dimension $e$. But $\Gamma^e_k(k[t]) = \mathrm{TS}^e_k(k[t]) = k[s_1, s_2, \dots, s_e]$ where $s_1, s_2, \dots, s_e$ are the elementary symmetric functions. □

*Remark* **(4.3.4).** If $X/S$ is smooth of relative dimension $\geq 2$ then $\Gamma^d(X/S)$ is not smooth for $d \geq 2$. This can be seen by an easy tangent space calculation. If $X/S$ is smooth of relative dimension 2 then on the other hand $\mathrm{Hilb}^d(X/S)$ is smooth and gives a resolution of $\Gamma^d(X/S)$ [Fog68, Cor. 2.6 and Thm. 2.9]. Moreover $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is a blow-up in this case [Hai98, ES04].

**Proposition (4.3.5).** *If $f : X \to Y$ has one of the following properties, then so has $f^d/\mathfrak{S}_d : \mathrm{Sym}^d(X/S) \to \mathrm{Sym}^d(Y/S)$:*

(i) *quasi-compact*
(ii) *closed*
(iii) *open*
(iv) *universally closed*
(v) *universally open*
(vi) *open immersion*

(vii) *affine*
(viii) *quasi-affine*
(ix) *integral*

*If $f$ has one of the above properties or one of the following*

(x) *closed immersion*
(xi) *immersion*

*then so has $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$.*

*Proof.* Use that $\Psi_X$ and $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ are universally closed, universally open, quasi-compact and surjective for (i)-(v). Property (vi) is well-known. For (vii) reduced to $Y/S$ affine using Proposition (3.4.2) and then use that $\Gamma^d(X/S)$ and $\mathrm{Sym}^d(X/S)$ are affine if $X/S$ is affine. The combination of (i), (vi) and (vii) gives (viii). Finally (ix) follows from (vii) and (iv). The last two properties for $f_*$ follow from Proposition (3.1.7). □

*Remark* **(4.3.6).** If $f$ has one of the properties (x) or (xi), then $f^d/\mathfrak{S}_d$ need not have that property. If $f$ has one of the properties

(i) finite
(ii) locally of finite type
(iii) locally of finite presentation
(iv) unramified
(v) flat
(vi) étale

then neither $f^d/\mathfrak{S}_d$ nor $f_*$ need to have that property.

**Corollary (4.3.7).** *The addition morphism $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \to \Gamma^{d+e}(X/S)$ is integral and universally open.*

*Proof.* The morphism $X \amalg X \to X$ is finite and étale and hence integral and universally open. Thus $\Gamma^{d+e}(X \amalg X/S) \to \Gamma^{d+e}(X/S)$ is integral and universally open by Proposition (4.3.5). As the addition morphism is the composition of the open and closed immersion $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \hookrightarrow \Gamma^{d+e}(X \amalg X/S)$ and the push-forward along $X \amalg X \to X$ the corollary follows. □

## Appendix A. Appendix

A.1. **The (AF) condition.** The (AF) condition has frequently been used as a natural setting for a wide range of problems. It guarantees the existence of finite quotients [SGA$_1$, Exp. V], push-outs [Fer03] and the Hilbert scheme of points [Ryd08b]. Moreover, under the (AF) condition, étale cohomology can be calculated using Čech cohomology [Art71, Cor. 4.2], [Sch03].

**Definition (A.1.1).** We say that a scheme $X/S$ is AF if it satisfies the following condition.

(AF) Every finite set of points $x_1, x_2, \dots, x_n \in X$ over the same point $s \in S$ is contained in an open subset $U \subseteq X$ such that $U \to S$ is quasi-affine.

*Remark* **(A.1.2).** Let $X/S$ and $Y/S$ be AF-schemes. Then $X \times_S Y/S$ is an AF-scheme. If $S' \to S$ is any morphism, then $X \times_S S'/S'$ is an AF-scheme. This is obvious as the class of quasi-affine morphisms is stable under products and base change. It is also clear that the (AF) condition is local on $S$ and that the subset $U$ in the condition can be chosen such that $U$ is an affine scheme. Moreover, if $S$ is quasi-separated, then we can replace the condition that $U \to S$ is quasi-affine with the condition that $U$ is affine.

**Proposition (A.1.3).** *Let $X$ be an $S$-scheme. If $X$ has an ample invertible sheaf $\mathcal{O}_X(1)$ relative to $S$ then $X/S$ is an* AF*-scheme. In particular, it is so if $X/S$ is (quasi-)affine or (quasi-)projective.*

*Proof.* Follows immediately from [$\text{EGA}_{\text{II}}$, Cor. 4.5.4] since we can assume that $S = \operatorname{Spec}(A)$ is affine. □

**Proposition (A.1.4).** *Let $X/S$ be an* AF*-scheme. Then $X/S$ is separated.*

*Proof.* Let $z$ be a point in the closure of $\Delta_{X/S}(X)$, where $\Delta_{X/S} : X \hookrightarrow X \times_S X$ is the diagonal morphism, and let $x_1, x_2 \in X$ be its two projections. Choose an affine neighborhood $U$ containing $x_1$ and $x_2$. Then $\Delta_{U/S} : U \hookrightarrow U \times_S U$ is closed and $\Delta_{U/S}$ is the pull-back of $\Delta_{X/S}$ along the open immersion $U \times_S U \subset X \times_S X$. Taking closure commutes with restricting to open subsets and thus $z \in U \subset X$. This shows that $\Delta_{X/S}(X)$ is closed and hence that $X/S$ is separated. □

The following conjecture was proved by Kleiman [Kle66].

**Theorem (A.1.5)** (Chevalley's conjecture)**.** *Let $X/k$ be a proper regular algebraic scheme. Then $X$ is projective if and only if $X/k$ is an* AF*-scheme.*

It is however not true that a proper singular scheme always is projective if it is AF. In fact, there are singular, proper but non-projective AF-surfaces [Hor71].

## A.2. **A theorem on integral morphisms.**

**Definition (A.2.1).** We say that a morphism $f : X \to Y$ has *topologically finite fibers* if the underlying topological space of every fiber is a finite set. We say that $f$ has *universally topologically finite fibers* if the base change of $f$ by any morphism $Y' \to Y$ has topologically finite fibers, equivalently the underlying topological space of every fiber is a finite set and each residue field extension has finite separable degree.

The purpose of this section is to prove the following theorem:

**Theorem (A.2.2).** *Let $f : X \to Y$ and $g : Y \to S$ be morphisms of algebraic spaces. If $g \circ f$ is integral with topologically finite fibers and $g$ is separated then the "schematic" image $Y'$ of $f$ exists and $Y' \to S$ is integral with topologically finite fibers.*

Let us first note that this is easy to proof when $g$ is locally of finite type:

**Proposition (A.2.3).** *Let $X$ and $Y$ be schemes locally of finite type and separated over the base scheme $S$. Let $f : X \to Y$ and $g : Y \to S$ be $S$-morphisms. If $g \circ f$ is finite then the schematic image $Y'$ of $f$ exists and $Y' \to S$ is finite.*

*Proof.* As $g \circ f$ is separated, $f$ is separated. As $g \circ f$ is quasi-compact and universally closed and $g$ is separated, $f$ is quasi-compact and universally closed. Thus the image $Y'$ exists [EGA$_{\mathrm{I}}$, Prop. 6.10.5] and $X \to Y'$ is surjective. As $g \circ f$ is universally closed and $X \to Y'$ is surjective it follows that $Y' \to S$ is universally closed. Further it is immediately seen that $Y' \to S$ has discrete fibers. Thus $Y' \to S$ is quasi-finite, universally closed and separated. By Deligne's theorem [EGA$_{\mathrm{IV}}$, Cor. 18.12.4] this implies that $Y' \to S$ is finite. □

*Remark* **(A.2.4).** It is easy to generalize Proposition (A.2.3) to the case where $X$ and $Y$ are *algebraic spaces*. In [Knu71, Thm. 6.15] Deligne's theorem is proven for algebraic spaces under a finite presentation hypothesis. The full version of Deligne's theorem for algebraic spaces is given in [LMB00, Thm. A.2].

*Remark* **(A.2.5).** Now instead assume as in Theorem (A.2.2) that $X$ and $Y$ are arbitrary schemes and $g \circ f$ is integral with topologically finite fibers. The first part of the proof of Proposition (A.2.3) then shows as before that the schematic image $Y'$ exists and that $Y' \to S$ is separated and universally closed. It is further easily seen that every fiber $Y'_s$ is a discrete finite topological space.

Under the hypothesis that $Y/S$ is an AF-scheme it easily follows that $Y' \to S$ is integral. In fact, then $Y'/S$ is AF and any neighborhood of $Y'_s$ in $Y'$ contains an affine neighborhood of $Y'_s$. Thus $Y' \to S$ is affine by [EGA$_{\mathrm{IV}}$, Lem. 18.12.7.1] and therefore integral by [EGA$_{\mathrm{IV}}$, Prop. 18.12.8].

In general, note that $Y'_s$ is affine and hence integral over $k(s)$ as a morphism is integral if and only if it is universally closed and affine, cf. [EGA$_{\mathrm{IV}}$, Prop. 18.12.8]. Theorem (A.2.2) thus follows by the following conjecture of Grothendieck (for schemes):

**Conjecture (A.2.6)** ([EGA$_{\mathrm{IV}}$, Rem. 18.12.9])**.** *If $X \to S$ is a separated, universally closed morphism of algebraic spaces, such that $X_s$ is integral, then $X \to S$ is integral.*

This conjecture will be proved in [Ryd08a]. In the remainder of this appendix, we will give an independent proof of Theorem (A.2.2) without using Grothendieck's conjecture. We first establish the following preliminary results.

(i) If $X \to Y$ is integral, $X$ is a semi-local scheme and $Y$ is henselian, then $X$ is henselian, cf. Proposition (A.2.7).

(ii) Affineness is descended by (not necessarily quasi-compact) flat morphisms if we a priori know that the morphism in question is quasi-compact and quasi-separated, cf. Proposition (A.2.8).
(iii) A criterion for an algebraic space to be a scheme, cf. Lemma (A.2.12).

**Proposition (A.2.7).** *If $A$ is semi-local and henselian and $B$ is an integral semi-local $A$-algebra, then $B$ is henselian. In particular $B$ is a finite direct product of local henselian rings.*

*Proof.* Follows immediately from [Ray70, Ch. XI, §2, Prop. 2]. □

**Proposition (A.2.8).** *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes with $g$ faithfully flat. Let $f' : X' \to Y'$ be the base-change of $f$ along $g$. Then*
(i) *$f$ is a homeomorphism if $f$ is quasi-compact and $f'$ is a homeomorphism.*
(ii) *$f$ is an isomorphism if and only if $f$ is quasi-compact and $f'$ is an isomorphism.*
(iii) *$f$ is affine if and only if $f$ is quasi-compact and quasi-separated and $f'$ is affine.*

*Proof.* The conditions in (ii) and (iii) are clearly necessary. Assume that $f'$ is a homeomorphism (resp. an isomorphism, resp. affine). Let $Y'' = \coprod_{y\in Y} \operatorname{Spec}(\mathcal{O}_{Y,y})$ and choose for every $y \in Y$ a point $y' \in g^{-1}(y)$. If we let $Y''' = \coprod_{y\in Y} \operatorname{Spec}(\mathcal{O}_{Y',y'})$ then $f'''$ is a homeomorphism (resp. an isomorphism, resp. affine) and we can factor $Y''' \to Y' \to Y$ through the natural faithfully flat and quasi-compact morphism $Y''' \to Y''$. As the statement of the proposition is true when $g$ is quasi-compact by [EGA$_{\mathrm{IV}}$, Prop. 2.6.2 (iv), Prop. 2.7.1 (viii), (xiii)] it follows that $f''$ is a homeomorphism (resp. an isomorphism, resp. affine). Replacing $Y'$ with $Y''$ we can thus assume that $Y' = \coprod_{y\in Y} \operatorname{Spec}(\mathcal{O}_{Y,y})$.

(i) In order to show that $f$ is a homeomorphism it is enough to show that $f$ is open since it is clearly bijective. As $f$ is generizing, see [EGA$_{\mathrm{I}}$, Def. 3.9.2], it follows by [EGA$_{\mathrm{I}}$, Thm. 7.3.1] that $f$ is open if and only it is open in the constructible topology [EGA$_{\mathrm{I}}$, 7.2.11]. But as $f$ is quasi-compact and bijective it follows from [EGA$_{\mathrm{I}}$, Prop. 7.2.12 (iv)] that $f$ is a homeomorphism in the constructible topology and in particular open.

(ii) From (i) it follows that $f$ is a homeomorphism and since $f'$ is an isomorphism, we have that $f$ is an isomorphism on the stalks. This shows that $f$ is an isomorphism.

(iii) Taking direct images along quasi-compact and quasi-separated morphisms commutes with flat pull-back by [EGA$_{\mathrm{IV}}$, Lem. 2.3.1]. Thus we have a cartesian diagram:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec}(f'_*\mathcal{O}_{X'}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d] \\ X \arrow[r] & \operatorname{Spec}(f_*\mathcal{O}_X) \arrow[r] & Y \end{tikzcd}$$

Since $f' : X' \to Y'$ is affine we have that $X' \to \mathrm{Spec}(f'_*\mathcal{O}_{X'})$ is an isomorphism and it is enough to show that $X \to \mathrm{Spec}(f_*\mathcal{O}_X)$ is an isomorphism. This follows from (ii). $\square$

**Definition (A.2.9).** We say that an algebraic space $X$ is *local* if there exist a point $x \in X$ such that every closed subset $Z \subseteq X$ contains $x$.

*Remark* **(A.2.10).** If $X$ is a local algebraic space then there is exactly one closed point $x \in X$. If $X$ is a local scheme then $X$ is the spectrum of a local ring and in particular affine.

**Lemma (A.2.11).** *Let $f : X \to Y$ be a closed surjective morphism of algebraic spaces. Let $y \in Y$ be a closed point such that $f^{-1}(y)$ is discrete and such that for any $x \in f^{-1}(y)$ we can write $X = X'_x \amalg X''$ where $X'_x$ is local and contains $x$. Then $Y = Y' \amalg Y''$ where $Y'$ is local and contains $y$. Furthermore $f^{-1}(Y') = \coprod_{x\in f^{-1}(y)} X'_x$.*

*Proof.* For every $x \in f^{-1}(y)$ let $X'_x \subseteq X$ be a local subspace containing $x$ and choose $X''$ such that $X = \left(\coprod_{x\in f^{-1}(y)} X'_x\right) \amalg X''$. Let $Y'$ be the subset of $Y$ consisting of every generization of $y$. As $f(X'')$ is closed and does not contain $y$, it does not intersect $Y'$. On the other hand $f(X'_x)$ is contained in $Y'$. Since $f$ is surjective this shows that $f(X'') = Y \setminus Y'$ and $f(\bigcup X'_x) = Y'$. Thus $Y'$ and $Y'' = Y \setminus Y'$ are both open and closed. $\square$

**Lemma (A.2.12).** *Let $X = \coprod_{\alpha\in\mathcal{I}} X_\alpha$ and $Y$ be algebraic spaces such that $X_\alpha$ is local with closed point $x_\alpha$ and $Y$ is local with closed point $y$. Let $f : X \to Y$ be a universally closed schematically dominant morphism such that $f^{-1}(y) = \{x_\alpha : \alpha \in \mathcal{I}\}$. If $X_\alpha$ is a henselian scheme for every $\alpha \in \mathcal{I}$ then $Y$ is affine.*

*Proof.* There is an étale quasi-compact separated surjective morphism $g : Y' \to Y$ such that $Y'$ is a scheme and such that there is a point $y' \in g^{-1}(y)$ with $k(y') = k(y)$. Let $X' = X \times_Y Y'$ with projections $h : X' \to X$ and $f' : X' \to Y'$. Similarly we let $X'_\alpha = X_\alpha \times_Y Y'$ and we have that $X' = \coprod_{\alpha\in\mathcal{I}} X'_\alpha$. As $k(y') = k(y)$ we have that $f'^{-1}(y') = \{x'_\alpha\}$ such that $x'_\alpha \in X'_\alpha$ and $h(x'_\alpha) = x_\alpha$.

Since $X_\alpha$ is henselian and $h$ is quasi-finite and separated it follows by [$\mathrm{EGA_{IV}}$, Thm. 18.5.11 c)] that $\mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \to X_\alpha$ is finite and that $\mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \subseteq X'$ is open and closed. Further as $X_\alpha$ is henselian, $k(x'_\alpha) = k(x_\alpha)$ and $X'_\alpha \to X_\alpha$ is étale it follows that $\mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \to X_\alpha$ is an isomorphism. By Lemma (A.2.11) we then have a decomposition $Y' = Y'_1 \amalg Y'_2$ where $Y'_1$ is local and $f'^{-1}(Y'_1) = \coprod_\alpha \mathrm{Spec}(\mathcal{O}_{X'_\alpha,x'_\alpha}) \cong X$. Thus we can, replacing $Y'$ with $Y'_1$, assume that $Y'$ is a local scheme and $X' \cong X$.

Let $Y'' = Y' \times_Y Y'$, which is a quasi-affine scheme, and $X'' = X \times_Y Y'' = X' \times_X X' \cong X$. Lemma (A.2.11) shows as before that $Y''$ is local and hence affine. Let $Y' = \mathrm{Spec}(A')$, $Y'' = \mathrm{Spec}(A'')$, $X' = \mathrm{Spec}(B')$ and $X'' = \mathrm{Spec}(B'')$

where $B'' = B'$. As $A' \hookrightarrow A''$ is faithfully flat it follows that $A''/A'$ is a flat $A'$-algebra. Further $A' \to B'$ is injective since $X \to Y$ is schematically dominant. Thus $A''/A' \hookrightarrow (A''/A') \otimes_{A'} B' = B''/B' = 0$ which shows that $A'' = A'$. This shows that $Y$ is the quotient of the étale equivalence relation $\mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$ where the two morphisms are the identity. Thus $Y = \mathrm{Spec}(A')$ is a local *scheme*. □

*Proof of Theorem* (A.2.2). As $g \circ f$ is separated, $f$ is separated. As $g \circ f$ is quasi-compact and universally closed and $g$ is separated, $f$ is quasi-compact and universally closed. Thus the image $Y'$ exists [$\mathrm{EGA_I}$, Prop. 6.10.5] and [Knu71, Prop. 4.6] and $X \to Y'$ is surjective. As $g \circ f$ is universally closed and $X \to Y'$ is surjective it follows that $Y' \to S$ is universally closed. Further it is obvious that $Y' \to S$ has topologically finite fibers.

Since the question is local over $S$, we can assume that $S$ is affine. Then $X$ is affine and we will show that $Y' \to S$ is *affine*. It then follows that $Y' \to S$ is integral since $\mathcal{O}_S \to g_*\mathcal{O}_{Y'} \hookrightarrow g_*f_*\mathcal{O}_X$ is integral.

Using Proposition (A.2.8) we are allowed to replace $S$ with the henselization $\mathrm{Spec}({}^{\mathrm{h}}\mathcal{O}_{S,s})$ at an arbitrary point $s$ and thus assume that $S$ is local and henselian. Then by Proposition (A.2.7) $X$ is henselian and a disjoint union of local schemes.

Let $x_1, x_2, \dots, x_n$ be the closed points of $X$ and $X = X_1 \amalg X_2 \amalg \dots \amalg X_n$ the corresponding partition into local henselian schemes. Then by Lemma (A.2.11) $Y = Y_1 \amalg Y_2 \amalg \dots \amalg Y_m$ where $Y_k$ is a local space with closed point $y_k \in f(x_j)$ for some $j$ depending on $k$. Further Lemma (A.2.12) shows that $Y_k$ is a local *scheme* and hence affine. □

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# Paper II

# FAMILIES OF ZERO-CYCLES AND DIVIDED POWERS: II. THE UNIVERSAL FAMILY

DAVID RYDH

Abstract. In this paper, we continue the study of the scheme of divided powers $\Gamma^d(X/S)$. In particular, we construct the universal family of $\Gamma^d(X/S)$ as a family of cycles supported on $\Gamma^{d-1}(X/S) \times_S X$ and discuss the "Hilbert-Chow" morphism. We also give a description of the $k$-points of $\Gamma^d(X/S)$ as effective zero-cycles with certain *rational* coefficients and give an alternative description of families of zero-cycles as *multivalued* morphisms. Finally, we construct sheaves of divided powers and a generalized norm functor.

## Introduction

Let $X/S$ be a separated algebraic space. In [**I**], a natural functor $\underline{\Gamma}^d_{X/S}$ from $S$-schemes to sets parameterizing effective zero-cycles of degree $d$ was introduced and shown to be an algebraic space — the space of divided powers $\Gamma^d(X/S)$. This is a globalization of the algebra of divided powers and the "correct" Chow scheme of points on $X/S$. Indeed, the space of divided powers commutes with base change and coincides with the symmetric product $\mathrm{Sym}^d(X/S)$ in characteristic zero or when $X/S$ is flat, e.g., when $X = \mathbb{P}^n_S$. In particular, we obtain a functorial description of $\mathrm{Sym}^d(X/S)$ in the flat case.

We let $\Gamma^d_1(X/S) = \Gamma^{d-1}(X/S) \times_S X$. A geometric point of $\Gamma^d_1(X/S)$ is a zero-cycle of degree $d$ with one marked point. It is thus expected that the addition morphism $\Phi_{X/S} : \Gamma^d_1(X/S) \to \Gamma^d(X/S)$, which forgets the marked point, should be related to the universal family of $\Gamma^d(X/S)$. When the addition morphism $\Phi_{X/S}$ is *flat*, then it has a tautological family of cycles given by the norm. Iversen [Ive70, Thm. II.3.4] showed that if $\Phi_{X/S}$ is flat, then $\Phi_{X/S}$ together with the norm family is the universal family. It should be noted that $\Phi_{X/S}$ is rarely flat, the notable exception being when $X/S$ is a *smooth curve*. The main result of this paper is a generalization of Iversen's result to arbitrary $X/S$ for which $\Phi_{X/S}$ need not be flat. More precisely, we construct a family of zero-cycles on $\Phi_{X/S}$, that is, a morphism $\varphi_{X/S} : \Gamma^d(X/S) \to \Gamma^d(\Gamma^d_1(X/S))$, and show that it is the universal family.

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2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 14C25.

*Key words and phrases.* Families of cycles, zero-cycles, divided powers, universal family, Weil restriction, norm functor, Hilbert scheme.

**Multiplicative polynomial laws.** To define the universal family, we need a couple of results on multiplicative laws. Firstly, we show in §1 that it is enough to consider the category of polynomial $A$-algebras in the definition of a multiplicative law $B \to C$ of $A$-algebras. Secondly, we define the norm law of a locally free algebra in §3. Thirdly, we construct *universal shuffle laws* in §4. These are canonical multiplicative laws $\Gamma_A^{d_1}(B) \otimes \Gamma_A^{d_2}(B) \to \Gamma_A^{d_1+d_2}(B)$ of degrees $((d_1, d_2))$ for positive integers $d_1$ and $d_2$. Apparently, it is difficult to directly define these laws. It is however easy to define canonical multiplicative laws $\mathrm{TS}_A^{d_1}(B) \otimes \mathrm{TS}_A^{d_2}(B) \to \mathrm{TS}_A^{d_1+d_2}(B)$ and we use these laws to define the universal shuffle laws. The universal shuffle law with $d_1 = d-1$ and $d_2 = 1$ will be of particular interest as this law gives a description of the universal family of $\Gamma_A^d(B)$, cf. Proposition (4.10).

**The universal family.** From the functorial description of $\Gamma^d(X/S)$ we have that the identity on $\Gamma^d(X/S)$ corresponds to a family of cycles on $X$ parameterized by $\Gamma^d(X/S)$ — the *universal family*. The image of the universal family is a closed subspace $Z_{\mathrm{univ}}$ of $\Gamma^d(X/S) \times_S X$ which is integral over $\Gamma^d(X/S)$. The nilpotent structure of this subspace is difficult to describe and we do not accomplish this. However, in §5 we show that $Z_{\mathrm{univ}}$ is contained in the closed subscheme $\Gamma_1^d(X/S) := \Gamma^{d-1}(X/S) \times_S X \hookrightarrow \Gamma^d(X/S) \times_S X$ which has the same underlying topological space as $Z_{\mathrm{univ}}$. In fact, we construct a family of cycles on $\Gamma_1^d(X/S) \to \Gamma^d(X/S)$ and show that this induces the identity on $\Gamma^d(X/S)$. This result is a globalization of the universal shuffle law in §4 described above. When $\Gamma_1^d(X/S) \to \Gamma^d(X/S)$ is flat and generically étale then the scheme $\Gamma_1^d(X/S)$ completely determines the universal family.

**Relation with the Hilbert scheme.** In §6 we briefly mention the natural morphism from the Hilbert scheme of $d$ points on $X$ to $\Gamma^d(X/S)$. This morphism takes a flat family to its determinant law and is known as the *Grothendieck-Deligne norm map*. When $\Gamma_1^d(X/S)$ is flat and generically étale over $\Gamma^d(X/S)$, the morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism. In particular, it is an isomorphism over the non-degeneracy locus $\Gamma^d(X/S)_{\mathrm{nondeg}}$ and an isomorphism when $X/S$ is a family of smooth curves.

**Composition and products of families.** In Sections 7 and 8 we define and give the basic properties of compositions and products of families. To define the product we have to pass to the flat case and use symmetric products, similarly as when defining the multiplicative shuffle laws.

**Points of $\Gamma^d(X/S)$.** In §9 we describe the $k$-points of $\Gamma^d(X/S)$. If $k$ is a *perfect* field, then the $k$-points of $\Gamma^d(X/S)$ correspond to effective zero-cycles of degree $d$ on $X_k$ with integral coefficients. For an arbitrary field $k$ there is a similar correspondence if we also allow certain rational coefficients. The denominators of these coefficients are powers of the characteristic of $k$ and the maximal exponent allowed is explicitly determined. This result also follows from [Kol96, Thm. I.4.5], using that the $k$-points of the space of divided powers and the Chow variety coincide, but our proof is more direct.

**Multi-morphisms.** Let $X$ be a scheme such that any set of $d$ points is contained in an affine open subset, e.g., let $X$ be quasi-projective. There is then another striking description of families of zero-cycles of degree $d$ on $X$ parameterized by any space $T$, that is, of morphisms $T \to \Gamma^d(X/S)$. We show that a family can be described as a *multi-morphism* $f : T \to X$ of degree $d$. This consists of a multivalued map $f : T \to X$ together with a semi-local multiplicative law $\theta : \mathcal{O}_X \to f_*\mathcal{O}_T$. The formalism is very close to that of ordinary morphisms of schemes. The condition on $X$ is used to ensure that for every point $t \in T$ the *set* $f(t) \subseteq X$ is contained in an affine subset. Similarly, a morphism of algebraic spaces $f : T \to X$ cannot be described as a morphism of locally ringed spaces unless every point in $X$ has an affine neighborhood, that is, unless $X$ is a scheme.

**Norm functor and Weil restriction.** Let $f : X \to Y$ be a morphism. The Weil restriction $\mathbf{R}_{X/Y}$ is a functor from $X$-schemes to $Y$-schemes defined by the property $\mathrm{Hom}_Y(T, \mathbf{R}_{X/Y}(W)) = \mathrm{Hom}_X(T \times_Y X, W)$. The existence of the Weil restriction of $W$, under suitable conditions on $f$ and $W$, can be established using Hilbert schemes [FGA, BLR90, Ryd08]. The *norm functor* $N_{X/Y}$ is a closely related functor which can be defined not only for $X$-schemes but also for sheaves on $X$. The existence of the norm functor is shown using a space or a sheaf of divided powers. The classical setting is when $X/Y$ is flat of constant rank $d$ and $\mathcal{L}$ is an invertible sheaf on $X$ [$\mathrm{EGA_{II}}$, §6.5]. For affine schemes and $X/Y$ flat, the norm functor has been studied intensively by Ferrand [Fer98] and we generalize some of these results.

**Notation and conventions.** We denote a *closed* immersion of schemes or algebraic spaces with $X \hookrightarrow Y$. When $A$ and $B$ are rings or modules we use $A \hookrightarrow B$ for an injective homomorphism. We let $\mathbb{N}$ denote the set of non-negative integers $0, 1, 2, \ldots$ and use the notation $((a,b)) = \binom{a+b}{a}$ for binomial coefficients.

## Contents

## 1. Determination of a multiplicative law

Recall [Rob63] that a polynomial law $F : M \to N$ is a set of maps $F_{A'} : M \otimes_A A' \to N \otimes_A A'$ for every $A$-algebra $A'$ which are natural with respect to $A$-algebra homomorphisms. The law $F$ is *homogeneous* of degree $d$ if $F_{A'}(a'x') = a'^d F_{A'}(x')$ for every $A$-algebra $A'$ and elements $a' \in A'$ and $x' \in M \otimes_A A'$. If $M$ and $N$ are $A$-algebras, then we say that $F$ is multiplicative if $F_{A'}(1) = 1$ and $F_{A'}(x'y') = F_{A'}(x')F_{A'}(y')$ for every $A$-algebra $A'$ and every $x', y' \in M \otimes_A A'$.

In some cases, cf. §§3–4, it is not clear that a natural map $M \to N$ extends functorially to any base change. The following proposition shows that it is enough to consider polynomial base changes.

**Proposition (1.1).** *Let $M$ and $N$ be $A$-modules.*

(i) *In the definition of polynomial laws we can replace the category $A$–**Alg** of $A$-algebras with the full subcategory of polynomial rings over $A$. To be precise, there is a one-to-one correspondence between polynomial laws $F : M \to N$ and sets of maps*

$$F_n : M[t_1, t_2, \dots, t_n] \to N[t_1, t_2, \dots, t_n], \quad n \in \mathbb{N}$$

*such that $F_m \circ (\mathrm{id}_M \otimes \varphi) = (\mathrm{id}_N \otimes \varphi) \circ F_n$ for any $A$-algebra homomorphism $\varphi : A[t_1, t_2, \dots, t_n] \to A[t_1, t_2, \dots, t_m]$. This correspondence is given by $F \mapsto (F_{A[t_1,t_2,\dots,t_n]})_{n \in \mathbb{N}}$.*

(ii) *If $(F_n)$ is homogeneous of degree $d$, that is, if $F_n(az) = a^d F_n(z)$ for every $n \geq 0$, $a \in A[t_1, t_2, \dots, t_n]$ and $z \in M[t_1, t_2, \dots, t_n]$, then the corresponding polynomial law $F$ is homogeneous of degree $d$.*

*In particular, in the definition of (homogeneous) polynomial laws, it is enough to consider smooth $A$-algebras.*

*Proof.* (i) It is immediately seen that to give a set of maps $\{F_n\}_n$ for $n \in \mathbb{N}$ commuting with $A$-algebra homomorphisms $\varphi$ as in the proposition is equivalent to give a single map $F' : M[t_1, t_2, \dots] \to N[t_1, t_2, \dots]$ such that for every endomorphism $\varphi$ of $A[t_1, t_2, \dots]$ the diagram

$$\begin{tikzcd} M[t_1, t_2, \dots] \arrow[r, "\mathrm{id}_M \otimes \varphi"] \arrow[d, "F'"] & M[t_1, t_2, \dots] \arrow[d, "F'"] \\ N[t_1, t_2, \dots] \arrow[r, "\mathrm{id}_N \otimes \varphi"] & N[t_1, t_2, \dots] \end{tikzcd} \tag{1.1.1}$$

commutes. A map $F'$ such that (1.1.1) commutes, gives a unique polynomial law $F : M \to N$ such that $F' = F_{A[t_1,t_2,\dots]}$ [Rob63, Prop. IV.4, p. 271]. Moreover, if

$f : \Gamma_A(M) \to N$ is the corresponding homomorphism, then $f\big(\gamma^{d_1}(x_1) \times \gamma^{d_2}(x_2) \times \dots \times \gamma^{d_n}(x_n)\big)$ is the coefficient of $t_1^{d_1} t_2^{d_2} \dots t_n^{d_n}$ in $F'(x_1t_1 + x_2t_2 + \dots + x_nt_n)$.

(ii) Let $z = x_1t_1 + x_2t_2 + \dots + x_nt_n \in M[t_1, t_2, \dots, t_n]$ be a homogeneous polynomial of degree one. If $F_n$ is homogeneous of degree $d$ then we have that

$$t_{n+1}^d F'(z) = F'(t_{n+1}z) = (F' \circ (\mathrm{id}_M \otimes \varphi))(z) = (\mathrm{id}_N \otimes \varphi)(F'(z))$$

where $\varphi$ is given by $t_i \mapsto t_{n+1}t_i$. It follows that $F'(z) \in N[t_1, t_2, \dots, t_n]$ is homogeneous of degree $d$ and thus that $f : \Gamma_A(M) \to N$ factors through the projection $\Gamma_A(M) \to \Gamma_A^d(M)$. In particular, we have that $F$ is homogeneous of degree $d$. $\square$

**Proposition (1.2).** *Let $B$ and $C$ be $A$-algebras. In the correspondence between polynomial laws $F : B \to C$ and sets of maps $(F_n)$ as in Proposition* (1.1)*, multiplicative polynomial laws correspond to multiplicative maps, i.e., maps $(F_n)$ such that*

(i) $F_n(1_B) = 1_C$.

(ii) $F_n(xy) = F_n(x)F_n(y), \quad \forall x, y \in B[t_1, t_2, \dots, t_n]$.

*In particular, in the definition of a multiplicative polynomial law it is enough to consider smooth $A$-algebras.*

*Proof.* If $F$ is a multiplicative law, then $F_n = F_{A[t_1,t_2,\dots,t_n]}$ is multiplicative by definition. Conversely, assume that we are given a set $(F_n)$ of multiplicative maps. This set of maps corresponds to a polynomial law $F : B \to C$ such that $F_n = F_{A[t_1,t_2,\dots,t_n]}$ by Proposition (1.1). It is clear that $F(1_B) = 1_C$. Let $A'$ be an $A$-algebra and $x, y \in B \otimes_A A'$. Then there is a positive integer $n$, a homomorphism $A[t_1, t_2, \dots, t_n] \to A'$ and $x_n, y_n \in B[t_1, t_2, \dots, t_n]$ such that $x_n$ and $y_n$ are mapped to $x$ and $y$ respectively. The multiplicativity of $F_n$ implies that $F_{A'}(xy) = F_{A'}(x)F_{A'}(y)$. $\square$

## 2. Inhomogeneous families

It is sometimes convenient to work with families which do not have constant degree. We therefore make the following definition:

**Definition (2.1).** Let $X/S$ be a separated algebraic space. We let $\Gamma^\star(X/S) = \coprod_{d \geq 0} \Gamma^d(X/S)$ and let $\underline{\Gamma}^\star_{X/S}(-) = \mathrm{Hom}_S(-, \Gamma^\star(X/S))$ be the corresponding functor.

Thus, by definition, a morphism $\alpha : T \to \Gamma^\star(X/S)$ corresponds to an open and closed partition $T = \coprod_{d \geq 0} T_d$ and families $\alpha_d : T_d \to \Gamma^d(X/S)$. We let

$$\mathrm{Image}(\alpha) = \coprod_{d \geq 0} \mathrm{Image}(\alpha_d) \hookrightarrow \coprod_{d \geq 0} X \times_S T_d = X \times_S T$$

and $\mathrm{Supp}(\alpha) = \mathrm{Image}(\alpha)_{\mathrm{red}}$. We say that the degree of $\alpha$ at $t \in T$ is $d$ if $\alpha(t) \in \Gamma^d(X/S)$.

**Proposition (2.2)** ([Zip86, Prop. 1.7.9 a)]). *Let $F : B \to C$ be a multiplicative law of $A$-algebras. Then there is an integer $n$, a complete set of orthogonal idempotents*

*$e_0, e_1, \dots, e_n$ in $C$ and a canonical decomposition $F = F_0 + F_1 + F_2 + \dots + F_n$ where $F_d : B \to Ce_d$ is a homogeneous multiplicative law of degree $d$. Note that $e_d = 0$ is possible.*

Note that conversely if $e_0, e_1, \dots, e_n$ is a complete set of orthogonal idempotents and $\big(F_d : B \to Ce_d\big)_{d=0,1,\dots,n}$ are multiplicative laws of degrees $0, 1, \dots, n$, then $F = F_0 + F_1 + \dots + F_n$ is a multiplicative law. In fact, $F(1) = \sum_i e_i = 1$ and $F(x)F(y) = \sum_i F_i(x)F_i(y) = F(xy)$.

**Theorem (2.3).** *Let $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$ be affine schemes and let $T = \operatorname{Spec}(A')$ be an affine $S$-scheme. Then there is a one-to-one correspondence between multiplicative laws $B \to A'$ and inhomogeneous families $T \to \Gamma^\star(X/S)$. This correspondence takes $f : T \to \Gamma^\star(X/S)$ onto $\Gamma(f) \circ (\gamma^0, \gamma^1, \gamma^2, \dots) : B \to A'$. The expression $\Gamma(f)$ is the induced map $\Gamma(\Gamma^\star(X/S)) = \prod_{d \geq 0} \Gamma^d_A(B) \to \Gamma(T) = A'$ on global sections.*

*Proof.* As $T$ is quasi-compact, any morphism $f : T \to \Gamma^\star(X/S)$ factors through $\Gamma^{\leq n}(X/S) = \coprod_{d \leq n} \Gamma^d(X/S)$. The theorem thus follows from Proposition (2.2). $\square$

## 3. Determinant laws and étale families

Let $A$ be a ring, $B$ an $A$-algebra and $M$ a $B$-module which is free of rank $d$ as an $A$-module. We then have the determinant or norm map

$$\mathrm{N}_{B/A} : B \to \operatorname{End}_A(M) \to \operatorname{End}_A(\wedge^d M) = A$$

where the first map takes $b$ to the endomorphism on $M$ which is multiplication by $B$. This map extends to a homogeneous multiplicative polynomial law which we denote the *determinant law*. We can also extend this definition to $B$-modules $M$ which are locally free of rank $d$ over $A$ taking an open cover of $\operatorname{Spec}(A)$. Similarly, if $M$ is locally free but not of constant rank, then we obtain an inhomogeneous multiplicative law $\mathrm{N}_{B/A} : B \to A$.

Assume now that $A$ is an integral domain with fraction field $K$, that $B$ is an $A$-algebra and that $M$ is a $B$-module which is of finite type as an $A$-module but not necessarily flat. If we let $d$ be the generic rank of $M$ then we have the norm map

$$\mathrm{N}_{B/A} : B \to \operatorname{End}_A(M) \to \operatorname{End}_K(M \otimes_A K) \to \operatorname{End}_K\big(\wedge^d(M \otimes_A K)\big) = K$$

and according to [EGA$_{\text{II}}$, Prop. 6.4.3] the elements $\mathrm{N}_{B/A}(b)$ are integral over $A$. In particular, if $A$ is in addition *integrally closed* then $\mathrm{N}_{B/A}$ has image $A$. Under this assumption this map extends to a determinant law as it is enough to define the multiplicative polynomial law over the integrally closed polynomial rings $A[t_1, \dots, t_n]$ by Proposition (1.2).

**Definition (3.1).** Let $S$ be an algebraic space and $f : X \to S$ affine. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$ such that $f_*\mathcal{F}$ is a finite $\mathcal{O}_S$-module and one of the following conditions holds:

(i) $f_*\mathcal{F}$ is a locally free $\mathcal{O}_S$-module.

(ii) $S$ is normal.

To $\mathcal{F}$ we associate the canonical family $\mathcal{N}_{\mathcal{F}} : S \to \Gamma^\star(X/S)$ given by the determinant law. To abbreviate, we let $\mathcal{N}_X = \mathcal{N}_{\mathcal{O}_X}$ when this is defined.

**Proposition (3.2).** *Let $S$ be an algebraic space and let $X/S$ be finite and étale. Then $\mathcal{N}_X$ is the unique morphism $S \to \Gamma^\star(X/S)$ such that* $\mathrm{Supp}(\mathcal{N}_X) = X_{\mathrm{red}}$ *and such that the degree of $\mathcal{N}_X$ at a point $s \in S$ is the rank of $X/S$ at $s$. Furthermore we have that* $\mathrm{Image}(\mathcal{N}_X) = X$. *In particular, the image of $\mathcal{N}_X$ commutes with arbitrary base change.*

*Proof.* The question is local on $S$ so we can assume that $X/S$ is of constant rank $d$. Let $S' \to S$ be an étale cover such that $X' = X \times_S S' \to S'$ trivializes, i.e., such that $X' = S'^{\amalg d}$. It is clear that the only family $S' \to \Gamma^d(X'/S')$ with support $X'_{\mathrm{red}}$ is the family with multiplicity one on each component. This is given by the morphism $S' \cong \Gamma^1_{S'}(S')^{\times_{S'} d} \hookrightarrow \Gamma^d(X')$. The corresponding multiplicative law is the multiplication map $(\mathcal{O}_{S'})^d \to \mathcal{O}_{S'}$ which coincides with the determinant law. Thus $\mathcal{N}_{X'}$ is the unique family with support $X'_{\mathrm{red}}$. As the image commutes with étale base change, the last statement of the proposition follows. □

## 4. Universal shuffle laws

Recall that the $A$-algebra $\Gamma^d_A(B)$ represents multiplicative polynomial laws of degree $d$ [Fer98, Prop. 2.5.1]. We thus have a canonical bijection

$$\mathrm{Hom}_{A\text{-}\mathbf{Alg}}\big(\Gamma^d_A(B), A'\big) \to \mathrm{Pol}^d_A(B, A') = \mathrm{Pol}^d_{A'}(A' \otimes_A B, A')$$

and under this correspondence, the identity on $\Gamma^d_A(B)$ corresponds to the *universal law* $U : \Gamma^d_A(B) \otimes_A B \to \Gamma^d_A(B)$. There is a natural surjection, the *canonical homomorphism* of Iversen,

$$\omega : \Gamma^d_A(B) \otimes_A B \to \Gamma^{d-1}_A(B) \otimes_A B$$

and we will show that $U$ factors through $\omega$. For this purpose, we first construct the multiplicative shuffle law $\mathrm{SL} : \Gamma^{d-1}_A(B) \otimes_A B \to \Gamma^d_A(B)$.

**(4.1)** We recall [**I**, 1.2.14] that the *universal multiplication of laws*

$$\rho_{d_1,d_2} : \Gamma^{d_1+d_2}_A(M) \to \Gamma^{d_1}_A(M) \otimes_A \Gamma^{d_2}_A(M)$$

is the homomorphism corresponding to the law $x \mapsto \gamma^{d_1}(x) \otimes \gamma^{d_2}(x)$. In particular, we have that

$$\rho_{d_1,d_2}\big(\gamma^\nu(x)\big) = \sum_{\substack{\nu_1+\nu_2=\nu \\ |\nu_1|=d_1,\ |\nu_2|=d_2}} \gamma^{\nu_1}(x) \otimes \gamma^{\nu_2}(x). \tag{4.1.1}$$

**(4.2)** *The shuffle product* — For any $A$-module $M$, the product of $\Gamma_A(M)$ gives $A$-module homomorphisms

$$\times : \Gamma^{d_1}_A(M) \otimes_A \Gamma^{d_2}_A(M) \to \Gamma^{d_1+d_2}_A(M).$$

The composition of the universal multiplication of laws $\rho_{d_1,d_2}$ followed by $\times$ is multiplication by $((d_1,d_2))$. In particular, if $((d_1,d_2))$ is invertible in $A$, then $x\otimes y \mapsto ((d_1,d_2))^{-1}x\times y$ is a retraction of $\rho_{d_1,d_2}$. If $B$ is an $A$-algebra, then $\times$ is $\Gamma_A^{d_1+d_2}(B)$-linear.

**(4.3)** *The multiplicative shuffle law* — Let $M$ be a *flat* $A$-module. The product on $\Gamma_A(M)$ is then identified with the *shuffle product*:

$$\times\,:\,\mathrm{TS}_A^{d_1}(M)\otimes_A \mathrm{TS}_A^{d_2}(M)\to \mathrm{TS}_A^{d_1+d_2}(M)$$

which is given by

$$x\times y=\sum_{\sigma\in\mathfrak{S}_{d_1,d_2}}\sigma(x\otimes y)$$

where the sum is taken in $\mathrm{T}_A^{d_1+d_2}(M)$. If $B=M$ is a flat $A$-algebra we can replace the sum with a product. This gives a multiplicative map

$$\mathrm{SL}\,:\,\mathrm{TS}_A^{d_1}(B)\otimes_A \mathrm{TS}_A^{d_2}(B)\to \mathrm{TS}_A^{d_1+d_2}(B) \tag{4.3.1}$$

defined by

$$\mathrm{SL}(z)=\prod_{\sigma\in\mathfrak{S}_{d_1,d_2}}\sigma(z).$$

Indeed, the set $\mathfrak{S}_{d_1,d_2}$ is a set of representatives of the left cosets of the subgroup $\mathfrak{S}_{d_1}\times\mathfrak{S}_{d_2}\hookrightarrow\mathfrak{S}_{d_1+d_2}$. If $z\in\mathrm{TS}_A^{d_1}(B)\otimes_A\mathrm{TS}_A^{d_2}(B)$ then $\sigma(z)=\sigma'(z)$ if $\sigma$ and $\sigma'$ belongs to the same left coset. As left multiplication on $\mathfrak{S}_{d_1+d_2}$ permutes the cosets, it is clear that $\mathrm{SL}(z)$ is invariant under $\mathfrak{S}_{d_1+d_2}$.

The composition of $\rho_{d_1,d_2}$ followed by SL is taking $((d_1,d_2))^{\mathrm{th}}$ powers and SL extends to a multiplicative law which is homogeneous of degree $((d_1,d_2))$. In fact, by Proposition (1.2) it is enough to show that SL extends functorially to

$$\mathrm{SL}_n\,:\,\mathrm{TS}_A^{d_1}(B)\otimes_A\mathrm{TS}_A^{d_2}(B)[t_1,t_2,\dots,t_n]\to\mathrm{TS}_A^{d_1+d_2}(B)[t_1,t_2,\dots,t_n]$$

which is easily seen.

**Definition (4.4).** Let $B$ be a flat $A$-algebra. The *shuffle homomorphism* is the homomorphism

$$\Lambda^{d_1,d_2}\,:\,\Gamma^{((d_1,d_2))}_{\Gamma_A^{d_1+d_2}(B)}\big(\Gamma_A^{d_1}(B)\otimes_A\Gamma_A^{d_2}(B)\big)\to\Gamma_A^{d_1+d_2}(B)$$

which corresponds to the shuffle law constructed in (4.3).

**Proposition (4.5).** *Let $d_1,d_2$ be integers and $N=((d_1,d_2))$. The shuffle homomorphism, defined in* (4.4) *for flat $A$-algebras $B$, extends uniquely to a homomorphism*

$$\Lambda^{d_1,d_2}\,:\,\Gamma^{N}_{\Gamma_A^{d_1+d_2}(B)}\big(\Gamma_A^{d_1}(B)\otimes_A\Gamma_A^{d_2}(B)\big)\to\Gamma_A^{d_1+d_2}(B).$$

*for every $A$-algebra $B$ such that for any homomorphism $B \to C$ of $A$-algebras the following diagram is commutative*

$$\begin{array}{ccc}
\Gamma^N_{\Gamma^{d_1+d_2}_A(B)}\bigl(\Gamma^{d_1}_A(B) \otimes_A \Gamma^{d_2}_A(B)\bigr) & \xrightarrow{\Lambda^{d_1,d_2}_B} & \Gamma^{d_1+d_2}_A(B) \\
\Big\downarrow & & \Big\downarrow \\
\Gamma^N_{\Gamma^{d_1+d_2}_A(C)}\bigl(\Gamma^{d_1}_A(C) \otimes_A \Gamma^{d_2}_A(C)\bigr) & \xrightarrow{\Lambda^{d_1,d_2}_C} & \Gamma^{d_1+d_2}_A(C).
\end{array}$$

*Proof.* If $C$ is an arbitrary $A$-algebra and $B$ is a flat $A$-algebra with a surjection $B \twoheadrightarrow C$ then the vertical arrows of the square are surjective and the upper arrow $\Lambda^{d_1,d_2}_B$ is given by Definition (4.4). We will verify that the composition of the upper and right arrows factors through the left arrow and thus induces a unique homomorphism $\Lambda^{d_1,d_2}_C$. As the diagram is commutative for flat $A$-algebras, it is then easily seen that this definition of $\Lambda^{d_1,d_2}_C$ is independent on the choice of flat resolution $B \twoheadrightarrow C$ and that the diagram becomes commutative for any homomorphism $B \to C$.

Let $I$ be the kernel of $B \twoheadrightarrow C$. The kernel of the left arrow in the diagram

$$\Gamma^N_{\Gamma^{d_1+d_2}_A(B)}\bigl(\Gamma^{d_1}_A(B) \otimes_A \Gamma^{d_2}_A(B)\bigr) \twoheadrightarrow \Gamma^N_{\Gamma^{d_1+d_2}_A(C)}\bigl(\Gamma^{d_1}_A(C) \otimes_A \Gamma^{d_2}_A(C)\bigr)$$

is the $\Gamma^{d_1+d_2}_A(B)$-module generated by the elements

$$\gamma^a\bigl((\gamma^{b_1}(i) \times f) \otimes (\gamma^{b_2}(j) \times g)\bigr) \times h$$

with $a \geq 1$, $b_1 + b_2 \geq 1$, $i, j \in I$, $f \in \Gamma^{d_1-b_1}_A(B)$, $g \in \Gamma^{d_2-b_2}_A(B)$ and $h \in \Gamma^{N-a}_{\Gamma^{d_1+d_2}_A(B)}\bigl(\Gamma^{d_1}_A(B)\otimes_A\Gamma^{d_2}_A(B)\bigr)$ by [**I**, 1.2.10]. Furthermore, replacing $A$ with a faithfully flat extension we can assume that $f, g$ and $h$ are of the form $f = \gamma^{d_1-b_1}(x)$, $g = \gamma^{d_2-b_2}(y)$ and $h = \gamma^{N-a}(z)$ where $x, y \in B$ and $z \in \Gamma^{d_1}_A(B) \otimes_A \Gamma^{d_2}_A(B)$ [Fer98, Lem. 2.3.1]. Finally, replacing $A$ with $A[t, u, v]$, it is enough to show that the elements

$$\gamma^N\bigl(\gamma^{d_1}(i + tx) \otimes \gamma^{d_2}(j + uy) + vz\bigr)$$
$$\gamma^N\bigl(\gamma^{d_1}(tx) \otimes \gamma^{d_2}(uy) + vz\bigr)$$

of $\Gamma^N_{\Gamma^{d_1+d_2}_A(B)}\bigl(\Gamma^{d_1}_A(B) \otimes_A \Gamma^{d_2}_A(B)\bigr)$ have the same image in $\Gamma^{d_1+d_2}_A(C)$. This follows by an easy computation. □

**Corollary (4.6).** *There exists a canonical multiplicative law $F : \mathrm{T}^d_A(B) \to \Gamma^d_A(B)$, homogeneous of degree $d!$, such that the composition $\mathrm{T}^d_A(B) \to \Gamma^d_A(B) \to \mathrm{T}^d_A(B)$ maps $z \in \mathrm{T}^d_A(B)$ onto $\prod_{\sigma\in\mathfrak{S}_d} \sigma(z)$.*

*Proof.* Let $\mathrm{SL}^{d_1,d_2} : \Gamma^{d_1}_A(B) \otimes \Gamma^{d_2}_A(B) \to \Gamma^{d_1+d_2}_A(B)$ be the multiplicative law corresponding to $\Lambda^{d_1,d_2}$. Let $F_B : \mathrm{T}^d_A(B) \to \Gamma^d_A(B)$ be the composition of the laws $\mathrm{SL}^{1,1} \otimes_A \mathrm{T}^{d-2}_A(B)$, $\mathrm{SL}^{2,1} \otimes_A \mathrm{T}^{d-3}_A(B)$, $\dots$, $\mathrm{SL}^{d-1,1}$. This is a multiplicative law of degree $d!$. Let $P$ be a flat $A$-algebra with a surjection $P \twoheadrightarrow B$. Let

$F_P : \mathrm{T}^d_A(P) \to \Gamma^d_A(P)$ be the multiplicative law constructed similarly. Then there is a commutative diagram

$$\begin{tikzcd} \mathrm{T}^d_A(P) \arrow[r, two heads] \arrow[d, "F_P"] \arrow[dr, phantom, "\circ"] & \mathrm{T}^d_A(B) \arrow[d, "F_B"] \\ \Gamma^d_A(P) \arrow[r, two heads] & \Gamma^d_A(B). \end{tikzcd}$$

As $F_P(z) = \prod_{\sigma\in\mathfrak{S}_d} \sigma(z) \in \mathrm{TS}^d_A(P) \cong \Gamma^d_A(P)$ for any $z \in \mathrm{T}^d_A(P)$ by construction, it follows that for any $z \in \mathrm{T}^d_A(B)$, the image of $F_B(z)$ in $\mathrm{T}^d_A(B)$ is $\prod_{\sigma\in\mathfrak{S}_d} \sigma(z)$. □

**Corollary (4.7).** *Let $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ be the canonical homomorphism. There is a canonical multiplicative law $F : \mathrm{TS}^d_A(B) \to \Gamma^d_A(B)$ of degree $d!$ such that $\varphi \circ F$ and $F \circ \varphi$ are the trivial laws of degree $d!$. In particular, if $x \in \ker(\varphi)$ then $x^{d!} = 0$ and if $y \in \mathrm{TS}^d_A(B)$ then $y^{d!} \in \mathrm{im}(\varphi)$.*

**(4.8)** *Iversen's canonical homomorphism* — Next, we consider the canonical homomorphism defined by Iversen in [Ive70, Prop. I.1.5]. This is the homomorphism

$$\omega : \Gamma^d_A(B) \otimes_A B \to \Gamma^{d-1}_A(B) \otimes_A B$$

given by $\rho_{d-1,1} \otimes \mathrm{id}_B$ followed by the multiplication map. In particular $\omega(\gamma^d(f) \otimes g) = \gamma^{d-1}(f) \otimes fg$. Furthermore, we let

$$u : \Gamma^d_{\Gamma^d_A(B)}\bigl(\Gamma^d_A(B) \otimes_A B\bigr) \xrightarrow{\cong} \Gamma^d_A(B) \otimes_A \Gamma^d_A(B) \to \Gamma^d_A(B)$$

be the composition of the canonical base-change isomorphism followed by the multiplication map. This is the homomorphism corresponding to the universal law $U$ given in the beginning of this section.

**Proposition (4.9)** ([Ive70, Prop. I.1.5])**.** *The homomorphism $\omega$ is surjective.*

*Proof.* It is enough to show that elements of the form $\bigl(\gamma^{d-1-k}(1) \times x\bigr) \otimes 1$, where $0 \le k \le d-1$ and $x \in \Gamma^k_A(B)$, are in the image of $\omega$. When $k = 0$ this is clear. We proceed by induction on $k$. The element $\bigl(\gamma^{d-k}(1) \times x\bigr) \otimes 1 \in \Gamma^d_A(B) \otimes_A B$ is mapped onto an element of the form

$$\bigl(\gamma^{d-1-k}(1) \times x\bigr) \otimes 1 + \sum_\alpha \bigl(\gamma^{d-k}(1) \times y_\alpha\bigr) \otimes z_\alpha$$

by the formula (4.1.1). By the induction hypothesis it follows that the second term belongs to the image of $\omega$ and hence so does the first term. □

The following Proposition generalizes [Ive70, Prop. I.3.1].

**Proposition (4.10).** *We have that $u = \Lambda^{d-1,1} \circ \Gamma^d_A(\omega)$.*

*Proof.* Let $u' = \Lambda^{d-1,1} \circ \Gamma^d(\omega)$. As $u$ and $u'$ are $\Gamma^d_A(B)$-algebra homomorphisms, it is enough to show that $u$ and $u'$ coincides on elements of the form $\gamma^{d_1}(1 \otimes b_1) \times \cdots \times \gamma^{d_k}(1 \otimes b_k)$. Replacing $A$ with the polynomial ring $A[t_1, t_2, \dots, t_k]$, it is further enough to show that $u$ and $u'$ coincides on the element $\gamma^d(1\otimes b') = \gamma^d\big(1\otimes(t_1b_1 + t_2b_2 + \cdots + t_kb_k)\big)$. This is clear as $\omega(1\otimes b') = 1\otimes b'$ and $\Lambda^{d-1,1}(\gamma^d(1\otimes b')) = \gamma^d(b')$. □

## 5. The universal family

To abbreviate, we use the notation

$$\Gamma^d_1(X/S) = \Gamma^{d-1}(X/S)\times_S X.$$

as in the introduction. This should be thought of as the space parameterizing zero-cycles of degree $d$ with one marked point. The addition morphism $\Gamma^d_1(X/S)\to\Gamma^d(X/S)$, which we will denote by $\Phi^d_{X/S}$, corresponds to forgetting the marking of the point. We will denote the projection on the marked point $\Gamma^d_1(X/S)\to X$ by $\pi_d$. When $X/S$ is affine, we let $\varphi_{X/S}$ be the family of zero-cycles of degree $d$ on $\Gamma^d_1(X/S)$ parameterized by $\Gamma^d(X/S)$ given by the shuffle homomorphism $\Lambda^{d-1,1}$ of Proposition (4.5). If a geometric point $\alpha\in\Gamma^d(X/S)$ corresponds to the cycle $x_1+x_2+\cdots+x_d$ then $(\varphi_{X/S})_\alpha$ corresponds to the cycle $(x_2+\cdots+x_{d-1},x_1)+\cdots+(x_1+\cdots+x_{d-1},x_d)$.

**(5.1)** Let $X/S$ and $U/T$ be separated algebraic spaces. For any commutative diagram

(5.1.1)
$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr] & X_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ & T \arrow[r, "g"] & S \end{tikzcd}$$

there is a natural commutative diagram

(5.1.2)
$$\begin{tikzcd} \Gamma^d_1(U/T) \arrow[r, "\eta"] \arrow[dr, "\Phi_{U/T}"'] & (f_*)^*\Gamma^d_1(X_T/T) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma^d_1(X_T/T) \arrow[d, "(\Phi_{X/S})_T"] \\ & \Gamma^d(U/T) \arrow[r, "f_*"] & \Gamma^d(X_T/T) \end{tikzcd}$$

**Proposition (5.2).** *Let $X/S$ be a separated algebraic space. There is a unique family of cycles $\varphi_{X/S}$ of degree $d$ on $\Phi^d_{X/S}$ such that for any commutative diagram (5.1.1) with $T$ and $U$ affine, the pull-back of the family $\varphi_{X/S}$ to $\Gamma^d(U/T)$ coincides with the push-forward of $\varphi_{U/T}$ along $\eta$.*

*Proof.* In what follows, all spaces are over $T$. If $f : U\to X_T$ is any étale morphism then we let $\Gamma^d(U)_{\mathrm{reg}} := \Gamma^d(U/T)|_{\mathrm{reg}(f)}$ be the regular locus [**I**, Cor. 3.3.11]. When $f : U\to X_T$ is étale then the morphism $\eta$ of diagram (5.1.2) is an isomorphism over $\Gamma^d(U)_{\mathrm{reg}}$ by [**I**, Cor. 3.3.11]. We let $\Gamma^d_1(U)_{\mathrm{reg}} = \Phi_U^{-1}\big(\Gamma^d(U)_{\mathrm{reg}}\big)$. If $\coprod_\alpha U_\alpha\to X_T$ is an étale cover, then in the diagram

$$\begin{tikzcd}
\coprod_{\alpha,\beta} \Gamma^d_1(U_\alpha \times_{X_T} U_\beta)_{\mathrm{reg}} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\Phi^d_{U_\alpha \times_{X_T} U_\beta}|_{\mathrm{reg}}"] & \coprod_\alpha \Gamma^d_1(U_\alpha)_{\mathrm{reg}} \arrow[r] \arrow[d, "\Phi^d_{U_\alpha}|_{\mathrm{reg}}"] & \Gamma^d_1(X_T) \arrow[d, "\Phi^d_{X_T}"] \\
\coprod_{\alpha,\beta} \Gamma^d(U_\alpha \times_{X_T} U_\beta)_{\mathrm{reg}} \arrow[r, shift left] \arrow[r, shift right] & \coprod_\alpha \Gamma^d(U_\alpha)_{\mathrm{reg}} \arrow[r] & \Gamma^d(X_T)
\end{tikzcd}$$

the natural squares are cartesian [**I**, Cor. 3.3.11] and the horizontal sequences are étale equivalence relations [**I**, Cor. 3.3.16]. If we choose a covering such that the $U_\alpha$'s are affine, then we have families $\varphi^d_{U_\alpha \times U_\beta}|_{\mathrm{reg}}$ and $\varphi^d_{U_\alpha}|_{\mathrm{reg}}$ on each component of the two leftmost vertical arrows. By étale descent, we obtain a family $\varphi^d_{X_T}$ on the rightmost arrow. From the compatibility of $\varphi^d$ with respect to base change and morphisms stated in Proposition (4.5), we can glue the families $\varphi^d_{X_T}$ for every $T$ to a family $\varphi^d_X$ with the ascribed properties. $\square$

**Proposition (5.3).** *The morphism* $(\Phi_{X/S}, \pi_d) : \Gamma^d_1(X/S) \to \Gamma^d(X/S) \times_S X$ *is a closed immersion.*

*Proof.* Follows from Proposition (4.9). $\square$

**Proposition (5.4).** *Let $X/S$ be a separated algebraic space. The family of zero-cycles $\bigl(\Gamma^d_1(X/S), \varphi_{X/S}\bigr)$ is a representative for the universal family of $\Gamma^d(X/S)$.*

*Proof.* We have to prove that the composition of the maps

$$\begin{aligned}
\varphi_{X/S} &: \Gamma^d(X/S) \to \Gamma^d\bigl(\Gamma^d_1(X/S)/\Gamma^d(X/S)\bigr) \\
\Gamma^d(\Phi_{X/S}, \pi_d) &: \Gamma^d\bigl(\Gamma^d_1(X/S)\bigr) \hookrightarrow \Gamma^d\bigl(\Gamma^d(X/S) \times_S X\bigr) \\
\pi &: \Gamma^d\bigl(\Gamma^d(X/S) \times_S X\bigr) = \Gamma^d(X/S) \times_S \Gamma^d(X/S) \to \Gamma^d(X/S)
\end{aligned}$$

is the identity. This follows from Proposition (4.10). $\square$

*Remark* **(5.5).** In general, we do not have that $\Gamma^d_1(X/S) = \mathrm{Image}(\varphi_{X/S})$. It is easily seen however that $\Gamma^d_1(X/S)_{\mathrm{red}} = \mathrm{Supp}(\varphi_{X/S})$.

**Proposition (5.6).** *The universal family $\Phi^d_{X/S} : \Gamma^d_1(X/S) \to \Gamma^d(X/S)$ is étale of rank $d$ over $\Gamma^d(X/S)_{\mathrm{nondeg}}$.*

*Proof.* This is a special case of [**I**, Prop. 4.1.8]. $\square$

**Corollary (5.7).** *Let $X/S$ be a separated algebraic space, $T$ an $S$-space and $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ a family of cycles. If $\alpha$ is non-degenerate at $t \in T$ then there is an open neighborhood $U \ni t$ such that* $\mathrm{Image}(\alpha|_U) \to U$ *is étale of degree $d$. In particular, the non-degeneracy locus of $\alpha$ is open in $T$. Moreover, $\alpha|_U$ is given by the canonical family $\mathcal{N}_{\mathrm{Image}(\alpha|_U)}$ and the image of $\alpha|_U$ commutes with arbitrary base change.*

*Proof.* Follows immediately from Propositions (3.2) and (5.6). $\square$

**Proposition (5.8).** *Let $X/S$ be a separated family of smooth curves, i.e., $X/S$ is a separated algebraic space, smooth of relative dimension one. Then the universal family $\Phi^d_{X/S}$ is locally free of rank $d$ and generically étale.*

*Proof.* The spaces $\Gamma^d_1(X/S)$ and $\Gamma^d(X/S)$ are smooth of relative dimension $d$ over $S$ [**I**, Prop. 4.3.3]. In particular, they are flat over $S$ and we can check the statements about $\Phi^d_{X/S}$ on the fibers. Replacing $S$ with a point $s$ we can thus assume that $S$ is a point. Then $\Gamma^d(X/S)$ and $\Gamma^d_1(X/S)$ are regular and in particular Cohen-Macaulay. As $\Phi^d_{X/S}$ is finite it follows that $\Phi^d_{X/S}$ is flat, cf. [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 15.4.2], and hence locally free. Moreover the connected components of $(X/S)^d$ are irreducible and their generic points are outside the diagonals. Thus $\Phi^d_{X/S}$ is generically étale of rank $d$, cf. Proposition (5.6). It follows that $\Phi^d_{X/S}$ is locally free of constant rank $d$. □

## 6. The Grothendieck-Deligne norm map

In this section we briefly discuss the natural morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ taking a flat subscheme to its norm family. We will call this map the Grothendieck-Deligne norm map as it is introduced in [FGA, No. 221, §6] and [Del73, 6.3.4]. This morphism is closely related to the Hilbert-Chow morphism [GIT, 5.4] and the Hilbert-Sym morphism [Nee91] as discussed in [**III**].

**Definition (6.1).** Let $f : X \to S$ be a separated algebraic space and $T$ an $S$-space. Let $\mathrm{Qcohpf}(X/S)(T)$ be the set of isomorphism classes of quasi-coherent finitely presented $\mathcal{O}_X$-modules which are flat and have proper support over $T$. We let $\mathrm{Qcohpf}^d(X/S)(T)$ be the subset of $\mathrm{Qcohpf}(X/S)(T)$ consisting of modules $\mathcal{G}$ with support finite over $T$ such that $f_*\mathcal{G}$ is locally free of constant rank $d$.

The usual pull-back makes $\mathrm{Qcohpf}(X/S)$ and $\mathrm{Qcohpf}^d(X/S)$ into contravariant functors. It can be shown that $\mathrm{Qcohpf}(X/S)$ is the coarse functor to an algebraic stack [LMB00, Thm. 4.6.2.1] but we will not use this.

We have natural transformations

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Qcohpf}^d(X/S)$$

$$\mathrm{Quot}^d(\mathcal{F}/X/S) \to \mathrm{Qcohpf}^d(X/S)$$

$$\mathrm{Qcohpf}^d(X/S) \to \Gamma^d(X/S)$$

where the first two are forgetful morphisms and the last is given by $\mathcal{G} \mapsto \mathcal{N}_{\mathcal{G}}$. Here $\mathcal{N}_{\mathcal{G}}$ is the canonical family determined by $\mathcal{G}$ defined in (3.1). This gives morphisms $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ and $\mathrm{Quot}^d(\mathcal{F}/X/S) \to \Gamma^d(X/S)$.

When the canonical family is flat of rank $d$ and generically étale, the morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism [Ive70, Thm. II.3.4]. In particular $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over $\Gamma^d(X/S)_{\mathrm{nondeg}}$ and an isomorphism if $X/S$ is a family of smooth curves, cf. Propositions (5.6) and (5.8)

## 7. Composition of families and étale projections

**(7.1)** *Universal composition of laws* — Consider the law $M \mapsto \Gamma^e_A(\Gamma^d_A(M))$ given by $x \mapsto \gamma^e(\gamma^d(x))$. This law is homogeneous of degree $de$ and thus gives a homomorphism

$$\kappa_{d,e} : \Gamma^{de}_A(M) \to \Gamma^e_A(\Gamma^d_A(M)).$$

Let $M$, $N$ and $P$ be $A$-modules. Given polynomial laws $F : M \to N$ and $G : N \to P$ homogeneous of degrees $d$ and $e$ respectively, we form the composite polynomial law $G \circ F : M \to P$. If $f : \Gamma^d_A(M) \to N$, $g : \Gamma^e_A(N) \to P$ and $g * f : \Gamma^{de}_A(M) \to P$ are the corresponding homomorphisms, we have that $g * f = g \circ \Gamma^e(f) \circ \kappa_{d,e}$.

When $M$, $N$ and $P$ are $A$-algebras, then $\kappa_{d,e}$ is an algebra homomorphism as the polynomial law defining $\kappa_{d,e}$ is multiplicative. When $B$ is an $A$-algebra and $C$ a $B$-algebra, it is also convenient to let $\kappa_{d,e}$ be the natural map

$$\Gamma^{de}_A(C) \to \Gamma^e_A(\Gamma^d_A(C)) \to \Gamma^e_A(\Gamma^d_B(C)).$$

This is the universal composition of a multiplicative law $F : C \to B$ over $B$ which is homogeneous of degree $d$ and a multiplicative law $G : B \to A$ which is homogeneous of degree $e$.

**Definition (7.2).** Let $X/Y$ and $Y/S$ be separated algebraic spaces. Let $T$ be an $S$-space and $\alpha \in \underline{\Gamma}^d_{X/Y}(Y \times_S T)$ and $\beta \in \underline{\Gamma}^e_{Y/S}(T)$ be families of cycles. Let $Z_\alpha = \mathrm{Image}(\alpha) \hookrightarrow X \times_S T$ and $Z_\beta = \mathrm{Image}(\beta) \hookrightarrow Y \times_S T$. Composing the corresponding laws, we obtain a morphism

$$T \to \Gamma^{de}(Z_\alpha \times_{Y \times_S T} Z_\beta / T) \hookrightarrow \Gamma^{de}(X \times_S T/T)$$

and we let $\beta * \alpha \in \underline{\Gamma}^{de}_{X/S}(T)$ be the corresponding family. By definition $\mathrm{Image}(\beta * \alpha) \hookrightarrow \mathrm{Image}(\alpha) \times_{Y \times_S T} \mathrm{Image}(\beta)$. It is clear that the composition $(\alpha, \beta) \to \beta * \alpha$ is functorial in $T$ and hence we obtain a natural transformation

$$* : \underline{\Gamma}^e_{Y/S}(-) \times \underline{\Gamma}^d_{X/Y}(Y \times_S -) \to \underline{\Gamma}^{de}_{X/S}(-).$$

of functors from $S$-schemes to sets. We define $\beta * \alpha$ for inhomogeneous families similarly.

**Proposition (7.3).** *Let $X/Y$, $Y/S$ be separated algebraic spaces. Let $T$ be an $S$-space and let $\alpha \in \underline{\Gamma}^\star_{X/Y}(Y \times_S T)$ and $\beta \in \underline{\Gamma}^\star_{Y/S}(T)$ be families of cycles.*

(i) *If $f : X \to X'$ is a $Y$-morphism, then*

$$f_*(\beta * \alpha) = \beta * f_*\alpha.$$

(ii) *Let $g : Y' \to Y$ be an $S$-morphism and $g' : X' \to X$ be the pull-back of $g$ along $X/Y$. Let $\beta' \in \underline{\Gamma}^\star_{Y'/S}(T)$ be a family of cycles. Then*

$$(g_*\beta') * \alpha = g'_*(\beta' * g^*\alpha)$$

(iii) *If $\alpha' \in \underline{\Gamma}^\star_{X/Y}(Y \times_S T)$ and $\beta' \in \underline{\Gamma}^\star_{Y/S}(T)$ are families of cycles, then*

$$(\beta + \beta') * \alpha = \beta * \alpha + \beta' * \alpha$$

$$\beta * (\alpha + \alpha') = \beta * \alpha + \beta * \alpha'.$$

*Proof.* (i) and (ii) are easily verified and (iii) follows from (i) and (ii). □

*Remark* **(7.4).** Let $S = \operatorname{Spec}(\overline{k})$ where $\overline{k}$ is an algebraically closed field. Let $X/Y$ and $Y/S$ be algebraic spaces with families of cycles $\alpha$ and $\beta$ of degrees $d$ and $e$ respectively. Then $\beta = y_1 + y_2 + \cdots + y_e$ and $\beta * \alpha = \alpha_{y_1} + \alpha_{y_2} + \cdots + \alpha_{y_e}$.

The following proposition also follows from the existence of product families in Section 8.

**Proposition (7.5).** *Let $f : X \to S$ be a separated morphism, let $g : Y \to S$ be a finite and étale morphism and let $\alpha : S \to \Gamma^\star(X/S)$ be a family of zero-cycles. Then $\mathcal{N}_{Y/S} * g^*\alpha = \alpha * \mathcal{N}_{X\times_S Y/X}$.*

*Proof.* It is enough to show the equality after a faithfully flat base change. We can thus assume that $Y = S^{\amalg n}$ is a trivial cover. Then both sides of the identity are equal to $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ where $\alpha_i$ is the family $\alpha$ on the $i^{\text{th}}$ component of $X \times_S Y = X^{\amalg n}$. □

**Proposition (7.6).** *Let $Y \to S$ be a finite étale morphism and $X \to Y$ a separated morphism. Then the morphism of presheaves*

$$\underline{\Gamma}^\star_{X/Y}(Y \times_S -) \to \underline{\Gamma}^\star_{X/S}(-)$$

*given by $\alpha \mapsto \mathcal{N}_{Y\times_S -} * \alpha$, is an isomorphism. In particular, if $Y$ and $S$ are connected then the degree of any family $\alpha' \in \underline{\Gamma}^\star_{X/S}(T)$ is a multiple of the rank of $Y \to S$.*

*Proof.* As the presheaves are sheaves in the étale topology, we can replace $S$ with an étale cover and assume that $Y = S^{\amalg n}$ is a trivial étale cover. We then have a corresponding decomposition $X = \coprod_{i=1}^n X_i$ and any family $\alpha' \in \underline{\Gamma}^\star_{X/S}(T)$ decomposes as a sum $\alpha' = \sum_{i=1}^n \alpha'_i$ where $\alpha'_i$ is supported on $X_i \times_S T$. This gives a family $\alpha = (\alpha'_i) \in \underline{\Gamma}^\star_{X/Y}(Y \times_S T)$ which composed with the canonical family $\mathcal{N}_{Y\times_S T}$ is $\alpha'$. □

For completeness, we mention the globalization of (7.1).

**Definition (7.7)** (Universal composition of families)**.** Let $X/Y$ and $Y/S$ be separated algebraic spaces and let $d$ and $e$ be positive integers. Consider the natural projection morphisms

$$\begin{aligned}&\Gamma^e(\Gamma^d(X/Y)/S) \times_S \Gamma^d(X/Y) \times_Y X \\ &\qquad\qquad \to \Gamma^e(\Gamma^d(X/Y)/S) \times_S \Gamma^d(X/Y) \to \Gamma^e(\Gamma^d(X/Y)/S).\end{aligned}$$

On the first morphism, we have the family $\mathrm{id}_{\Gamma^e(\Gamma^d(X/Y)/S)} \times_S \Phi^d_{X/Y}$ and on the second we have the family $\Phi^e_{\Gamma^d(X/Y)/S}$. The composition of these families gives a morphism

$$\kappa' : \Gamma^e(\Gamma^d(X/Y)/S) \to \Gamma^{de}(\Gamma^d(X/Y) \times_Y X/S).$$

We let
$$\kappa^{d,e}_{X/Y/S} \,:\, \Gamma^e(\Gamma^d(X/Y)/S) \to \Gamma^{de}(X/S)$$
be $\kappa'$ followed by the push-forward along the projection on the second factor.

**Proposition (7.8).** *Let $X/Y$, $Y/S$ be separated algebraic spaces, $T$ an $S$-space and let $\alpha \in \underline{\Gamma}^d_{X/Y}(Y \times_S T)$ and $\beta \in \underline{\Gamma}^e_{Y/S}(T)$ be families of cycles. Then*
$$\beta * \alpha = \kappa^{d,e} \circ \Gamma^e(\alpha) \circ (\beta, \mathrm{id}_T).$$

*Proof.* Replacing $X$ and $Y$ with $X \times_S T$ and $Y \times_S T$ we can assume that $T = S$. Let $\widetilde{\beta}$ be the pull-back of the universal family $\Phi^e_{\Gamma^d(X/Y)/S}$ along $\Gamma^e(\alpha) \circ \beta$. Note that $\kappa' \circ \Gamma^e(\alpha) \circ \beta$ corresponds to the family $\widetilde{\beta} * \Phi^d_{X/Y}$. As $\widetilde{\beta}$ is the push-forward of $\beta$ along the closed immersion $\alpha : Y \to \Gamma^d(X/Y)$, we have that $\widetilde{\beta} * \Phi^d_{X/Y}$ is the push-forward of $\beta * \alpha$ along $\alpha \times_Y \mathrm{id}_X : X \hookrightarrow \Gamma^d(X/Y) \times_Y X$. As $\kappa^{d,e}$ is the push-forward of $\kappa'$ along the projection $\Gamma^d(X/Y) \times_Y X \to X$, this ends the demonstration. ☐

## 8. Products of families

Given two families $\alpha : T \to \Gamma^d(X/S)$ and $\beta : T \to \Gamma^e(Y/S)$ we construct a product family $\alpha \times \beta : T \to \Gamma^{de}(X \times_S Y/S)$. If $\alpha = \sum_i \alpha_i$ and $\beta = \sum_j \beta_j$ then $\alpha \times \beta = \sum_{i,j} \alpha_i \times \beta_j$. Moreover, $\alpha \times \beta = \alpha * (\beta \times_S X) = \beta * (\alpha \times_S Y)$.

**Lemma (8.1).** *Let $X/S$ and $Y/S$ be separated algebraic spaces. There is a natural action of $\mathfrak{S}_d \times \mathfrak{S}_e$ on $(X/S)^d \times_S (Y/S)^e$ permuting the factors. Let $\mathrm{Sym}^{d,e}(X,Y)$ be the geometric quotient* [Ryd07]. *If either both $X/S$ and $Y/S$ are flat or $S$ is a $\mathbb{Q}$-scheme, then*
$$\mathrm{Sym}^{d,e}(X,Y) = \mathrm{Sym}^d(X/S) \times_S \mathrm{Sym}^e(Y/S) = \Gamma^d(X/S) \times_S \Gamma^e(Y/S).$$

*Proof.* Let $n = d + e$ and consider the action of $\mathfrak{S}_n$ on $(X \amalg Y/S)^n$. For any decomposition $n = d' + e'$ there is an open and closed subset $(X^{d'} \times Y^{e'})^{\amalg((d,e))}$ which is invariant under the action of $\mathfrak{S}_n$. The quotient of this subset is the quotient of $X^{d'} \times Y^{e'}$ by $\mathfrak{S}_{d'} \times \mathfrak{S}_{e'}$. If $X/S$ and $Y/S$ are flat or $S$ is a $\mathbb{Q}$-scheme, then $\Gamma = \mathrm{Sym}$ and we have that
$$\mathrm{Sym}^n(X \amalg Y) = \coprod_{d'+e'=n} \mathrm{Sym}^{d'}(X/S) \times_S \mathrm{Sym}^{e'}(Y/S).$$
The identity of the lemma then follows. ☐

Consider the morphism
$$\tau \,:\, (X/S)^d \times (Y/S)^e \to (X \times_S Y)^{de}$$
given by $(\pi_i, \pi_j)_{1\leq i\leq d,\, 1\leq j\leq e}$. The composition of $\tau$ with the quotient map $(X \times_S Y)^{de} \to \mathrm{Sym}^{de}(X \times_S Y)$ is invariant under the action of $\mathfrak{S}_d \times \mathfrak{S}_e$. Thus, under the

assumptions of the lemma, there is an induced morphism

$$\rho : \mathrm{Sym}^d(X/S) \times_S \mathrm{Sym}^e(Y/S) \to \mathrm{Sym}^{de}(X \times_S Y/S).$$

**Proposition (8.2).** *The morphism $\rho$ takes a pair of families $(\alpha, \beta)$ to the family $\alpha * (\beta \times_S X) = \beta * (\alpha \times_S Y)$.*

*Proof.* Working étale-locally, we can assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(C)$ are affine. As $\mathrm{TS}^d_A(B) \otimes_A \mathrm{TS}^e_A(C) \to \mathrm{T}^d_A(B) \otimes_A \mathrm{T}^e_A(C)$ is injective by the lemma, it is enough to show that the description of the morphism $\rho$ is correct for families factoring through $(X/S)^d \to \mathrm{Sym}^d(X/S)$ and $(Y/S)^e \to \mathrm{Sym}^e(Y/S)$. Let $a$ and $b$ be $T$-points of $(X/S)^d$ and $(Y/S)^e$. Then $a$ (resp. $b$) corresponds to sections $x_1, x_2, \dots, x_d$ (resp. $y_1, y_2, \dots, y_e$) of $X \times_S T/T$ (resp. $Y \times_S T/T$). The image by $\tau$ of $(a, b)$ is a $T$-point of $(X \times_S Y)^{de}$ corresponding to the sections $(x_i, y_j)_{ij}$. Passing to the quotient, $(a, b)$ is mapped to the pair of families $(\sum_i x_i, \sum_j y_j)$ and the image of this pair under $\rho$ is $\sum_{ij}(x_i, y_j)$. As the composition commutes with addition of cycles, it is now enough to show the equality for $d = e = 1$ and this case is trivial. □

**Theorem (8.3).** *There is a canonical morphism $\Gamma^d(X/S) \times_S \Gamma^e(Y/S) \to \Gamma^{de}(X \times_S Y/S)$ taking $(\alpha, \beta)$ to $\alpha \times \beta := \alpha * (\beta \times_S X) = \beta * (\alpha \times_S Y)$.*

*Proof.* Working étale-locally, we can assume that $S$, $X$ and $Y$ are affine. Let $X \hookrightarrow X'$ and $Y \hookrightarrow Y'$ be closed immersions into schemes which are flat over $S$. We have two morphisms $\Gamma^d(X/S) \times_S \Gamma^e(Y/S) \to \Gamma^{de}(X \times_S Y/S)$ given by $(\alpha, \beta) \mapsto \alpha * (\beta \times_S X)$ and $(\alpha, \beta) \mapsto \beta * (\alpha \times_S Y)$ respectively. To show that these coincide, it is enough to show that the compositions of these two morphisms with $\Gamma^{de}(X \times_S Y/S) \hookrightarrow \Gamma^{de}(X' \times_S Y'/S)$ coincide. This follows from Proposition (8.2) as the composition commutes with push-forward. □

## 9. Families of zero-cycles over reduced parameter spaces

The geometric points of $\Gamma^d(X/S)$ correspond to cycles of degree $d$. To be precise, if $k$ is an algebraically closed field and $s$ is a $k$-point of $S$, then the $k$-points of $\Gamma^d(X/S)$ over $s$ corresponds to the effective zero-cycles of degree $d$ on $(X_s)_{\mathrm{red}}$ [**I**, Cor. 3.1.9]. To determine the $k$-points for an arbitrary field $k$, we have to characterize the $\overline{k}$-points which descends to $k$. If $k$ is perfect, these points are the ones corresponding to cycles invariant under the action of the Galois group $\mathrm{Gal}(\overline{k}/k)$. The $k$-points of $\Gamma^d(X/S)$ are thus effective zero-cycles of degree $d$ on $(X_s)_{\mathrm{red}}$ where the degree is counted with multiplicity. The inseparable case is slightly more complicated.

**Definition (9.1).** Let $k \hookrightarrow K$ be a finite algebraic extension. There is then a canonical factorization into a separable extension $k \hookrightarrow k_s$ and a purely inseparable extension $k_s \hookrightarrow K$. The *separable degree* of $K/k$ is $[k_s : k]$ and the *inseparable degree* is $[K : k_s]$. The *exponent* of $K/k$ is the smallest positive integer $n$ such that $K^n k$ is separable over $k$, i.e., the smallest positive integer $n$ such that $K^n \subseteq k_s$.

We let the *quasi-degree* of $K/k$ be the product of the separable degree and the exponent. We let the *inseparable discrepancy* be the quotient of the inseparable degree with the exponent.

*Remark* **(9.2).** If $k$ is of characteristic zero, then the inseparable degree, the exponent and the inseparable discrepancy are all one. If $k$ is of characteristic $p$, then the inseparable degree, the exponent and the inseparable discrepancy are powers of $p$. Let $d_s$ be the separable degree, $d_i$ the inseparable degree, $p^e$ the exponent, $d = [K : k]$ the degree, $d_q$ the quasi-degree and $\delta$ the inseparable discrepancy. Then

$$d = d_s d_i, \quad d_i = p^e \delta, \quad d_q = d_s p^e, \quad d = d_q \delta.$$

The inseparable discrepancy is one if and only if $k_s \hookrightarrow K$ is generated by one element, or equivalently, if and only if $k \hookrightarrow K$ is generated by one element.

**Example (9.3).** The standard example of a field extension with exponent different from the inseparable degree is the following: Let $k = \mathbb{F}_p(s,t)$ and $K = k^{1/p} = k(s^{1/p}, t^{1/p})$. Then $K/k$ has inseparable degree $p^2$ and exponent $p$.

**Lemma (9.4).** *Let $k \hookrightarrow K$ be a finite algebraic extension of fields of characteristic $p$. The exponent of $K/k$ is the smallest power $p^e$ such that $k^{p^{-e}} \hookrightarrow k^{p^{-e}}K$ is separable.*

*Proof.* Standard results on $p$-bases, cf. [Mat86, Thm. 26.7], show that if $k \hookrightarrow k'$ is a separable algebraic extension then $k^{p^{-e}}k' = k'^{p^{-e}}$. Thus $k^{p^{-e}} \hookrightarrow k^{p^{-e}}K$ is separable if and only if $k_s^{p^{-e}} \hookrightarrow k_s^{p^{-e}}K$ is separable. This is equivalent to $K^{p^e} \subseteq k_s$, i.e., that $K/k$ has exponent at most $p^e$. □

The following proposition is a reinterpretation of [Kol96, Thm. I.4.5] as will be seen in Proposition (9.13).

**Proposition (9.5).** *Let $k \hookrightarrow K$ be a finite algebraic extension with quasi-degree $d$. Then $k$ is equal to the intersection of all purely inseparable extensions $k'/k$ such that $k' \hookrightarrow Kk'$ has degree at most $d$.*

*Proof.* Let $d_s$ and $p^e$ be the separable degree and exponent of $K/k$. Let $k_1$ be the intersection of all fields $k'$ such that $k'/k$ is purely inseparable and $k' \hookrightarrow Kk'$ has degree at most $d = d_s p^e$. If $k \neq k_1$ we can find an element $x \in k_1 \setminus k$ such that $x^p \in k$. Let $k'$ be a maximal purely inseparable extension of $k$ such that $x \notin k'$. Then $k^{p^{-e}}k' \subseteq k'(x^{p^{-e}})$ by [Kol96, Main Lemma I.4.5.5]. In particular the degree of $k^{p^{-e}}k'/k'$ is at most $p^e$. Note that by Lemma (9.4) we have that $k^{p^{-e}} \hookrightarrow k^{p^{-e}}K$ is separable and hence has degree $d_s$. Thus $Kk'/k'$ has degree at most $d_s p^e$. This implies that $x \notin k_1$ which is a contradiction. □

**Proposition (9.6).** *Let $k \hookrightarrow K$ be a finite field extension. Then $\Gamma^d(K/k)$ has at most one $k$-point. It has a $k$-point if and only if the quasi-degree of $K/k$ divides $d$.*

*This $k$-point corresponds to the composition of the polynomial laws*

$$\begin{aligned} F_{\mathrm{insep}} &: K \to k_s, \quad b \mapsto b^{d/d_s} \\ F_{\mathrm{sep}} &: k_s \to k, \quad b \mapsto \mathrm{N}_{k_s/k}(b) \end{aligned}$$

*where $d_s$ is the separable degree of $K/k$ and $\mathrm{N}_{k_s/k} : k_s \to k$ is the norm, cf. §3. In particular, there is a $k$-point if $[K:k] \mid d$.*

*Proof.* Let $d_s$ and $p^e$ be the separable degree and the exponent of $K/k$ and $k_s$ its separable closure. By Proposition (7.6) there is a one-to-one correspondence between $k$-points of $\Gamma^d(K/k)$ and $k_s$-points of $\Gamma^{d/d_s}(K/k_s)$. Replacing $k$ with $k_s$ and $d$ with $d/d_s$ we can thus assume that $K/k$ is separably closed.

Let $F : K \to k$ be a polynomial law, homogeneous of degree $d$. Then $K^{p^e} \subseteq k$ and as $F$ is multiplicative we have that $F(b)^{p^e} = F\left(b^{p^e}\right) = \left(b^{p^e}\right)^d$ for any $b \in K$. As $p^{\mathrm{th}}$ roots are unique in $k$ it follows that $F(b) = b^d \in k$. As $K/k$ is purely inseparable, it follows that $p^e \mid d$. □

**Definition (9.7).** Let $X/S$ be a separated algebraic space. Given a family of zero-cycles $\alpha$ on $X/S$ parameterized by an $S$-space $T$, we define the *multiplicity* of $\alpha$ at a point $x \in X \times_S T$, denoted $\mathrm{mult}_x(\alpha)$, as follows. Let $t \in T$ be the image of $x$ in $T$. The pull-back of the family to $k(t)$ is then supported at $\mathrm{Image}(\alpha_t) = \mathrm{Supp}(\alpha_t) = \{x_1, x_2, \dots, x_n\}$ and given by the morphism

$$\alpha_t : \mathrm{Spec}\big(k(t)\big) \to \Gamma^d\big(\mathrm{Supp}(\alpha_t)\big) = \coprod_{d_1+d_2+\cdots+d_n=d} \mathop{\times}_{i=1}^{n} \Gamma^{d_i}\big(\mathrm{Spec}(k(x_i))\big).$$

As each of the schemes $\Gamma^{d_i}\big(\mathrm{Spec}(k(x_i))\big)$ has at most one $k(t)$-point by Proposition (9.6), the morphism $\alpha_t$ is uniquely determined by the decomposition $d = d_1 + d_2 + \cdots + d_n$. The multiplicity at $x_i$ is defined to be $d_i/[k(x_i) : k(t)]$ and zero at points outside $\mathrm{Supp}(\alpha)$. As the support commutes with base change we have that

$$\mathrm{Supp}(\alpha) = \{x \in X \times_S T \ : \ \mathrm{mult}_x(\alpha) > 0\}.$$

**Definition (9.8).** Let $X/S$ be a separated algebraic space and let $T$ be a $S$-space. Given a family of zero-cycles $\alpha$ on $X/S$ parameterized by $T$, we let its *fundamental cycle* $[\alpha]$ be the cycle on $X \times_S T$ with coefficients in $\mathbb{Q}$ given by

$$[\alpha] = \sum_{x \in X \times_S (T_{\max})} \mathrm{mult}_x(\alpha) \left[\overline{\{x\}}\right]$$

where $T_{\max}$ is the set of generic points of $T$.

**Proposition (9.9).** *Let $X/S$ be a separated algebraic space and $T$ a* reduced *$S$-space. A family of zero-cycles $\alpha \in \underline{\Gamma}^d_{X/S}(T)$ is then uniquely determined by its fundamental cycle $[\alpha]$. Moreover* $\mathrm{Supp}(\alpha) = \mathrm{Supp}\big([\alpha]\big)$.

*Proof.* As every component of $Z = \mathrm{Supp}(\alpha)$ dominates a component of $T$ [**I**, Thm. 2.4.6], the support of $[\alpha]$ coincides with the support of $\alpha$. As $T$ is reduced, the morphism $\alpha : T \to \Gamma^d(X/S)$ is determined by its restriction to the generic points of $T$. If $\xi \in T_{\max}$ then $\alpha_\xi : k(\xi) \to \Gamma^d(\mathrm{Supp}(\alpha_\xi))$ is determined by the multiplicities at the points of $\mathrm{Supp}(\alpha_\xi)$ by Proposition (9.6). □

**Definition (9.10).** Let $k$ be a field and $X/k$ a separated algebraic space. Let $\mathcal{Z}$ be a zero-cycle on $X$ with coefficients in $\mathbb{Q}$. The *degree* of $\mathcal{Z}$ at a point $z \in \mathrm{Supp}(\mathcal{Z})$ is the product of the multiplicity of $\mathcal{Z}$ at $z$ and $[k(z) : k]$. We say that $\mathcal{Z}$ is quasi-integral if for any $z \in \mathrm{Supp}(\mathcal{Z})$ the following two equivalent conditions are satisfied

(i) The product of $\mathrm{mult}_z(\mathcal{Z})$ and the inseparable discrepancy of $k(z)/k$ is an integer.
(ii) The degree of $\mathcal{Z}$ at $z$ is an integer multiple of the quasi-degree of $k(z)/k$.

Note that if $k$ is perfect then $\mathcal{Z}$ is quasi-integral if and only if it has integral coefficients.

**Proposition (9.11).** *Let $k$ be a field and $X/k$ a separated algebraic space. There is a one-to-one correspondence between $k$-points of $\Gamma^d(X/k)$ and quasi-integral effective zero-cycles on $X$ of degree $d$. This correspondence takes a family of zero-cycles $\alpha$ onto its fundamental cycle $[\alpha]$.*

*Proof.* Follows from the definitions and Proposition (9.6). □

**Definition (9.12).** Let $k$ be a field and $X/k$ a separated algebraic space. Let $\mathcal{Z} = \sum_{i=1}^n a_i[Z_i]$ be a zero-cycle on $X$ with coefficients in $\mathbb{Q}$. For a field extension $k'/k$ we define the cycle $\mathcal{Z}_{k'}$ on $X_{k'} = X \times_k \mathrm{Spec}(k')$ as

$$\mathcal{Z}_{k'} = \sum_{i=1}^n a_i[Z_i \times_k \mathrm{Spec}(k')]$$

where $[Z_i \times_k \mathrm{Spec}(k')]$ is the fundamental cycle of $Z_i \times_k \mathrm{Spec}(k')$, i.e., the sum of the irreducible components of $Z_i \times_k \mathrm{Spec}(k')$ weighted by the lengths of the local rings at their generic points.

If $\alpha \in \Gamma^d(X/k)$ and $k'/k$ is a field extension, then $[\alpha]_{k'} = [\alpha_{k'}]$.

**Proposition (9.13)** ([Kol96, Thm. I.4.5])**.** *Let $k$ be a field and $X/k$ a separated algebraic space. Let $\mathcal{Z}$ be a zero-cycle on $X$ with coefficients in $\mathbb{Q}$. Then $\mathcal{Z}$ is quasi-integral if and only if $k$ is the intersection of all purely inseparable field extensions $k' \supseteq k$ such that $\mathcal{Z}_{k'}$ has integral coefficients.*

*Proof.* Follows immediately from Proposition (9.5). □

*Remark* **(9.14).** It is reasonable that an effective zero-cycle on $X$ with integral coefficients should give a family of zero-cycles on $X/k$. The above proposition explains why fractional coefficients are also sometimes allowed. Indeed, let $\mathcal{Z}$ be an effective zero-cycle on $X_{\overline{k}}$ with integral coefficients and let $\alpha$ be the corresponding point in $\Gamma^d(X/k)$. If $k'/k$ is a field extensions such that $\mathcal{Z}$ decends to $X \times_k \mathrm{Spec}(k')$ with integral coefficients, then $\alpha$ is defined over $k'$. Thus the residue field of $\alpha$

has at least to be small enough to be contained in all such field extensions $k'$. Proposition (9.13) states that the residue field is not smaller than this.

## 10. Families of zero-cycles as multivalued morphisms

In this section, we give an alternative description of families of zero-cycles on AF-schemes as "multi-morphisms".

**Definition (10.1).** A *multivalued map* $f : X \to Y$ is a map which to every $x \in X$ assigns a finite subset $f(x) \subseteq Y$. The *inverse image* of $W \subseteq Y$ with respect to $f$ is

$$f^{-1}(W) = \{x \in X \ : \ f(x) \subseteq W\}.$$

A multivalued map $f : X \to Y$ of topological spaces is *continuous* if $f^{-1}(U)$ is open for every open subset $U \subseteq Y$. A multivalued map $f : X \to Y$ is of *degree at most* $d$ if $|f(x)| \leq d$ for every $x \in X$.

Note that it is allowed for $f(x)$ to be the empty set.

**Definition (10.2).** Let $X$ be a topological space. A *$d$-cover* of $X$ is an open cover $\{U_\alpha\}$ of $X$ such that any set of at most $d$ points of $X$ is contained in one of the $U_\alpha$'s. A *$d$-sheaf* on $X$ is a presheaf $\mathcal{F}$ on $X$ such that

$$\mathcal{F}(U) \longrightarrow \textstyle\prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \textstyle\prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact for any open subset $U \subseteq X$ and every $d$-cover $\{U_\alpha\}$ of $U$. In other words, a $d$-sheaf is a sheaf in the Grothendieck topology on $X$ where the covers are $d$-covers. A 1-sheaf is an ordinary sheaf.

**Definition (10.3).** Let $f : X \to Y$ be a continuous multivalued map of degree at most $d$. If $\mathcal{F}$ is a presheaf of sets on $X$ we let $f_*\mathcal{F}$ be the presheaf $U \mapsto \mathcal{F}(f^{-1}(U))$ for every open subset $U \subseteq Y$. If $\mathcal{F}$ is a $k$-sheaf then $f_*\mathcal{F}$ is a $dk$-sheaf. If $\mathcal{F}$ is a $dk$-sheaf of sets on $Y$ we let $f^*\mathcal{F}$ be the associated $k$-sheaf to the presheaf $U \mapsto \varinjlim_{V \supseteq f(U)} \mathcal{F}(V)$, where $U \subseteq X$ is open and the limit is over all open subsets $V \subseteq Y$ containing $f(U)$. If $\mathcal{F}$ is a presheaf on $X$ and $Z \subseteq X$ is a finite subset, we denote by

$$\mathcal{F}_Z = \varinjlim_{V \supseteq Z} \mathcal{F}(V)$$

the *stalk* at $Z$.

It is not difficult to see, as in the single-valued case, that if $f : X \to Y$ is a continuous multivalued map of degree at most $d$, then $f^*$ and $f_*$ are adjoint functors between the categories of $k$-sheaves on $X$ and $kd$-sheaves on $Y$ and $(f^*\mathcal{F})_x = \mathcal{F}_{f(x)}$.

**Definition (10.4).** Let $X$ and $Y$ be ringed spaces. A *multi-morphism* from $X$ to $Y$ is a pair $(f, \theta)$ consisting of a multivalued continuous map $f : X \to Y$ and a multiplicative law (of presheaves) $\theta : \mathcal{O}_Y \to f_*\mathcal{O}_X$ over $\mathbb{Z}$. We say that $(f, \theta)$ is of degree $d$ if $\theta$ is homogeneous of degree $d$.

*Remark* **(10.5).** An ordinary morphism of ringed spaces is a multi-morphism of degree 1. Given multi-morphisms $f : X \to Y$ and $g : Y \to Z$ we can form the composition $g \circ f : X \to Z$. If $f$ and $g$ has degrees $d$ and $e$ respectively, then $g \circ f$ has degree $de$.

**Proposition (10.6).** *Let $(f, \theta) : X \to Y$ be a multi-morphism of ringed spaces. There is a canonical partition $X = \coprod_{d \geq 0} X_d$ of open and closed subsets $X_d \subseteq X$ such that $f|_{X_d}$ is a multi-morphism of degree $d$.*

*Proof.* This follows easily from Proposition (2.2). □

**Definition (10.7).** Let $A$ be a semi-local ring and $B$ be a local ring. A multiplicative $\mathbb{Z}$-law $A \to B$ is called *semi-local* if the kernel of the composite law $A \to B \to B/\mathfrak{m}_B$ is the Jacobson radical of $A$.

Note that if $Y$ is an AF-scheme and $Z \subseteq Y$ is finite, then the stalk $\mathcal{O}_{Y,Z}$ is semi-local.

**Definition (10.8).** Let $X$ and $Y$ be schemes. A *multi-morphism* from $X$ to $Y$ is a multi-morphism of ringed spaces $(f, \theta)$ such that $\mathcal{O}_{Y,f(x)}$ is semi-local and the law $\theta^\sharp_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is semi-local for every $x \in X$.

*Remark* **(10.9).** If $f : X \to Y$ and $g : Y \to Z$ are multi-morphisms of schemes, then $g \circ f : X \to Z$ is a multi-morphism of schemes if $\mathcal{O}_{Z,g(f(x))}$ is semi-local for every $x \in X$.

**Proposition (10.10).** *Let $(f, \theta) : X \to Y$ be a multi-morphism of schemes. If $(f, \theta)$ has degree $d$, then the multivalued map $f$ is of degree at most $d$. In particular, there is a one-to-one correspondence between multi-morphisms of degree one and ordinary morphisms of schemes.*

*Proof.* If $\theta$ is of degree $d$ then so is $\theta^\sharp_x$. The kernel of $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}/\mathfrak{m}_x$ is by assumption the Jacobson radical $\mathfrak{r}$ of $\mathcal{O}_{Y,f(x)}$. Let $B = \mathcal{O}_{Y,f(x)}/\mathfrak{r}$. We thus have a non-degenerate multiplicative law $B \to k(x)$ of degree $d$. This law factors through a homomorphism $B \to B \otimes_{\mathbb{Z}} k(x) \twoheadrightarrow B'$ where $B'$ is a product of at most $d$ fields [**I**, Thm. 2.4.6]. As $B \to k(x)$ is non-degenerate, we have by definition that $B \to B'$ is injective and thus $B$ is a product of at most $d$ fields. □

**Definition (10.11).** Let $f = (f, \theta) : X \to Y$ be a multi-morphism of schemes and let $n$ be a positive integer. We denote by $n \cdot f$ the multi-morphism $(f, \theta^n)$ from $X$ to $Y$ where $\theta^n$ is the homomorphism $\theta$ followed by taking the $n^{\text{th}}$ power. If $f$ has degree $d$ then $n \cdot f$ has degree $nd$. More generally, if $f_1, f_2 : X \to Y$ are multi-morphisms, we can define their sum $f_1 + f_2 : X \to Y$ as the multi-morphism $(f_1 \cup f_2, \theta_1 \theta_2)$. If $\mathcal{O}_{Y, f_1(x) \cup f_2(x)}$ is semi-local, this is a multi-morphism of schemes. If $f_1$ and $f_2$ have degrees $d_1$ and $d_2$ respectively, then $f_1 + f_2$ has degree $d_1 + d_2$.

**Definition (10.12).** Let $X$ and $Y$ be $S$-schemes with structure morphisms $\varphi_X : X \to S$ and $\varphi_Y : Y \to S$. We say that a multi-morphism $(f, \theta) : X \to Y$

is an $S$-multi-morphism if $\varphi_Y \circ f = \varphi_X$ as multivalued maps and $\theta : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a $\varphi_Y^*\mathcal{O}_S$-law. Here the $\varphi_Y^*\mathcal{O}_S$-algebra structure on $f_*\mathcal{O}_X$ is given by the homomorphism $\mathcal{O}_S \to (\varphi_X)_*\mathcal{O}_X = (\varphi_Y)_*f_*\mathcal{O}_X$ and adjointness.

**Proposition (10.13).** *Let $\varphi_X : X \to S$ and $\varphi_Y : Y \to S$ be $S$-schemes. If $f : X \to Y$ is an $S$-multi-morphism of degree $d$, then $\varphi_Y \circ f = d \cdot \varphi_X$.*

*Proof.* The law defining $\varphi_Y \circ f$ is $\psi : \mathcal{O}_S \to (\varphi_Y)_*\mathcal{O}_Y \to (\varphi_Y)_*f_*\mathcal{O}_X$. As $(\varphi_Y)_*\mathcal{O}_Y \to (\varphi_Y)_*f_*\mathcal{O}_X$ is an $\mathcal{O}_S$-law, it follows that $\psi$ is the $d^{\text{th}}$ power of $\mathcal{O}_S \to (\varphi_X)_*\mathcal{O}_X$. □

It is *not* true, unless $X$ is reduced, that if $f : X \to Y$ is a multi-morphism of $S$-schemes such that $\varphi_Y \circ f = d \cdot \varphi_X$, then $f$ is a $S$-multi-morphism. This is demonstrated by the following example.

**Example (10.14).** Let $A = \mathbb{Z}[x]$, $B = A[y]$, $C = A[\epsilon]/\epsilon^2$. Then it can be shown that

$$\Gamma^2_{\mathbb{Z}}(B) = \frac{\mathbb{Z}[x_p, x_s, y_p, y_s, x \times y]}{(x \times y)^2 - x_s y_s (x \times y) + x_p y_s^2 + x_s^2 y_p - 4 x_p y_p}$$

where $x_p = \gamma^2(x)$, $x_s = x \times 1$, $y_p = \gamma^2(y)$ and $y_s = y \times 1$. Let $F : B \to C$ be a multiplicative $\mathbb{Z}$-law of degree 2 and let $f : \Gamma^2_{\mathbb{Z}}(B) \to C$ be the corresponding homomorphism. That the composite law $A \to B \to C$ is $a \mapsto a^2 \cdot 1_C$ is equivalent to $f(\gamma^2(x)) = x^2$ and $f(x_s) = 2x$. This implies that

$$\big(f(x \times y) - x f(y_s)\big)^2 = 0.$$

In particular, $f(y_p) = f(y_s) = 0$ and $f(x \times y) = \epsilon$ defines a homomorphism such that $A \to B \to C$ is taking squares. It is clear that $F$ is not an $A$-law as this would imply that $f(x \times y) = x f(y_s) = 0$.

**Theorem (10.15).** *Let $X/S$ be any scheme and $Y/S$ be an* AF*-scheme. There is a one-to-one correspondence between $S$-multi-morphisms $f : X \to Y$ and families of zero-cycles, i.e., morphisms $\alpha : X \to \Gamma^\star(Y/S)$. In this correspondence a family of cycles $\alpha$ corresponds to the multi-morphism $(f, \theta)$ such that*

(i) *For every $x \in X$, the image $f(x)$ is the projection of the support* $\mathrm{Supp}(\alpha \times_X \mathrm{Spec}(k(x))) \hookrightarrow Y \times_S \mathrm{Spec}(k(x))$ *onto $Y$.*
(ii) *For any affine open subsets $V \subseteq S$ and $U \subseteq Y \times_S V$, the law*

$$\theta(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$

*corresponds to the morphism*

$$\alpha|_{\Gamma^\star(U/V)} : \alpha^{-1}(\Gamma^\star(U/V)) \to \Gamma^\star(U/V).$$

*Proof.* To begin with, note that for any open $U \subseteq Y$, we have that $f^{-1}(U) = \alpha^{-1}(\Gamma^{\geq 1}(U/S))$. In particular, $f$ is continuous. It is further clear that $\theta$ is a morphism of presheaves and that $\theta_x^\sharp$ is the law corresponding to $\mathrm{Spec}(\mathcal{O}_{X,x}) \to \Gamma^d(\mathrm{Spec}(\mathcal{O}_{Y,f(x)}))$ where $d$ is the degree of $\alpha$ at $x$. This law is semi-local by the definition of $f$.

We will now construct an inverse to the mapping $\alpha \to (f, \theta)$. For this, we can assume that $S$ is affine and that $\theta$ is homogeneous of degree $d$. As $Y$ is an AF-scheme, there is an open affine cover $\{U_\beta\}$ of $Y$ such that any $d$ points of $Y$ lie in some $U_\beta$. This induces an open affine cover $\{\Gamma^d(U_\beta/S)\}$ of $\Gamma^d(Y/S)$ [**I**, Prop. 3.1.10]. The laws $\theta(U_\beta)$ correspond to morphisms $\alpha_\beta : f^{-1}(U_\beta) \to \Gamma^d(U_\beta/S)$. The semi-locality of $\theta$ ensures that if $x \in f^{-1}(U_\beta)$ then the projection of $\mathrm{Supp}(\alpha_\beta)_x$ onto $U_\beta$ is $f(x)$. In particular, $\alpha_\beta^{-1}\big(\Gamma^d(U_{\beta'}/S)\big) = f^{-1}(U_\beta \cap U_{\beta'})$. Thus the $\alpha_\beta$'s glue to a morphism $\alpha : X \to \Gamma^d(Y/S)$ which corresponds to $(f, \theta)$. $\square$

*Remark* **(10.16).** Let $X/S$, $Y/S$ and $T/S$ be schemes. Given two multi-morphisms $f : T \to X$ and $g : T \to Y$ over $S$, there is an induced multi-morphism $(f, g) : T \to X \times_S Y$. This is given by taking the product $\alpha \times \beta$ of the corresponding families $\alpha$ and $\beta$. If $f$ and $g$ have degrees $d$ and $e$, then $(f, g)$ has degree $de$ and the composition of $(f, g)$ and the first (resp. second) projection is $e \cdot f$ (resp. $d \cdot g$). In particular, $\pi_1 \circ (f, g) = f$ and $\pi_2 \circ (f, g) = g$ as topological maps.

## 11. Sheaves of divided powers

Let $X/S$ be a separated algebraic space and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. In this section we construct a canonical quasi-coherent sheaf $\Gamma^d(\mathcal{F})$ on $\Gamma^d(X/S)$. This is a globalization of the construction of the $\Gamma^d_A(B)$-module $\Gamma^d_A(M)$ for an $A$-algebra $B$ and a $B$-module $M$. The sheaf $\Gamma^d(\mathcal{F})$ has been constructed by Deligne when $X/S$ is flat [Del73, 5.5.29].

**Proposition (11.1).** *Let $X/S$ be a separated algebraic space and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. There is then a canonical quasi-coherent sheaf $\Gamma^d(\mathcal{F})$ on $\Gamma^d(X/S)$. If $f : X' \to X$ is étale, then there is a canonical isomorphism*

$$\Gamma^d(f^{-1}\mathcal{F})|_{\mathrm{reg}(f)} \to (f_*|_{\mathrm{reg}})^{-1}\,\Gamma^d(\mathcal{F}).$$

*If $X/S$ is affine, then $\Gamma^d(\mathcal{F})$ is canonically isomorphic to $\Gamma^d_{\mathcal{O}_S}(\mathcal{F})$.*

*Proof.* We will construct $\Gamma^d(\mathcal{F})$ through étale descent via the étale equivalence relation

$$\coprod_{\alpha,\beta} \Gamma^d(U_\alpha \times_X U_\beta/S)|_{\mathrm{reg}} \rightrightarrows \coprod_{\alpha} \Gamma^d(U_\alpha/S)|_{\mathrm{reg}} \longrightarrow \Gamma^d(X/S)$$

for an étale covering $\{U_\alpha \to X\}$ [**I**, 3.3.16.1]. If the $U_\alpha$'s are affine then so are the $U_\alpha \times_X U_\beta$'s. The proposition thus follows after we have showed that

$$\Gamma^d_{\mathcal{O}_S}(f^{-1}\mathcal{F})|_{\mathrm{reg}(f)} \to (f_*|_{\mathrm{reg}})^{-1}\Gamma^d_{\mathcal{O}_S}(\mathcal{F}) \tag{11.1.1}$$

is an isomorphism for any étale morphism $f : X' \to X$ of *affine* schemes. Let $Y = \mathrm{V}(\mathcal{F}) = \mathrm{Spec}(\mathrm{S}(\mathcal{F}))$. Then

$$\Gamma^d(Y \times_X X'/S)|_{\mathrm{reg}(f)} \to (f_*|_{\mathrm{reg}})^{-1}\Gamma^d(Y/S) \tag{11.1.2}$$

is an isomorphism [**I**, Cor. 3.3.11]. As $\mathcal{F}$ is a direct summand of $\mathcal{O}_Y$, it follows from (11.1.2) that (11.1.1) is an isomorphism. $\square$

## 12. Weil restriction and the norm functor

In this section, we globalize and generalize the results of Ferrand on the norm functor [Fer98]. Let $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be a finite faithfully flat and finitely presented morphism of constant rank $d$. In this situation Ferrand constructs a norm functor $N_{B/A}$ from $B$-modules to $A$-modules which is uniquely determined by the following properties:

(i) $N_{B/A}(B) = A$ and the image of the multiplication by $b$ in $B$ is the multiplication by $\mathrm{N}_{B/A}(b)$ in $A$, cf. §3.

(ii) The norm functor commutes with base change, i.e., for any $A$-algebra $A'$, denoting $B' = B \otimes_A A'$, we have that the functors

$$M \mapsto N_{B/A}(M) \otimes_A A' \quad \text{and} \quad M \mapsto N_{B'/A'}(M \otimes_B B')$$

are isomorphic.

The functoriality gives a polynomial law $\nu : M \to N_{B/A}(M)$, homogeneous of degree $d$, which is compatible with the polynomial law $\mathrm{N}_{B/A}$. If $C$ is a $B$-algebra then $N_{B/A}(C)$ is an $A$-algebra. Ferrand constructs $N_{B/A}(M)$ as the tensor product $\Gamma^d_A(M) \otimes_{\Gamma^d_A(B)} A$ where the $\Gamma^d_A(B)$-algebra structure of $A$ is given by the determinant law $\mathrm{N}_{B/A} : B \to A$.

Given algebraic spaces $X/S$ and $Y/S$ together with a family of cycles $\alpha : Y \to \Gamma^\star(X/S)$ we will construct a norm functor $N_\alpha : \mathcal{C}_X \to \mathcal{C}_Y$. Here $\mathcal{C}$ is one of the following fibered categories over the category of algebraic spaces:

- The category of quasi-coherent modules **QCoh**.
- The category of affine schemes **Aff**.
- The category of separated algebraic spaces **AlgSp**.

In Ferrand's setting, $S = Y$ is affine, $X/S$ is finite flat of constant rank $d$ and $\alpha = \mathcal{N}_{X/S}$ is the canonical family given by the determinant, cf. Definition (3.1). We construct the generalized norm functor in the obvious way:

**Definition (12.1).** With notation as above, we let $N_\alpha(W) = \alpha^*\Gamma^\star(W/S)$ where $W \in \mathcal{C}_X$. If $W$ is an algebraic space, we let $\nu_\alpha(W)$ be the induced family of cycles $\nu_\alpha(W) : N_\alpha(W) \to \Gamma^\star(W/S)$ as in the diagram below:

$$\begin{tikzcd} \Gamma^\star(W/S) \arrow[d] & N_\alpha(W) \arrow[l, "\nu_\alpha(W)"'] \arrow[d] \arrow[dl, phantom, "\square"] \\ \Gamma^\star(X/S) & Y. \arrow[l, "\alpha"] \end{tikzcd}$$

When $W$ is a quasi-coherent $\mathcal{O}_X$-module, we let $\nu_\alpha(W)$ be the induced homomorphism $\Gamma^\star(W) \to \alpha_* N_\alpha(W)$ on $\Gamma^\star(X/S)$.

*Remark* **(12.2).** When $W/X$ is étale (or unramified) it is possible to define a "regular norm functor" using $N_\alpha(W)_{\mathrm{reg}} = \alpha^*\big(\Gamma^\star(W/S)_{\mathrm{reg}}\big)$.

*Remark* **(12.3).** If $Z/S$ is a third space and $\beta : Z \to \Gamma^e(Y/S)$ is a family of cycles, there is a functorial morphism $N_\beta(N_\alpha(W)) \to N_{\alpha\circ\beta}(W)$ but this is not always an isomorphism, cf. [Fer98, Ex. 4.4].

When $W/X$ is a space, it is useful to think of $N_\alpha(W)$ as the pull-back of $W$ along the multi-morphism $\alpha$ as in the following diagram:

$$\begin{tikzcd}
W \arrow[d] & N_\alpha(W) \arrow[l, dashed, "\nu_\alpha(W)"'] \arrow[d] \\
X & Y. \arrow[l, dashed, "\alpha"']
\end{tikzcd}$$

**Proposition (12.4).** *With notation as above, let $W/X$ be a space and $Y'$ be a $Y$-scheme. The $Y'$-points of $N_\alpha(W)$ corresponds to the set of liftings of the family of cycles $\alpha \times_Y Y'$ to a family of cycles $\beta : Y' \to \Gamma^\star(W/S)$. In other words, it is the set of liftings of the multi-morphism $\alpha \times_Y Y'$ to a multi-morphism $\beta$ in the diagram*

$$\begin{tikzcd}
W \arrow[d] & N_\alpha(W) \arrow[l, dashed, "\nu_\alpha(W)"'] \arrow[d] & N_\alpha(W) \times_Y Y' \arrow[l] \arrow[d] \\
X & Y \arrow[l, dashed, "\alpha"'] & Y'. \arrow[l] \arrow[ull, dashed, "\beta"']
\end{tikzcd}$$

*If $\alpha$ is non-degenerate, the lifting $\beta$ is non-degenerate and the $Y'$-points of $N_\alpha(W)$ correspond to sections of $W \to X$ over* $\mathrm{Image}(\alpha) \times_Y Y' \hookrightarrow X \times_S Y'$. *If $W/X$ is unramified, the $Y'$-points of $N_\alpha(W)_{\mathrm{reg}}$ correspond to sections of $W \to X$ over* $\mathrm{Image}(\alpha \times_Y Y')$.

*Proof.* The correspondence follows from the construction of $N_\alpha(W)$. The last two assertions are immediate consequences of the definitions of non-degenerate families and regular families [**I**, Defs. 4.1.6 and 3.3.3] taking into account that $\mathrm{Image}(\alpha \times_Y Y') = \mathrm{Image}(\alpha) \times_Y Y'$ when $\alpha$ is non-degenerate, cf. Corollary (5.7). □

**Definition (12.5).** Let $X \to Y$ and $W \to X$ be morphism of algebraic spaces. The *Weil restriction* $\mathbf{R}_{X/Y}(W)$ is the functor from $Y$-schemes to sets that takes an $Y$-scheme $Y'$ to the set of sections of $W \times_Y Y' \to X \times_Y Y'$.

**Corollary (12.6)** ([Fer98, Prop. 6.2.2])**.** *Let $X \to Y$ be a morphism and $\alpha : Y \to \Gamma^\star(X/Y)$ a family of cycles. Let $W$ be an algebraic space separated over $X$. There is then a canonical morphism $\mathbf{R}_{X/Y}(W) \to N_\alpha(W)$ which is functorial in $W$. Assume that $X \to Y$ is finite and étale and that $\alpha = \mathcal{N}_{X/Y}$ is the canonical family given by the determinant. Then the above functor is an isomorphism.*

*Proof.* Follows immediately from Proposition (12.4) as $\alpha$ is non-degenerate and hence $\mathrm{Image}(\alpha \times_Y Y') = \mathrm{Image}(\alpha) \times_Y Y' = X \times_Y Y'$. □

**Corollary (12.7).** *Let $f : X \to Y$ be a finitely presented morphism such that there exists a family of zero-cycles $\alpha : Y \to \Gamma^\star(X/Y)$ with* $\mathrm{Supp}(\alpha) = X_{\mathrm{red}}$, *e.g., $f$*

*finite and flat, or $Y$ normal and $f$ finite and open. If $W$ is an étale and separated scheme over $X$, then $N_\alpha(W)_{\mathrm{reg}}$ coincides with the Weil restriction $\mathbf{R}_{X/Y}(W)$. In particular, the canonical morphism $\mathbf{R}_{X/Y}(W) \to N_\alpha(W)$ is an open immersion.*

*Proof.* Note that $\mathrm{Image}(\alpha \times_Y Y')$ has the same support as $X \times_Y Y'$. As $W/X$ is étale, any section of $W/X$ over $\mathrm{Supp}(\alpha \times_Y Y')$ thus lifts to a unique section over $X \times_Y Y'$. $\square$

**Example (12.8).** The following counter-example, due to Ferrand [Fer98, 6.4], shows that even if $W/X$ is finite and étale and $X/Y$ is finite and flat, but not étale, it may happen that $N_\alpha(W)_{\mathrm{reg}} \subseteq N_\alpha(W)$ is not an isomorphism and that $N_\alpha(W) \to Y$ is not étale.

Let $X = \mathrm{Spec}(L) \to Y = \mathrm{Spec}(K)$ correspond to an inseparable field extension $K \subseteq L$ of degree $d$. Let $W = X^{\amalg d}$. Then there is a closed point in $N_\alpha(W)$ with residue field $L$. This point corresponds to the family $s_1 + s_2 + \cdots + s_l$ where $s_i : \mathrm{Spec}(L) \to W$ is the inclusion of the $i^{\text{th}}$ copy. Thus $N_\alpha(W) \to Y$ is not étale and as $N_\alpha(W)_{\mathrm{reg}} \to Y$ is étale the subset $N_\alpha(W)_{\mathrm{reg}} \subseteq N_\alpha(W)$ is proper.

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# Paper III

# HILBERT AND CHOW SCHEMES OF POINTS, SYMMETRIC PRODUCTS AND DIVIDED POWERS

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Abstract. Let $X$ be a quasi-projective $S$-scheme. We explain the relation between the Hilbert scheme of $d$ points on $X$, the $d^{\text{th}}$ symmetric product of $X$, the scheme of divided powers of $X$ of degree $d$ and the Chow scheme of zero-cycles of degree $d$ on $X$ with respect to a given projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$. The last three schemes are shown to be universally homeomorphic with isomorphic residue fields and isomorphic in characteristic zero or outside the degeneracy loci. In arbitrary characteristic, the Chow scheme coincides with the scheme of divided powers for a sufficiently ample projective embedding.

## Introduction

Let $X$ be a quasi-projective $S$-scheme. The purpose of this article is to explain the relation between

a) The Hilbert scheme of points $\operatorname{Hilb}^d\big(X/S\big)$ parameterizing zero-dimensional subschemes of $X$ of degree $d$.
b) The $d^{\text{th}}$ symmetric product $\operatorname{Sym}^d(X/S)$.
c) The scheme of divided powers $\Gamma^d(X/S)$ of degree $d$.
d) The Chow scheme $\operatorname{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ parameterizing zero dimensional cycles of degree $d$ on $X$ with a given projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

If $X/S$ is not quasi-projective then none of these objects need exist as schemes but the first three do exist in the category of algebraic spaces separated over $S$ [Ryd08, Ryd07b, **I**]. Classically, only the Chow *variety* of $X \hookrightarrow \mathbb{P}(\mathcal{E})$ is defined but we will show that for zero-cycles there is a natural Chow *scheme* whose underlying reduced scheme is the Chow variety.

There are canonical morphisms

$$\operatorname{Hilb}^d(X/S) \to \operatorname{Sym}^d(X/S) \to \Gamma^d(X/S) \to \operatorname{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^k)\big)$$

where $k \geq 1$ and $X \hookrightarrow \mathbb{P}(\mathcal{E}^k)$ is the Veronese embedding. The last two of these are universal homeomorphisms with trivial residue field extensions and are isomorphisms if $S$ is a $\mathbb{Q}$-scheme. If $S$ is arbitrary and $X/S$ is flat then the second morphism is an isomorphism. For arbitrary $X/S$ the third morphism is not an

2000 *Mathematics Subject Classification.* 14C05, 14L30.

*Key words and phrases.* Divided powers, symmetric powers, Chow scheme, Hilbert scheme, Weighted projective spaces.

isomorphism. In fact, the Chow scheme may depend on the chosen embedding as shown by Nagata [Nag55]. However, we will show that the third morphism is an isomorphism for *sufficiently large k*. Finally, we show that all three morphisms are isomorphisms outside the degeneracy locus.

The comparison between the last three schemes uses weighted projective structures. Given a projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$, there is an induced weighted projective structure on the symmetric product $\mathrm{Sym}^d(X)$. This follows from standard invariant theory, using the Segre embedding $(X/S)^d \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes d})$. In characteristic zero, this weighted projective structure on $\mathrm{Sym}^d(X)$ is actually projective, i.e., all generators have degree one. In positive characteristic, the weighted projective structure is "almost projective": the sheaf $\mathcal{O}(1)$ on $\mathrm{Sym}^d(X)$ is ample and generated by its global sections. This is a remarkable fact as for a general quotient, the sheaf $\mathcal{O}(1)$ of the weighted projective structure is usually not even invertible.

Since $\mathcal{O}(1)$ is generated by its global sections, we obtain a projective morphism $\mathrm{Sym}^d(X) \to \mathbb{P}(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{E}))$. In characteristic $p$, the ample sheaf $\mathcal{O}(1)$ is not always very ample, and thus this morphism is usually not a closed immersion. It is however a *universal homeomorphism* onto its image — the Chow variety. In summary, the reason that the Chow variety depends on the choice of projective embedding is that it is the image of an invariant object, the symmetric product, under a projective morphism induced by an ample sheaf which is not always very ample.

Given a projective embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$, the scheme of divided powers $\Gamma^d(X)$ has a similar weighted projective structure with similar properties. We define the Chow *scheme* to be the image of the analogous morphism $\Gamma^d(X) \to \mathbb{P}(\Gamma^d_{\mathcal{O}_S}(\mathcal{E}))$. For a sufficiently ample projective embedding, e.g., take the Veronese embedding $X \hookrightarrow \mathbb{P}(\mathrm{S}^k(\mathcal{E}))$ for a sufficiently large $k$, this morphism is a closed immersion. In particular, it follows that $\Gamma^d(X)$ is projective.

## Contents

## 1. The algebra of divided powers and symmetric tensors

We begin this section by briefly recalling the definition of the algebra of divided powers $\Gamma_A(M)$ and the multiplicative structure of $\Gamma^d_A(B)$. We then give a sufficient and necessary condition for $\Gamma^d_A(M)$ to be generated by $\gamma^d(M)$. This generalizes the sufficiency condition given by Ferrand [Fer98, Lem. 2.3.1]. The condition is essentially that every residue field of $A$ should have at least $d$ elements. Similar conditions on the residue fields reappear in Sections 3.1 and 5.2. Finally, we recall some explicit degree bounds on the generators of $\Gamma^d_A(A[x_1, x_2, \dots, x_r])$.

1.1. **Divided powers and symmetric tensors.** This section is a quick review of the results needed from [Rob63, Rob80]. Also see [Fer98, **I**].

**Notation (1.1.1).** Let $A$ be a ring and $M$ an $A$-algebra. We denote the $d^{\text{th}}$ tensor product of $M$ over $A$ by $\mathrm{T}^d_A(M)$. We have an action of the symmetric group $\mathfrak{S}_d$ on $\mathrm{T}^d_A(M)$ permuting the factors. The invariant ring of this action is the algebra of symmetric tensors which we denote by $\mathrm{TS}^d_A(M)$. By $\mathrm{T}_A(M)$ and $\mathrm{TS}_A(M)$ we denote the graded $A$-modules $\bigoplus_{d\geq 0} \mathrm{T}^d_A(M)$ and $\bigoplus_{d\geq 0} \mathrm{TS}^d_A(M)$ respectively.

**(1.1.2)** Let $A$ be a ring and let $M$ be an $A$-module. Then there exists a graded $A$-algebra, the algebra of divided powers, denoted $\Gamma_A(M) = \bigoplus_{d\geq 0} \Gamma^d_A(M)$ equipped with maps $\gamma^d : M \to \Gamma^d_A(M)$ such that, denoting the multiplication with $\times$ as in [Fer98], we have that for every $x, y \in M$, $a \in A$ and $d, e \in \mathbb{N}$

$$\Gamma^0_A(M) = A,\ \gamma^0(x) = 1,\ \Gamma^1_A(M) = M,\ \text{and } \gamma^1(x) = x \tag{1.1.2.1}$$
$$\gamma^d(ax) = a^d\gamma^d(x)$$
$$\gamma^d(x+y) = \textstyle\sum_{d_1+d_2=d} \gamma^{d_1}(x) \times \gamma^{d_2}(y)$$
$$\gamma^d(x) \times \gamma^e(x) = \binom{d+e}{d}\gamma^{d+e}(x)$$

Using (1.1.2.1) we identify $A$ with $\Gamma^0_A(M)$ and $M$ with $\Gamma^1_A(M)$. If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a set of elements of $M$ and $\nu \in \mathbb{N}^{(\mathcal{I})}$ then we let

$$\gamma^\nu(x) = \mathop{\times}_{\alpha\in\mathcal{I}} \gamma^{\nu_\alpha}(x_\alpha)$$

which is an element of $\Gamma^d_A(M)$ with $d = |\nu| = \sum_{\alpha\in\mathcal{I}} \nu_\alpha$.

**(1.1.3)** *Functoriality* — $\Gamma_A(\cdot)$ is a covariant functor from the category of $A$-modules to the category of graded $A$-algebras [Rob63, Ch. III §4, p. 251].

**(1.1.4)** *Base change* — If $A'$ is an $A$-algebra then there is a natural isomorphism $\Gamma_A(M) \otimes_A A' \to \Gamma_{A'}(M \otimes_A A')$ mapping $\gamma^d(x) \otimes_A 1$ to $\gamma^d(x \otimes_A 1)$ [Rob63, Thm. III.3, p. 262].

**(1.1.5)** *Multiplicative structure* — When $B$ is an $A$-algebra then the multiplication of $B$ induces a multiplication on $\Gamma^d_A(B)$ which we will denote by juxtaposition [Rob80]. This multiplication is such that $\gamma^d(x)\gamma^d(y) = \gamma^d(xy)$.

**(1.1.6)** *Universal property* — If $M$ is an $A$-module, then the $A$-module $\Gamma^d_A(M)$ represents *polynomial laws* which are homogeneous of degree $d$ [Rob63, Thm. IV.1, p. 266]. If $B$ is an $A$-algebra, then the $A$-algebra $\Gamma^d_A(B)$ represents *multiplicative polynomial laws* which are homogeneous of degree $d$ [Fer98, Prop. 2.5.1].

**(1.1.7)** *Basis* — If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a set of generators of $M$, then $\big(\gamma^\nu(x)\big)_{\nu\in\mathbb{N}^{(\mathcal{I})}}$ is a set of generators of $\Gamma_A(M)$. If $(x_\alpha)_{\alpha\in\mathcal{I}}$ is a basis of $M$ then $\big(\gamma^\nu(x)\big)_{\nu\in\mathbb{N}^{(\mathcal{I})}}$ is a basis of $\Gamma_A(M)$ [Rob63, Thm. IV.2, p. 272].

**(1.1.8)** *Presentation* — Let $M = G/R$ be a presentation of the $A$-module $M$. Then $\Gamma_A(M) = \Gamma_A(G)/I$ where $I$ is the ideal of $\Gamma_A(G)$ generated by the images in $\Gamma_A(G)$ of $\gamma^d(x)$ for every $x \in R$ and $d \geq 1$ [Rob63, Prop. IV.8, p. 284].

**(1.1.9)** $\Gamma$ *and* TS — The homogeneous polynomial law $M \to \mathrm{TS}^d_A(M)$ of degree $d$ given by $x \mapsto x^{\otimes_A d} = x \otimes_A \cdots \otimes_A x$ corresponds by the universal property (1.1.6) to an $A$-module homomorphism $\varphi : \Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ [Rob63, Prop. III.1, p. 254].

When $M$ is a free $A$-module then $\varphi$ is an isomorphism [Rob63, Prop. IV.5, p. 272]. The functors $\Gamma^d_A$ and $\mathrm{TS}^d_A$ commute with filtered direct limits [**I**, 1.1.4, 1.2.11]. Since any flat $A$-module is the filtered direct limit of free $A$-modules [Laz69, Thm. 1.2], it thus follows that $\varphi : \Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ is an isomorphism for any flat $A$-module $M$.

Moreover by [Rob63, Prop. III.3, p. 256], there is a diagram of $A$-modules

$$\begin{tikzcd}
\mathrm{TS}^d_A(M) \arrow[r, hook] & \mathrm{T}^d_A(M) \arrow[d, two heads] \\
\Gamma^d_A(M) \arrow[u] & \mathrm{S}^d_A(M) \arrow[l]
\end{tikzcd}$$

such that going around the square is multiplication by $d!$. Thus if $d!$ is invertible then $\Gamma^d_A(M) \to \mathrm{TS}^d_A(M)$ is an isomorphism. In particular, this is the case when $A$ is purely of characteristic zero, i.e., contains the field of rationals.

Let $B$ be an $A$-algebra. As the law $B \to \mathrm{TS}^d_A(B)$ given by $x \mapsto x^{\otimes_A d}$ is multiplicative, it follows that the homomorphism $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is an $A$-algebra homomorphism. In Section 4.1 we study the properties of $\varphi$ more closely.

1.2. **When is $\Gamma^d_A(M)$ generated by $\gamma^d(M)$?** $\Gamma^d_A(M)$ is not always generated by $\gamma^d(M)$ but a result due to Ferrand [Fer98, Lem. 2.3.1], cf. Proposition (1.2.4), shows that there is a finite free base change $A \hookrightarrow A'$ such that $\Gamma^d_{A'}(M \otimes_A A')$ is generated by $\gamma^d(M \otimes_A A')$. We will prove a slightly stronger statement in Proposition (1.2.2).

We let $\bigl(\gamma^d(M)\bigr)$ denote the $A$-submodule of $\Gamma^d_A(M)$ generated by the subset $\gamma^d(M)$.

**Lemma (1.2.1).** *Let $A$ be a ring and $M$ an $A$-module. There is a commutative diagram*

$$\begin{tikzcd}
\bigl(\gamma^d_A(M)\bigr) \otimes_A A' \arrow[r] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & \Gamma^d_A(M) \otimes_A A' \arrow[d, "\psi"', "\cong"] \\
\bigl(\gamma^d_{A'}(M \otimes_A A')\bigr) \arrow[r, phantom, "\subseteq"] & \Gamma^d_{A'}(M \otimes_A A')
\end{tikzcd}$$

*where $\psi$ is the canonical isomorphism of* (1.1.4). *If $A \to A'$ is a surjection or a localization then $\varphi$ is surjective. In particular, if in addition $\bigl(\gamma^d_{A'}(M \otimes_A A')\bigr) = \Gamma^d_{A'}(M \otimes_A A')$ then $\bigl(\gamma^d_A(M)\bigr) \otimes_A A' \to \Gamma^d_A(M) \otimes_A A'$ is surjective.*

*Proof.* The morphism $\varphi$ is well-defined as $\psi\bigl(\gamma^d(x) \otimes_A a'\bigr) = a'\gamma^d(x \otimes_A 1)$ if $x \in M$ and $a' \in A'$. If $A' = A/I$ then $\varphi$ is clearly surjective. If $A' = S^{-1}A$ is a localization then $\varphi$ is surjective since any element of $M \otimes_A A'$ can be written as $x \otimes_A (1/f)$ and $\varphi\bigl(\gamma^d(x) \otimes_A 1/f^d\bigr) = \gamma^d\bigl(x \otimes_A (1/f)\bigr)$. $\square$

**Proposition (1.2.2).** *Let $M$ be an $A$-module. The $A$-module $\Gamma^d_A(M)$ is generated by the subset $\gamma^d(M)$ if the following condition is satisfied*

(*) *For every $\mathfrak{p} \in \mathrm{Spec}(A)$ the residue field $k(\mathfrak{p})$ has at least $d$ elements or $M_\mathfrak{p}$ is generated by one element.*

*If $M$ is of finite type, then this condition is also necessary.*

*Proof.* Lemma (1.2.1) gives that $\bigl(\gamma^d_A(M)\bigr) = \Gamma^d_A(M)$ if and only if $\bigl(\gamma^d_{A_\mathfrak{p}}(M_\mathfrak{p})\bigr) = \Gamma^d_{A_\mathfrak{p}}(M_\mathfrak{p})$ for every $\mathfrak{p} \in \mathrm{Spec}(A)$. We can thus assume that $A$ is a local ring and only need to consider the condition (*) for the maximal ideal $\mathfrak{m}$. If $M$ is generated by one element then it is obvious that $\bigl(\gamma^d_A(M)\bigr) = \Gamma^d_A(M)$.

Further, any element in $\Gamma^d_A(M)$ is the image of an element in $\Gamma^d_A(M')$ for some submodule $M' \subseteq M$ of finite type. It is thus sufficient, but not necessary, that $\Gamma^d_A(M')$ is generated by $\gamma^d(M')$ for every submodule $M' \subseteq M$ of finite type. We can thus assume that $M$ is of finite type. Lemma (1.2.1) applied with $A \twoheadrightarrow A/\mathfrak{m} = k(\mathfrak{m})$ together with Nakayama's lemma then shows that $\bigl(\gamma^d_A(M)\bigr) = \Gamma^d_A(M)$ if and only if $\bigl(\gamma^d_{A/\mathfrak{m}}(M/\mathfrak{m}M)\bigr) = \Gamma^d_{A/\mathfrak{m}}(M/\mathfrak{m}M)$. We can thus assume that $A = k$ is a field.

We will prove by induction on $e$ that $\Gamma^e_k(M)$ is generated by $\gamma^e(M)$ when $0 \leq e \leq d$ if and only if either $\mathrm{rk}\, M \leq 1$ or $|k| \geq e$. Every element in $\Gamma^e_k(M)$ is a linear combination of elements of the form

$$\gamma^\nu(x) = \gamma^{\nu_1}(x_1) \times \gamma^{\nu_2}(x_2) \times \cdots \times \gamma^{\nu_m}(x_m).$$

where $x_i \in M$ and $|\nu| = e$. By induction $\gamma^{\nu_2}(x_2)\times\cdots\times\gamma^{\nu_m}(x_m) \in \bigl(\gamma^{e-\nu_1}(M)\bigr)$ and we can thus assume that $m = 2$ and it is enough to show that $\gamma^i(x) \times \gamma^{e-i}(y) \in \bigl(\gamma^e(M)\bigr)$ for every $x, y \in M$ and $0 \leq i \leq e$ if and only if either $\operatorname{rk} M \leq 1$ or $|k| \geq e$. If $x$ and $y$ are linearly dependent this is obvious. Thus we need to show that for $x$ and $y$ linearly independent, we have that $\gamma^i(x)\times\gamma^{e-i}(y) \in \bigl(\gamma^e(kx\oplus ky)\bigr)$ if and only if $|k| \geq e$. A basis for $\Gamma^e_k(kx \oplus ky)$ is given by $z_0, z_1, \dots, z_e$ where $z_i = \gamma^i(x) \times \gamma^{e-i}(y)$, see (1.1.7). For any $a, b \in k$ we let

$$\xi_{a,b} := \gamma^e(ax + by) = \sum_{i=0}^{e} \gamma^i(ax) \times \gamma^{e-i}(by) = \sum_{i=0}^{e} a^i b^{e-i} z_i.$$

Then $\bigl(\gamma^e_k(kx \oplus ky)\bigr) = \Gamma^e_k(kx \oplus ky)$ if and only if $\sum_{(a,b)\in k^2} k\xi_{a,b} = \bigoplus_{i=0}^{e} kz_i$. Since $\xi_{\lambda a,\lambda b} = \lambda^e \xi_{a,b}$ this is equivalent to $\sum_{(a:b)\in\mathbb{P}^1_k} k\xi_{a,b} = \bigoplus_{i=0}^{e} kz_i$. It is thus necessary that $\bigl|\mathbb{P}^1_k\bigr| = |k| + 1 \geq e + 1$. On the other hand if $a_1, a_2, \dots, a_e \in k^*$ are distinct then $\xi_{a_1,1}, \xi_{a_2,1}, \dots, \xi_{a_e,1}, \xi_{1,0}$ are linearly independent. In fact, this amounts to $(1, a_i, a_i^2, a_i^3, \dots, a_i^e)_{i=1,2,\dots,e}$ and $(0, 0, \dots, 0, 1)$ being linearly independent in $k^{e+1}$. If they are dependent then there exist a non-zero $(c_0, c_1, \dots, c_{e-1}) \in k^e$ such that $c_0 + c_1 a_i + c_2 a_i^2 + \cdots + c_{e-1} a_i^{e-1} = 0$ for every $1 \leq i \leq e$ but this is impossible since $c_0 + c_1 x + \cdots + c_{e-1} x^{e-1} = 0$ has at most $e - 1$ solutions. $\square$

**Lemma (1.2.3).** *Let $\Lambda_d = \mathbb{Z}[T]/P_d(T)$ where $P_d(T)$ is the unitary polynomial $\prod_{0\leq i<j\leq d}(T^i - T^j) - 1$. Then every residue field of $\Lambda_d$ has at least $d+1$ elements. In particular, if $A$ is any algebra, then $A \hookrightarrow A' = A \otimes_{\mathbb{Z}} \Lambda_d$ is a faithfully flat finite extension such that every residue field of $A'$ has at least $d+1$ elements.*

*Proof.* The Vandermonde matrix $(T^{ij})_{0\leq i,j\leq d}$ is invertible in $\operatorname{End}_{\Lambda_d}(\Lambda_d^{d+1})$ since it has determinant one. Let $k$ be a field and $\varphi : \Lambda_d \to k$ be any homomorphism. If $t = \varphi(T)$ then $(t^{ij})_{0\leq i,j\leq d}$ is invertible in $\operatorname{End}_k(k^{d+1})$ and it follows that $1, t, t^2, \dots, t^d$ are all distinct and hence that $k$ has at least $d+1$ elements. $\square$

**Proposition (1.2.4).** [Fer98, Lem. 2.3.1] *Let $\Lambda_d$ be as in Lemma* (1.2.3)*. If $A$ is a $\Lambda_d$-algebra then $\Gamma^d_A(M)$ is generated by $\gamma^d(M)$. In particular, for every $A$ there is a finite faithfully flat extension $A \to A'$, independent of $M$, such that $\Gamma^d_{A'}(M')$ is generated by $\gamma^d(M')$.*

*Proof.* Follows immediately from Proposition (1.2.2) and Lemma (1.2.3). $\square$

1.3. **Generators of the ring of divided powers.** In this section we will recall some results on the degree of the generators of $\Gamma^d_A(B)$. For our purposes the results of Fleischmann [Fle98] is sufficient and we will not use the more precise and stronger statements of [Ryd07a] even though some bounds would be slightly improved then.

**Definition (1.3.1)** (Multidegree)**.** Let $B = A[x_1, x_2, \dots, x_r]$. We define the *multidegree* of a monomial $x^\alpha \in B$ to be $\alpha$. This makes $B$ into a $\mathbb{N}^r$-graded ring

$$B = \bigoplus_{\alpha\in\mathbb{N}^r} B_\alpha = \bigoplus_{\alpha\in\mathbb{N}^r} Ax^\alpha$$

Let $\mathcal{M}$ be the $A$-module basis of $B$ consisting of the monomials. Recall from paragraph (1.1.7) that a basis of $\Gamma_A(B)$ is given by the elements $\gamma^\nu(x) = \times_\alpha \gamma^{\nu_\alpha}(x^\alpha)$ for $\nu \in \mathbb{N}^{(\mathcal{M})}$. We let $\operatorname{mdeg}\bigl(\gamma^k(x^\alpha)\bigr) = k\alpha$ and $\operatorname{mdeg}(f \times g) = \operatorname{mdeg}(f) + \operatorname{mdeg}(g)$ for $f, g \in \Gamma_A(B)$. Then

$$\operatorname{mdeg}\bigl(\times_\alpha \gamma^{\nu_\alpha}(x^\alpha)\bigr) = \sum_{x^\alpha\in\mathcal{M}} \nu_\alpha \operatorname{mdeg}(x^\alpha) = \sum_{\alpha\in\mathbb{N}^r} \nu_\alpha \alpha.$$

We let $\Gamma^d_A(B)_\alpha$ be the $A$-module generated by basis elements $\gamma^\nu(x)$ of multidegree $\alpha$. This makes $\Gamma^d_A(B) = \bigoplus_{\alpha\in\mathbb{N}^r} \Gamma^d_A(B)_\alpha$ into a $\mathbb{N}^r$-graded ring.

**Definition (1.3.2)** (Degree)**.** Let $B = A[x_1, x_2, \dots, x_r] = \bigoplus_{k\geq 0} B_k$ with the usual grading, i.e., $B_k$ are the homogeneous polynomials of degree $k$. The graded $A$-algebra $C = \bigoplus_{k\geq 0} \Gamma^d_A(B_k)$ is a subalgebra of $\Gamma^d_A(B)$. If an element $f \in \Gamma^d_A(B)$ belongs to $C_k = \Gamma^d_A(B_k)$ we say that $f$ is *homogeneous of degree* $k$. The *degree* of an arbitrary element $f \in \Gamma^d_A(B)$ is the smallest natural number $n$ such that $f \in \Gamma^d_A\left(\bigoplus_{k=0}^n B_k\right)$.

*Remark* **(1.3.3).** Let $B = A[x_0, x_1, \dots, x_r]$ and let $C = \bigoplus_{k\geq 0} \Gamma^d_A(B_k)$ be the graded subring of $\Gamma^d_A(B)$. The degree in the previous definition is such that there is a relation between the degree of elements in $C$ and the degree of an element in the graded localization $C_{(\gamma^d(s))}$ for $s \in B_1$. To see this, note that

$$C_{(\gamma^d(s))} = \Gamma^d_A\left(B_{(s)}\right) = \Gamma^d_A\bigl(A[x_0/s, \dots, x_r/s]\bigr).$$

We let $A[x_0/s, \dots, x_r/s]$ be graded such that $x_i/s$ has degree 1. An element $f \in \Gamma^d_A(A[x_0/s, \dots, x_r/s])$ of degree $n$ can then be written as $g/\gamma^d(s)^n$ where $g \in \Gamma^d_A(B_n)$ is homogeneous of degree $n$.

**Theorem (1.3.4)** ([Ric96, Prop. 2], [Ryd07a, Cor. 8.4])**.** *If $d!$ is invertible in $A$ then $\Gamma^d_A(A[x_1, \dots, x_r])$ is generated by the elementary multisymmetric functions $\gamma^{d_1}(x_1)\times\gamma^{d_2}(x_2)\times\cdots\times\gamma^{d_r}(x_r)\times\gamma^{d-d_1-d_2-\cdots-d_r}(1)$, $d_i \in \mathbb{N}$ and $d_1+d_2+\cdots+d_r \leq d$.*

**Theorem (1.3.5)** ([Fle98, Thm. 4.6, 4.7], [Ryd07a, Cor. 8.6])**.** *For an arbitrary ring $A$, the $A$-algebra $\Gamma^d_A(A[x_1, \dots, x_r])$ is generated by $\gamma^d(x_1), \gamma^d(x_2), \dots, \gamma^d(x_r)$ and the elements $\gamma^k(x^\alpha)\times\gamma^{d-k}(1)$ with $k\alpha \leq (d-1, d-1, \dots, d-1)$. Further, there is no smaller multidegree bound and if $d = p^s$ for some prime $p$ not invertible in $A$, then $\Gamma^d_A(A[x_1, \dots, x_r])$ is not generated by elements of strictly smaller multidegree.*

Theorems (1.3.4) and (1.3.5) give the following degree bound:

**Corollary (1.3.6).** *Let $A$ be a ring and $B = k[x_1, x_2, \dots, x_r]$. Then $\Gamma^d_A(B)$ is generated by elements of degree at most* $\max\bigl(1, r(d-1)\bigr)$. *If $d!$ is invertible in $A$, then $\Gamma^d_A(B)$ is generated by elements of degree one.*

## 2. Weighted projective schemes and quotients by finite groups

In this section we review the definition and the basic results on weighted projective schemes. We will in particular focus on weighted projective structures which are *covered in degree one*. By this, we mean that the sections of $\mathcal{O}(1)$ give an affine cover of the weighted projective scheme. We then recall the construction, using invariant theory, of a geometric quotient of a projective scheme by a finite group. Finally, we discuss the failure of a geometric quotient to commute with arbitrary base change and closed immersions.

2.1. **Remarks on projectivity.** We will follow the definitions in EGA. In particular, very ample, ample, quasi-projective and projective will have the meanings of [EGA$_{\mathrm{II}}$, §4.4, §4.6, §5.3, §5.5]. By definition, a morphism $q : X \to S$ is quasi-projective if it is of finite type and there exists an invertible $\mathcal{O}_X$-sheaf $\mathcal{L}$ ample with respect to $q$. Note that this does *not* imply that $X$ is a subscheme of $\mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$. However, if $S$ is quasi-compact and quasi-separated then there is a quasi-coherent $\mathcal{O}_S$-module of finite type $\mathcal{E}$ and an immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$ [EGA$_{\mathrm{II}}$, Prop. 5.3.2]. Similarly, a projective morphism is always quasi-projective and proper but the converse only holds if $S$ is quasi-compact and quasi-separated.

Furthermore, if $q : X \to S$ is a projective morphism and $\mathcal{L}$ a very ample invertible sheaf on $X$ then $\mathcal{L}$ does not necessarily correspond to a closed embedding into a projective space over $S$. We always have a closed embedding $X \hookrightarrow \mathbb{P}(q_*\mathcal{L})$ as $q$ is proper [EGA$_{\mathrm{II}}$, Prop. 4.4.4] but $q_*\mathcal{L}$ need not be of finite type. If $S$ is *locally noetherian* however, then $q_*\mathcal{L}$ is of finite type [EGA$_{\mathrm{III}}$, Thm. 3.2.1]. If $S$ is quasi-compact and quasi-separated then we can find a sub-$\mathcal{O}_S$-module of finite type $\mathcal{E}$ of $q_*\mathcal{L}$ such that we have a closed immersion $i : X \hookrightarrow \mathbb{P}(\mathcal{E})$ and such that $\mathcal{L} = i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.

We will also need the following stronger notion of projectivity introduced by Altman and Kleiman in [AK80, §2]. Our definition differs slightly from theirs as we do not require strongly projective morphisms to be of finite *presentation*.

**Definition (2.1.1).** A morphism $X \to S$ is *strongly projective* (resp. *strongly quasi-projective*) if it is of finite type and factors through a closed immersion (resp. an immersion) $X \hookrightarrow \mathbb{P}_S(\mathcal{L})$ where $\mathcal{L}$ is a locally free $\mathcal{O}_S$-module of constant rank.

*Remark* **(2.1.2).** A strongly (quasi-)projective morphism is (quasi-)projective and the converse holds when $S$ is quasi-compact, quasi-separated and admits an ample sheaf, e.g., $S$ affine [AK80, Ex. 2.2 (i)]. In fact, in this case there is an embedding $X \hookrightarrow \mathbb{P}^n_S$ and thus the notions of projective and strongly projective also agree with the definitions in [Har77].

### 2.2. Weighted projective schemes.

**Definition (2.2.1).** Let $S$ be a scheme. A *weighted projective scheme* over $S$ is an $S$-scheme $X$ together with a quasi-coherent graded $\mathcal{O}_S$-algebra $\mathcal{A}$ of *finite type*, not necessarily generated by degree one elements, such that $X = \mathrm{Proj}_S(\mathcal{A})$. We let as usual $\mathcal{O}_X(n) = \widetilde{\mathcal{A}(n)}$ for any $n \in \mathbb{Z}$.

If $\mathcal{A}$ is generated by degree one elements then the sheaves $\mathcal{O}_X(n)$ are invertible for any integer $n$ and very ample if $n$ is positive. Furthermore, there is then a canonical isomorphism $\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n)$. All these properties may fail if $\mathcal{A}$ is not generated by degree one elements.

It can however be shown, cf. Corollary (2.2.4), that if $S$ is quasi-compact then $q : X \to S$ is projective. To be precise, there is a positive integer $n$ such that $\mathcal{O}_X(n)$ is invertible, the homomorphism $q^*\mathcal{A}_n \to \mathcal{O}_X(n)$ is surjective and $i_n : X \to \mathbb{P}(\mathcal{A}_n)$ is a closed immersion. In particular, $\mathcal{O}_X(n) = i_n^*\mathcal{O}_{\mathbb{P}(\mathcal{A}_n)}(1)$ is very ample. Another consequence is that if $X$ is a weighted projective scheme over an arbitrary scheme $S$ then $X \to S$ is proper.

We will give a demonstration of the projectivity of $X \to S$ when $S$ is quasi-compact and also show some properties of the sheaves $\mathcal{O}_X(n)$. The results are somewhat weaker than those in [BR86, §4] but we also give stronger results when $X$ is covered in degree one.

The following lemma is an explicit form of [EGA$_{\mathrm{II}}$, Lem. 2.1.6].

**Lemma (2.2.2).** *If $B$ is a graded $A$-algebra generated by elements $f_1, f_2, \dots, f_s \in B$ of degrees $d_1, d_2, \dots, d_s$ and $l$ is the least common multiple of $d_1, d_2, \dots, d_s$ then*

(i) *$B_{n+l} = (B_n B_l)$ for every $n \geq (s-1)(l-1)$.*
(ii) *$B_{kn} = (B_n)^k$ for every $k \geq 0$ if $n = al$ with $a \geq s-1$.*

*Proof.* Clearly $B_k$ is generated by $f_1^{a_1} f_2^{a_2} \dots f_s^{a_s}$ such that $\sum_i a_i d_i = k$. Let $g_i = f_i^{l/d_i} \in B_l$. If $k \geq s(l-1)+1$ and $f = f_1^{a_1} f_2^{a_2} \dots f_s^{a_s} \in B_k$ then $g_i | f$ for some $i$ which shows (i). (ii) follows easily from (i). □

**Proposition (2.2.3)** (cf. [BR86, Cor. 4A.5, Thm. 4B.7])**.** *Let $A$ be a ring and let $B$ be a graded $A$-algebra generated by a finite number of elements $f_1, f_2, \dots, f_s$ of degrees $d_1, d_2, \dots, d_s$. Let $l$ be the least common multiple of the $d_i$'s. Let $S = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(B)$ and $\mathcal{O}_X(n) = \widetilde{B(n)}$. Then*

(i) *$X = \bigcup_{f \in B_n} \mathrm{D}_+(f)$ if $n = al$ and $a \geq 1$.*
(ii) *$\mathcal{O}_X(n)$ is invertible if $n = al$ and $a \in \mathbb{Z}$.*
(iii) *$\mathcal{O}_X(n)$ is ample and generated by global sections if $n = al$ and $a \geq 1$.*
(iv) *The canonical homomorphism $\mathcal{O}_X(al) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(al+n)$ is an isomorphism for every $a, n \in \mathbb{Z}$.*
(v) *If $n = al$ with $a \geq 1$ then there is a canonical morphism $i_n : X \to \mathbb{P}(B_n)$. If $a \geq \max\{1, s-1\}$ then $i_n$ is a closed immersion and $\mathcal{O}_X(n) = i_n^*\mathcal{O}_{\mathbb{P}(B_n)}(1)$ is very ample relative to $S$.*
(vi) *$\mathcal{O}_X(n)$ is generated by global sections if $n \geq (s-1)(l-1)$.*

*Proof.* (i) is trivial as $X = \bigcup_{i=1}^{s} \mathrm{D}_+(f_i) = \bigcup_{i=1}^{s} \mathrm{D}_+\big(f_i^{al/d_i}\big)$ if $a \geq 1$, cf [EGA$_{\mathrm{II}}$, Cor. 2.3.14]. Note that if $f \in B_l$ then

$$B_f = \big(B_{(f)} \oplus B(1)_{(f)} \oplus \dots \oplus B(l-1)_{(f)}\big)[f, f^{-1}]. \tag{2.2.3.1}$$

Thus $\Gamma\big(\mathrm{D}_+(f), \mathcal{O}_X(al)\big) = B(al)_{(f)} = B_{(f)}f^a$ is a free $B_{(f)}$-module of rank one which shows (ii).

(iii) If $a \geq 1$ then $\big(\mathrm{D}_+(f)\big)_{f \in B_{al}}$ is an affine cover of $X$. As $\mathcal{O}_X(al)$ is an invertible sheaf it is thus generated by global sections and ample by definition, cf. [EGA$_{\mathrm{II}}$, Def. 4.5.3 and Thm. 4.5.2 a$'$)].

(iv) It is enough to show that the homomorphism $\mathcal{O}_X(al) \otimes \mathcal{O}_X(n) \to \mathcal{O}_X(al+n)$ is an isomorphism locally over $\mathrm{D}_+(f)$ with $f \in B_l$. Locally this homomorphism is $B(al)_{(f)} \otimes_{B_{(f)}} B(n)_{(f)} \to B(al+n)_{(f)}$ which is an isomorphism by equation (2.2.3.1)

(v) If $n = al$ with $a \geq 1$ then by (i) the morphism $i_n : X \to \mathbb{P}(B_n)$ is everywhere defined. If in addition $a \geq s-1$ then $B^{(n)}$ is generated by degree one elements by Lemma (2.2.2) (ii). Thus we have a closed immersion $X = \mathrm{Proj}(B) \cong \mathrm{Proj}(B^{(n)}) \hookrightarrow \mathbb{P}(B_n)$.

(vi) Assume that $n \geq (s-1)(l-1)$, then $B_{n+kl} = (B_n B_l^k)$ for any positive integer $k$ by Lemma (2.2.2) (i). If $f \in B_l$ and $b \in B(n)_{(f)}$, then $b = b'/f^k$ for some $b' \in B_{n+kl} = \big(B_n B_l^k\big)$ and thus $b \in \big(B_{(f)} B_n\big)$. This shows that $\mathcal{O}_X(n)$ is generated by global sections as $B_n \subseteq \Gamma\big(\mathrm{D}_+(f), \mathcal{O}_X(n)\big)$. ☐

**Corollary (2.2.4)** ([EGA$_{\mathrm{II}}$, Cor. 3.1.11])**.** *If $S$ is quasi-compact and $X = \mathrm{Proj}_S(\mathcal{A})$ is a weighted projective scheme then there exists a positive integer $n$ such that $X \to \mathbb{P}(\mathcal{A}_n)$ is everywhere defined and a closed immersion. In particular $X$ is projective and $\mathcal{O}_X(n)$ is very ample relative to $S$.*

*Proof.* Let $\{S_i\}$ be a finite affine cover of $S$ and let $A_i = \Gamma(S_i, \mathcal{O}_S)$ and $B_i = \Gamma(S_i, \mathcal{A})$. Then as $B_i$ is a finitely generated graded $A_i$-algebra, there is by Proposition (2.2.3) a positive integer $n_i$ such that $X \times_S S_i \to \mathbb{P}\big((B_i)_{n_i}\big)$ is defined and a closed immersion. Choosing $n$ as the least common multiple of the $n_i$'s we obtain a closed immersion $X \hookrightarrow \mathbb{P}(\mathcal{A}_n)$. ☐

*Remark* **(2.2.5).** Note that (2.2.3) (iv), (v), (vi) implies that the following are equivalent:

(i) $\mathcal{O}_X(n)$ is invertible for all $0 < n < l$.
(ii) $\mathcal{O}_X(n)$ is invertible for all $n$.
(iii) $\mathcal{O}_X(n)$ is very ample for all sufficiently large $n$.

As (i) is easily seen to not hold in many examples in particular (iii) is not always true.

The following condition will be important later on as it is satisfied for $\mathrm{Sym}^d(X/S)$ for $X/S$ quasi-projective. Note that in the remainder of this section we do not assume that $\mathcal{A}$ is finitely generated. In particular, $\mathrm{Proj}_S(\mathcal{A})$ need not be a weighted projective space.

**Definition (2.2.6).** Let $S$ be a scheme, $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra and $X = \mathrm{Proj}_S(\mathcal{A})$. If there is an affine cover $(S_\alpha)$ of $S$ such that $X \times_S S_\alpha$ is covered by $\bigcup_{f \in \Gamma(S_\alpha, \mathcal{A}_1)} \mathrm{D}_+(f)$, then we say that $X/S$ is *covered in degree one.*

**Proposition (2.2.7).** *Let $A$ be a ring and let $B$ be a graded $A$-algebra generated by elements of degree $\leq d$. Let $S = \mathrm{Spec}(A)$, $X = \mathrm{Proj}(B)$ and $\mathcal{O}_X(n) = \widetilde{B(n)}$. If $X/S$ is covered in degree one then*

(i) $X = \bigcup_{f \in B_n} \mathrm{D}_+(f)$ *if* $n \geq 1$.
(ii) $\mathcal{O}_X(n)$ *is invertible for $n \in \mathbb{Z}$ and ample and generated by global sections if $n \geq 1$.*
(iii) $\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$ *for every $m, n \in \mathbb{Z}$.*
(iv) *The canonical morphism $i_n : X \to \mathbb{P}(B_n)$ is defined for every $n \geq 1$. If $n \geq d$ then $i_n$ is a closed immersion and $\mathcal{O}_X(n) = i_n^* \mathcal{O}_{\mathbb{P}(B_n)}(1)$ is very ample relative $S$.*

*Proof.* (i) is equivalent to $X/S$ being covered in degree one. Using the cover $X = \bigcup_{f \in B_1} \mathrm{D}_+(f)$ instead of the cover $X = \bigcup_{f \in B_l} \mathrm{D}_+(f)$ we may then prove (ii) and (iii) exactly as (ii), (iii) and (iv) in Proposition (2.2.3).

(iv) Let $n \geq d$ and let $B'$ be the sub-$A$-algebra of $B$ generated by $B_n$. It is enough to show that the inclusion $B' \hookrightarrow B$ induces an isomorphism $\mathrm{Proj}(B) \cong \mathrm{Proj}(B')$. We will show this using the cover $X = \bigcup_{f \in B_1} \mathrm{D}_+(f^n)$. Let $f \in B_1$ and $g \in B_{(f^n)}$ such that $g = b/f^{nk}$ for some $b \in B_{nk}$. To show that $g \in B'_{(f^n)}$ we can assume that $b = b_1 b_2 \dots b_s$ is a product of elements of degrees $d_i \leq d$, as every element of $B_{nk}$ are sums of such. Then $g = \left(\prod_{i=1}^{s} b_i f^{n-d_i}\right) / f^{ns}$ which is an element of $B'_{(f^n)}$. □

**Corollary (2.2.8).** *Let $S$ be any scheme and $\mathcal{A}$ a graded quasi-coherent $\mathcal{O}_S$-algebra such that $\mathcal{A}$ is generated by elements of degree at most $d$. Let $X = \mathrm{Proj}(\mathcal{A})$, $\mathcal{O}_X(n) = \widetilde{\mathcal{A}(n)}$ and assume that $X/S$ is covered in degree one. Then*

(i) $\mathcal{O}_X(n) = \mathcal{O}_X(1)^{\otimes n}$ *is invertible for every $n \in \mathbb{Z}$.*
(ii) *If $n \geq 1$ then $\mathcal{O}_X(n)$ is ample and $q^* \mathcal{A}_n \to \mathcal{O}_X(n)$ is surjective.*
(iii) *For every $n \geq 1$ the canonical morphism $i_n : X \to \mathbb{P}(\mathcal{A}_n)$ is everywhere defined. If $n \geq d$ it is a closed immersion.*

*In particular, if $X = \mathrm{Proj}_S(\mathcal{A})$ also is a weighted projective scheme, i.e., if $\mathcal{A}$ is of finite type, then $X$ is projective.*

**Example (2.2.9)** (Standard weighted projective spaces)**.** Let $A = k$ be an algebraically closed field of characteristic zero and $B = k[x_0, x_1, \dots, x_r]$. Choose positive integers $d_0, d_1, \dots, d_r$ and consider the action of $G = \mu_{d_0} \times \cdots \times \mu_{d_r} \cong \mathbb{Z}/d_0\mathbb{Z} \times \cdots \times \mathbb{Z}/d_r\mathbb{Z}$ on $B$ given by $(n_0, n_1, \dots, n_r) \cdot x_i = \xi_{d_i}^{n_i} x_i$ where $\xi_{d_i}$ is a ${d_i}^{\text{th}}$ primitive root of unity. Then $B^G = k[x_0^{d_0}, x_1^{d_1}, \dots, x_r^{d_r}]$ and $\mathrm{Proj}\left(B^G\right)$ is a *weighted projective space* of type $(d_0, d_1, \dots, d_r)$.

It can be seen, cf. Proposition (2.3.4), that $\mathrm{Proj}\left(B^G\right)$ is the geometric quotient of $\mathrm{Proj}(B) = \mathbb{P}^r$ by $G$. More generally, if $S$ is a noetherian scheme and $X/S$ is projective with an action of a finite group $G$ linear with respect to a very ample sheaf $\mathcal{O}_X(1)$, then a geometric quotient $X/G$ exists and can be given a structure as a weighted projective scheme.

The weighted projective space $\operatorname{Proj}\left(B^G\right)$ is often denoted $\mathbb{P}(d_0, d_1, \dots, d_r)$. It can also be constructed as the quotient of $\mathbb{A}^{r+1} - 0$ by $\mathbb{G}_m$ where $\mathbb{G}_m$ acts on $\mathbb{A}^{r+1}$ by $\lambda \cdot x_i = \lambda^{d_i} x_i$. The closed points of $\mathbb{P}(d_0, d_1, \dots, d_r)$ are thus $\{x = (x_0 : x_1 : \dots : x_r)\} = k^{r+1}/\sim$ where $x \sim y$ if there is a $\lambda \in k^*$ such that $\lambda^{d_i} x_i = y_i$ for every $i$.

2.3. **Quotients of projective schemes by finite groups.** Let $X$ be an $S$-scheme and $G$ a discrete group acting on $X/S$, i.e., there is a group homomorphism $G \to \operatorname{Aut}_S(X)$. In the category of ringed spaces we can construct a quotient $Y = (X/G)_{\mathrm{rs}}$ as following. Let $Y$ as a topological space be $X/G$ with the quotient topology, and quotient map $q : X \to Y$. Further let the sheaf of sections $\mathcal{O}_Y$ be the subsheaf $(q_*\mathcal{O}_X)^G \hookrightarrow q_*\mathcal{O}_X$ of $G$-invariant sections. Note that $G$ acts on $q_*\mathcal{O}_X$ since for any open subset $U \subseteq Y$ the inverse image $q^{-1}(U)$ is $G$-stable and hence has an induced action of $G$. Thus we obtain a ringed $S$-space $(Y, \mathcal{O}_Y)$ together with a morphism of ringed $S$-spaces $q : X \to Y$. The ringed space $(Y, \mathcal{O}_Y)$ is not always a scheme, in fact not always even a locally ringed space. But when it exists as a scheme it is called the *geometrical quotient* and is also the *categorial quotient* in the category of schemes over $S$. For general existence results we refer to [Ryd07b]. The existence of a geometric quotient of an affine schemes by a finite group is not difficult to show:

**Proposition (2.3.1)** ([SGA$_1$, Exp. V, Prop. 1.1, Cor. 1.5]). *Let $S$ be a scheme, $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras and $X = \operatorname{Spec}_S(\mathcal{A})$. An action of $G$ on $X/S$ induces an action of $G$ on $\mathcal{A}$. If $G$ is a finite group then $Y = \operatorname{Spec}_S\left(\mathcal{A}^G\right)$ is the geometric quotient of $X$ by $G$. If $S$ is* locally noetherian *and $X \to S$ is of finite type, then $Y \to S$ is of finite type.*

From this local result it is not difficult to show the following result:

**Theorem (2.3.2)** ([SGA$_1$, Exp. V, Prop. 1.8]). *Let $f : X \to S$ be a morphism of arbitrary schemes and $G$ a finite discrete group acting on $X$ by $S$-morphisms. Assume that every $G$-orbit of $X$ is contained in an affine open subset. Then the geometrical quotient $q : X \to Y = X/G$ exists as a scheme.*

It can also be shown, from general existence results, that if $X/S$ is separated then this is also a necessary condition [Ryd07b, Rmk. 4.9].

*Remark* **(2.3.3).** If $X \to S$ is quasi-projective, then every $G$-orbit is contained in an affine open set. In fact, we can assume that $S = \operatorname{Spec}(A)$ is affine and thus that we have an embedding $X \hookrightarrow \mathbb{P}^n_S$. For any orbit $Gx$ we can then choose a section $f \in \mathcal{O}_{\mathbb{P}^n}(m)$ for some sufficiently large $m$ such that $\mathrm{V}(f)$ does not intersect $Gx$. The affine subset $\mathrm{D}(f)$ then contains the orbit $Gx$. More generally [EGA$_{\mathrm{II}}$, Cor. 4.5.4] shows that every finite set, in particulary every $G$-orbit, is contained in an affine open set if $X/S$ is such that there exists an ample invertible sheaf on $X$ relative to $S$.

In Corollary (2.3.6) we will show that if $S$ is a *noetherian* scheme and $X \to S$ is (quasi-)projective, then so is $X/G \to S$. In fact if $X$ is projective we will give a weighted projective structure on $X/G$.

**Proposition (2.3.4).** *Let $S$ be a scheme and let $\mathcal{A} = \bigoplus_{d\geq 0} \mathcal{A}_d$ be a graded quasi-coherent $\mathcal{O}_S$-algebra, generated by degree one elements. Let $G$ be a finite group acting on $\mathcal{A}$ by graded $\mathcal{O}_S$-algebra automorphisms. Then $G$ acts on $X = \mathrm{Proj}_S(\mathcal{A})$ linearly with respect to $\mathcal{O}_X(1)$. As $X$ admits a very ample invertible sheaf relative to $S$, a geometric quotient $Y = X/G$ exists, cf. Remark* (2.3.3)*. There is an isomorphism $Y \cong \mathrm{Proj}_S\left(\mathcal{A}^G\right)$ and under this isomorphism, the quotient map $q : X \to Y$ is induced by $\mathcal{A}^G \hookrightarrow \mathcal{A}$.*

*Proof.* Everything is local over $S$ so we can assume that $S = \mathrm{Spec}(A)$, $\mathcal{A} = \widetilde{B}$ and $X = \mathrm{Proj}(B)$. We can cover $X$ by $G$-stable affine subsets of the form $\mathrm{D}_+(f)$ with $f \in B^G$ homogeneous. In fact, if $Z$ is a $G$-orbit of $X$ then the demonstration of [EGA$_{\mathrm{II}}$, Cor. 4.5.4] shows that there is a homogeneous $f' \in B$ such that $Z \subseteq \mathrm{D}_+(f')$. If we let $f = \prod_{\sigma\in G} \sigma(f')$, then $Z \subseteq \mathrm{D}_+(f)$ and $f \in B^G$ is homogeneous. Over such an open set we have that

$$X|_{\mathrm{D}_+(f)}/G = \mathrm{Spec}\big(\left(B_{(f)}\right)^G\big) = \mathrm{Spec}\big((B^G)_{(f)}\big) = \mathrm{Proj}\left(B^G\right)|_{\mathrm{D}_+(f)}.$$

It is thus clear that $Y = \mathrm{Proj}\left(B^G\right)$. □

*Remark* **(2.3.5).** Note that $\mathcal{A}^G$ is not always generated by $\mathcal{A}_1^G$ even though $\mathcal{A}$ is generated by $\mathcal{A}_1$. Also, if $S = \mathrm{Spec}(A)$ is affine and $\mathcal{A} = \widetilde{B}$, we may not be able to cover $X = \mathrm{Proj}(B)$ with $G$-stable affine subsets of the form $\mathrm{D}_+(f)$ with $f \in B_1^G$. This is demonstrated by example (2.2.9) if we choose $d_i > 1$ for some $i$.

**Corollary (2.3.6)** ([Knu71, Ch. IV, Prop. 1.5])**.** *Let $S$ be* noetherian*, $X \to S$ be projective (resp. quasi-projective) and $G$ a finite group acting on $X$ by $S$-morphisms. Then the geometrical quotient $X/G$ is projective (resp. quasi-projective).*

*Proof.* Let $X \hookrightarrow (X/S)^m = X \times_S X \times_S \cdots \times_S X$ be the closed immersion given by $x \to (\sigma_1 x, \sigma_2 x, \dots, \sigma_m x)$ where $G = \{\sigma_1, \sigma_2, \dots, \sigma_m\}$. As $X \to S$ is quasi-projective and $S$ is noetherian, there is an immersion $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module of finite type $\mathcal{E}$, see [EGA$_{\mathrm{II}}$, Prop. 5.3.2]. This immersion together with the immersion $X \hookrightarrow (X/S)^m$ given above, gives a $G$-equivariant immersion $X \hookrightarrow \left(\mathbb{P}_S(\mathcal{E})/S\right)^n$ if we let $G$ permute the factors of $\left(\mathbb{P}_S(\mathcal{E})/S\right)^n$. Following this immersion by the Segre embedding we get a $G$-equivariant immersion $f : X \hookrightarrow \mathbb{P}_S(\mathcal{E}^{\otimes m})$ where $G$ acts linearly on $\mathbb{P}_S(\mathcal{E}^{\otimes m})$, i.e., by automorphisms of $\mathcal{E}^{\otimes m}$.

Let $Y = \overline{f(X)}$ be the schematic image of $f$. As $Y$ is clearly $G$-stable we have an action of $G$ on $Y$ and a geometric quotient $q : Y \to Y/G$. Then, as $X \hookrightarrow Y$ is an open immersion and $q$ is open, we have that $X/G = (Y/G)|_{q(Y)}$. Thus it is enough to show that $Y/G$ is projective. Let $\mathcal{A} = \mathrm{S}(\mathcal{E}^{\otimes m})/I$ such that $Y = \mathrm{Proj}(\mathcal{A})$. Then there is an action of $G$ on $\mathcal{A}_1$ which induces the action $Y$. By Proposition (2.3.4) we have that $Y/G = \operatorname{Proj}\left(\mathcal{A}^G\right)$. The scheme $Y/G$ is a weighted projective scheme as $\mathcal{A}^G$ is an $\mathcal{O}_S$-algebra of finite type by Proposition (2.3.1). It then follows by Corollary (2.2.4) that $Y/G$ is projective. □

2.4. **Finite quotients, base change and closed subschemes.** A geometric quotient is always *uniform*, i.e., it commutes with flat base change [GIT, Rmk. (7), p. 9]. It is also a *universal topological quotient*, i.e., the fibers corresponds to the orbits and the quotient has the quotient topology and this holds after any base change. However, in positive characteristic a geometric quotient is not necessarily a *universal* geometric quotient, i.e., it need not commute with arbitrary base change. This is shown by the following example:

**Example (2.4.1).** Let $X = \operatorname{Spec}(B)$, $S = \operatorname{Spec}(A)$, $S' = \operatorname{Spec}(A/I)$ with $A = k[\epsilon]/\epsilon^2$ where $k$ is a field of characteristic $p > 0$, $B = k[\epsilon, x]/(\epsilon^2, \epsilon x)$ and $I = (\epsilon)$. We have an action of $G = \mathbb{Z}/p = \langle \tau \rangle$ on $B$ given by $\tau(x) = x + \epsilon$ and $\tau(\epsilon) = \epsilon$. Then $\tau(x^n) = x^n$ for all $n \geq 2$ and thus $B^G = k[\epsilon, x^2, x^3]/(\epsilon^2, \epsilon x^2, \epsilon x^3)$. Further, we have that $(B \otimes_A A')^G = k[x]$ and $B^G \otimes_A A' = k[x^2, x^3]$.

Recall that a morphism of schemes is a *universal homeomorphism* if the underlying morphism of topological spaces is a homeomorphism after any base change.

**Proposition (2.4.2)** ([EGA$_{\text{IV}}$, Cor. 18.12.11]). *Let $f : X \to Y$ be a morphism of schemes. Then $f$ is a universal homeomorphism if and only if $f$ is integral, universally injective and surjective.*

**Proposition (2.4.3).** *Let $X/S$ be a scheme with an action of a finite group $G$ such that every $G$-orbit of $X$ is contained in an affine open subset. Let $S' \to S$ be any morphism and let $X' = X \times_S S'$. Then geometric quotients $q : X \to X/G$ and $r : X' \to X'/G$ exists. Let $(X/G)' = (X/G) \times_S S'$. As $r$ is a categorical quotient we have a canonical morphism $X'/G \to (X/G)'$. This morphism is a universal homeomorphism.*

*Proof.* The geometric quotients $q$ and $r$ exists by Theorem (2.3.2). As $q$ and $r$ are universal topological quotients it follows that $X'/G \to (X/G)'$ is universally bijective. As $X' \to X'/G$ is surjective and $X' \to (X/G)'$ is universally open it follows that $X'/G \to (X/G)'$ is universally open and hence a universal homeomorphism. □

If $G$ acts on $X$ and $U \subseteq X$ is a $G$-stable open subscheme, then $U/G$ is an open subscheme of $X/G$. In fact, $U/G$ is the image of $U$ by the open morphism $q : X \to X/G$. If $Z \hookrightarrow X$ is a closed $G$-stable subscheme, then $Z/G$ is not always the image of $Z$ by $q$. In fact, $Z/G$ need not even be a subscheme of $X/G$. We have the following result:

**Proposition (2.4.4).** *Let $G$ be a finite group, $X/S$ a scheme with an action of $G$ such that the geometric quotient $q : X \to X/G$ exists. Let $Z \hookrightarrow X$ be a closed $G$-stable subscheme. Then the geometric quotient $r : Z \to Z/G$ exist. Let*

*$q(Z)$ be the scheme-theoretic image of the morphism $Z \hookrightarrow X \to X/G$. As $r$ is a categorical quotient, the morphism $Z \to q(Z) \hookrightarrow X/G$ factors canonically as $Z \to Z/G \to q(Z) \hookrightarrow X/G$. The morphism $Z/G \to q(Z)$ is a* schematically dominant universal homeomorphism.

*Proof.* As $Z/G$ and $q(Z)$ both are universal topological quotients of $Z$, the canonical morphism $Z/G \to q(Z)$ is universally bijective. Since $Z \to q(Z)$ is universally open and $Z \to Z/G$ is surjective we have that $Z/G \to q(Z)$ is universally open and thus a universal homeomorphism. Further as $Z \to q(Z)$ is schematically dominant the morphism $Z/G \to q(Z)$ is also schematically dominant. □

**Corollary (2.4.5).** *Let $G$ and $X/S$ be as in Proposition* (2.4.4)*. There is a canonical universal homeomorphism $(X_{\mathrm{red}})/G \to (X/G)_{\mathrm{red}}$.*

We can say even more about the exact structure of $Z/G \to q(Z)$. For ease of presentation we state the result in the affine case.

**Proposition (2.4.6).** *Let $A$ be a ring with an action by a finite group $G$ and let $I \subset A$ be a $G$-stable ideal. Let $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$. Then $Z/G = \mathrm{Spec}\big((A/I)^G\big)$ and $q(Z) = \mathrm{Spec}\left(A^G/I^G\right)$. We have an injection $A^G/I^G \hookrightarrow (A/I)^G$. If $f \in (A/I)^G$ then there is an $n \mid \mathrm{card}(G)$ such that $f^n \in A^G/I^G$. To be more precise we have that*

(i) *If $A$ is a $\mathbb{Z}_{(p)}$-algebra with $p$ a prime, e.g., a local ring with residue field $k$ or a $k$-algebra with $\mathrm{char}\, k = p$, then $n$ can be chosen as a power of $p$.*
(ii) *If $A$ is purely of characteristic zero, i.e., a $\mathbb{Q}$-algebra, then $A^G/I^G \hookrightarrow (A/I)^G$ is an isomorphism.*

*Proof.* Let $f \in A$ such that its image $\overline{f} \in A/I$ is $G$-invariant. To show that $\overline{f}^n \in A^G/I^G$ for some positive integer $n$ it is enough to show that $\overline{f}^n \in A^G/I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z})$. As $\mathbb{Z} \to \mathbb{Z}_{\mathfrak{p}}$ is flat, we have that

$$\begin{aligned}
A^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\
I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\
A^G/I^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G/(I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G \\
(A/I)^G \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}} &= (A/I \otimes_{\mathbb{Z}} \mathbb{Z}_{\mathfrak{p}})^G.
\end{aligned}$$

Thus we can assume that $A$ is a $\mathbb{Z}_{\mathfrak{p}}$-algebra.

Let $q$ be the characteristic exponent of $\mathbb{Z}_{\mathfrak{p}}/\mathfrak{p}\mathbb{Z}_{\mathfrak{p}}$, i.e., $q = p$ if $\mathfrak{p} = (p)$, $p > 0$ and $q = 1$ if $\mathfrak{p} = (0)$. Choose positive integers $k$ and $m$ such that $\mathrm{card}(G) = q^k m$ and $q \nmid m$ if $q \neq 1$. Then choose a Sylow subgroup $H$ of $G$ of order $q^k$, or $H = (e)$ if $q = 1$, and let $\sigma_1 H, \sigma_2 H, \dots, \sigma_m H$ be its cosets. Then

$$g = \frac{1}{m} \sum_{i=1}^{m} \prod_{\sigma \in \sigma_i H} \sigma(f)$$

is $G$-invariant and its image $\overline{g} \in A^G/I^G$ maps to $\overline{f}^{q^k} \in (A/I)^G$. □

Proposition (2.4.6) can also be extended to the case where $G$ is any reductive group [GIT, Lem. A.1.2].

*Remark* **(2.4.7).** Let $X/S$ be a scheme with an action of a finite group $G$ with geometric quotient $q : X \to X/G$. Then

(i) If $S$ is a $\mathbb{Q}$-scheme and $S' \to S$ is any morphism then $(X \times_S S')/G = X/G \times_S S'$.
(ii) If $S$ is arbitrary and $U \subseteq X$ is an open immersion then $U/G = q(U)$.
(iii) If $S$ is a $\mathbb{Q}$-scheme and $Z \hookrightarrow X$ is a closed immersion then $Z/G = q(Z)$.
(iv) If $S$ is a $\mathbb{Q}$-scheme then $(X/G)_{\mathrm{red}} = X_{\mathrm{red}}/G$.

(ii) follows from the uniformity of geometric quotients, (i) and (iv) follows from the universality of geometric quotients in characteristic zero and (iii) follows from Proposition (2.4.6).

Statement (iii) can also be proven as follows. We can assume that $X = \mathrm{Spec}(A)$ is affine. Then the homomorphism $A^G \hookrightarrow A$ has an $A^G$-module retraction, the Reynolds-operator $R$, given by $R(a) = \frac{1}{\mathrm{card}(G)} \sum_{\sigma \in G} \sigma(a)$. This implies that $A^G \hookrightarrow A$ is *universally injective*, i.e., injective after tensoring with any $A$-module $M$. In particular $A^G \hookrightarrow A$ is *cyclically pure*, i.e., $I^G A = I$, where $I^G = I \cap A^G$, for any ideal $I \subseteq A$. If we let $S = \mathrm{Spec}\left(A^G\right)$ and $S' = \mathrm{Spec}\left(A^G/I^G\right)$ then $Z = X \times_S S' = \mathrm{Spec}(A/I)$ and (iii) follows from (i).

## 3. The parameter spaces

In this section we define the symmetric product $\mathrm{Sym}^d(X/S)$, the scheme of divided powers $\Gamma^d(X/S)$, and the Chow scheme of zero-cycles $\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E}))$. We show that when $X/S$ is a projective bundle, then $\mathrm{Sym}^d(X/S)$ is projective and we essentially obtain a bound on the generators of the natural weighted projective structure, cf. Theorem (3.1.12). As a corollary, we show that if $X/S$ is projective then so is $\Gamma^d(X/S)$, cf. Theorem (3.2.10). If $\mathcal{A}$ is a graded $\mathcal{O}_S$-algebra and $X = \mathrm{Proj}(\mathcal{A})$ then $\Gamma^d(X/S) = \mathrm{Proj}(\bigoplus_{k\geq 0} \Gamma^d(\mathcal{A}_k))$ in analogy with the description $\mathrm{Sym}^d(X) = \mathrm{Proj}(\bigoplus_{k\geq 0} \mathrm{TS}^d(\mathcal{A}_k))$. The Chow scheme of $X = \mathrm{Proj}(\mathcal{A})$ is defined as the projective scheme corresponding to the subalgebra of $\bigoplus_{k\geq 0} \Gamma^d(\mathcal{A}_k)$ generated by degree one elements. The reduction of this scheme is the classical Chow variety.

### 3.1. **The symmetric product.**

**Definition (3.1.1).** Let $X$ be a scheme over $S$ and $d$ a positive integer. We let the symmetric group on $d$ letters $\mathfrak{S}_d$ act by permutations on $(X/S)^d = X \times_S X \times_S \cdots \times_S X$. When $X/S$ is separated, the geometric quotient of $(X/S)^d$ by the action of $\mathfrak{S}_d$ exists as an algebraic space [Ryd07b] and we denote it by $\mathrm{Sym}^d(X/S) := (X/S)^d/\mathfrak{S}_d$. The scheme (or algebraic space) $\mathrm{Sym}^d(X/S)$ is called the $d^{\text{th}}$ *symmetric product* of $X$ over $S$ and is also denoted $\mathrm{Symm}^d(X/S)$, $(X/S)^{(d)}$ or $X^{(d)}$ by some authors.

**Definition (3.1.2).** Let $X/S$ be a scheme. We say that $X/S$ is AF if the following condition is satisfied:

(AF) Every finite set of points $x_1, x_2, \dots, x_n$ over the same point $s \in S$ is contained in an affine open subset of $X$.

*Remark* **(3.1.3).** If $X$ has an ample sheaf relative to $S$, then $X/S$ is AF, cf. [EGA$_{\text{II}}$, Cor. 4.5.4]. It is also clear from [EGA$_{\text{II}}$, Cor. 4.5.4] that if $X/S$ is AF then so is $X \times_S S'/S'$ for any base change $S' \to S$. It can further be seen that if $X/S$ is AF then $X/S$ is separated.

*Remark* **(3.1.4).** Let $X/S$ be an AF-scheme and let $d$ be a positive integer. By Theorem (2.3.2) the geometric quotient $\mathrm{Sym}^d(X/S)$ is then a scheme. Let $(S_\alpha)$ be an affine cover of $S$ and let $(U_{\alpha\beta})$ be an affine cover of $X \times_S S_\alpha$ such that any set of $d$ points of $X$ lying over the same point $s \in S_\alpha$ is included in some $U_{\alpha\beta}$. Then $(U_{\alpha\beta}/S_\alpha)^d$ is an open cover of $(X/S)^d$ by affine schemes. Thus $\coprod_{\alpha,\beta} \mathrm{Sym}^d(U_{\alpha\beta}/S_\alpha) \to \mathrm{Sym}^d(X/S)$ is an open covering by affines.

In the remainder of this section we will study the symmetric product when $S = \mathrm{Spec}(A)$ is an affine scheme and $X/S$ is projective. We will use the following notation:

**Notation (3.1.5).** Let $A$ be a ring and let $B = \bigoplus_{k \geq 0} B_k$ be a graded $A$-algebra finitely generated by elements in degree one. Let $S = \mathrm{Spec}(A)$ and $X = \mathrm{Proj}(B)$ with very ample sheaf $\mathcal{O}_X(1) = \widetilde{B(1)}$ and canonical morphism $q : X \to S$.

Further we let $C = \bigoplus_{k \geq 0} \mathrm{T}^d_A(B_k) \subset \mathrm{T}^d_A(B)$. Then $(X/S)^d = \mathrm{Proj}(C)$ and $\mathrm{Proj}(C) \hookrightarrow \mathbb{P}(C_1) = \mathbb{P}\big(\mathrm{T}^d_A(B_1)\big)$ is the Segre embedding of $(X/S)^d$ corresponding to the embedding $X = \mathrm{Proj}(B) \hookrightarrow \mathbb{P}(B_1)$. The permutation of the factors induces an action of the symmetric group $\mathfrak{S}_d$ on $C$ and we let $D = C^{\mathfrak{S}_d} = \bigoplus_{k \geq 0} \mathrm{TS}^d_A(B_k)$ be the graded invariant ring.

By Proposition (2.3.4) we have that $\mathrm{Sym}^d(X/S) := \mathrm{Proj}(C)/\mathfrak{S}_d = \mathrm{Proj}(D)$. If $A$ is noetherian, then $D$ is finitely generated and $\mathrm{Sym}^d(X/S)$ is a weighted projective space. In general, however, we do not know that $D$ is finitely generated. We do know that $\mathrm{Sym}^d(X/S) \to S$ is universally closed though, as $(X/S)^d \to S$ is projective.

**Lemma (3.1.6).** *Let $x_1, x_2, \dots, x_d \in X$ be points such that $q(x_1) = q(x_2) = \dots = q(x_d) = s$. Then there exists a positive integer $n$ and an element $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ such that $x_1, x_2, \dots, x_d \in X_f = \mathrm{D}_+(f)$. If the residue field $k(s)$ has at least $d$ elements then it is possible to take $n = 1$.*

*Proof.* The existence of $f$ for some $n$ follows from [EGA$_{\text{II}}$, Cor. 4.5.4]. For the last assertion, assume that $k(s)$ has at least $d$ elements. As we can lift any element $\overline{f} \in B_n \otimes_A k(s)$ to an element $f \in B_n$ after multiplying with an invertible element of $k(s)$, we can assume that $A = k(s)$. Replacing $B$ with the symmetric algebra

$\mathrm{S}(B_1) = k[t_0, t_1, \dots, t_r]$ we can further assume that $B$ is a polynomial ring and $X = \mathbb{P}^r_{k(s)}$.

An element of $B_1 = \Gamma\big(X, \mathcal{O}_X(1)\big)$ is then a linear form $f = a_0t_0+a_1t_1+\dots+a_rt_r$ with $a_i \in k(s)$ and can be thought of as a $k(s)$-rational point of $(\mathbb{P}^r_{k(s)})^\vee$. The linear forms which are zero in one of the $x_i$'s form a proper closed linear subset of all linear forms. Thus if $k(s)$ is infinite then there is a $k(s)$-rational point corresponding to a linear form non-zero in every $x_i$. If $k = k(s)$ is finite, then at most $(|k|^r - 1)/|k^*|$ linear forms are zero at a certain $x_i$ and equality holds when $x_i$ is $k$-rational. Thus at most

$$\begin{aligned} d(|k|^r - 1)/(|k| - 1) &\leq (|k|^{r+1} - |k|)/(|k| - 1) \\ &= (|k|^{r+1} - 1)/(|k| - 1) - 1 \end{aligned}$$

linear forms contain at least one of the $x_1, x_2, \dots, x_d$ and hence there is at least one linear form which does not vanish on any of the points. □

**Proposition (3.1.7).** *The product $X^d = X \times_S X \times_S \dots \times_S X$ is covered by $\mathfrak{S}_d$-stable affine open subsets of the form $X_f \times_S X_f \times_S \dots \times_S X_f$ where $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ for some $n$. If every residue field of $S$ has at least $d$ elements then the open subsets with $f \in B_1 \subseteq \Gamma\big(X, \mathcal{O}_X(1)\big)$ cover $X^d$.*

*Proof.* Follows immediately from Lemma (3.1.6). □

**Corollary (3.1.8).** *The symmetric product $\mathrm{Sym}^d(X/S)$ is covered by open affine subsets $\mathrm{Sym}^d\big(X_f/S\big)$ with $f \in B_n \subseteq \Gamma\big(X, \mathcal{O}_X(n)\big)$ for some $n$. If every residue field of $S$ has at least $d$ elements then those affine subsets with $n = 1$ cover $\mathrm{Sym}^d(X/S)$.*

**Corollary (3.1.9).** *The symmetric product $Y = \mathrm{Sym}^d(X/S) = \mathrm{Proj}(D)$ is covered by $Y_g$ where $g \in D_1 \subseteq \Gamma\big(Y, \mathcal{O}_Y(1)\big)$, i.e., $Y = \mathrm{Proj}(D)$ is covered in degree one.*

*Proof.* Let $A \hookrightarrow A'$ be a finite flat extension such that every residue field of $A'$ has at least $d$ elements, e.g., the extension $A' = A \otimes_{\mathbb{Z}} \Lambda_d$ suffices by Lemma (1.2.3). Let $B' = B \otimes_A A'$ and $C' = C \otimes_A A'$ and let $D' = D \otimes_A A' = \bigoplus_{n\geq 0} \mathrm{TS}^d_A(B_n) \otimes_A A'$. Then $D' = \bigoplus_{n\geq 0} \mathrm{TS}^d_{A'}(B'_n)$ as $A \hookrightarrow A'$ is flat. Note that if $f' \in B'_n$ then $g' = f' \otimes f' \otimes \dots \otimes f' \in D'_n$ and $\mathrm{Sym}^d(X'_{f'}/S) = \mathrm{D}_+(g')$ as open subsets of $\mathrm{Sym}^d(X'/S')$. Thus Corollary (3.1.8) shows that $\sqrt{D'_1 D'_+} = D'_+$. As $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective it follows that $\sqrt{D_1 D_+} = D_+$. □

We now use the degree bound on the generators of $\mathrm{TS}^d_A(A[x_1, \dots, x_r])$ obtained in Corollary (1.3.6) to get something very close to a degree bound on the generators of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ when $B = A[x_0, x_1, \dots, x_r]$ is the polynomial ring.

**Proposition (3.1.10).** *Let $N$ be a positive integer and $D_{\leq N}$ be the subring of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ generated by elements of degree at most $N$. Then the inclusion $D_{\leq N} \hookrightarrow D$ induces a morphism $\psi_N : \mathrm{Proj}(D) \to \mathrm{Proj}(D_{\leq N})$. Further we have that:*

(i) *If $B = A[x_0, x_1, \dots, x_r]$ is a polynomial ring and $N \geq r(d-1)$ then $\psi_N$ is an isomorphism.*
(ii) *If $A$ is purely of characteristic zero, i.e., a $\mathbb{Q}$-algebra, then $\psi_N$ is an isomorphism for any $N$.*

*Proof.* By Corollary (3.1.9) the morphism $\psi_N$ is everywhere defined for $N \geq 1$. Let $A \hookrightarrow A'$ be a finite flat extension such that every residue field of $A'$ has at least $d$ elements, e.g., $A' = A \otimes_{\mathbb{Z}} \Lambda_d$ as in Lemma (1.2.3). If we let $C' = C \otimes_A A'$ then we have that $D' = D \otimes_A A' = C'^{\mathfrak{S}_d}$ and $D'_{\leq N} = D_{\leq N} \otimes_A A'$ as $A \hookrightarrow A'$ is flat. If $\psi'_N : \operatorname{Proj}(D') \to \operatorname{Proj}(D'_{\leq N})$ is an isomorphism then so is $\psi_N$ as $A \hookrightarrow A'$ is faithfully flat. Replacing $A$ with $A'$, it is thus enough to prove the corollary when every residue field of $S$ has at least $d$ elements. Hence we can assume that we have a cover of $\operatorname{Proj}(D)$ by $\mathrm{D}_+\big(f^{\otimes d}\big)$ with $f \in B_1$ by Corollary (3.1.8).

We have that $D_{(f^{\otimes d})} = \mathrm{TS}^d_A(B_{(f)})$ and this latter ring is generated by elements of degree $\leq \max\{r(d-1), 1\}$ for arbitrary $A$ and by elements of degree one when $A$ is purely of characteristic zero by Corollary (1.3.6). As noted in Remark (1.3.3) this implies that $D_{(f^{\otimes d})} = (D_{\leq N})_{(f^{\otimes d})}$ which shows (i) and (ii). □

**Corollary (3.1.11).** *Let $N$ be a positive integer and $D_N$ be the subring of $D^{(N)} = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_{Nk})$ generated by $\mathrm{TS}^d_A(B_N)$. Then the inclusion $D_N \hookrightarrow D^{(N)}$ induces a morphism $\psi_N : \operatorname{Proj}(D) \to \operatorname{Proj}(D_N)$. Further we have that:*

(i) *If $B = A[x_0, x_1, \dots, x_r]$ is a polynomial ring and $N \geq r(d-1)$ then $\psi_N$ is an isomorphism.*
(ii) *If $A$ is purely of characteristic zero, i.e., a $\mathbb{Q}$-algebra, then $\psi_N$ is an isomorphism for any $N$.*

*Proof.* Let $D_{\leq N}$ be the subring of $D = \bigoplus_{k\geq 0} \mathrm{TS}^d_A(B_k)$ generated by elements of degree at most $N$. As $\operatorname{Proj}(D)$ is covered in degree one by Corollary (3.1.9) then so is $\operatorname{Proj}(D_{\leq N})$. In fact, as $\operatorname{Proj}(D) \to \operatorname{Spec}(A)$ is universally closed, it follows that $\operatorname{Proj}(D) \to \operatorname{Proj}(D_{\leq N})$ is surjective. By Proposition (2.2.7) (iv) it then follows that $D_N \hookrightarrow (D_{\leq N})^{(N)}$ induces an isomorphism $\operatorname{Proj}(D^{(N)}_{\leq N}) \to \operatorname{Proj}(D_N)$. The corollary thus follows from Proposition (3.1.10). □

**Theorem (3.1.12).** *Let $S$ be any scheme and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-sheaf of finite type. Then for any $N \geq 1$, there is a canonical morphism*

$$\operatorname{Sym}^d(\mathbb{P}(\mathcal{E})/S) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N \mathcal{E})\big).$$

*If $\mathcal{L}$ is a locally free $\mathcal{O}_S$-sheaf of constant rank $r+1$ then the canonical morphism $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N \mathcal{L})\big)$ is a closed immersion for $N \geq r(d-1)$. In particular, it follows that $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \to S$ is strongly projective.*

*Proof.* The existence of the morphism follows by Corollary (3.1.11). Part (i) of the same corollary shows that $\operatorname{Sym}^d(\mathbb{P}(\mathcal{L})/S) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N \mathcal{L})\big)$ is a closed immersion when $N \geq r(d-1)$. As $\mathrm{S}^N\mathcal{L}$ is locally free of constant rank it follows by paragraph (1.1.7) that $\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{L})$ is locally free of constant rank which shows that $\mathrm{Sym}^d(\mathbb{P}(\mathcal{L})/S)$ is strongly projective. □

In section 3.3, we will show that the canonical morphism $\mathrm{Sym}^d(\mathbb{P}(\mathcal{E})/S) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^N\mathcal{E})\big)$ is a universal homeomorphism onto its image which is defined to be the Chow scheme $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$.

3.2. **The scheme of divided powers.** Let $S$ be any scheme and $\mathcal{A}$ a quasi-coherent sheaf of $\mathcal{O}_S$-algebras. As the construction of $\Gamma^d_A(B)$ commutes with localization with respect to multiplicatively closed subsets of $A$ we may define a quasi-coherent sheaf of $\mathcal{O}_S$-algebras $\Gamma^d_{\mathcal{O}_S}(\mathcal{A})$. We let $\Gamma^d(\mathrm{Spec}(\mathcal{A})/S) = \mathrm{Spec}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{A})\big)$. The scheme $\Gamma^d(X/S)$ is thus defined for any scheme $X$ affine over $S$. Similarly we obtain for any homomorphism of quasi-coherent $\mathcal{O}_S$-algebras $\mathcal{A} \to \mathcal{B}$ a morphism of schemes $\Gamma^d(\mathrm{Spec}(\mathcal{B})/S) \to \Gamma^d(\mathrm{Spec}(\mathcal{A})/S)$. This defines a covariant functor $X \mapsto \Gamma^d(X/S)$ from affine schemes over $S$ to affine schemes over $S$.

It is more difficult to define $\Gamma^d(X/S)$ for any $X$-scheme $S$ since $\Gamma^d_A(B)$ does not commute with localization with respect to $B$. In fact, it is not even a $B$-algebra. In [**I**, 3.1] a certain functor $\underline{\Gamma}^d_{X/S}$ is defined which is represented by $\Gamma^d(X/S)$ when $X/S$ is affine. When $X/S$ is quasi-projective, or more generally an AF-scheme, cf. Definition (3.1.2), then $\underline{\Gamma}^d_{X/S}$ is represented by a scheme [**I**, Thm. 3.1.11]. If $X/S$ is a separated algebraic space, then $\underline{\Gamma}^d_{X/S}$ is represented by a separated algebraic space [**I**, Thm. 3.4.1].

The object representing $\underline{\Gamma}^d_{X/S}$ will be denoted by $\Gamma^d(X/S)$. We briefly state some facts about $\Gamma^d(X/S)$ used in the other sections. We then show that $\Gamma^d(X/S)$ is (quasi-)projective when $X/S$ is (quasi-)projective.

**(3.2.1)** *The space of divided powers* — For any algebraic scheme $X$ separated above $S$, there is an algebraic space $\Gamma^d(X/S)$ over $S$ with the following properties:

(i) For any morphism $S' \to S$, there is a canonical base-change isomorphism $\Gamma^d(X/S) \times_S S' \cong \Gamma^d(X \times_S S'/S')$.

(ii) If $X/S$ is an AF-scheme, then $\Gamma^d(X/S)$ is an AF-scheme.

(iii) If $\mathcal{A}$ is a quasi-coherent sheaf on $S$ such that $X = \mathrm{Spec}_S(\mathcal{A})$ is affine $S$, then $\Gamma^d(X/S) = \mathrm{Spec}_S\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{A})\big)$ is affine over $S$.

(iv) If $X = \coprod_{i=1}^n X_i$ then $\Gamma^d(X/S)$ is the disjoint union

$$\coprod_{\substack{d_1,d_2,\dots,d_n \geq 0 \\ d_1+d_2+\dots+d_n=d}} \Gamma^{d_1}(X_i/S) \times_S \Gamma^{d_2}(X_2/S) \times_S \dots \times_S \Gamma^{d_n}(X_n/S).$$

(v) If $X \to S$ has one of the properties: finite type, finite presentation, locally of finite type, locally of finite presentation, quasi-compact, finite, integral, flat; then so has $\Gamma^d(X/S) \to S$.

This is Thm. 3.1.11, Prop. 3.1.4, Prop. 3.1.8 and Prop. 4.3.1 of [**I**].

**(3.2.2)** *Push-forward of cycles* — Let $f : X \to Y$ be any morphism of algebraic schemes separated over $S$. There is then a natural morphism, push-forward of cycles, $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ which for affine schemes is given by the covariance of the functor $\Gamma^d_A(\cdot)$. If $f : X \to Y$ is an immersion (resp. a closed immersion, resp. an open immersion) then $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ is an immersion (resp. closed immersion, resp. open immersion) [**I**, Prop. 3.1.7].

**(3.2.3)** *Addition of cycles* — Let $d, e$ be positive integers. The composition of the open and closed immersion $\Gamma^d(X/S) \times_S \Gamma^e(X/S) \hookrightarrow \Gamma^{d+e}(X \amalg X/S)$ given by (3.2.1) (iv) and the push-forward $\Gamma^{d+e}(X \amalg X/S) \to \Gamma^{d+e}(X/S)$ along the canonical morphism $X \amalg X \to X$ is called addition of cycles [**I**, Def. 4.1.1].

**(3.2.4)** *The Sym-Gamma morphism* — Let $X/S$ be a separated algebraic space and let $(X/S)^d = X \times_S X \times_S \cdots \times_S X$. There is an integral surjective morphism $\Psi_X : (X/S)^d \to \Gamma^d(X/S)$, given by addition of cycles, invariant under the permutation of the factors. This gives a factorization $(X/S)^d \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ and we denote the second morphism by $\mathrm{SG}_X$ [**I**, Prop. 4.1.5].

**(3.2.5)** *Local description of the scheme of divided powers* — If $U \subseteq X$ is an open subset, then we have already seen that $\Gamma^d(U/S) \subseteq \Gamma^d(X/S)$ is an open subset. Moreover, there is a cartesian diagram

$$\begin{tikzcd}
(U/S)^d \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sym}^d(U/S) \arrow[r, "\mathrm{SG}_U"] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma^d(U/S) \arrow[d] \\
(X/S)^d \arrow[r] & \mathrm{Sym}^d(X/S) \arrow[r, "\mathrm{SG}_X"] & \Gamma^d(X/S).
\end{tikzcd}$$

If $X/S$ is an AF-scheme, then there are Zariski-covers $S = \bigcup S_\alpha$ and $X = \bigcup U_\alpha$ of affine schemes such that $\Gamma^d(X/S) = \bigcup \Gamma^d(U_\alpha/S_\alpha)$. This gives a local description of the SG-map. For an arbitrary separated algebraic space $X/S$ there is a similar étale-local description of SG. If $U \to X$ is an étale morphism, then there is a cartesian diagram

$$\begin{tikzcd}
(U/S)^d|_{\mathrm{fpr}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Sym}^d(U/S)|_{\mathrm{fpr}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \Gamma^d(U/S)|_{\mathrm{reg}} \arrow[d] \\
(X/S)^d \arrow[r] & \mathrm{Sym}^d(X/S) \arrow[r] & \Gamma^d(X/S)
\end{tikzcd}$$

where the vertical arrows are étale [**I**, Prop. 4.2.4]. Here fpr and reg denotes the open locus where the corresponding maps are *fixed-point reflecting* and *regular*. If $\coprod U_\alpha \to X$ is an étale cover, then $\coprod \Gamma^d(U_\alpha/S)|_{\mathrm{fpr}} \to \Gamma^d(X/S)$ is an étale cover. Thus, we have an étale-local description of SG in affine schemes.

We now give a similar treatment of $\Gamma^d(X/S)$ for $X = \mathrm{Proj}(B)$ projective as that given for $\mathrm{Sym}^d(X/S)$ in the previous section.

**Proposition (3.2.6).** *Let $S = \mathrm{Spec}(A)$ where $A$ is affine and let $X = \mathrm{Proj}(B)$ where $B$ is a graded $A$-algebra finitely generated in degree one. Then $\Gamma(X/S)$ is covered by open subsets of the form $\Gamma^d(X_f/S)$ with $f \in B_n$ for some $n$. If every residue field of $S$ has at least $d$ elements, then is enough to consider open subsets with $n = 1$.*

*Proof.* Follows from Proposition (3.1.7) using that $\Psi_X((X_f/S)^d) = \Gamma^d(X_f/S)$. $\square$

**Proposition (3.2.7).** *Let $A$ be a ring and $B$ a graded $A$-algebra finitely generated in degree one. Let $W = \mathrm{Proj}\big(\bigoplus_{k\geq 0}\Gamma^d_A(B_k)\big)$. Then $W$ is covered by the open subsets of the form $W_{\gamma^d(b)}$ where $b \in B_n$ for some $n$. If every residue field of $S = \mathrm{Spec}(A)$ has at least $d$ elements, then is enough to consider open subsets with $b \in B_1$.*

*Proof.* Let $F$ be a graded flat $A$-algebra with a surjection $F \twoheadrightarrow B$. Consider the induced surjective homomorphism $\bigoplus_{k\geq 0}\Gamma^d_A(F_k) \twoheadrightarrow \bigoplus_{k\geq 0}\Gamma^d_A(B_k)$ and the corresponding closed immersion $W \hookrightarrow W' = \mathrm{Proj}\big(\bigoplus_{k\geq 0}\Gamma^d_A(F_k)\big)$. The open subset of $W' = \mathrm{Sym}^d(\mathrm{Proj}(F)/S)$ given by $\gamma^d(f) = 0$, where $f \in F_n$, coincides with the open subset $\mathrm{Sym}^d(\mathrm{Proj}(F)_f/S)$. These subsets cover $W'$ by Corollary (3.1.8). The proposition follows immediately. $\square$

**Corollary (3.2.8).** *Let $S$ be any scheme and let $\mathcal{A}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Then $\Gamma^d\big(\mathrm{Proj}(\mathcal{A})/S\big)$ and $W = \mathrm{Proj}\big(\bigoplus_{k\geq 0}\Gamma^d_A(\mathcal{A}_k)\big)$ are canonically isomorphic. Under this isomorphism, the open subset $\Gamma^d\big(\mathrm{Proj}(\mathcal{A})_f\big)$ is identified with $W_{\gamma^d(f)}$ for any homogeneous element $f \in \mathcal{A}$.*

*Proof.* By Propositions (3.2.6) and (3.2.7) the open subsets $\Gamma^d\left(\mathrm{Spec}\left(\mathcal{A}_{(f)}\right)\right)$ and $W_{\gamma^d(f)}$ for $f \in \mathcal{A}_n$, covers $\Gamma^d(\mathrm{Proj}(\mathcal{A}))$ and $W$ respectively. As these subsets are canonically isomorphic the corollary follows. $\square$

**Proposition (3.2.9).** *Let $S$ be any scheme and let $\mathcal{A}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Let $\mathcal{D} = \bigoplus_{k\geq 0}\Gamma^d_A(\mathcal{A}_k)$. Let $N$ be a positive integer and let $\mathcal{D}_N$ be the subring of $\mathcal{D}^{(N)} = \bigoplus_{k\geq 0}\Gamma^d_A(\mathcal{A}_{Nk})$ generated by $\Gamma^d_A(\mathcal{A}_N)$. The inclusion $\mathcal{D}_N \hookrightarrow \mathcal{D}^{(N)}$ induces a morphism $\psi_N : \mathrm{Proj}(\mathcal{D}) \to \mathrm{Proj}(\mathcal{D}_N)$. Furthermore*

(i) *If $\mathcal{A}$ is locally generated by at most $r+1$ elements and $N \geq r(d-1)$ then $\psi_N$ is an isomorphism.*
(ii) *If $S$ is purely of characteristic zero, i.e., a $\mathbb{Q}$-scheme, then $\psi_N$ is an isomorphism for every $N$.*

*Proof.* The statements are local on $S$ so we may assume that $S = \mathrm{Spec}(A)$ is affine and $\mathcal{A} = \widetilde{B}$ where $B$ is a graded $A$-algebra finitely generated in degree one. Choose a surjection $B' = A[x_0, x_1, \dots, x_r] \twoheadrightarrow B$. Let $D = \bigoplus_{k\geq 0}\Gamma^d_A(B_k)$, $D' = \bigoplus_{k\geq 0}\Gamma^d_A(B'_k)$ and let $D_N$ and $D'_N$ be the subrings of $D^{(N)}$ and $D'^{(N)}$ generated by degree one elements. Then we have a commutative diagram

$$
\begin{tikzcd}
D'_N \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\circ"] & D_N \arrow[d, hook] \\
D'^{(N)} \arrow[r, two heads] & D^{(N)}.
\end{tikzcd}
\tag{3.2.9.1}
$$

By Corollary (3.1.11) the inclusion $D'_N \hookrightarrow D'^{(N)}$ induces a morphism

$$\psi'_N : \operatorname{Proj}\left(D'^{(N)}\right) \to \operatorname{Proj}(D'_N)$$

having properties (i) and (ii). From the commutative diagram (3.2.9.1) it follows that the inclusion $D_N \hookrightarrow D^{(N)}$ induces a morphism $\psi_N : \operatorname{Proj}(D^{(N)}) \to \operatorname{Proj}(D_N)$ with the same properties. □

**Theorem (3.2.10).** *If $X \to S$ is strongly projective (resp. strongly quasi-projective) then $\Gamma^d(X/S) \to S$ is strongly projective (resp. strongly quasi-projective). If $X \to S$ is projective (resp. quasi-projective) and $S$ is quasi-compact and quasi-separated then $\Gamma^d(X/S) \to S$ is projective (resp. quasi-projective).*

*Proof.* In the strongly projective (resp. strongly quasi-projective) case we immediately reduce to the case where $X = \mathbb{P}_S(\mathcal{L})$ for some locally free $\mathcal{O}_S$-module $\mathcal{L}$ of finite rank $r+1$, using the push-forward (3.2.2), and the result follows from Theorem (3.1.12).

If $S$ is quasi-compact and quasi-separated and $X \to S$ is projective (resp. quasi-projective) then there is a closed immersion (resp. immersion) $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. It is enough to show that $\Gamma(\mathbb{P}_S(\mathcal{E}))$ is projective. As $\Gamma(\mathbb{P}_S(\mathcal{E})) = \operatorname{Proj}\bigl(\bigoplus_{k\geq 0} \Gamma^d(\mathrm{S}^k(\mathcal{E})\bigr)$ by Corollary (3.2.8), this follows from Proposition (3.2.9) and the quasi-compactness of $S$. □

3.3. **The Chow scheme.** Let $k$ be a field and let $E$ be a vector space over $k$ with basis $x_0, x_1, \dots, x_n$. Let $E^\vee$ be the dual vector space with dual basis $y_0, y_1, \dots, y_n$. Let $X = \mathbb{P}(E) = \mathbb{P}^n_k$. If $k'/k$ is a field extension then a point $x : \operatorname{Spec}(k') \to X$ is given by coordinates $(x_0 : x_1 : \dots : x_n)$ in $k'$. To $x$ we associate the *Chow form* $F_x(y_0, y_1, \dots, y_n) = \sum_{i=0}^n x_i y_i \in k'[y_0, y_1, \dots, y_n]$ which is defined up to a constant.

A zero-cycle on $X = \mathbb{P}^n_k$ is a formal sum of closed points. To any zero-dimensional subscheme $Z \hookrightarrow X$ we associate the zero-cycle $[Z]$ defined as the sum of its points with multiplicities. If $\mathcal{Z} = \sum_j a_j [z_j]$ is a zero-cycle on $X$ and $k'/k$ a field extension then we let $\mathcal{Z}_{k'} = \mathcal{Z} \times_k k' = \sum_j a_j [z_j \times_k k']$. It is clear that if $Z \hookrightarrow X$ is a zero-dimensional subscheme then $[Z] \times_k k' = [Z \times_k k']$.

We say that a cycle is effective if its coefficients are positive. The degree of a cycle $\mathcal{Z} = \sum_j a_j [z_j]$ is defined as $\deg(\mathcal{Z}) = \sum_j a_j \deg\bigl(k(z_j)/k\bigr)$. It is clear that $\deg(\mathcal{Z}_{k'}) = \deg(\mathcal{Z})$ for any field extension $k'/k$.

Let $\mathcal{Z}$ be an effective zero-cycle on $X$ and choose a field extension $k'/k$ such that $\mathcal{Z}_{k'} = \sum_j a_j [z'_j]$ is a sum of $k'$-points, i.e., $k(z'_j) = k'$. We then define its Chow form as $F_{\mathcal{Z}} = \prod_j F_{z'_j}^{a_j}$. It is easily seen that

(i) $F_{\mathcal{Z}}$ does not depend on the choice of field extension $k'/k$.
(ii) $F_{\mathcal{Z}}$ has coefficients in $k$.
(iii) The degree of $F_{\mathcal{Z}}$ coincides with the degree of $\mathcal{Z}$.
(iv) $\mathcal{Z}$ is determined by $F_{\mathcal{Z}}$.

Further, if $k$ is perfect there is a correspondence between zero-cycles of degree $d$ on $X$ and Chow forms of degree $d$, i.e., homogeneous polynomials, $F \in k[y_0, y_1, \dots, y_n]$ which splits into $d$ linear forms after a field extension. The Chow forms of degree $d$ with coefficients in $k$ is a subset of the linear forms on $\mathbb{P}\big(\mathrm{S}^d(E^\vee)\big)$ and thus a subset of the $k$-points of $\mathbb{P}\big(\mathrm{S}^d(E^\vee)^\vee\big) = \mathbb{P}\big(\mathrm{TS}^d(E)\big)$.

**(3.3.1)** *The Chow variety* — Classically it is shown that for $r \geq 0$ and $d \geq 1$ there is a closed subset of $\mathbb{P}\big(\mathrm{T}^{r+1}(\mathrm{TS}^d(E))\big)$ parameterizing $r$-cycles of degree $d$ on $\mathbb{P}(E)$. The Chow variety $\mathrm{Chow}_{r,d}\big(\mathbb{P}(E)\big)$ is then taken as the *reduced* scheme corresponding to this subset. More generally, if $S$ is any scheme and $\mathcal{L}$ is a locally free sheaf then there is a closed subset of $\mathbb{P}_S\big(\mathrm{T}^{r+1}(\mathrm{TS}^d(\mathcal{L}))\big)$ parameterizing $r$-cycles of degree $d$ on $\mathbb{P}_S(\mathcal{L})$.

We will now show that the classical Chow variety parameterizing zero-cycles of degree $d$ has a canonical closed *subscheme* structure. We begin with the case where $S$ is the spectrum of a field.

**(3.3.2)** *The Chow scheme for* $\mathbb{P}(E)/k$ — Let $k'/k$ be a field extension such that $k'$ is algebraically closed. As $(\mathbb{P}(E)/k)^d \to \mathrm{Sym}^d(\mathbb{P}(E)/k)$ is integral, it is easily seen that a $k'$-point of $\mathrm{Sym}^d(\mathbb{P}(E)/k)$ corresponds to an unordered tuple $(x_1, x_2, \dots, x_d)$ of $k'$-points of $\mathbb{P}(E)$. Assigning such a tuple the Chow form of the cycle $[x_1]+[x_2]+\dots+[x_d]$ gives a map $\mathrm{Hom}\big(k', \mathrm{Sym}^d(\mathbb{P}(E)/k)\big) \to \mathrm{Hom}\big(k', \mathbb{P}(\mathrm{TS}^d(E))\big)$. It is easily seen to be compatible with the homomorphism of algebras

$$\bigoplus_{k\geq 0} \mathrm{S}^k\big(\mathrm{TS}^d(E)\big) \to \bigoplus_{k\geq 0} \mathrm{TS}^d\big(\mathrm{S}^k(E)\big)$$

and thus extends to a morphism of schemes

$$\mathrm{Sym}^d\big(\mathbb{P}(E)/k\big) \to \mathbb{P}\big(\mathrm{TS}^d(E)\big).$$

It is further clear that the image of this morphism consists of the Chow forms of degree $d$ and that $\mathrm{Sym}^d\big(\mathbb{P}(E)/k\big) \to \mathbb{P}\big(\mathrm{TS}^d(E)\big)$ is universally injective and hence a universal homeomorphism onto its image as $\mathrm{Sym}^d\big(\mathbb{P}(E)/k\big)$ is projective. We let $\mathrm{Chow}_{0,d}\big(\mathbb{P}(E)\big)$ be the scheme-theoretical image of this morphism.

More generally, we define $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})/S\big)$ for any locally free sheaf $\mathcal{L}$ on $S$ as follows:

**Definition-Proposition (3.3.3).** *Let $S$ be a scheme and $\mathcal{L}$ a locally free $\mathcal{O}_S$-sheaf of finite type. Then the homomorphism $\bigoplus_{k\geq 0} \mathrm{S}^k\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L}) \to \bigoplus_{k\geq 0} \mathrm{TS}^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{L})$ induces a morphism*

$$\varphi_{\mathcal{L}} : \mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\big)$$

*which is a universal homeomorphism onto its image. We let* $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big)$ *be its scheme-theoretic image.*

*Proof.* The question is local so we can assume that $S = \mathrm{Spec}(A)$ and $\mathcal{L} = \widetilde{M}$ where $M$ is a free $A$-module of finite rank. Corollary (3.1.11), with $N = 1$ and $B = \bigoplus_{k\geq 0} \mathrm{S}^k M$, shows that $\bigoplus_{k\geq 0} \mathrm{S}^k \mathrm{TS}^d_A(M) \to \bigoplus_{k\geq 0} \mathrm{TS}^d_A(\mathrm{S}^k M)$ induces a well-defined morphism $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\big)$.

To show that $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\big)$ is a universal homeomorphism onto its image it is enough to show that it is universally injective as $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to S$ is universally closed. As $\mathcal{L}$ is flat over $S$ the symmetric product commutes with base change and it is enough to show that $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathbb{P}\big(\mathrm{TS}^d_{\mathcal{O}_S}(\mathcal{L})\big)$ is injective when $S$ is a field. This was discussed above. $\square$

If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is a closed immersion (resp. an immersion) then the subset of $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big)$ parameterizing cycles with support in $X$ is closed (resp. locally closed). In fact, it is the image of the morphism

$$\mathrm{Sym}^d(X/S) \to \mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big). \tag{3.3.3.1}$$

As $\mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) = \Gamma^d\big(\mathbb{P}(\mathcal{L})/S\big)$, this morphism factors through $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$. Moreover, as $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a homeomorphism [**I**, Cor. 4.2.5], the morphism

$$\Gamma^d(X/S) \to \mathrm{Sym}^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big). \tag{3.3.3.2}$$

has the same image as (3.3.3.1). Since $\Gamma^d$ is more well-behaved, e.g., commutes with base change $S' \to S$, the following definition is reasonable:

**Definition (3.3.4).** Let $S$ be any scheme and $\mathcal{L}$ a locally free sheaf on $S$. If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is a closed immersion we let $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{L})\big)$ be the scheme-theoretic image of $\Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{L})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big)$. If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is an immersion we let $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{L})\big)$ be the open subscheme of $\mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}(\mathcal{L})\big)$ corresponding to cycles with support in $X$.

*Remark* **(3.3.5).** Classically $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{L})\big)$ is defined as the reduced subscheme of $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big) \hookrightarrow \mathbb{P}\big(\mathrm{TS}^d(\mathcal{L})\big)$ parameterizing zero-cycles of degree $d$ with support in $X$. It is clear that this is the reduction of the scheme $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{L})\big)$ as defined in Definition (3.3.4).

*Remark* **(3.3.6).** If $\mathcal{L}$ is a locally free sheaf on $S$ of finite type then by definition $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{L})\big)$ is $\mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is the image of

$$\bigoplus_{k\geq 0} \mathrm{S}^k\big(\mathrm{TS}^d(\mathcal{L})\big) \to \bigoplus_{k\geq 0} \mathrm{TS}^d\big(\mathrm{S}^k(\mathcal{L})\big)$$

i.e., $\mathcal{B}$ is the subalgebra of $\bigoplus_{k\geq 0} \mathrm{TS}^d\big(\mathrm{S}^k(\mathcal{L})\big)$ generated by degree one elements. If $X \hookrightarrow \mathbb{P}(\mathcal{L})$ is a closed immersion then $X = \mathrm{Proj}(\mathcal{A})$ where $\mathcal{A}$ is a quotient of $\mathrm{S}(\mathcal{L})$.

The Chow scheme $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{L})\big)$ is then $\mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is the subalgebra of $\bigoplus_{k\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathcal{A}_k)$ generated by degree one elements, cf. Corollary (3.2.8).

**Proposition (3.3.7).** *Let $S$ be any scheme and let $\mathcal{B}$ be a graded quasi-coherent $\mathcal{O}_S$-algebra of finite type generated in degree one. Then there is a canonical morphism*

$$\varphi_{\mathcal{B}} : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big) \to \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\big)$$

*which is a universal homeomorphism onto its image. This morphism commutes with base change $S' \to S$ and surjections $\mathcal{B} \twoheadrightarrow \mathcal{B}'$.*

*Proof.* The existence of the morphism follows from Proposition (3.2.9). That $\varphi_{\mathcal{B}}$ is universally injective can be checked on the fibers and this is done in the beginning of this section. The last statements follows from the corresponding statements of the algebra of divided powers. $\square$

Remark (3.3.6) and Proposition (3.3.7) shows that there is a natural extension of the definition of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}_S(\mathcal{L})\big)$ which includes the case where $\mathcal{L}$ need not be locally free. In particular, we obtain a definition valid for arbitrary projective schemes:

**Definition (3.3.8).** Let $X/S$ be quasi-projective morphism of schemes and let $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ be an immersion for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Let $\overline{X}$ be the scheme-theoretic image of $X$ in $\mathbb{P}_S(\mathcal{E})$ which can be written as $\overline{X} = \mathrm{Proj}(\mathcal{B})$ where $\mathcal{B}$ is a quotient of $\mathrm{S}(\mathcal{E})$. We let $\mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\big)$ be the scheme-theoretic image of $\varphi_{\mathcal{B}} : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big) \to \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\big)$ or equivalently, the scheme-theoretic image of

$$\varphi_{\overline{X},\mathcal{E}} : \Gamma^d\big(\mathrm{Proj}(\mathcal{B})/S\big) \to \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{B}_1)\big) \hookrightarrow \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathcal{E})\big).$$

We let $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\big)$ be the open subscheme of $\mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\big)$ given by the image of

$$\Gamma^d(X/S) \subseteq \Gamma^d(\overline{X}/S) \to \mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}_S(\mathcal{E})\big).$$

This is indeed an open subscheme as $\Gamma^d\big(\overline{X}/S\big) \to \mathrm{Chow}_{0,d}\big(\overline{X} \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a homeomorphism by Corollary (3.3.7).

*Remark* **(3.3.9).** Let $S$ be any scheme, $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module and $X \hookrightarrow \mathbb{P}(\mathcal{E})$ an immersion. Let $S' \to S$ be any morphism and let $X' = X \times_S S'$ and $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'}$. There is a commutative diagram

$$\begin{tikzcd}
\Gamma^d(X'/S') \arrow[rr, "\varphi_{X',\mathcal{E}'}"] \arrow[d, "\cong"] & & \mathbb{P}_{S'}(\mathcal{E}') \arrow[d, "\cong"] \\
\Gamma^d(X/S) \times_S S' \arrow[rr, "\varphi_{X,\mathcal{E}} \times_S \mathrm{id}_{S'}"] \arrow[urr, phantom, "\circ"] & & \mathbb{P}_S(\mathcal{E}) \times_S S'
\end{tikzcd}$$

i.e., $\varphi_{X',\mathcal{E}'} = \varphi_{X,\mathcal{E}} \times_S \mathrm{id}_{S'}$. As the underlying sets of $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ and $\mathrm{Chow}_{0,d}\bigl(X' \hookrightarrow \mathbb{P}(\mathcal{E}')\bigr)$ are the images of $\varphi_{X,\mathcal{E}}$ and $\varphi_{X',\mathcal{E}'}$ it follows that the canonical morphism

$$\mathrm{Chow}_{0,d}\bigl(X' \hookrightarrow \mathbb{P}_{S'}(\mathcal{E}')\bigr) \hookrightarrow \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr) \times_S S' \tag{3.3.9.1}$$

is a nil-immersion, i.e., a bijective closed immersion. As the scheme-theoretic image commutes with flat base change [$\mathrm{EGA}_{\mathrm{IV}}$, Lem. 2.3.1] the morphism (3.3.9.1) is an isomorphism if $S' \to S$ is flat.

If $Z \hookrightarrow X$ is an immersion (resp. a closed immersion, resp. an open immersion) then there is an immersion (resp. a closed immersion, resp. an open immersion)

$$\mathrm{Chow}_{0,d}\bigl(Z \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr) \hookrightarrow \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}_S(\mathcal{E})\bigr).$$

**Proposition (3.3.10).** *Let $S = \mathrm{Spec}(A)$ where $A$ is affine and such that every residue field of $S$ has at least $d$ elements. Let $X = \mathrm{Proj}(B)$ where $B$ is a graded $A$-algebra finitely generated in degree one. Let $D = \bigoplus_{k\geq 0} \Gamma^d_A\bigl(B_k\bigr)$ and let $E \hookrightarrow D$ be the subalgebra generated by elements of degree one. Then $\mathrm{Chow}\bigl(X \hookrightarrow \mathbb{P}(B_1)\bigr) = \mathrm{Proj}(E)$ is covered by open subsets of the form $\mathrm{Spec}(E_{\gamma^d(f)})$ with $f \in B_1$. Furthermore, $E_{\gamma^d(f)}$ is the subalgebra of $\Gamma^d\bigl(B_{(f)}\bigr)$ generated by elements of degree one, i.e., elements of the form $\times_{i=1}^n \gamma^{d_i}(b_i/f)$ with $b_i \in B_1$.*

*Proof.* The first statement follows immediately from Proposition (3.2.7) taking into account that the inclusion $E \hookrightarrow D$ induces a surjective morphism $\mathrm{Proj}(D) \to \mathrm{Proj}(E)$ by Proposition (3.3.7). The last statement is obvious. $\square$

## 4. The relations between the parameter spaces

In this section we show that the morphisms

$$\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$$

are universal homeomorphisms with trivial residue field extensions. That the first morphism is a universal homeomorphisms with trivial residue field extensions is shown in [**I**]. We also briefly mention the construction of the morphism $\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S)$.

### 4.1. The Sym-Gamma morphism.

In this section we discuss some properties of the canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ defined in (3.2.4). Recall the following basic result:

**Proposition (4.1.1).** [**I**, Cor. 4.2.5] *Let $X/S$ be a separated algebraic space. The canonical morphism $\mathrm{SG}_X : \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$ is a universal homeomorphism with trivial residue field extensions. If $S$ is purely of characteristic zero or $X/S$ is flat, then $\mathrm{SG}_X$ is an isomorphism.*

From Proposition (4.1.1) we obtain the following results which only concerns $\mathrm{Sym}^d(X/S)$ but relies on the existence of the well-behaved functor $\Gamma^d$ and the morphism $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S)$.

**Corollary (4.1.2).** *Let $S \to S'$ be a morphism of schemes and $X/S$ a separated algebraic space. The induced morphism $\mathrm{Sym}^d(X'/S') \to \mathrm{Sym}^d(X/S) \times_S S'$ is a universal homeomorphism with trivial residue field extensions. If $S'$ is of characteristic zero then this morphism is an isomorphism. If $X'/S'$ is flat then the morphism is a nil-immersion.*

*Proof.* Follows from Proposition (4.1.1) and the commutative diagram

$$\begin{tikzcd} \mathrm{Sym}^d(X'/S') \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Sym}^d(X/S) \times_S S' \arrow[d] \\ \Gamma^d(X'/S') \arrow[r, "\cong"] & \Gamma^d(X/S) \times_S S'. \end{tikzcd}$$

□

**Corollary (4.1.3).** *Let $X/S$ be a separated algebraic space and $Z \hookrightarrow X$ a closed subscheme. Let $q : (X/S)^d \to \mathrm{Sym}^d(X/S)$ be the quotient morphism. The induced morphism $\mathrm{Sym}^d(Z/S) \to q\big((Z/S)^d\big)$ is a universal homeomorphism with trivial residue field extensions. If $S$ is of characteristic zero then this morphism is an isomorphism. If $Z/S$ is flat then the morphism is a nil-immersion.*

*Proof.* Follows from Proposition (4.1.1) and the commutative diagram

$$\begin{tikzcd} \mathrm{Sym}^d(Z/S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathrm{Sym}^d(X/S) \arrow[d] \\ \Gamma^d(Z/S) \arrow[r, hook] & \Gamma^d(X/S). \end{tikzcd}$$

□

Let us also mention the following result.

**Theorem (4.1.4).** *Let $A$ be any ring, let $B$ be an $A$-algebra and let $d$ be a positive integer. Let $\varphi : \Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ be the canonical homomorphism. Then*

(i) *If $x \in \ker(\varphi)$ then $d!x = 0$ and $x^{d!} = 0$.*
(ii) *If $y \in \mathrm{TS}^d_A(B)$ then $d!y \in \mathrm{im}(\varphi)$ and $y^{d!} \in \mathrm{im}(\varphi)$.*

*Proof.* This is (1.1.9) and [**II**, Cor. 4.7]. □

It is not difficult to prove that if every prime but $p$ is invertible in $A$, then $d!$ in the theorem can be replaced with the highest power of $p$ dividing $d!$.

**Examples (4.1.5).** The following examples are due to C. Lundkvist [Lun08]:

(i) An $A$-algebra $B$ such that $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is not injective
(ii) An $A$-algebra $B$ such that $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B)$ is not surjective
(iii) A surjection $B \to C$ of $A$-algebras such that $\mathrm{TS}^d_A(B) \to \mathrm{TS}^d_A(C)$ is not surjective
(iv) An $A$-algebra $B$ such that $\Gamma^d_A(B)_{\mathrm{red}} \hookrightarrow \mathrm{TS}^d_A(B)_{\mathrm{red}}$ is not an isomorphism

(v) An $A$-algebra $B$ and a base change $A \to A'$ such that the canonical homomorphism $\mathrm{TS}^d_A(B) \otimes_A A' \to \mathrm{TS}^d_{A'}(B')$ is not injective.
(vi) An $A$-algebra $B$ and a base change $A \to A'$ such that the canonical homomorphism $\mathrm{TS}^d_A(B) \otimes_A A' \to \mathrm{TS}^d_{A'}(B')$ is not surjective.

*Remark* **(4.1.6).** The *seminormalization* of a scheme $X$ is a universal homemomorphism with trivial residue fields $X^{\mathrm{sn}} \to X$ such that any universal homeomorphism with trivial residue field $X' \to X$ factors uniquely through $X^{\mathrm{sn}} \to X$ [Swa80]. If $X^{\mathrm{sn}} = X$ then we say that $X$ is seminormal. If $X \to Y$ is a morphism and $X$ is seminormal then $X \to Y$ factors canonically through $Y^{\mathrm{sn}} \to Y$.

Using Proposition (4.1.1) it can be shown that $\mathrm{Sym}^d(X/S)^{\mathrm{sn}} = \mathrm{Sym}^d(X^{\mathrm{sn}}/S^{\mathrm{sn}})^{\mathrm{sn}}$. Corollaries (4.1.2) and (4.1.3) then show that in the fibered category of seminormal schemes $\mathbf{Sch}^{\mathrm{sn}}$, taking symmetric products commutes with arbitrary base change and closed subschemes. This is a special property for $\mathrm{Sym}^d$ which does not hold for arbitrary quotients.

4.2. **The Gamma-Chow morphism.** Let us first restate the contents of Proposition (3.2.9) taking into account the definition of $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$.

**Proposition (4.2.1).** *Let $S$ be a scheme, $q : X \to S$ quasi-projective and $\mathcal{E}$ a quasi-coherent $\mathcal{O}_S$-module of finite type such that there is an immersion $X \hookrightarrow \mathbb{P}(\mathcal{E})$. Let $k \geq 1$ be an integer. Then*

(i) *The canonical map*
$$\mathrm{S}\big(\Gamma^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})\big) \to \bigoplus_{i\geq 0} \Gamma^d_{\mathcal{O}_S}(\mathrm{S}^{ki}\mathcal{E})$$
*induces a morphism*
$$\varphi_{\mathcal{E},k} : \Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathbb{P}\big(\Gamma^d_{\mathcal{O}_S}(\mathrm{S}^k\mathcal{E})\big)$$
*which is a universal homeomorphism onto its image. The scheme-theoretical image of $\varphi_{\mathcal{E},k}$ is by definition $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$.*

(ii) *Assume that either $\mathcal{E}$ is locally generated by at most $r+1$ elements and $k \geq r(d-1)$ or $S$ has pure characteristic zero, i.e., is a $\mathbb{Q}$-scheme. Then $\varphi_{\mathcal{E},k}$ is a closed immersion and $\Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\big)$ is an isomorphism.*

*Remark* **(4.2.2).** As $\Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is a universal homeomorphism, the *topology* of the Chow scheme does not depend on the chosen embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$.

In higher dimension, it is well-known that the Chow variety $\mathrm{Chow}_{r,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ does not depend on the embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ as a *set*. This follows from the fact that a geometric point corresponds to an $r$-cycle of degree $d$ [Sam55, §9.4d,h]. The invariance of the topology is also well-known, cf. [Sam55, §9.7]. This implies that the *weak normalization* of the Chow variety does not depend on the embedding in the analytic case, cf. [AN67]. This also follows from functorial descriptions of the

Chow variety over weakly normal schemes as in [Gue96] over $\mathbb{C}$ or more generally in [Kol96, §1.3]. We will now show that the residue fields of $\mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ do not depend on the embedding.

**Proposition (4.2.3).** *Let $S$, $q : X \to S$ and $\mathbb{P}(\mathcal{E})$ as in Proposition* (4.2.1)*. The morphism $\varphi_{\mathcal{E}} : \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ is a universal homeomorphism with trivial residue field extensions.*

*Proof.* We have already seen that the morphism $\varphi_{\mathcal{E}}$ is a universal homeomorphism. It is thus enough to show that it has trivial residue field extensions. To show this it is enough to show that for every point $a : \mathrm{Spec}(k) \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr)$ with $k = k^{\mathrm{sep}}$ there exists a, necessarily unique, point $b : \mathrm{Spec}(k) \to \Gamma^d(X/S)$ lifting $a$, i.e., the diagram

$$\begin{tikzcd} \Gamma^d(X/S) \arrow[r, "\varphi_{\mathcal{E}}"] & \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E})\bigr) \\ & \mathrm{Spec}(k) \arrow[u, "a"'] \arrow[ul, dotted, "b"] \end{tikzcd}$$

has a unique filling. By (3.2.1) (i) and Remark (3.3.9) the schemes $\Gamma^d(X/S)$ and $\bigl(\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E}))\bigr)_{\mathrm{red}}$ commute with base change, i.e.,

$$\Gamma^d(X/S) \times_S S' = \Gamma^d(X \times_S S'/S')$$
$$\bigl(\mathrm{Chow}_{0,d}(X \hookrightarrow \mathbb{P}(\mathcal{E})) \times_S S'\bigr)_{\mathrm{red}} = \mathrm{Chow}_{0,d}\bigl(X \times_S S' \hookrightarrow \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_S} \mathcal{O}_{S'})\bigr)_{\mathrm{red}}$$

for any $S' \to S$. We can thus assume that $S = \mathrm{Spec}(k)$ and hence that the image of $a$ is a closed point.

Let $r+1$ be the rank of $\mathcal{E}$. The point $a$ then corresponds to a Chow form $F_a \in k[y_0, y_1, \dots, y_r]$ which is homogeneous of degree $d$. Over $\overline{k} = k^{p^{-\infty}}$ this form factors into linear forms

$$F_a = F_1^{d_1} F_2^{d_2} \dots F_n^{d_n}$$

where $d = d_1 + d_2 + \dots + d_n$. Let $F_j = \sum_j x_i^{(j)} y_i$ and let $k\bigl(x^{(j)}\bigr) = k\bigl(x_0^{(j)}, \dots, k_r^{(j)}\bigr)$. If we let $d = p^e m$ such that $p \nmid m$, then $k\bigl(x^{(j)}\bigr)^{p^e} \subseteq k$ as $F_i^{d_i}$ is $k$-rational. Thus the exponent of $k\bigl(x^{(j)}\bigr)/k$ is at most $p^e$ and it follows by [**II**, Prop. 7.6] that $\Gamma^{d_j}(X/S)$ has a unique $k$-point $b_j$ corresponding to $F_j$. The $k$-point $b = b_1 + b_2 + \dots + b_n$ is a lifting of $a$. $\square$

*Remark* **(4.2.4).** Proposition (4.2.3) also follows from the following fact. Let $k$ be a field, $E$ a $k$-vector space and $X \hookrightarrow \mathbb{P}_k(E)$ a subscheme. Let $\mathcal{Z}$ be an $r$-cycle on $X$. The residue field of the point corresponding to $\mathcal{Z}$ in the Chow variety $\mathrm{Chow}_{r,d}\bigl(X \hookrightarrow \mathbb{P}(E)\bigr)$, the *Chow field* of $\mathcal{Z}$, does not depend on the embedding $X \hookrightarrow \mathbb{P}_k^n$ [Kol96, Prop-Def. I.4.4].

As $\varphi_{\mathcal{E}^{\otimes k}} : \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\bigl(X \hookrightarrow \mathbb{P}(\mathcal{E}^{\otimes k})\bigr)$ is an isomorphism for sufficiently large $k$ by Proposition (4.2.1) the Chow field coincides with the corresponding residue field of $\Gamma^d(X/S)$.

4.3. **The Hilb-Sym morphism.** The Hilbert-Chow morphism can be constructed in several different ways. Mumford [GIT, Ch. 5 §4] constructs a morphism

$$\mathrm{Hilb}^d(\mathbb{P}^n) \to \mathrm{Div}^d((\mathbb{P}^n)^\vee) \cong \mathbb{P}(\mathcal{O}_{(\mathbb{P}^n)^\vee}(d))$$

from the Hilbert scheme of $d$ points to Cartier divisors of degree $d$ on the dual space. This construction is a generalization of the construction of the Chow variety and it follows immediately that the image of this morphism is the Chow variety of $\mathbb{P}^n$. As the Chow variety $\mathrm{Chow}_{0,d}(\mathbb{P}^n)$ does not coincide with $\mathrm{Sym}^d(\mathbb{P}^n) = \Gamma^d(\mathbb{P}^n)$ in general, it is not clear that this construction lifts to a morphism to $\mathrm{Sym}^d(\mathbb{P}^n)$ or to $\Gamma^d(\mathbb{P}^n)$. Neeman solved this [Nee91] constructing a morphism $\mathrm{Hilb}^d(\mathbb{P}^n) \to \mathrm{Sym}^d(\mathbb{P}^n)$ directly without using Chow forms.

There is a natural way of constructing the "Hilbert-Chow" morphism due to Grothendieck [FGA] and Deligne [Del73]. There is a natural map $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ taking a flat family $Z \to T$ to its *norm family* [**II**]. We will call this map the *Grothendieck-Deligne norm map* and denote it with $\mathrm{HG}_X$. Using that the symmetric product coincide with the space of divided powers for $X/S$ flat, it follows by functoriality that the morphism $\mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ factors through the symmetric product. To be precise, we have the following natural transformation:

**Definition (4.3.1).** Let $X/S$ be a separated algebraic space. We let $\mathrm{HS}_X$ : $\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S)$ be the following morphism. Let $T$ be an $S$-scheme and $f : T \to \mathrm{Hilb}^d(X/S)$ a $T$-point. Then $f$ corresponds to a subscheme $Z \hookrightarrow X \times_S T$ which is flat and finite over $T$. There is a commutative diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[dr, "{(f,\mathrm{id}_T)}"'] & \mathrm{Hilb}^d(Z/T) \arrow[r, "\mathrm{HG}_Z"] \arrow[d, hook] & \Gamma^d(Z/T) \arrow[d, hook] & \mathrm{Sym}^d(Z/T) \arrow[l, "\mathrm{SG}_Z", "\cong"'] \arrow[d] \\
 & \mathrm{Hilb}^d(X/S) \times_S T \arrow[r, "\mathrm{HG}_X"] & \Gamma^d(X/S) \times_S T & \mathrm{Sym}^d(X/S) \times_S T \arrow[l, "\mathrm{SG}_X"']
\end{tikzcd}$$

and we let $\mathrm{HS}_X(f)$ be the composition $T \to \mathrm{Sym}^d(X/S) \times_S T \to \mathrm{Sym}^d(X/S)$.

The morphisms $\mathrm{HS}_X$ and $\mathrm{HG}_X$ are isomorphisms when $X/S$ is a smooth curve [**I**]. They are also both isomorphisms over the non-degeneracy locus as shown in the next section. In [ES04, RS07], it is shown that the closure of the non-degeneracy locus of $\mathrm{Hilb}^d(X/S)$ — the good component — is a blow-up of either $\Gamma^d(X/S)$ or $\mathrm{Sym}^d(X/S)$.

## 5. Outside the degeneracy locus

In this section we will prove that the morphisms

$$\mathrm{Hilb}^d(X/S) \to \mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$$

are all isomorphisms over the open subset parameterizing "non-degenerated families" of points. That the morphism $\mathrm{HG}_X : \mathrm{Hilb}^d(X/S) \to \Gamma^d(X/S)$ is an isomorphism over the non-degeneracy locus is shown in [**II**]. It is thus enough to show that the last two morphisms are isomorphisms outside the degeneracy locus.

5.1. **Families of cycles.** Let $\overline{k}$ be an algebraically closed field and fix a geometric point $s : \mathrm{Spec}(\overline{k}) \to S$. Let $\alpha : \mathrm{Spec}(\overline{k}) \to \mathrm{Sym}^d(X/S)$ be a geometric point above $s$. As $(X/S)^d \to \mathrm{Sym}^d(X/S)$ is integral, we have that $\alpha$ lifts (non-uniquely) to a geometric point $\beta : \mathrm{Spec}(\overline{k}) \to (X/S)^d$. Let $\pi_i : (X/S)^d \to X$ be the $i^{\mathrm{th}}$ projection and let $x_i = \pi_i \circ \beta$. It is easily seen that the different liftings $\beta$ of $\alpha$ corresponds to the permutations of the $d$ geometric points $x_i : \mathrm{Spec}(\overline{k}) \to X$. This gives a correspondence between $\overline{k}$-points of $\mathrm{Sym}^d(X/S)$ and effective zero-cycles of degree $d$ on $X_s$.

As $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ are universal homeomorphisms, there is a bijection between their geometric points. It is thus reasonable to say that $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ parameterize effective zero-cycles of degree $d$. Moreover, as $\mathrm{Sym}^d(X/S) \to \Gamma^d(X/S) \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ have trivial residue field extensions, there is a bijection between $k$-points for any field $k$.

**Definition (5.1.1).** Let $X$, $S$, $\overline{k}$ and $s$ be as above. Let $\mathcal{Z}$ be an effective zero-cycle of degree $d$ on $X_s$. The residue field of the corresponding point in $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ or $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is called the *Chow field* of $\mathcal{Z}$.

**Definition (5.1.2).** Let $k$ be a field and $X$ a scheme over $k$. Let $k'/k$ and $k''/k$ be field extensions of $k$. Two cycles $\mathcal{Z}'$ and $\mathcal{Z}''$ on $X \times_k k'$ and $X \times_k k''$ respectively, are said to be equivalent if there is a common field extension $K/k$ of $k'$ and $k''$ such that $\mathcal{Z}' \times_{k'} K = \mathcal{Z}'' \times_{k''} K$. If $\mathcal{Z}'$ is a cycle on $X \times_S k'$ equivalent to a cycle on $X \times_S k''$ then we say that $\mathcal{Z}'$ is *defined over* $k''$.

*Remark* **(5.1.3).** If $\mathcal{Z}$ is a cycle on $X \times_S k$ then the corresponding morphism $\mathrm{Spec}(\overline{k}) \to \mathrm{Sym}^d(X/S)$ factors through $\mathrm{Spec}(\overline{k}) \to \mathrm{Spec}(k)$. Thus if $\mathcal{Z}$ is defined over a field $K$ then the Chow field is contained in $K$. Conversely it can be shown that $\mathcal{Z}$ is defined over an inseparable extension of the Chow field. Thus, in characteristic zero the Chow field of $\mathcal{Z}$ is the unique minimal field of definition of $\mathcal{Z}$. In positive characteristic, it can be shown that the Chow field of $\mathcal{Z}$ is the intersection of all minimal fields of definitions of $\mathcal{Z}$, cf. [Kol96, Thm. I.4.5] and [**II**, Prop. 7.13].

Let $T$ be any scheme and $f : T \to \mathrm{Sym}^d(X/S)$, $f : T \to \Gamma^d(X/S)$ or $f : T \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ a morphism. A geometric $k$-point of $T$ then corresponds to a zero-cycle of degree $d$ on $X \times_S k$. The following definition is therefore natural.

**Definition (5.1.4).** A family of cycles parameterized by $T$ is a $T$-point of either $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ or $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$. We use the notation $\mathcal{Z} \to T$ to denote a family of cycles parameterized by $T$ and let $\mathcal{Z}_t$ be the cycle over $t$, i.e., the cycle corresponding to $\overline{k(t)} \to T \to \mathrm{Sym}^d(X/S)$, etc.

As $\Gamma^d(X/S)$ commutes with base change and has other good properties it is the "correct" parameter scheme and the morphisms $T \to \Gamma^d(X/S)$ are the "correct" families of cycles.

### 5.2. **Non-degenerated families.**

**(5.2.1)** *Non-degenerate families of subschemes* — Let $k$ be a field and $X$ be a $k$-scheme. If $Z \hookrightarrow X$ is a closed subscheme then it is natural say that $Z$ is non-degenerate if $Z_{\overline{k}}$ is reduced, i.e., if $Z \to k$ is geometrically reduced. If $Z$ is of dimension zero then $Z$ is non-degenerate if and only if $Z \to k$ is étale. Similarly for any scheme $S$, a finite flat morphism $Z \to S$ of finite presentation is called a non-degenerate family if every fiber is non-degenerate, or equivalently, if $Z \to S$ is étale.

Let $Z \to S$ be a family of zero dimensional subschemes, i.e., a finite flat morphism of finite presentation. The subset of $S$ consisting of points $s \in S$ such that the fiber $Z_s \to k(s)$ is non-degenerate is open [$\mathrm{EGA_{IV}}$, Thm. 12.2.1 (viii)]. Thus, there is an open subset $\mathrm{Hilb}^d(X/S)_{\mathrm{nd}}$ of $\mathrm{Hilb}^d(X/S)$ parameterizing non-degenerate families.

**(5.2.2)** *Non-degenerate families of cycles* — A zero-cycle $\mathcal{Z} = \sum_i a_i[z_i]$ on a $k$-scheme $X$ is called non-degenerate if every point in the support of $\mathcal{Z}_{\overline{k}}$ has multiplicity one. Equivalently the multiplicities $a_i$ are all one and the field extensions $k(z_i)/k$ are separable. It is clear that there is a one-to-one correspondence between non-degenerate zero-cycles on $X$ and non-degenerated zero-dimensional subschemes of $X$.

Given a family of cycles $\mathcal{Z} \to S$, i.e., a morphism $S \to \mathrm{Sym}^d(X/S)$, $S \to \Gamma^d(X/S)$ or $S \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$, we say that it is non-degenerate family if $\mathcal{Z}_s$ is non-degenerate for every $s \in S$.

**(5.2.3)** *Degeneracy locus of cycles* — Let $X \to S$ be a morphism of schemes and let $\Delta \hookrightarrow (X/S)^d$ be the big diagonal, i.e., the union of all diagonals $\Delta_{ij} : (X/S)^{d-1} \to (X/S)^d$. It is clear that the image of $\Delta$ by $(X/S)^d \to \mathrm{Sym}^d(X/S)$ parameterizes degenerate cycles and that the open complement parameterizes non-degenerate cycles. We let $\mathrm{Sym}^d(X/S)_{\mathrm{nd}}$, $\Gamma^d(X/S)_{\mathrm{nd}}$ and $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)_{\mathrm{nd}}$ be the open subschemes of $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ respectively, parameterizing non-degenerate cycles.

We will now give an explicit cover of the degeneracy locus of $\mathrm{Sym}^d(X/S)$, $\Gamma^d(X/S)$ and $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$. Some of the notation is inspired by [ES04, 2.4 and 4.1] and [RS07].

**Definition (5.2.4).** Let $A$ be a ring and $B$ an $A$-algebra. Let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in B^d$. We define the symmetrization and anti-symmetrization operators from $B^d$ to

$\mathrm{T}^d_A(B)$ as follows

$$s(\mathbf{x}) = \sum_{\sigma\in\mathfrak{S}_d} x_{\sigma(1)} \otimes_A x_{\sigma(2)} \otimes_A \cdots \otimes_A x_{\sigma(d)}$$
$$a(\mathbf{x}) = \sum_{\sigma\in\mathfrak{S}_d} (-1)^{|\sigma|} x_{\sigma(1)} \otimes_A x_{\sigma(2)} \otimes_A \cdots \otimes_A x_{\sigma(d)}.$$

As $s$ and $a$ are $A$-multilinear, $s$ is symmetric and $a$ is alternating it follows that we get induced homomorphisms, also denoted $s$ and $a$

$$s\ :\ \mathrm{S}^d_A(B) \to \mathrm{TS}^d_A(B)$$
$$a\ :\ \textstyle\bigwedge^d_A(B) \to \mathrm{T}^d_A(B).$$

*Remark* **(5.2.5).** If $d$ is invertible in $A$, then sometimes the symmetrization and anti-symmetrization operators are defined as $\frac{1}{d!}s$ and $\frac{1}{d!}a$. We will never use this convention. In [ES04] the tensor $a(\mathbf{x})$ is denoted by $\nu(\mathbf{x})$ and referred to as a norm vector.

**Definition (5.2.6).** Let $A$ be a ring and $B$ an $A$-algebra. Let $\mathbf{x} = (x_1, x_2, \dots, x_d) \in B^d$ and $\mathbf{y} = (y_1, y_2, \dots, y_d) \in B^d$. We define the following element in $\Gamma^d_A(B)$

$$\delta(\mathbf{x},\mathbf{y}) = \det\bigl(\gamma^1(x_i y_j) \times \gamma^{d-1}(1)\bigr)_{ij}.$$

Following [RS07] we call the ideal $I = I_A = \bigl(\delta(\mathbf{x},\mathbf{y})\bigr)_{\mathbf{x},\mathbf{y}\in B^d}$, the canonical ideal. As $\delta$ is multilinear and alternating in both arguments we extend the definition of $\delta$ to a function

$$\delta\ :\ \textstyle\bigwedge^d_A(B) \times \bigwedge^d_A(B) \to \mathrm{S}^2_A\bigl(\bigwedge^d_A(B)\bigr) \to \Gamma^d_A(B).$$

**Proposition (5.2.7)** ([ES04, Prop. 4.4])**.** *Let $A$ be a ring, $B$ an $A$-algebra and $\mathbf{x},\mathbf{y} \in B^d$. The image of $\delta(\mathbf{x},\mathbf{y})$ by $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B) \hookrightarrow \mathrm{T}^d_A(B)$ is $a(\mathbf{x})a(\mathbf{y})$. In particular, $a(\mathbf{x})a(\mathbf{y})$ is symmetric.*

**Lemma (5.2.8)** ([ES04, Lem. 2.5])**.** *Let $A$ be a ring and let $B$ and $A'$ be $A$-algebras. Let $B' = B \otimes_A A'$. Denote by $I_A \subset \Gamma^d_A(B)$ and $I_{A'} \subset \Gamma^d_{A'}(B') = \Gamma^d_A(B) \otimes_A A'$ the canonical ideals corresponding to $B$ and $B'$. Then $I_A A' = I_{A'}$.*

**Lemma (5.2.9).** *Let $S$ be a scheme and $X$ and $S'$ be $S$-schemes. Let $X' = X \times_S S'$. Let $\varphi : \Gamma^d(X'/S') = \Gamma^d(X/S) \times_S S' \to \Gamma^d(X/S)$ be the projection morphism. The inverse image by $\varphi$ of the degeneracy locus of $\Gamma^d(X/S)$ is the degeneracy locus of $\Gamma^d(X'/S')$.*

*Proof.* Obvious as we know that a geometric point $\mathrm{Spec}(k) \to \Gamma^d(X/S)$ corresponds to a zero-cycle of degree $d$ on $X \times_S \mathrm{Spec}(k)$. $\square$

**Lemma (5.2.10).** *Let $k$ be a field and let $B$ be a $k$-algebra generated as an algebra by the $k$-vector field $V \subseteq B$. Let $k'/k$ be a field extension and let $x_1, x_2, \dots, x_d$ be $d$ distinct $k'$-points of $\mathrm{Spec}(B \otimes_k k')$. If $k$ has at least $\binom{d}{2}$ elements then there is an element $b \in V$ such that the values of $b$ at $x_1, x_2, \dots, x_d$ are distinct.*

*Proof.* For a vector space $V_0 \subseteq V$ we let $B_0 \subseteq B$ be the sub-algebra generated by $V_0$. There is a finite dimensional vector space $V_0 \subseteq V$ such that the images of $x_1, x_2, \dots, x_d$ in $\mathrm{Spec}(B_0 \otimes_k k')$ are distinct. Replacing $V$ and $B$ with $V_0$ and $B_0$ we can thus assume that $V$ is finite dimensional. It is further clear that we can assume that $B = \mathrm{S}(V)$. The points $x_1, x_2, \dots, x_d$ then corresponds to vectors of $V^\vee \otimes_k k'$ and we need to find a $k$-rational hyperplane which does not contain the $\binom{d}{2}$ difference vectors $x_i - x_j$. A similar counting argument as in the proof of Lemma (3.1.6) shows that if $k$ has at least $\binom{d}{2}$ elements then this is possible. □

**Proposition (5.2.11).** *Let $A$ be a ring and $B$ an $A$-algebra. Let $V \subset B$ be an $A$-submodule such that $B$ is generated by $V$ as an algebra. Consider the following three ideals of $\Gamma^d_A(B)$*

(i) *The canonical ideal $I_1 = \big(\delta(\mathbf{x}, \mathbf{y})\big)_{\mathbf{x},\mathbf{y} \in B^d}$.*
(ii) $I_2 = \big(\delta(\mathbf{x}, \mathbf{x})\big)_{\mathbf{x} \in B^d}$.
(iii) $I_3 = \big(\delta(\mathbf{x}, \mathbf{x})\big)_{\mathbf{x} = (1, b, b^2, \dots, b^{d-1}),\ b \in V}$.

*The closed subsets determined by $I_1$ and $I_2$ coincide with the degeneracy locus of $\Gamma^d\big(\mathrm{Spec}(B)/\mathrm{Spec}(A)\big) = \mathrm{Spec}\big(\Gamma^d_A(B)\big)$. If every residue field of $A$ has at least $\binom{d}{2}$ elements then so does the closed subset determined by $I_3$.*

*Proof.* The discussion in (5.2.3) shows that it is enough to prove that the image of the ideals $I_k$ by the homomorphism $\Gamma^d_A(B) \to \mathrm{TS}^d_A(B) \hookrightarrow \mathrm{T}^d_A(B)$ set-theoretically defines the big diagonal of $\mathrm{Spec}\big(\mathrm{T}^d_A(B)\big)$. By Proposition (5.2.7) the image of $\delta(\mathbf{x}, \mathbf{y})$ is $a(\mathbf{x})a(\mathbf{y})$. Thus the radicals of the images of $I_1$ and $I_2$ equals the radical of $J = \big(a(\mathbf{x})\big)_{\mathbf{x} \in B^d}$. It is further easily seen that $J$ is contained in the ideal of every diagonal of $\mathrm{Spec}\big(\mathrm{T}^d_A(B)\big)$. Equivalently, the closed subset corresponding to $J$ contains the big diagonal.

By Lemmas (5.2.8) and (5.2.9) it is enough to show the first part of the proposition after any base change $A \to A'$ such that $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective. We can thus assume that every residue field of $A$ has at least $\binom{d}{2}$ elements. Both parts of the proposition then follows if we show that the closed subset corresponding to the ideal

$$K = \big(a(1, b, b^2, \dots, b^{d-1})\big)_{b \in V} \subseteq \mathrm{T}^d_A(B)$$

is contained in the big diagonal. As the formation of the ideal $K$ commutes with base changes $A \to A'$ which are either surjections or localizations we can replace $A$ with one of its residue fields and assume that $A$ is a field with at least $\binom{d}{2}$ elements.

Let $\mathrm{Spec}(k) : x \to \mathrm{Spec}\big(\mathrm{T}^d_A(B)\big)$ be a point corresponding to $d$ distinct $k$-points $x_1, x_2, \dots, x_d$ of $\mathrm{Spec}(B \otimes_A k)$. Lemma (5.2.10) shows that there is an element $b \in V$ which takes $d$ distinct values $a_1, a_2, \dots, a_d \in k$ on the $d$ points. The value of

$a(1,b,b^2,\dots,b^{d-1})$ at $x$ is then

$$\sum_{\sigma\in\mathfrak{S}_d}(-1)^{|\sigma|}a_1^{\sigma(1)-1}a_2^{\sigma(2)-1}\dots a_d^{\sigma(d)-1}=\det\begin{pmatrix}1 & a_1 & a_1^2 & \dots & a_1^{d-1}\\ 1 & a_2 & a_2^2 & \dots & a_2^{d-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & a_d & a_d^2 & \dots & a_d^{d-1}\end{pmatrix}=\prod_{j<i}(a_i-a_j)$$

which is non-zero. Thus $x$ is not contained in the zero-set of $K$. This shows that zero-set of $K$ is contained in the big diagonal and hence that zero-set defined by $K$ is the big diagonal. □

5.3. **Non-degenerated symmetric tensors and divided powers.**

**Lemma (5.3.1).** *Let $A$ be a ring, $B$ an $A$-algebra and $x,y\in\bigwedge_A^d(B)$. Then $\Gamma_A^d(B)_{\delta(x,y)}\to\mathrm{TS}_A^d(B)_{\delta(x,y)}$ is an isomorphism.*

*Proof.* Denote the canonical homomorphism $\Gamma_A^d(B)\to\mathrm{TS}_A^d(B)$ with $\varphi$. Let $f\in\mathrm{TS}_A^d(B)$. As the anti-symmetrization operator $a:\mathrm{T}_A^d(B)\to\mathrm{T}_A^d(B)$ is a $\mathrm{TS}_A^d(B)$-module homomorphism we have that $fa(x)=a(fx)$. By Proposition (5.2.7)

$$f\varphi\big(\delta(x,y)\big)=fa(x)a(y)=a(fx)a(y)=\varphi\big(\delta(fx,y)\big)$$

which shows that $\varphi$ is surjective after localizing in $\delta(x,y)$.

To show that $\varphi_{\delta(x,y)}$ is injective, it is enough to show that the composition

$$\Gamma_A^d(B)_{\delta(x,y)}\to\mathrm{TS}_A^d(B)_{\delta(x,y)}\hookrightarrow\mathrm{T}_A^d(B)_{\delta(x,y)}$$

is injective. Choose a surjection $F\twoheadrightarrow B$ with $F$ a flat $A$-algebra and let $I$ be the kernel of $F\twoheadrightarrow B$. Let $J$ be the kernel of $\mathrm{T}_A^d(F)\twoheadrightarrow\mathrm{T}_A^d(B)$.

Let $f\in J^G$. As $f\in J$ we can write $f$ as a sum $f_1+f_2+\dots+f_n$ such that for every $i$ we have that $f_i=f_{i1}\otimes f_{i2}\otimes\dots\otimes f_{id}\in\mathrm{T}_A^d(F)$ with $f_{ij}\in I$ for some $j$. Choose liftings $x',y'\in\bigwedge_A^d(F)$ of $x,y\in\bigwedge_A^d(B)$. Identifying $\Gamma_A^d(F)$ and $\mathrm{TS}_A^d(F)$, we have that $f\delta(x',y')=\delta(fx',y')$. This is a sum of determinants with elements in $\Gamma_A^d(F)$ such that in every determinant there is a row in which every element is in the ideal $\gamma^1(I)\times\gamma^{d-1}(1)$. Thus $\delta(fx',y')$ is in the kernel of $\Gamma_A^d(F)\twoheadrightarrow\Gamma_A^d(B)$ by (1.1.8). The image of $f$ in $\Gamma_A^d(B)$ is thus zero after multiplying with $\delta(x,y)$. Consequently $\varphi$ is injective after localizing in $\delta(x,y)$. □

**Theorem (5.3.2).** *Let $X/S$ be a separated algebraic space. Then $\mathrm{Sym}^d(X/S)_{\mathrm{nd}}\to\Gamma^d(X/S)_{\mathrm{nd}}$ is an isomorphism.*

*Proof.* We can assume that $S$ and $X$ are affine (3.2.5). The theorem then follows from Proposition (5.2.11) and Lemma (5.3.1). □

**Definition (5.3.3).** Let $A$ be any ring and $B=A[x_1,x_2,\dots,x_r]$. We call the elements $f\in\Gamma_A^d(B)$ of degree one, see Definition (1.3.2), *multilinear* or *elementary multisymmetric functions*. These are elements of the form

$$\gamma^{d_1}(x_1)\times\gamma^{d_2}(x_2)\times\dots\times\gamma^{d_n}(x_n)\times\gamma^{d-d_1-\dots-d_n}(1).$$

We let $\Gamma^d_A(A[x_1,x_2,\dots,x_n])_{\text{mult.lin.}}$ denote the subalgebra of $\Gamma^d_A(A[x_1,x_2,\dots,x_n])$ generated by multi-linear elements.

*Remark* **(5.3.4).** If the characteristic of $A$ is zero or more generally if $d!$ is invertible in $A$, then $\Gamma^d_A(A[x_1,x_2,\dots,x_n])_{\text{mult.lin.}} = \Gamma^d_A(A[x_1,x_2,\dots,x_n])$ by Theorem (1.3.4).

**Lemma (5.3.5).** *Let $A$ be a ring and $B = A[x_1,x_2,\dots,x_n]$. Let $b \in B_1$ and let $\mathbf{x} = (1, b, b^2, \dots, b^{d-1})$. Then $\big(\Gamma^d_A(B)_{\text{mult.lin.}}\big)_{\delta(\mathbf{x},\mathbf{x})} \hookrightarrow \Gamma^d_A(B)_{\delta(\mathbf{x},\mathbf{x})}$ is an isomorphism.*

*Proof.* Let $f \in \Gamma^d_A(B) = \mathrm{TS}^d_A(B)$. We will show that $f$ is a sum of products of multilinear elements after multiplication by a power of $\delta(\mathbf{x},\mathbf{x})$. As $f\delta(\mathbf{x},\mathbf{x}) = \delta(f\mathbf{x},\mathbf{x})$ and the latter is a sum of products of elements of the type $\gamma^1(c)\times\gamma^{d-1}(1)$ we can assume that $f$ is of this type. As $c \mapsto \gamma^1(c)\times\gamma^{d-1}(1)$ is linear we can further assume that $c = x^\alpha$ for some non-trivial monomial $x^\alpha \in B$. It will be useful to instead assume that $c = x^\alpha b^k$ with $|\alpha| \geq 1$ and $k \in \mathbb{N}$. We will now proceed on induction on $|\alpha|$.

Assume that $|\alpha| = 1$. If $k = 0$ then $f = \gamma^1(x^\alpha b^k)\times\gamma^{d-1}(1)$ is multilinear. We continue with induction on $k$ to show that $f \in \Gamma^d_A(B)_{\text{mult.lin.}}$. We have that

$$\begin{aligned} f = \gamma^1(x^\alpha b^k)\times\gamma^{d-1}(1) = \big(\gamma^1(x^\alpha b^{k-1})\times\gamma^{d-1}(1)\big)\big(\gamma^1(b)\times\gamma^{d-1}(1)\big)& \\ - \gamma^1(x^\alpha b^{k-1})\times\gamma^1(b)\times\gamma^{d-2}(1)& \end{aligned}$$

and by induction it is enough to show that the last term is in $\Gamma^d_A(B)_{\text{mult.lin.}}$. Similar use of the relation

$$\begin{aligned} \gamma^1(x^\alpha b^{k-\ell})\times\gamma^\ell(b)\times\gamma^{d-\ell-1}(1) = \big(\gamma^1(x^\alpha b^{k-\ell-1})\times\gamma^{d-1}(1)\big)\big(\gamma^{\ell+1}(b)\times\gamma^{d-\ell-1}(1)\big)& \\ - \gamma^1(x^\alpha b^{k-\ell-1})\times\gamma^{\ell+1}(b)\times\gamma^{d-\ell-2}(1)& \end{aligned}$$

with $1 \leq l \leq d-2$ and $l \leq k-1$ shows that it is enough to consider either $\gamma^1(x^\alpha)\times\gamma^k(b)\times\gamma^{d-k-1}(1)$ if $k \leq d-1$ or $\gamma^1(x^\alpha b^{k-d+1})\times\gamma^{d-1}(b)$ if $k > d-1$. The first element of these is multilinear and the second is the product of the multilinear element $\gamma^d(b)$ and $\gamma^1(x^\alpha b^{k-d})\times\gamma^{d-1}(1)$ which by the induction on $k$ is in $\Gamma^d_A(B)_{\text{mult.lin.}}$.

If $|\alpha| > 1$ then $x^\alpha = x^{\alpha'}x^{\alpha''}$ for some $\alpha',\alpha''$ such that $|\alpha'|,|\alpha''| < |\alpha|$. We have that

$$\begin{aligned} f = \gamma^1(c)\times\gamma^{d-1}(1) = \big(\gamma^1(x^{\alpha'}b^k)\times\gamma^{d-1}(1)\big)\big(\gamma^1(x^{\alpha''})\times\gamma^{d-1}(1)\big)& \\ - \gamma^1(x^{\alpha'}b^k)\times\gamma^1(x^{\alpha''})\times\gamma^{d-2}(1).& \end{aligned}$$

By induction it is enough to show that the last term is a sum of products of multilinear elements, after suitable multiplication by $\delta(\mathbf{x},\mathbf{x})$. Let $g = \gamma^1(x^{\alpha'}b^k)\times\gamma^1(x^{\alpha''})\times\gamma^{d-2}(1)$. Then $g\delta(\mathbf{x},\mathbf{x}) = \delta(g\mathbf{x},\mathbf{x})$ which is a sum of products of elements of the kind $\gamma^1(x^{\alpha'}b^{t'})\times\gamma^{d-1}(1)$ and $\gamma^1(x^{\alpha''}b^{t''})\times\gamma^{d-1}(1)$. By induction on $|\alpha|$ these are in $\big(\Gamma^d_A(B)_{\text{mult.lin.}}\big)_{\delta(\mathbf{x},\mathbf{x})}$. □

**Theorem (5.3.6).** *Let $X/S$ be quasi-projective morphism of schemes and let $X \hookrightarrow \mathbb{P}_S(\mathcal{E})$ be an immersion for some quasi-coherent $\mathcal{O}_S$-module $\mathcal{E}$ of finite type. Then $\Gamma^d(X/S)_{\mathrm{nd}} \to \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)_{\mathrm{nd}}$ is an isomorphism.*

*Proof.* As $\Gamma$ commutes with arbitrary base change and Chow commutes with flat base change we may assume that $S$ is affine and, using Lemma (1.2.3), that every residue field of $S$ has at least $\binom{d}{2}$ elements. If $\mathcal{E}' \twoheadrightarrow \mathcal{E}$ is a surjection of $\mathcal{O}_S$-modules then $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big) = \mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E}')\big)$ by Definition (3.3.8) and we may thus assume that $\mathcal{E}$ is free. Further as $\mathrm{Chow}_{0,d}\big(X \hookrightarrow \mathbb{P}(\mathcal{E})\big)$ is the schematic image of $\Gamma^d(X/S) \hookrightarrow \Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ we may assume that $X = \mathbb{P}(\mathcal{E}) = \mathbb{P}^n$.

By Proposition (3.3.10) and the assumption on the residue fields of $S = \mathrm{Spec}(A)$, the scheme $\mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ is covered by affine open subsets over which the morphism $\Gamma^d\big(\mathbb{P}(\mathcal{E})/S\big) \to \mathrm{Chow}_{0,d}\big(\mathbb{P}(\mathcal{E})\big)$ corresponds to the inclusion of rings

$$\Gamma^d_A(A[x_1, x_2, \dots, x_n])_{\mathrm{mult.lin.}} \hookrightarrow \Gamma^d_A(A[x_1, x_2, \dots, x_n]).$$

The theorem now follows from Proposition (5.2.11) and Lemma (5.3.5). $\square$

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# Paper IV

# FAMILIES OF CYCLES

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Abstract. Let $X$ be an algebraic space, locally of finite type over an *arbitrary* scheme $S$. We give a definition of *relative cycles* on $X/S$. When $S$ is reduced and of characteristic zero, this definition agrees with the definitions of Barlet, Kollár and Suslin-Voevodsky when these are defined. Relative multiplicity-free cycles and relative Weil-divisors over arbitrary parameter schemes are then studied more closely. We show that relative *normal* cycles are given by flat subschemes, at least in characteristic zero. In particular, the morphism $\mathrm{Hilb}^{\mathrm{equi}}_r(X/S) \to \mathrm{Chow}_r(X/S)$ taking a subscheme, equidimensional of dimension $r$, to its relative fundamental cycle of dimension $r$, is an isomorphism over normal subschemes.

When $S$ is of characteristic zero, any relative cycle induces a unique relative fundamental class. The set of Chow classes in the sense of Angéniol constitute a subset of the classes corresponding to relative cycles. When $\alpha$ is a relative cycle such that either $S$ is reduced, $\alpha$ is multiplicity-free, or $\alpha$ is a relative Weil-divisor, then its relative fundamental class is a Chow class. In particular, the corresponding Chow functors agree in these cases.

## Introduction

The Chow variety $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$, parameterizing families of cycles of dimension $r$ and degree $d$ on a projective variety $X$, was constructed in the first half of the twentieth century [CW37, Sam55]. The main goal of this paper is to define a natural contravariant functor $\mathrm{Chow}_{r,d}(X)$ from schemes to sets, such that its restriction to reduced schemes is represented by $\mathrm{ChowVar}_{r,d}(X)$. Here $\mathrm{ChowVar}_{r,d}(X)$ is a reduced variety coinciding with $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ for a sufficiently ample projective embedding $X \hookrightarrow \mathbb{P}^n$ [Hoy66]. In characteristic zero, the Chow variety $\mathrm{ChowVar}_{r,d}(X \hookrightarrow \mathbb{P}^n)$ is independent on the embedding [Bar75] but this is not the case in positive characteristic [Nag55].

We will first define a notion of *relative cycles* on $X/S$. This definition is given in great generality without any assumptions on $S$ and only assuming that $X/S$ is

2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 14C25.
*Key words and phrases.* Chow scheme, Hilbert scheme, divided powers, cycles, Weil divisors.

locally of finite type. This definition includes non-equidimensional and even non-separated relative cycles. We then let

$$\mathit{Cycl}(X/S) = \{\text{relative cycles on } X/S\}$$

$$\mathrm{Chow}_r(X/S)(T) = \left\{\begin{matrix}\text{proper relative cycles which are equi-}\\ \text{dimensional of dimension } r \text{ on } X \times_S T/T\end{matrix}\right\}.$$

If $X$ is projective, the functor $\mathrm{Chow}_r(X)$ is a disjoint union of the subfunctors $\mathrm{Chow}_{r,d}(X)$ parameterizing cycles of a fixed degree. We also let

$$\mathrm{Chow}(X)(T) = \{\text{proper equidimensional relative cycles on } X \times_S T/T\}.$$

A similar Chow-functor, which we will denote by $\mathrm{Ang}_r(X)$, has been constructed by Angéniol [Ang80] in characteristic zero and we will present some evidence indicating that Chow = Ang in characteristic zero. In fact, there is a natural monomorphism $\mathrm{Ang}_r(X) \to \mathrm{Chow}_r(X)$ which is a bijection when restricted to reduced $T$ and between the open subfunctors parameterizing multiplicity-free cycles.

It is known [Bar75, Gue96, Kol96, SV00] that if $T$ is a normal scheme of *characteristic zero*, then there is a one-to-one correspondence between $T$-points of the Chow variety and cycles $\mathcal{Z}$ on $X \times T$ which are equidimensional of relative dimension $r$ and whose generic fiber has degree $d$. Thus when $S$ is normal and of characteristic zero we define a relative $r$-cycle on $X/S$ to be a cycle on $X$ which is equidimensional of relative dimension $r$ and the definition of $\mathrm{Chow}_r(X)(T)$ for $T$ normal follows. There is a subtle point here concerning the pull-back $\mathrm{Chow}_r(X)(T) \to \mathrm{Chow}_r(X)(T')$ for a morphism $T' \to T$ between normal schemes. If $t \in T$ is a point, then the naïve fiber $\mathcal{Z}_t$ does not necessarily coincide with the cycle corresponding to the morphism $\mathrm{Spec}(k(t)) \to T \to \mathrm{ChowVar}_{r,d}(X)$.

This problem is due to the fact that $\mathcal{Z}$ is not "flat" over $T$. As an illustration, let $T = \mathrm{Spec}(k[s,t])$ be the affine plane and consider the family of zero-cycles on $X = \mathrm{Spec}(k[x,y])$ of degree two given by the primitive cycle $\mathcal{Z} = [Z]$ where $Z \hookrightarrow X \times T$ is the subvariety given by the ideal $(x^2 - s^3t, y^2 - st^3, xy - s^2t^2, tx - sy)$. On the open subset $T \setminus (0,0)$, we have that $Z$ is flat of rank two, but the fiber over the origin is the subscheme defined by $(x^2, y^2, xy)$ which has rank three. The naïve fiber in this case would be three times the origin of $X$ while the correct fiber is two times the origin.

If $T$ is a smooth curve, the above "pathology" does not occur as then every cycle is flat. If $T$ is a smooth variety, the correct fiber $\mathcal{Z}_t$ can be defined through intersection theory [Ful98, Ch. 10]. If $T$ is a normal variety, then the correct fiber $\mathcal{Z}_t$ can be defined through Samuel multiplicities [SV00, Thm. 3.5.8]. For an arbitrary reduced scheme $T$, the fiber of a cycle $\mathcal{Z}$ on $X \times T$ at $t$ can be defined by taking the "limit cycle" along a curve passing through $t$ as defined by Kollár [Kol96] and Suslin-Voevodsky [SV00]. This construction may depend on the choice of the curve, but if $T$ is normal and of characteristic zero the limit cycle is well-defined. If $T$ is weakly normal, then $\mathcal{Z}$ will be a relative cycle if and only if the limit cycle is well-defined for every point $t \in T$. In positive characteristic, even if $T$ is normal, the limit cycle may have rational coefficients [SV00, Ex. 3.5.10].

It is natural to include cycles with certain rational coefficients in positive characteristic [**II**] and we will call these cycles *quasi-integral*. A relative cycle over a *perfect* field will always have integral coefficients. The denominator of the multiplicity of a subvariety is bounded by its *inseparable discrepancy*.

The limit-cycle condition only gives the correct functor for semi-normal schemes. The problem is easily illustrated by taking a geometrically unibranch but non-normal parameter scheme $S$, such as a cuspidal curve. The normalization $X \to S$ then satisfies the limit-cycle condition — the limit-cycle of the singular point of $S$ is the corresponding point of $X$ with multiplicity one. We thus obtain a "relative" zero-cycle of degree one $X \to S$ but this does not correspond to a morphism $S \to \mathrm{ChowVar}_{0,1}(X) \cong X$.

**Definition of the Chow functor.** The definition of the Chow functor is based upon the assumption that $\mathrm{Chow}_{0,d}(X)$ should be represented by the scheme of divided powers $\Gamma^d(X/S)$. This is in agreement with the conditions on $\mathrm{Chow}_{0,d}(X)$ imposed at the beginning as $\Gamma^d(X/S)_{\mathrm{red}} = \mathrm{ChowVar}_{0,d}(X/S)$, cf. [**III**]. If $X/S$ is flat or the characteristic of $S$ is zero, then $\Gamma^d(X/S) = \mathrm{Sym}^d(X/S)$ [**I**].

We let $\Gamma^\star(X/S) = \coprod_{d\geq 0} \Gamma^d(X/S)$ which thus represents $\mathrm{Chow}_0(X)$. A relative proper zero-cycle on $X/S$ is then a morphism $S \to \Gamma^\star(X/S)$. If $S$ is reduced, a relative proper zero-cycle is represented by a quasi-integral cycle on $X$, such that its support is proper and equidimensional of dimension zero over $S$. For a reduced scheme $S$, we then make the following definition of a higher-dimensional relative cycle:

If $S$ is reduced, then a relative cycle on $X/S$ of dimension $r$ is a cycle $\mathcal{Z}$ on $X$ which is equidimensional of dimension $r$ over $S$ and such that for any *smooth projection* $(U, B, T, p, g, \varphi)$ consisting of a diagram

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "\varphi"] & X \arrow[dd] \\ B \arrow[d] \arrow[dr, phantom, "\circ"] & \\ T \arrow[r, "g"] & S \end{tikzcd}$$

where $U \to X$, $B \to T$ and $T \to S$ are smooth, $U \to X \times_S T$ is étale, and $\varphi : U \to B$ is finite over the support of $p^*\mathcal{Z}$, we have that $p^*\mathcal{Z}$ is a relative (proper) zero-cycle over $B$, i.e., given by a morphism $B \to \Gamma^\star(U/B)$.

We will show that when $S$ is of characteristic zero, the above definition agrees with the definition given by Barlet [Bar75] in the complex-analytic case and the definition given by Angéniol [Ang80] in the algebraic setting. The functor $\mathrm{Chow}_r(X)_{\mathrm{red}}$ is then an algebraic space which coincides with $\mathrm{ChowVar}_r(X)$ when $X$ is projective. Over semi-normal schemes, we recover the definitions with limit-cycles due to Kollár and Suslin-Voevodsky.

**Definition over arbitrary parameter schemes.** Over a general scheme $S$ it is more complicated to define what a relative cycle on $X/S$ is. A main obstacle is the fact that relative cycles on $X/S$ are not usually represented by cycles on $X$. The course taken by El Zein and Angéniol [AEZ78, Ang80] is to represent a relative cycle by its *relative fundamental class*. This is a cohomology class in $c \in \operatorname{Ext}^{-r}_{Z_m}(\Omega^r_{Z_m/S}, \mathcal{D}^{\bullet}_{Z_m/S})$, where $j_m : Z_m \hookrightarrow X$ is an infinitesimal neighborhood of the support $Z$ of the relative cycle. Such a cohomology class induces, by duality, a class in $\operatorname{Ext}^{-r}_X((j_m)_*\mathcal{O}_{Z_m} \otimes \Omega^r_{X/S}, \mathcal{D}^{\bullet}_{X/S})$ and, if $X/S$ is smooth, a class in $\mathrm{H}^{-r}_Z(X, (\Omega^r_{X/S})^{\vee} \otimes \mathcal{D}^{\bullet}_{X/S}) = \mathrm{H}^{n-r}_Z(X, \Omega^{n-r}_{X/S})$.

The connection with cycles is as follows. A class $c$, supported at $Z \subseteq X$, in one of the above cohomology groups, induces for any projection $(U, B, T, p, g, \varphi)$ with $U \to X$, $B \to T$ and $T \to S$ smooth and $\varphi : U \to B$ finite over $p^{-1}(Z)$, a trace homomorphism $\operatorname{tr}(c) : \varphi_*\Omega^r_{p^{-1}(Z_m)/T} \to \Omega^r_{B/T}$. This homomorphism extends uniquely to a homomorphism $\operatorname{tr}(c) : \varphi_*\Omega^{\bullet}_{p^{-1}(Z_m)/T} \to \Omega^{\bullet}_{B/T}$, commuting with the differentials, and in particular we obtain a trace map $\operatorname{tr}(c) : \varphi_*\mathcal{O}_{p^{-1}(Z_m)} \to \mathcal{O}_B$. In characteristic zero, a family of zero-cycles $B \to \Gamma^d(p^{-1}(Z_m)/B)$ is determined by its trace $\varphi_*\mathcal{O}_{p^{-1}(Z_m)} \to \mathcal{O}_B$.

In characteristic zero, Angéniol [Ang80, Thm. 1.5.3] gives a condition characterizing the homomorphisms $\varphi_*\mathcal{O}_{p^{-1}(Z_m)} \to \mathcal{O}_B$ which are the traces of families $B \to \Gamma^d(p^{-1}(Z_m)/B)$. He then generalizes this condition to a condition on $\operatorname{tr}(c) : \varphi_*\Omega^{\bullet}_{p^{-1}(Z_m)/T} \to \Omega^{\bullet}_{B/T}$ which is stable under the choice of projection [Ang80, Prop. 2.3.5]. Thus, if $\operatorname{tr}(c)$ satisfies this condition, then the induced trace for any projection comes from a family of zero-cycles. It is not clear whether the converse is true, i.e., if a class such that the induced trace on any projection comes from a family of zero-cycles satisfies Angéniol's condition.

In positive characteristic, some kind of "crystalline duality" would be required to accomplish a similar description and we do not follow this line. Our definition is more straight-forward. We define a relative cycle, supported on a subset $Z \subseteq X$, to be a collection of relative zero-cycles $B \to \Gamma^{\star}(U/B)$ for every projection $(U, B, Z)$ of $X/S$ such that $p^{-1}(Z) \to B$ is finite. We further impose natural compatibility conditions on the zero-cycles of different projections. At first glance, this looks impractical to work with but we describe situations in which the relative cycles are easier to describe.

- (A1) If $S$ is reduced, then every relative cycle is induced by an ordinary cycle on $X$ as described above, cf. Corollary (8.7).
- (A2) If $\alpha$ is a multiplicity-free relative cycle on $X/S$, i.e., if the pull-back cycles $\alpha_s$ are without multiplicities for every geometric point $s \to S$, then $\alpha$ is induced by a subscheme of $X$ which is flat over a fiberwise dense subset, cf. Corollary (9.9).
- (A3) If $\alpha$ is a relative Weil-divisor on $X/S$, i.e., if $X/S$ is equidimensional of dimension $r+1$, and if $X/S$ is flat with $(\mathrm{R}_1)$-fibers, e.g., if $X/S$ has normal

fibers, then $\alpha$ is induced by a subscheme of $X$ which is a relative Cartier divisor over a fiberwise dense subset, cf. Corollary (9.16).

In positive characteristic, the descriptions (A2) and (A3) are unfortunately conjectural except when $S$ is reduced. Note that in these three cases we obtain an object which represents the cycle $\alpha$ but not all such objects induce a relative cycle. There are however correspondences as follows:

- (B1) $S$ *normal.* Relative cycles over $S$ correspond to effective quasi-integral cycles on $X$ with universally open support, cf. Theorem (10.1).
- (B2) $S$ *semi-normal.* Relative cycles over $S$ correspond to ordinary cycles on $X$ with universally open support such that the limit-cycle for every point $s \in S$ is well-defined and quasi-integral, cf. Theorem (10.17).
- (B3) $S$ arbitrary. *Smooth* relative cycles correspond to subschemes which are smooth, cf. Theorem (9.8).
- (B4) $S$ arbitrary. *Normal* relative cycles correspond to subschemes which are flat with normal fibers, cf. Theorem (12.8).
- (B5) $S$ arbitrary, $X/S$ smooth. *Relative Weil-divisors* on $X/S$ correspond to relative Cartier-divisors on $X/S$, cf. Theorem (9.15).
- (B6) $S$ arbitrary, $X/S$ flat with geometrically $(\mathrm{R}_2)$-fibers. *Relative Weil-divisors* on $X/S$ correspond to Weil-divisors $Z$ on $X/S$ such that $Z$ is a relative Cartier-divisor over an open subset of $Z$ containing all points of relative codimension at most one, cf. Theorem (11.7).
- (B7) $S$ arbitrary. *Multiplicity-free* relative cycles which are geometrically $(\mathrm{R}_1)$ correspond to subschemes which are flat with geometrically $(\mathrm{R}_1)$-fibers over an open subset containing all points of relative codimension at most one, cf. Theorem (11.5).
- (B8) $S$ reduced, $X/S$ flat with geometrically $(\mathrm{R}_1)$-fibers. *Relative Weil-divisors* on $X/S$ correspond to Weil-divisors $Z$ on $X/S$ which are relative Cartier-divisors over an open fiberwise dense subset of $Z$, cf. Theorem (11.7).
- (B9) $S$ reduced. *Multiplicity-free* relative cycles correspond to subschemes which are flat with reduced fibers over an open fiberwise dense subset, cf. Theorem (11.5).

Here (B3)–(B7) are conjectural in positive characteristic.

In characteristic zero, it follows from Bott's theorem on grassmannians, similarly as in [AEZ78], that a relative cycle induces a fundamental class $c$ in one of the cohomology groups discussed above. It is however not clear that $c$ satisfies the conditions imposed by Angéniol except for relative cycles as in (A1)–(A3).

**Push-forward and pull-back.** If $f : X \to Y$ is a *finite* morphism of $S$-schemes, then there is a natural functor — the push-forward — from relative cycles on $X/S$ to relative cycles on $Y/S$. When $S$ is reduced, the push-forward of relative cycles coincides with the ordinary push-forward of cycles.

It is reasonable to believe that for any proper morphism $f : X \to Y$ there should be a push-forward functor $f_* : \mathit{Cycl}(X/S) \to \mathit{Cycl}(Y/S)$ coinciding with the ordinary push-forward of cycles when $S$ is reduced. Recall that if $V$ is a subvariety of $X$, then the push-forward $f_*([V])$ of the cycle $[V]$ is $\deg\bigl(k(V)/k(f(V))\bigr)[f(V)]$ if $f|_V$ is generically finite and zero otherwise. The push-forward for arbitrary cycles is then defined by linearity. If $\alpha$ is a relative cycle on $X/S$ then it is straight-forward to define $f_*\alpha$ on a dense subset of its support, but the verification that this cycle extends to a cycle, necessarily unique, on the whole support is only accomplished in the cases (A1) and (B1)–(B9) above using flatness.

If $f : U \to X$ is a flat morphism which is equidimensional of dimension $r$, then it is again reasonable that there should be a pull-back functor $f^* : \mathit{Cycl}(X/S) \to \mathit{Cycl}(Y/S)$ coinciding with the flat pull-back of cycles when $S$ is reduced. Contrary to the case with the push-forward it is not even clear how $f^*$ should be defined in general. It is possible to partially define the pull-back by giving families of zero-cycles on certain projections. In the cases (A1)–(A3), it is clear how the pull-back should be defined generically on any projection but it is only in the cases (B1)–(B9) that it is shown that the generic pull-back extends to a relative cycle.

If $f : U \to X$ is *smooth* of relative dimension $r$, then it is possible to construct a pull-back $f^*(c)$ for the cohomological description of relative cycles in characteristic zero. We will show that smooth pull-back exists when $S$ is reduced but in general this is as problematic as the flat pull-back. This motivates the following alternative definition of relative cycles. A relative cycle $\alpha$ on $X/S$ with support $Z$ consists of relative zero-cycles $\alpha_{U/B}$ on $U/B$ for any commutative square

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ B \arrow[r, "g"] & X \end{tikzcd}$$

such that $p$ and $g$ are smooth and $p^{-1}(Z) \to B$ is finite. These zero-cycles are required to satisfy natural compatibility conditions. Every relative cycle of the new definition determines a unique relative cycle of the first definition. With the new definition, it is at least clear that smooth pull-backs exist.

**Products and intersections.** If $\alpha$ and $\beta$ are relative cycles on $X/S$ and $Y/S$ respectively, then it is reasonable to demand that there should be a natural relative cycle $\alpha \times \beta$ on $X \times_S Y/S$. This relative cycle is only defined under the same conditions as the flat pull-back.

If $\alpha$ is a relative cycle on $X/S$ and $D$ is a relative Cartier divisor on $X/S$ meeting $\alpha$ properly in every fiber, then there is a relative cycle $D \cap \alpha$ on $X/S$. If two relative cycles $\alpha$ and $\beta$ on a smooth scheme $X/S$ meet properly in every fiber, then under the assumption that $\alpha \times \beta$ is defined, we then define $\alpha \cap \beta$ as the intersection of $\alpha \times \beta$ with the diagonal $\Delta_{X/S}$.

**Overview of contents.** The paper is naturally divided into three parts. In the first part, Sections 1–6, we give the foundations on relative cycles. In the second part, Sections 7–12, we treat relative cycles which are flat, multiplicity-free, normal or smooth, relative Weil divisor and relative cycles over reduced parameter schemes. In the third part, Sections 13–17, we discuss proper push-forward, flat pull-back and intersections of cycles, compare our definition of relative cycles with Angéniol's definition and mention the classical construction of the Chow variety via Grassmannians. The third part is very brief and many results are only sketched.

In Section 1, we briefly recall the results on proper relative zero-cycles from [**I**, **II**]. We also show that the definition of a proper relative zero-cycle on $X/S$ is local on $X$ with respect to finite étale coverings.

In Section 2, we define *non-proper* relative zero-cycles. This is done by working étale-locally on the carrier scheme $X$. A non-proper relative zero-cycle is a proper relative zero-cycle if and only if its support is proper. This gives a new étale-local definition of proper relative zero-cycles.

In Section 3, a topological condition (T) on morphisms is introduced. This condition is closely related to open morphisms. In fact, if $S$ is locally noetherian and $f : X \to S$ is locally of finite type, then $f$ is universally open if and only if it is universally (T). Universally open morphisms and equidimensional morphisms satisfy (T).

In Section 4, we define higher-dimensional relative cycles. A priori, the support of a higher-dimensional relative cycle only satisfies (T), but we show that the support is universally open.

In Section 5, we show that in the definition of a relative cycle, it is enough to consider *smooth* projections. We then briefly discuss how the definition of a relative cycle can be modified so that pull-back by smooth morphisms exist.

In Section 6, conditions for when a relative cycle on an open subset $U \subseteq X$ extends to a relative cycle on $X$ are given. We also give a slightly generalized version of Chevalley's theorem on universally open morphisms.

In Section 7, we show that any flat and finitely presented sheaf $\mathcal{F}$ induces a relative cycle, the *norm family*. We thus obtain morphisms from the Hilbert and the Quot functors to the Chow functor.

In Section 8, we associate an ordinary cycle $\operatorname{cycl}(\alpha)$ on $X$ to any relative cycle $\alpha$ on $X/S$. If $S$ is reduced, this cycle uniquely determines $\alpha$. This is (A1).

In Section 9, we show that smooth relative cycles correspond to smooth subschemes and that relative Weil divisors on smooth carrier schemes correspond to relative Cartier divisors. This is (B3) and (B5). Assuming only generic smoothness, we obtain the descriptions (A2) and (A3). These results are only shown in characteristic zero but are presumably valid in arbitrary characteristic.

In Section 10, we study relative cycles over reduced parameter schemes and obtain the characterizations (B1) and (B2). We also describe the pull-back of cycles via Samuel multiplicities.

In Section 11, we introduce $n$-flat and $n$-smooth morphisms and give the characterizations (B6)–(B9). In Section 12, we prove a generalized Hironaka lemma. Together with (B7), this result yields (B4). In particular, the Hilb-Chow morphism is an isomorphism over the locus parameterizing normal subschemes. All these results depend upon (B3) but are otherwise characteristic-free.

In Sections 13–15, proper push-forward, flat pull-back and intersections of relative cycles are discussed. In Section 16, we indicate the existence of a relative fundamental class to any relative cycle and compare our functor with Angéniol's and Barlet's functors. Finally, in Section 17, we discuss the incidence correspondence and the classical Chow-construction.

In the Appendix, an overview of duality and (relative) fundamental classes is given.

**Terminology and assumptions.** As families of cycles are defined étale-locally, the natural choice is to use algebraic spaces instead of schemes. In fact, all results are true for algebraic spaces. For convenience, we only treat relative cycles on $X/S$ where $S$ is a scheme, but this is no restriction as the definition is étale-local on both $S$ and $X$.

We allow relative cycles to have *non-closed* support. The reason for this is that if $\alpha$ is a relative cycle on $X/S$ then it decomposes as a sum $\alpha_0 + \alpha_1 + \cdots + \alpha_r$ where $\alpha_i$ is supported on the locally closed subset consisting of points of relative dimension $i$. It is likely that the assumption that a relative cycle has closed support is missing in some statements and the reader may choose to assume that all relative cycles have closed support (except in the example above).

Usually, a cycle is a finite formal sum of equidimensional closed subvarieties. As we treat relative cycles which are not equidimensional and also not necessarily closed, we make the following definition. A *cycle* $\alpha$ on $X$ is a locally closed subset $Z \subseteq X$ together with rational numbers $(m_{Z_i})$ indexed by the irreducible components $\{Z_i\}$ of $Z$. The irreducible sets $\{Z_i\}$ are the *components* of $\alpha$ and the numbers $(m_{Z_i})$ are the *multiplicities* of $\alpha$. When the $m_{Z_i}$'s are integers, we say that $\alpha$ is *integral*. When the $m_{Z_i}$'s are non-negative, we say that $\alpha$ is *effective*. Every cycle is uniquely represented as a formal sum $\sum_i m_{Z_i}[Z_i]$. This sum is locally finite if $X$ is locally noetherian. Note that this definition excludes cycles with embedded components. Through-out this paper we will only consider effective cycles.

Let $X/S$ be an algebraic space locally of finite type. We say that $x \in X$ has *relative codimension* $n$ if the codimension of $\overline{\{x\}}$ in its fiber $X_s$ is $n$. A useful fact is that if $X \to B$ is a quasi-finite morphism, $B \to S$ is smooth and $x \in X$ has relative codimension $n$ over a point of depth $m$ in $S$, then its image $b \in B$ has depth $n + m$. This is why the characterizations (A2)–(A3) and (B6)–(B9) only involves the codimension.

Noetherian assumptions are eliminated in many instances but often only with a brief sketch in the proof. Sometimes we use the notion of associated points on a non-noetherian schemes. In the terminology of Bourbaki these are the points corresponding to *weakly* associated prime ideals. These satisfy the usual properties of associated points, e.g., an open (retro-compact) subset $U \subseteq X$ is schematically dense if and only if $U$ contains all associated points. Recall that on a locally noetherian scheme $X$, a point $x \in X$ is associated if and only if $X$ has depth zero at $x$. In the non-noetherian case the condition that $S$ has (locally) a finite number of irreducible components appears frequently.

## Contents



## 1. Proper relative zero-cycles

We recall the main results of [**I**, **II**], here stated only for schemes locally of finite type. We then show that the definition of proper relative zero-cycles on $X/S$ is étale-local on $X$. At the end, the underlying cycle of a relative cycle is briefly discussed.

**Definition (1.1).** Let $f : Z \to S$ be affine. Then we let $\Gamma^d(Z/S)$ be the spectrum of the algebra of divided powers $\Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_Z)$.

**Definition (1.2).** Let $X/S$ be a separated algebraic space locally of finite type. A *relative zero-cycle of degree* $d$ on $X$ consists of a closed subscheme $Z \hookrightarrow X$ such that $Z \hookrightarrow X \to S$ is finite, together with a morphism $\alpha : S \to \Gamma^d(Z/S)$. Two relative zero-cycles $(Z_1, \alpha_1)$ and $(Z_2, \alpha_2)$ are *equivalent* if there is a closed subscheme $Z$ of both $Z_1$ and $Z_2$ and a morphism $\alpha : S \to \Gamma^d(Z/S)$ such that $\alpha_i$ is the composition of $\alpha$ and the morphism $\Gamma^d(Z/S) \hookrightarrow \Gamma^d(Z_i/S)$ for $i = 1, 2$.

If $g : S' \to S$ is a morphism of spaces and $(Z, \alpha)$ is a relative cycle on $X/S$, we let $g^*(Z, \alpha) = \bigl(g^{-1}(Z), g^*\alpha\bigr)$ be the pull-back along $g$.

If $(Z, \alpha)$ is a relative zero-cycle, then there is a unique minimal closed subscheme $\mathrm{Image}(\alpha) \hookrightarrow Z$ such that $(Z, \alpha)$ is equivalent to a relative zero-cycle $(\mathrm{Image}(\alpha), \alpha')$. The subscheme $\mathrm{Image}(\alpha)$ is called the *image* of $\alpha$ and its reduction $\mathrm{Supp}(\alpha) := \mathrm{Image}(\alpha)_{\mathrm{red}}$ is the *support* of $\alpha$. The image commutes with smooth base change but not with arbitrary base change. The support commutes with arbitrary base change in the sense that $\mathrm{Supp}(\alpha \times_S S') = (\mathrm{Supp}(\alpha) \times_S S')_{\mathrm{red}}$.

**Notation (1.3).** If $\alpha : S \to \Gamma^d(X/S)$ is a morphism then we will by abuse of notation often write $\alpha$ for the first map in the canonical factorization $S \to \Gamma^d(\mathrm{Image}(\alpha)/S) \hookrightarrow \Gamma^d(X/S)$ of $\alpha$.

**Definition (1.4).** Let $X/S$ be a separated algebraic space locally of finite type. We let $\underline{\Gamma}^d_{X/S}$ be the contravariant functor from $S$-schemes to sets defined as follows. For any $S$-scheme $T$ we let $\underline{\Gamma}^d_{X/S}(T)$ be the set of equivalence classes of relative zero cycles $(Z, \alpha)$ of degree $d$ on $X \times_S T/T$. For any morphism $g : T' \to T$ of $S$-schemes, the map $\underline{\Gamma}^d_{X/S}(g)$ is the pull-back of relative cycles as defined above.

An element of $\underline{\Gamma}^d_{X/S}(T)$ will be called a *family of zero cycles of degree* $d$ on $X/S$ parameterized by $T$. By abuse of notation, henceforth a relative cycle will always denote an equivalence class of relative cycles. The main result of [**I**] is that $\underline{\Gamma}^d_{X/S}$ is representable by a separated algebraic space $\Gamma^d(X/S)$ — the scheme of divided powers — which coincides with the scheme in Definition (1.1) when $X/S$ is affine. If $X$ is locally of finite type (resp. locally of finite presentation) over $S$ then so is $\Gamma^d(X/S)$.

**Definition (1.5)** ([**II**, Def. 2.1]). Let $X/S$ be a separated scheme (or algebraic space), locally of finite type over $S$. We let $\Gamma^\star(X/S) = \coprod_{d\geq 0} \Gamma^d(X/S)$. A *proper family of zero-cycles* on $X/S$ parameterized by $T$ is a morphism $\alpha : T \to \Gamma^\star(X/S)$. A *proper relative zero-cycle* on $X/S$ is a morphism $\alpha : S \to \Gamma^\star(X/S)$. If the image of a point $s \in S$ by $\alpha$ lies in $\Gamma^d(X/S)$ then we say that $\alpha$ has degree $d$ at $s$.

If $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ are affine, then $\Gamma^\star(X/S)$ represents multiplicative laws which are not necessarily homogeneous [**II**, Thm. 2.3]. To be precise, if $T = \mathrm{Spec}(A')$ then $\mathrm{Hom}_S(T, \Gamma^\star(X/S))$ is the set of multiplicative laws $B \to A'$.

We will use the following results and constructions from [**I**, **II**]:

(i) If $f : X \to Y$ is a morphism, then the *push-forward* along $f$ is the morphism $f_* : \Gamma^d(X/S) \to \Gamma^d(Y/S)$ taking a family $(Z, \alpha)$ onto the family $(f_T(Z), f_*\alpha)$. Here $f_*\alpha$ is the composition of $\alpha : T \to \Gamma^d(Z/T)$ and $\Gamma^d(Z/T) \to \Gamma^d(f_T(Z)/T)$. The image does not commute with the push-forward in general, but the support does, i.e., $\mathrm{Supp}(f_*(\alpha)) = f_T(\mathrm{Supp}(\alpha))$ [**I**, 3.3.7].

(ii) The image $\mathrm{Image}(\alpha) \hookrightarrow X \times_S T$ of a proper family of cycles $\alpha : T \to \Gamma^\star(X/S)$ is finite and universally open over $T$ [**I**, 2.4.6, 2.5.7].

(iii) If $T$ is a reduced scheme, then the image $\mathrm{Image}(\alpha) \hookrightarrow X \times_S T$ of a family $\alpha : T \to \Gamma^\star(X/S)$ is reduced [**I**, 2.4.6].

(iv) If $k$ is an algebraically closed field, then there is a one-to-one correspondence between $k$-points of $\Gamma^d(X/S)$ and effective zero cycles of degree $d$ on $X \times_S \mathrm{Spec}(k)$ [**I**, 3.1.9].

(v) If $f : Z \to S$ is finite and flat of finite presentation, i.e., such that $f_*\mathcal{O}_Z$ is a locally free $\mathcal{O}_S$-module, then there is a canonical family $\mathcal{N}_{Z/S} : S \to \Gamma^\star(Z/S)$, the *norm* of $f$. The support of $\mathcal{N}_{Z/S}$ is $Z_{\mathrm{red}}$ but in general the image of $\mathcal{N}_{Z/S}$ can be smaller than $Z$. If $f$ is in addition *étale* then $\mathrm{Image}(\mathcal{N}_{Z/S}) = Z$ and the image commutes with arbitrary base change [**II**, Prop. 3.2]. More generally, if $X/S$ is affine, then a norm family $\mathcal{N}_{\mathcal{F}/S} : S \to \Gamma^\star(X/S)$ is defined for any quasi-coherent sheaf $\mathcal{F}$ on $X$ such that $f_*\mathcal{F}$ is locally free.

(vi) If $\alpha$ is a family of degree $d$ parameterized by $T$ such that for any algebraically closed field $k$ and point $t : \mathrm{Spec}(k) \to T$ the support of $\alpha_t$ has (at least) $d$ points then we say that $\alpha$ is *non-degenerate*. Then $Z = \mathrm{Image}(\alpha)$ is étale of constant rank $d$ and $\alpha = \mathcal{N}_{Z/T}$ [**II**, Cor. 5.7].

(vii) If $X/S$ is a smooth curve, then $\Gamma^d(X/S) \cong \mathrm{Hilb}^d(X/S)$, i.e., for any relative cycle $\alpha$ on $X/S$ there is a unique subscheme $Z \hookrightarrow X$, flat and finite over $S$, such that $\mathcal{N}_Z = \alpha$ [**II**, Prop. 5.8]. Note however that $\mathrm{Image}(\alpha)$ does not always equal $Z$.

(viii) Let $U/X$ be a separated algebraic space. If $\alpha$ is a proper relative zero-cycle on $X/S$ and $\beta$ is a proper relative zero-cycle on $U/X$, then there is a proper relative zero-cycle $\alpha * \beta$ on $U/S$. If $\alpha$ and $\beta$ have degrees $d$ and $e$ respectively, then $\alpha * \beta$ has degree $de$. If $T$ is the spectrum of an algebraically closed field and $\alpha$ corresponds to the cycle $[x_1] + [x_2] + \cdots + [x_d]$, then $\alpha * \beta$ corresponds to the cycle $\beta_{x_1} + \beta_{x_2} + \cdots + \beta_{x_d}$ [**II**, §7].

We will now show that proper relative zero-cycles of $X/S$ can be defined étale-locally on $X$.

**Definition (1.6).** Let $X/S$ be a separated algebraic space and let $f : U \to X$ be a separated and étale morphism. Let $\alpha : S \to \Gamma^\star(X/S)$ be a proper family of zero-cycles on $X$ and assume that $f$ is proper over the support of $\alpha$. Then $f$ is étale and finite over $Z = \mathrm{Image}(\alpha)$ and we let $f^*(\alpha) = \alpha * \mathcal{N}_{f|_Z}$ which we by push-forward consider as a family on $U/S$. When $f$ is an open immersion, then we let $\alpha|_U = f^*\alpha$.

The notation $f^*(\alpha)$ is reasonable in view of the following results:

**Proposition (1.7)** ([**II**, Prop. 7.5])**.** *Let $X/S$ be a separated algebraic space and let $\alpha$ be a proper relative cycle on $X/S$. Let $f : S' \to S$ be a finite étale morphism and denote by $g : X' \to X$ the pull-back of $f$ along $X \to S$. Then $g^*\alpha = \mathcal{N}_f * f^*\alpha$.*

**Lemma (1.8).** *Let $X/S$ be a separated algebraic space, $\alpha$ a proper relative zero-cycle on $X/S$ and $p : U \to X$ an étale morphism, finite over* $\mathrm{Supp}(\alpha)$. *Then* $\mathrm{Image}(p^*\alpha) = p^{-1}(\mathrm{Image}(\alpha))$.

*Proof.* As the image and composition commutes with étale base change, it is enough to show the equality on an étale cover $S' \to S$. Since the image of $\alpha$ is finite over $S$, we can thus assume that $p|_{\mathrm{Image}(\alpha)}$ is a trivial étale cover. Then both sides of the equality become disjoint unions of copies of $\mathrm{Image}(\alpha)$. □

**Proposition (1.9)** (Étale descent)**.** *Let $X/S$ be a separated algebraic space and let $p : U \to X$ be an étale surjective morphism. Let $\pi_1$ and $\pi_2$ be the projections of $U \times_X U$ onto the two factors. Let $\beta$ be a proper relative cycle on $U/S$ such that the $\pi_i$'s are finite over the support of $\beta$ and $\pi_1^*\beta = \pi_2^*\beta$. Then there is a unique proper relative cycle $\alpha$ on $X/S$ such that $\beta = p^*\alpha$.*

*Proof.* Let $W \hookrightarrow U$ be the image of $\beta$. Then $\pi_1^{-1}(W) = \pi_2^{-1}(W)$ by Lemma (1.8) and thus we obtain by étale descent, a closed subspace $Z \hookrightarrow X$ such that $p^{-1}(Z) = W$. Replacing $X$ with $Z$ we can thus assume that $X/S$ is finite and that $p$ is finite and étale.

As $X/S$ is finite, there is, for any point $s \in S$, an étale neighborhood $S' \to S$ of $s$ such that $U \times_S S' \to X \times_S S'$ has a section $s$ which is an open and closed immersion. We define $\alpha' : S' \to \Gamma^\star(X/S)$ as $s^*(\beta \times_S S')$. As $\pi_1^*\beta = \pi_2^*\beta$, it follows that $\alpha'$ is independent on the choice of section. Furthermore, it follows that the pull-backs of $\alpha'$ along the two projections of $S' \times_S S'$ coincide. By étale descent we obtain, locally around $s$, a unique family $\alpha : S \to \Gamma^\star(X/S)$ as in the proposition. □

**Definition (1.10).** Let $S$ be the spectrum of a field $k$ and let $\alpha$ be a relative zero-cycle on $X/S$ and let $x \in X$ be a point. Let $Z = \mathrm{Supp}(\alpha)$. If $x \notin Z$, then we let $\deg_x \alpha = \mathrm{mult}_x \alpha = 0$. If $x \in Z$, then we let $\deg_x \alpha$ be the degree of $\alpha|_U$ for any open neighborhood $U \subseteq X$ such that $U \cap Z = \{x\}$ and we let $\mathrm{mult}_x \alpha$ be the rational number such that $(\mathrm{mult}_x \alpha) \deg\big(k(x)/k\big) = \deg_x \alpha$. The *geometric multiplicity* of $\alpha$ at $x$, denoted $\mathrm{geom.mult}_x \alpha$ is the multiple of $\mathrm{mult}_x$ and $\deg_{\mathrm{insep}}\big(k(x)/k\big)$).

Let $S$ be an arbitrary scheme and let $\alpha$ be a proper relative zero-cycle on $X/S$. Let $x \in X$ be a point with image $s \in S$. Then we let the degree (resp. multiplicity, resp. geometric multiplicity) of $\alpha$ at $x$, be the degree (resp. multiplicity, resp. geometric multiplicity) of $\alpha_s$ at $x$. Here $\alpha_s$ denotes the relative zero-cycle $s^*\alpha$ on $X_s/\mathrm{Spec}(k(s))$.

**Definition (1.11).** Let $S$ be an arbitrary scheme and let $\alpha$ be a relative zero-cycle on $X/S$. The *underlying cycle* of $\alpha$ is the zero-cycle

$$\mathrm{cycl}(\alpha) = \sum_{x \in \mathrm{Supp}(\alpha)_{\max}} \mathrm{mult}_x(\alpha) \left[\overline{\{x\}}\right].$$

*Remark* **(1.12).** If $\alpha$ is a relative zero-cycle on $X/\operatorname{Spec}(k)$, then

$$\deg(\operatorname{cycl}(\alpha)) = \sum_{x\in\operatorname{Supp}(\alpha)} \operatorname{mult}_x(\alpha)\deg\bigl(k(x)/k\bigr) = \sum_{x\in\operatorname{Supp}(\alpha)} \deg_x\alpha = \deg(\alpha).$$

The assignment $\alpha \mapsto \operatorname{cycl}(\alpha)$ induces a one-to-one correspondence between relative zero-cycles on $X/k$ and cycles with *quasi-integral* coefficients [**II**, Prop. 9.11].

**Proposition (1.13).** *Let $k$ be a field, let $\alpha$ be a relative zero-cycle on $X/\operatorname{Spec}(k)$ and let $x \in X$ be a point. Let $k'/k$ be a field extension and $\alpha'$ be the relative zero-cycle on $X_{k'}/k'$ given by pull-back. Then*

(i) *The degree of $\alpha$ at $x$ equals the sum of the degrees of $\alpha'$ at the points above $x$.*
(ii) *The geometric multiplicities of $\alpha$ at $x$ and of $\alpha'$ at any point $x'$ above $x$ coincide.*
(iii) *Taking the underlying cycle commutes with the base change $k'/k$, that is, $\operatorname{cycl}(\alpha)_{k'} = \operatorname{cycl}(\alpha')$.*

*Let $p : U \to X$ be an étale morphism and $u \in U$ a point mapping to $x$. Then*

(iv) *The multiplicity of $\alpha$ at $x$ and of $p^*\alpha$ at $u$ coincide.*
(v) *Taking the underlying cycle commutes with the pull-back along $p$, that is, $p^*\operatorname{cycl}(\alpha) = \operatorname{cycl}(p^*\alpha)$.*

*Let $k/k_0$ be a field extension, then*

(vi) *The multiplicity of $\alpha$ at $x$ and the multiplicity of the family $\mathcal{N}_{k/k_0} * \alpha$ on $X/\operatorname{Spec}(k_0)$ at $x$ coincide.*

*Proof.* Follows easily from Remark (1.12) and the observation that the degree of $\mathcal{N}_{k/k_0} * \alpha$ at $x$ is $\deg(k/k_0)\deg_x\alpha$. □

**Definition (1.14)** (Trace)**.** Let $f : X \to S$ be an affine morphism and let $\alpha : S \to \Gamma^\star(X/S)$ be a proper relative zero-cycle on $X/S$. The *trace* of $\alpha$ is the $\mathcal{O}_S$-module homomorphism $\operatorname{tr}(\alpha) : f_*\mathcal{O}_X \to \mathcal{O}_S$ given as the composition of

$$f_*\mathcal{O}_X \to \Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_X), \quad g \mapsto \gamma^1(g) \times \gamma^{d-1}(1)$$

and $\alpha^* : \prod_d \Gamma^d_{\mathcal{O}_S}(f_*\mathcal{O}_X) \to \mathcal{O}_S$.

If $Z = \operatorname{Image}(\alpha)$, then the trace of $\alpha$ factors through $f_*\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_Z$. If $\mathcal{F}$ is a sheaf on $X$ such that $f_*\mathcal{F}$ is locally free, then $\operatorname{tr}(\mathcal{N}_{\mathcal{F}})$ is the usual trace of the representation $f_*\mathcal{O}_X \to \operatorname{End}_{\mathcal{O}_S}(f_*\mathcal{F})$.

## 2. Non-proper relative zero-cycles

We now introduce the notion of *non-proper* relative zero-cycles, or equivalently, non-proper families of zero-cycles, as a first step towards the generalization to higher dimensions. We define non-proper families in great generality, including non-separated schemes and families with support which is not closed.

A non-proper family of zero-cycles should be viewed as an analog of a subscheme $Z \hookrightarrow X \times_S T$ which is flat, locally quasi-finite and locally of finite presentation over

$T$, but not necessarily proper. Every such subscheme $Z$ also defines a non-proper family. More generally, we assign a non-proper family to any coherent sheaf which has finite flat dimension in Section 7. For an étale morphism $p : U \to X$ we define the pull-back $p^*$ of relative zero-cycles which is the ordinary inverse image for flat subschemes.

**Definition (2.1).** Let $X$ be an algebraic space locally of finite type over $S$ and let $Z \hookrightarrow X$ be a locally closed subset such that $Z \to S$ is locally quasi-finite. A *neighborhood of $X/S$ adapted to $Z$* is a commutative square

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d] \arrow[dr, phantom, "\circ"] & X \arrow[d] \\ T \arrow[r, "g"] & S \end{tikzcd}$$

such that $U \to X \times_S T$ is étale and $p^{-1}(Z) \to T$ is *finite*. We will denote such a neighborhood with $(U, T, p, g)$. If $g$ is étale (resp. smooth) then we say that $(U, T, p, g)$ is an étale (resp. smooth) neighborhood.

A morphism of neighborhoods $(U_1, T_1, p_1, g_1) \to (U_2, T_2, p_2, g_2)$ is a pair of morphisms $p : U_1 \to U_2$ and $g : T_1 \to T_2$ such that

$$\begin{tikzcd} U_1 \arrow[r, "p"'] \arrow[d] \arrow[rr, bend left, "p_1"] & U_2 \arrow[r, "p_2"'] \arrow[d] & X \arrow[d] \\ T_1 \arrow[r, "g"] \arrow[rr, bend right, "g_1"'] & T_2 \arrow[r, "g_2"] & S \end{tikzcd}$$

is commutative.

*Remark* **(2.2).** If $(p, g)$ is a morphism of neighborhoods as in the definition, then $(U_1, T_1, p, g)$ is a neighborhood of $U_2/T_2$ adapted to $p_2^{-1}(Z)$. In fact, as $U_1 \to X$ and $U_2 \to X$ are étale it follows that $U_1 \to U_2 \times_{T_2} T_1$ is étale. Moreover, $U_1 \to U_2 \times_{T_2} T_1$ is proper over $p_2^{-1}(Z) \times_{T_2} T_1$.

Recall that a subset $Z \subseteq X$ is *retro-compact* if $Z \cap U$ is quasi-compact for any quasi-compact open subset $U \subseteq X$ [$\mathrm{EGA}_{\mathrm{III}}$, Def. 0.9.1.1]. If $X$ is locally noetherian, then any subset $Z \subseteq X$ is retro-compact.

**Definition (2.3).** Let $X$ be an algebraic space locally of finite type over $S$. A (non-proper) relative zero-cycle on $X/S$ consists of the following data

(i) A locally closed retro-compact subset $Z$ of $X$ — the *support* of the cycle.
(ii) For every neighborhood $(U, T, p, g)$ of $X/S$ adapted to $Z$, a proper family of zero-cycles $\alpha_{U/T} : T \to \Gamma^\star(U/T)$ with support $p^{-1}(Z)$.

satisfying the following conditions:

(a) The support $Z \to S$ is equidimensional of relative dimension zero, i.e., locally quasi-finite and every irreducible component of $Z$ dominates an irreducible component of $S$.

(b) For every morphism $(p,g) : (U_1, T_1, p_1, g_1) \to (U_2, T_2, p_2, g_2)$ of neighborhoods we have that
$$\alpha_{U_1/T_1} = p'^*(g^* \alpha_{U_2/T_2})$$
where $p' : U_1 \to U_2 \times_{T_2} T_1$ is the canonical étale morphism.

A *non-proper family of zero-cycles* on $X/S$ parameterized by an $S$-scheme $T$, is a relative zero-cycle on $X \times_S T/T$.

*Remark* **(2.4).** If $X/S$ is separated, then every proper relative zero-cycle $\alpha$ on $X/S$ determines a unique non-proper relative zero-cycle with the same support and such that $\alpha_{X/S} = \alpha$. Conversely, a non-proper relative zero-cycle is proper if and only if its support is proper over $S$. In fact, if $Z/S$ is proper then for every neighborhood $(U,T,p,g)$, we have that $\alpha_{U/T}$ is determined by $\alpha_{X/S}$ according to condition (b).

**Definition (2.5).** We let $Cycl_0(X/S)$ be the set of relative zero-cycles on $X/S$. We also let $Cycl^0_{X/S}$ denote the functor from $S$-schemes to sets such that $Cycl^0_{X/S}(T) = Cycl^0(X \times_S T)$ and the pull-back is the natural map.

As $\Gamma^\star(U/T)$ is representable, it immediately follows that $Cycl^0_{X/S}$ is an fppf-sheaf.

**(2.6)** *Push-forward* — If $f : X \hookrightarrow Y$ is a quasi-compact immersion, and $\alpha$ is a non-proper relative zero-cycle on $X/S$, then there is an induced non-proper relative zero-cycle $f_*\alpha$ on $Y$. More generally, if $f : X \to Y$ is a morphism, and $\alpha$ is a non-proper relative zero-cycle on $X/S$ such that $f(\mathrm{Supp}(\alpha))$ is locally closed and $f|_{\mathrm{Supp}(\alpha)}$ is proper onto its image, then we can define $f_*\alpha$. In particular, this is the case if $f$ is proper and $\mathrm{Supp}(\alpha) \subseteq X$ is closed or if $\mathrm{Supp}(\alpha)/S$ is proper and $Y/S$ is separated.

**(2.7)** *Addition of cycles* — Let $\alpha$ and $\beta$ be relative zero-cycles on $X/S$ with supports $Z_\alpha$ and $Z_\beta$. If $Z_\alpha$ and $Z_\beta$ are closed in $Z_\alpha \cup Z_\beta$, e.g., if $Z_\alpha$ and $Z_\beta$ are closed in $X$, then there is a relative zero-cycle $\alpha+\beta$ on $X/S$ with support $Z_\alpha \cup Z_\beta$ defined by $(\alpha+\beta)_{U/T} = \alpha_{U/T} + \beta_{U/T}$ for any neighborhood $(U,T)$ adapted to $Z_\alpha \cup Z_\beta$.

The condition on $Z_\alpha$ and $Z_\beta$ is equivalent with the condition that $f : Z_\alpha \amalg Z_\beta \to Z_\alpha \cup Z_\beta$ is proper. This is necessary to ensure that a neighborhood adapted to $Z_\alpha \cup Z_\beta$ also is adapted to $Z_\alpha$ and $Z_\beta$. This condition also implies that $Z_\alpha \cup Z_\beta$ is locally closed. We have that $\alpha + \beta = f_*(\alpha \amalg \beta)$.

**(2.8)** *Flat zero-cycles* — If $Z/S$ is locally quasi-finite, flat and locally of finite presentation, then there is a non-proper family $\mathcal{N}_{Z/S}$ of zero-cycles on $Z/S$ with support $Z_{\mathrm{red}}$. This family is defined by $(\mathcal{N}_{Z/S})_{U/T} = \mathcal{N}_{p^{-1}(Z)/T}$ on any projection $(U,T,p)$. The compatibility condition (b) follows from the functoriality of the norm.

**(2.9)** *Relative cycles on smooth curves* — If $X/S$ is a smooth curve, then any relative cycle $\alpha$ on $X/S$ is the norm family $\mathcal{N}_{Z/S}$ of a unique subscheme $Z \hookrightarrow X$.

**(2.10)** *Pull-back* — If $\alpha$ is a relative zero-cycle on $X/S$ and $f : U \to X$ is an étale morphism, then we define the relative zero-cycle $f^*\alpha$ on $U/S$ as follows. The support of $f^*\alpha$ is $f^{-1}(\mathrm{Supp}(\alpha))$ and for any neighborhood $(V, T, p, g)$ adapted to $f^{-1}(\mathrm{Supp}(\alpha))$ we let $(f^*\alpha)_{V/T} = \alpha_{V/T, f\circ p, g}$.

**Lemma (2.11)** (Existence of étale neighborhoods)**.** *Let $X/S$ be an algebraic space, locally of finite type. Let $Z \hookrightarrow X$ be a locally closed subspace such that $Z \to S$ is locally quasi-finite. Then for every point $z \in Z$, there exists an étale neighborhood $(U, T, p, g)$ of $X/S$ adapted to $Z$ such that there exists $u \in U$ such that $z = p(u)$ and $k(z) \to k(u)$ is an isomorphism. Furthermore, we can assume that $u$ is the only point in its fiber $U_t \cap p^{-1}(Z)$. If $X/S$ is a scheme or a separated algebraic space, then we can furthermore choose $U \to X \times_S T$ as an open immersion.*

*Proof.* Replacing $X$ with an étale neighborhood of $z$ in $X$, we can assume that $X$ is a scheme [Knu71, Thm. II.6.4] and that $z$ is the only point in its fiber over $S$. It then follows from [$\mathrm{EGA}_{\mathrm{IV}}$, Thm. 18.12.1 and Rmk. 18.12.2] that there is an étale morphism $g : T \to S$, a point $z' \in Z_T = Z \times_S T$ above $z$ such that $k(z) \cong k(z')$ and an open neighborhood $V$ of $z'$ in $Z_T$ such that $V \to T$ is finite. Any open subscheme $U$ of $X \times_S T$ such that $U \cap Z_T = V$ gives an étale neighborhood as in the lemma.

If $X/S$ is a scheme, then the last statement follows immediately, as we can skip the first step in the construction of $U$. If $X/S$ is a separated algebraic space then nevertheless $Z/S$ is a scheme [LMB00, Thm. A.2] and the statement follows. □

*Remark* **(2.12).** If $X/S$ is a *locally separated* algebraic space, then we can choose $U \to X \times_S T$ as an open immersion if we drop the condition that $k(u)/k(z)$ is a trivial extension. This follows from the fact that if $S$ is a *strictly* henselian local scheme and if $Z \to S$ is a locally separated quasi-finite morphism, then $Z \to S$ is finite over an open subset containing the closed fiber. This can be shown similarly as [LMB00, Lem. A.1].

**Proposition (2.13).** *The support of a relative zero-cycle is universally open and hence universally equidimensional of relative dimension zero.*

*Proof.* This follows immediately from Lemma (2.11) as the support of a proper relative zero-cycle is universally open. □

**Definition (2.14).** Let $X/S$ be locally of finite type and let $\alpha$ be a relative zero-cycle on $X/S$. Let $x \in X$. The degree (resp. multiplicity, resp. geometric multiplicity) of $\alpha$ at $x$ is the corresponding number of $(\alpha_s)|_U$ at $x$ for any neighborhood $U \subseteq X_s$ of $x$ such that $\mathrm{Supp}(\alpha_s)|_U$ is finite. We say that $\alpha$ is *non-degenerate* or *étale* at $x$ if $\mathrm{geom.\,mult}_x(\alpha) = 1$.

**Proposition (2.15).** *Let $X/S$ be an algebraic space, locally of finite type, and let $\alpha$ be a relative zero-cycle on $X/S$. Then the function* $\operatorname{geom.mult} : \operatorname{Supp}(\alpha) \to \mathbb{N}$, $x \mapsto \operatorname{geom.mult}_x(\alpha)$ *is upper semi-continuous. In particular, $\alpha$ is étale at an open subset of* $\operatorname{Supp}(\alpha)$.

*Proof.* Let $x \in \operatorname{Supp}(\alpha)$ be a point with geometric multiplicity $m$. We have to show that the geometric multiplicity is at most $m$ in a neighborhood of $x$. This can be checked on any étale neighborhood and we can thus assume that $\operatorname{Supp}(\alpha) \to S$ is finite, that $x$ is the only point in its fiber $\operatorname{Supp}(\alpha)_s$ and that $k(x)/k(s)$ is purely inseparable. Then $m$ is the degree of $\alpha$ at $s$. As the degree of $\alpha$ is $m$ in a neighborhood of $S$, the geometric multiplicity of $\alpha$ is at most $m$ in a neighborhood of $x$. $\square$

**Definition (2.16).** Let $X/\operatorname{Spec}(k)$ be locally of finite type and let $\alpha$ be a relative zero-cycle on $X/\operatorname{Spec}(k)$. If $\operatorname{Supp}(\alpha)$ is finite, then $\alpha$ is a proper relative zero-cycle and $\deg(\alpha)$ is defined. If $\operatorname{Supp}(\alpha)$ is infinite, then we let $\deg(\alpha) = \infty$.

**Proposition (2.17).** *Let $X/S$ be a* separated *algebraic space, locally of finite type, and let $\alpha$ be a relative zero-cycle on $X/S$. Then the function* $\deg : S \to \mathbb{N} \cup \{\infty\}$, $s \mapsto \deg(\alpha_s)$ *is lower semi-continuous, i.e., for every $d \in \mathbb{N}$, the subset of $S$ where* deg *is at most $d$ is closed. The relative cycle $\alpha$ is* proper *if and only if* deg *is finite and locally constant.*

*Proof.* Let $s \in S$ such that $d = \deg(\alpha_s)$ is finite. Let $Z = \operatorname{Supp}(\alpha)$. Let $(S', s') \to (S, s)$ be an étale neighborhood such that $Z \times_S S' = Z'_1 \amalg Z'_2$ where $Z'_1 \to S'$ is finite of rank $d$ and $(Z'_2)_{s'}$ is empty [$\mathrm{EGA_{IV}}$, Thm. 18.12.1]. Then $\deg(\alpha \times_S S') \geq d$ over the image of $Z'_1 \to S'$ which is an open neighborhood of $s'$ as $Z \to S$ is universally open. Hence deg is lower semi-continuous.

Assume that $\deg(s) = d$ for all $s \in S$. Then in the above construction it follows that the images of $Z'_1$ and $Z'_2$ does not intersect. It follows that over the image of $Z'_1$, which is open, $\operatorname{Supp}(\alpha \times_S S')$ is finite. By étale descent, so is $\operatorname{Supp}(\alpha)$ in a neighborhood of $s$. $\square$

**Proposition (2.18)** (Étale descent)**.** *Let $X/S$ be an algebraic space and let $p : U \to X$ be an étale morphism. Let $\pi_1$ and $\pi_2$ be the projections of $U \times_X U$ onto the first and second factors. Let $\beta$ be a relative zero-cycle on $U/S$ such that $\pi_1^*\beta = \pi_2^*\beta$. Then there is a unique relative zero-cycle $\alpha$ on $X/S$ with support contained in $p(U)$ such that $\beta = p^*\alpha$.*

*Proof.* As $\pi_1^{-1}(\operatorname{Supp}(\beta)) = \pi_2^{-1}(\operatorname{Supp}(\beta))$ we obtain by étale descent of quasi-compact immersions [$\mathrm{SGA_1}$, 5.5 and 7.9], a locally closed retro-compact subscheme $Z \hookrightarrow X$ such that $p^{-1}(Z) = \operatorname{Supp}(\beta)$ and $Z$ is contained in $p(U)$. The support of $\alpha$ will be $Z$.

Let $(V, T, q, g)$ be a neighborhood of $X/S$ adapted to $Z$. We will construct a canonical proper family on $V/T$ which is compatible with $\beta$. We let $W = U \times_X V$

such that

$$\begin{tikzcd} W \times_V W \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & W \arrow[r] \arrow[d, "r"] & V \arrow[d, "q'"] \\ U_T \times_{X_T} U_T \arrow[r, shift left] \arrow[r, shift right] & U_T \arrow[r, "p"] & X_T \end{tikzcd}$$

is cartesian. The family $r^*(\beta \times_S T)$ is compatible with respect to the two projections of $W \times_V W$. Replacing $S$, $X$ and $U$ with $T$, $V$ and $W$ respectively, we can thus assume that $X/S$ itself is adapted to $Z$.

The support of $\beta$ is $p^{-1}(Z)$. Lemma (2.11) gives an étale neighborhood $(V, T, q, g)$ of $U/S$ adapted to $p^{-1}(Z)$ such that $p^{-1}(Z)$ is contained in the image of $q : V \to U$. If we construct a unique proper family $\alpha' : T \to \Gamma^\star(X/S)$ then the existence of the proper family $\alpha : S \to \Gamma^\star(X/S)$ follows by étale descent. We can thus replace $S$ with $T$ and assume that there is an étale neighborhood $(V, S, q, g)$ of $U/S$ adapted to $p^{-1}(Z)$. By the compatibility of the family $\beta$, we can finally replace $U$ with $V$. Then $\beta$ is proper and the result follows from Proposition (1.9). □

*Remark* **(2.19).** An easy special case of the proposition is the following situation. Let $X/S$ be an algebraic space and let $X = \bigcup_i U_i$ be an open covering. Given non-proper families $\alpha_i$ on $U_i/S$ which coincide on the intersections, there is then a unique family $\alpha$ on $U/S$ such that $\alpha_{U_i/S} = \alpha_i$.

**Corollary (2.20).** *In the definition of non-proper relative zero-cycles, it is enough to only consider étale neighborhoods $(U, T, p, g)$ of $X/S$, i.e., neighborhoods such that $g : T \to S$ is étale. Furthermore, we can require that $U$ and $T$ are affine.*

*Proof.* Follows immediately from Lemma (2.11) and Proposition (2.18). □

**(2.21)** *Composition of relative zero-cycles* — Let $X/Y$ and $Y/S$ be algebraic spaces locally of finite type. Let $\alpha$ be a relative zero-cycle on $Y/S$ and $\beta$ a relative zero-cycle on $f : X \to Y$. Then there is a natural relative zero-cycle $\alpha * \beta$ on $X/S$ with support $Z = f^{-1}\mathrm{Supp}(\alpha) \cap \mathrm{Supp}(\beta)$ such that when $f : X \to Y$ is étale, we have that $f^*\alpha = \alpha * \mathcal{N}_f$. Also $*$ will be associative. We define $\alpha * \beta$ as follows.

It is by Proposition (2.18) enough to define $\alpha*\beta$ on an étale cover of $X$. Replacing $X$ and $Y$ with étale covers, we can thus assume that $X$ and $Y$ are separated. Let $Z = \mathrm{Supp}(\alpha * \beta)$ and let $(U, T, p, g)$ be a neighborhood of $X/S$ adapted to $Z$. Let $p' : U \to X_T$ be the induced morphism. Let $W = f_T(p'(p^{-1}(Z))) \subseteq \mathrm{Supp}(\alpha) \times_S T$ which is an open subset as $\mathrm{Supp}(\beta) \to Y$ is universally open. Choose an open subset $V \subseteq Y_T$ restricting to $W$ and let $U' = p'^{-1}(f_T^{-1}(V)) \subseteq U$. Then $p^{-1}(Z) \subseteq U'$ and it is enough to define $(\alpha * \beta)_{U'/T}$.

As $p^{-1}(Z) \to W$ is surjective, we have that $W \to T$ is proper and thus $\alpha_{V/T}$ is defined. As $p^{-1}(Z) \to W$ is proper $\alpha_{U'/\mathrm{Image}(\alpha_{V/T})}$ is also defined. We let $(\alpha * \beta)_{U'/T} = \alpha_{V/T} * \alpha_{U'/\mathrm{Image}(\alpha_{V/T})}$.

**Proposition (2.22).** *Let $X/S$ be an algebraic space locally of finite type and let $\alpha$ be a relative zero-cycle on $X/S$. Let $f : S' \to S$ be an étale morphism and denote by $g : X' \to X$ the pull-back of $f$ along $X \to S$. Then $g^*\alpha = \mathcal{N}_f * f^*\alpha$.*

*Proof.* This follows from the construction of $*$ for non-proper relative cycles and the proper case, Proposition (1.7). □

The compatibility condition (b) of Definition (2.3) implies the following compatibility.

**Corollary (2.23).** *Let $X \to S$ be an algebraic space, locally of finite type which factors through an étale morphism $h : S' \to S$. If $\alpha$ is a relative zero-cycle on $X/S$ then $\mathcal{N}_{S'/S} * \alpha_{X/S'} = \alpha_{X/S}$. In particular, there is a one-to-one correspondence between relative zero-cycles on $X/S'$ and relative zero-cycles on $X/S$.*

*Proof.* Let $p : X' \to X$ be the pull-back of $h : S' \to S$. The factorization $X \to S' \to S$ induces an open section $s : X \to X'$ of $p$. Then

$$\begin{aligned}\alpha_{X/S} &= s^*p^*(\alpha_{X/S}) = s^*(\mathcal{N}_{S'/S} * h^*(\alpha_{X/S})) \\ &= s^*(\mathcal{N}_{S'/S} * \alpha_{X'/S'}) = \mathcal{N}_{S'/S} * s^*\alpha_{X'/S'} = \mathcal{N}_{S'/S} * \alpha_{X/S'}\end{aligned}$$

by Proposition (2.22). □

**Proposition (2.24).** *Let $\alpha$ be a relative zero-cycle on $X/S$ with support $Z \hookrightarrow X$. There is then a unique locally closed subspace* $\operatorname{Image}(\alpha) \hookrightarrow X$ *such that for any neighborhood $(U', S', p, g)$ we have that* $\operatorname{Image}(\alpha_{U'/S'}) \subseteq p^{-1}(\operatorname{Image}(\alpha))$ *with equality if $g$ is smooth. Moreover,* $\operatorname{Image}(\alpha)_{\mathrm{red}} = Z$.

*Proof.* Let $z \in Z$ and let $(U', S', p, g)$ be a smooth neighborhood such that $z$ is in the image of $p(U')$. Such a neighborhood exists by Lemma (2.11). Let $S'' = S' \times_S S'$, $X' = X \times_S S'$, $X'' = X \times_S S''$ and let $\pi_1, \pi_2$ be the two projections $X'' = X' \times_X X' \to X'$. Let $U''_i = \pi_i^* U'$, $i = 1, 2$ and $U'' = U' \times_X U' = U''_1 \times_{X''} U''_2$.

The image $W' = \operatorname{Image}(\alpha_{U'/S'})$ is an infinitesimal neighborhood of $Z' = p^{-1}(Z)$. As the image of a proper family of zero-cycles commutes with smooth base change it follows that

$$W''_i = \pi_i^{-1}(W') = \operatorname{Image}(\alpha_{U''_i/S''}).$$

Let $W'' = \operatorname{Image}(\alpha_{U''/S''})$. By the compatibility of $\alpha$ we have that $\alpha_{U''/S''} = \pi_i^* \alpha_{U''_i/S''}$. Furthermore, as $U''/U''_i$ is étale we have by Lemma (1.8) that the inverse image of $W''_i$ along $U'' \to U''_i$ is $W''$.

Thus $W' \hookrightarrow U'$ is a closed subscheme with support $Z'$ such that the inverse images of $W'$ along the projections of $U'' \to U' \times_X U'$ coincide. By fppf descent it thus follows that there is a closed subscheme $W \hookrightarrow p(U')$ such that $W' = p^{-1}(W)$. In particular, we have that $W_{\mathrm{red}} = Z \cap p(U')$. As it is obvious that $W$ does not depend on the choice of smooth neighborhood, there is a unique locally closed subspace $\operatorname{Image}(\alpha)$ such that $p^{-1}(\operatorname{Image}(\alpha)) = W$. □

If $\alpha$ is a relative zero-cycle on $X/S$ with image $Z$, then $\alpha$ is the push-forward of a relative zero-cycle on $Z/S$ along the immersion $Z \hookrightarrow X$. Also note that if $\alpha$ is étale with image $Z$, then $Z/S$ is étale and $\alpha = \mathcal{N}_{Z/S}$.

**(2.25)** *Trace* — Let $\alpha$ be a relative zero-cycle on $X/S$. Let $Z$ be the *image* of $\alpha$. For every neighborhood $(U, T, p, g)$ we obtain a *trace map* $h_* \mathcal{O}_{p^{-1}(Z)} \to \mathcal{O}_T$, cf. Definition (1.14). Here $h$ denotes the morphism $p^{-1}(Z) \to U \to T$.

**(2.26)** *Fundamental class* — Let $S$ be locally noetherian and let $X/S$ be separated and locally of finite type. Let $\alpha$ be a relative zero-cycle on $X/S$ and $Z \hookrightarrow X$ its image. Let $(U, T, p, g)$ be an étale neighborhood. By duality, cf. Appendix A, the trace map corresponds to a class in

$$\mathrm{H}^0(p^{-1}(Z), h^! \mathcal{O}_T) = \mathrm{H}^0(p^{-1}(Z), p^* \mathcal{D}^\bullet_{Z/S})$$

By the compatibility condition on $\alpha$, it follows that that this class is the restriction of a unique class in $\mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/S})$, the *relative fundamental class* of $\alpha$, cf. [AEZ78, Prop. II.2].

Let $j : Z \hookrightarrow X$ be the inclusion and assume that $j$ is closed. By duality, we then also have that

$$\mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/S}) = \mathrm{H}^0(Z, j^! \mathcal{D}^\bullet_{X/S}) = \mathrm{Ext}^0_X(j_* \mathcal{O}_Z, \mathcal{D}^\bullet_{X/S}).$$

This gives a unique class in $\mathrm{H}^0_{|Z|}(X, \mathcal{D}^\bullet_{X/S})$. In particular, if $X/S$ is smooth of relative dimension $n$, then this is a class in $\mathrm{H}^n_{|Z|}(X, \Omega^n_{X/S})$.

When $S$ is of characteristic zero, or the characteristic of $k(z)$ exceeds the geometric multiplicity of $\alpha$ at $z$ for every $z \in Z$, then the relative fundamental class, in either $\mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/S})$ or $\mathrm{H}^0_{|Z|}(X, \mathcal{D}^\bullet_{X/S})$, uniquely determines $\alpha$.

**(2.27)** *Fundamental class II* — Let $S$ be locally noetherian, let $q : B \to S$ be smooth of relative dimension $r$ and let $f : X \to B$ be separated and locally of finite type. Let $\alpha$ be a relative zero-cycle on $X/B$. Then we have the relative fundamental class $c_\alpha \in \mathrm{H}^0(Z, \mathcal{D}^\bullet_{Z/B})$. This gives an element

$$\begin{aligned}
\mathrm{Ext}^0_Z\big(h^*(\Omega^r_{B/S}), h^! \mathcal{O}_B \otimes_{\mathcal{O}_Z} h^*(\Omega^r_{B/S})\big) &= \mathrm{Ext}^0_Z\big(h^*(\Omega^r_{B/S}), h^!(\Omega^r_{B/S})\big) \\
&= \mathrm{Ext}^{-r}_Z\big(h^*(\Omega^r_{B/S}), h^! \mathcal{D}^\bullet_{B/S}\big) \\
&= \mathrm{Ext}^{-r}_Z\big(h^*(\Omega^r_{B/S}), \mathcal{D}^\bullet_{Z/S}\big).
\end{aligned}$$

If $\alpha$ is induced from a relative cycle of dimension $r$ on $X/S$, then this class is induced by a class in $\mathrm{Ext}^{-r}_Z\big(\Omega^r_{Z/S}, \mathcal{D}^\bullet_{Z/S}\big)$ as will be shown in Theorem (16.1).

**(2.28)** *Interpretation with multiplicative laws* — If $h : U = \mathrm{Spec}(B) \to T = \mathrm{Spec}(A)$ is a morphism of affine schemes, then a morphism $\alpha_{U/T} : T \to \Gamma^\star(U/T)$ corresponds to a multiplicative $A$-law $B \to A$ [**II**, Thm. 2.3]. Such a law corresponds to multiplicative *maps* $h'_* \mathcal{O}_{U \times_T T'} \to \mathcal{O}_{T'}$ for every *smooth* $T$-scheme $T'$ (it is enough to take $T' = \mathbb{A}^n_T$) such that for any morphism $\varphi : T'_1 \to T'_2$ the following diagram commutes

$$\begin{array}{ccc}(h_1')_*\mathcal{O}_{U\times_T T_1'} & \longrightarrow & (h_1')_*\varphi^*\mathcal{O}_{U\times_T T_2'} \\ \downarrow & & \downarrow \\ \mathcal{O}_{T_1'} & \longrightarrow & \varphi^*\mathcal{O}_{T_2'}.\end{array}$$

In the definition of a relative zero-cycle, we can thus instead of giving a proper zero-cycle $\alpha_{U/T}$ on every neighborhood $(U,T)$ instead give a multiplicative map $h_*\mathcal{O}_U \to \mathcal{O}_T$ with support on $p^{-1}(Z)$ such that these maps satisfy a similar compatibility condition.

## 3. Condition (T)

In this section we give a topological condition on a morphism closely related to conditions such as equidimensional and universally open.

**Definition (3.1).** Let $f : X \to S$ be a morphism. An irreducible component $X_i \hookrightarrow X$ is *dominating* over $S$ if $\overline{f(X_i)}$ is an irreducible component of $S$. We let $X_{\mathrm{dom}/S} \subseteq X$ be the union of the irreducible components which are dominating over $S$. If $X = X_{\mathrm{dom}/S}$ then we say that $f$ is *componentwise dominating*.

*Remark* **(3.2).** Let $X \to S$ be a morphism. If $S$ has a finite number of irreducible components with generic points $\xi_1, \xi_2, \dots, \xi_n$, then $X_{\mathrm{dom}/S}$ is the underlying set of the schematic closure of $X \times_S \coprod_i \operatorname{Spec}(\mathcal{O}_{S,\xi_i})$ in $X$. If $X \to S$ is open, then $X_{\mathrm{dom}/S} = X$.

**Definition (3.3).** Let $f : X \to S$ be a morphism locally of finite type. We let $X_{\dim_S = r}$ (resp. $X_{\dim_S > r}$) be the subset of $X$ consisting of points $x \in X$ with $\dim_x\big(X_{f(x)}\big) = r$ (resp. $> r$). By Chevalley's theorem [EGA$_{\mathrm{IV}}$, Thm. 13.1.3], this is a locally closed (resp. closed) subset.

Let $f : X \to S$ be locally of finite type. Recall [EGA$_{\mathrm{IV}}$, 13.3, Err$_{\mathrm{IV}}$, 35] that $f$

(i) is *equidimensional* if $f$ is componentwise dominating, and locally on $S$ there exists an integer $r$ such that the fibers of $f$ are equidimensional of dimension $r$,
(ii) is *equidimensional at* $x \in X$ if $f|_U$ is equidimensional for some open neighborhood $U$ of $x$,
(iii) is *locally equidimensional* if $f$ is equidimensional at every point $x \in X$.

**Proposition (3.4).** *Let $f : X \to S$ be a morphism locally of finite type. The following conditions are equivalent:*

(i) *For every integer $r$, the subscheme $X_{\dim_S = r}$ is equidimensional of dimension $r$ over $S$.*
(ii) *For every integer $r$, every irreducible component of $X_{\dim_S = r}$ dominates an irreducible component of $S$.*

(iii) *Every point* $x \in X$ *is contained in an irreducible component* $W$ *of* $X$ *which is equidimensional over* $S$ *at* $x$.
(iv) *Every point* $x \in X$ *which is generic in its fiber* $X_{f(x)}$ *is contained in an irreducible component* $W$ *of* $X$ *which is equidimensional over* $S$ *at* $x$.

*Moreover, these conditions are satisfied if* $f$ *is universally open or if the irreducible components of* $X$ *are equidimensional over* $S$*, e.g., if* $f$ *is locally equidimensional.*

*Proof.* By definition, (i) is equivalent to (ii) and trivially (iii) implies (iv). It is obvious that (i) implies (iii). If (iv) is satisfied, then any irreducible component of $X_{\dim_S = r}$ is contained in, and hence equal to, an irreducible component which is equidimensional of dimension $r$. This shows that (iv) implies (i).

If $f$ is universally open, then (iv) is satisfied by [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 14.3.13]. If $f$ is locally equidimensional, then (i) is satisfied. □

**Definition (3.5).** We say that $X/S$ satisfies condition (T) when the equivalent conditions of Proposition (3.4) are satisfied. We say that $X/S$ satisfies (T) universally if $X \times_S S'/S'$ satisfies (T) for any base change $S' \to S$.

Note that if $X/S$ satisfies (T), then $X'/S'$ satisfies (T) for any *flat* base change $S' \to S$, cf. [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 13.3.8].

**Proposition (3.6).** *Let* $f : X \to S$ *be locally of finite type. The following are equivalent.*

(i) $f$ *satisfies* (T) *universally.*
(ii) $f' : X' \to S'$ *is componentwise dominating for every morphism* $S' \to S$.
(iii) $f' : X' \to S'$ *is componentwise dominating for every morphism* $S' \to S$ *where* $S'$ *is the spectrum of a valuation ring.*

*If* $S$ *is locally noetherian, then these statements are equivalent with:*

(iv) $f' : X' \to S'$ *is componentwise dominating for every morphism* $S' \to S$ *where* $S'$ *is the spectrum of a discrete valuation ring.*
(v) $f$ *is universally open.*

*Proof.* Is is clear that (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (iv). If $S'$ is the spectrum of a valuation ring, then $f'$ satisfies condition (T) if and only if $f'$ is componentwise dominating by [$\mathrm{EGA}_{\mathrm{IV}}$, Lem. 14.3.10]. An easy argument then shows that (iii) implies (i). That (iv) implies (v) is [$\mathrm{EGA}_{\mathrm{IV}}$, Cor. 14.3.7] and finally (v) implies (i) by [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 14.3.13]. □

## 4. Families of higher-dimensional cycles

In this section, we define higher-dimensional relative cycles. The support of a cycle will be universally open, Proposition (4.7), but a priori, the support only satisfies the weaker condition (T) of the previous section. We do not require that the support of a relative cycle is equidimensional, nor that its irreducible components are equidimensional. In the sequel, we will often use the following two results.

(i) If $B \to S$ is a smooth morphism, then for every $b \in B$, there is an open neighborhood $U \ni b$ and an étale morphism $U \to \mathbb{A}^r_S$ [$\mathrm{EGA_{IV}}$, Cor. 17.11.4].

(ii) If $Z \to B$ is open (or equidimensional), $B \to S$ is smooth and the composition is flat and locally of finite presentation with Cohen-Macaulay fibers (e.g. smooth), then $Z \to B$ is flat [$\mathrm{EGA_{IV}}$, Thm. 11.3.10, Prop. 15.4.2].

As in previous sections, we work with algebraic spaces $X/S$ locally of finite type. It may appear more natural to assume that $X/S$ is locally of finite presentation, and indeed this is required in several statements. However, even if $X/S$ is of finite presentation, the support, image, representing scheme, etc., of a relative cycle is a subscheme of $X$ which need not be of finite presentation. Of course, it is expected that any relative cycle $\alpha$ on $X/S$ is of finite presentation, i.e., that there exists $X_0/S_0$ of finite presentation and a relative cycle $\alpha_0$ on $X_0/S_0$ which pull-backs to $\alpha$. If this is the case, then locally there are infinitesimal neighborhoods of the support, image, representing scheme, etc., which are finitely presented. Unfortunately, these neighborhoods are not canonical and do not glue.

**Definition (4.1).** Let $X$ be an algebraic space, locally of finite type over $S$ and let $Z \hookrightarrow X$ be a locally closed subset. A *projection of $X/S$ adapted to $Z$* (resp. quasi-adapted to $Z$) is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "\varphi"] & X \arrow[dd] \\ B \arrow[d] \arrow[dr, phantom, "\circ"] & \\ T \arrow[r, "g"] & S \end{tikzcd}$$

such that $U \to X \times_S T$ is étale, $p^{-1}(Z) \to B$ is finite (resp. quasi-finite) and $B \to T$ is smooth. We will denote such a projection with $(U, B, T, p, g, \varphi)$. If $g$ is étale (resp. smooth) then we say that $(U, B, T, p, g, \varphi)$ is an étale (resp. smooth) projection.

A morphism of projections $(U_1, B_1, T_1, p_1, g_1, \varphi_1) \to (U_2, B_2, T_2, p_2, g_2, \varphi_2)$ is a triple of morphisms $p : U_1 \to U_2$, $q : B_1 \to B_2$ and $g : T_1 \to T_2$ such that

$$\begin{tikzcd} U_1 \arrow[r, "p"'] \arrow[rr, bend left, "p_1"] \arrow[d, "\varphi_1"] & U_2 \arrow[r, "p_2"'] \arrow[d, "\varphi_2"] & X \arrow[dd] \\ B_1 \arrow[r, "q"] \arrow[d] & B_2 \arrow[d] & \\ T_1 \arrow[r, "g"] \arrow[rr, bend right, "g_1"'] & T_2 \arrow[r, "g_2"] & S \end{tikzcd}$$

is commutative.

**Definition (4.2).** Let $X/S$ be an algebraic space, locally of finite type over $S$. A *relative cycle* $\alpha$ on $X/S$ consists of the following:

(i) A locally closed retro-compact subset $Z$ of $X$ — the *support* of $\alpha$.
(ii) For every projection $(U,B,T,p,g)$ of $X/S$ adapted to $Z$, a proper family of zero-cycles $\alpha_{U/B/T} : B \to \Gamma^\star(U/B)$ with support $p^{-1}(Z)_{\mathrm{dom}/B}$.

satisfying the following conditions:

(a) The support $Z$ satisfies (T).
(b) For every morphism $(p,q,g) : (U_1,B_1,T_1,p_1,g_1) \to (U_2,B_2,T_2,p_2,g_2)$ of projections, we have that

$$\mathcal{N}_{B_1/g^*B_2} * \alpha_{U_1/B_1/T_1} = g^*\alpha_{U_2/B_2/T_2} * \mathcal{N}_{U_1/g^*U_2}.$$

A relative cycle is locally equidimensional (resp. equidimensional of dimension $r$) if $Z/S$ is locally equidimensional (resp. equidimensional of relative dimension $r$).

Let us show that condition (b) makes sense. First note that $U_1 \to g^*U_2$ is étale and thus the right-hand side is defined. To define the left-hand side, we can replace the $B_i$'s with the respective images of the universally open morphisms $p_i^{-1}(Z)_{\mathrm{dom}/B_i} \to B_i$. Then as $U_1 \to g^*U_2$ is universally open, it follows that $B_1 \to B_2$ is universally open. Thus $B_1 \to g^*B_2$ is *flat* and of finite presentation [$\mathrm{EGA_{IV}}$, Thm. 11.3.10, Prop. 15.4.2]. As $B_1 \to g^*B_2$ is quasi-finite $\mathcal{N}_{B_1/g^*B_2}$ is thus defined.

*Remark* **(4.3).** As $B \to T$ is smooth, there is an open and closed partition of $B$ such that $B \to T$ is equidimensional. It is thus clear that in Definition (4.2), we can assume that $B \to T$ is equidimensional and that $\mathrm{Supp}(\alpha_{U/B/T}) \to B$ is surjective. If the support $Z$ is equidimensional of dimension $r$, then it is enough to consider projections with $B/T$ smooth of dimension $r$.

*Remark* **(4.4).** It is easily seen if $\alpha$ is a relative cycle on $X/S$, then for any projection $(U,B,T,p,g)$ *quasi-adapted* to $\mathrm{Supp}(\alpha)$ there is a unique *non-proper* relative zero-cycle $\alpha_{U/B/T}$ with support $p^{-1}(Z)_{\mathrm{dom}/B}$. The compatibility condition (b) is then also satisfied for morphisms of quasi-adapted projections.

**Proposition (4.5).** *Let $\alpha$ be a relative cycle on $X/S$ with support $Z$. Let $Z_r = Z_{\dim_S = r}$. If $(U,B,T,p,g)$ is a projection such that $B/T$ has relative dimension $r$, then* $\mathrm{Supp}(\alpha_{U/B/T}) \subseteq p^{-1}(Z_r)$ *with equality if $p^{-1}(Z_r) \to T$ is componentwise dominating. The collection of $\alpha_{U/B/T}$ for which $B/T$ has dimension $r$, determines a unique equidimensional relative cycle $\alpha_r$ with support $Z_r$.*

*Proof.* Let $(U,B,T,p,g)$ be a neighborhood adapted to $Z$. As $p^{-1}(Z)$ is finite over $B$, it follows that every point of $p^{-1}(Z)$ has dimension at most $r$ relative to $T$ and that $\mathrm{Supp}(\alpha_{U/B/T}) = p^{-1}(Z)_{\mathrm{dom}/B} \subseteq p^{-1}(Z)_{\dim_T = r} = p^{-1}(Z_r)$.

Let $(U,B,T,p,g)$ be a neighborhood adapted to $Z_r$ and let $p' : U' \to X$ be the restriction of $p$ to the open subset $X \setminus \overline{Z_{\dim_S > r}}$. Then $(U',B,T,p',g)$ is a neighborhood adapted to $Z$ and $\alpha_{U'/B/T}$ determines $(\alpha_r)_{U/B/T}$ uniquely. □

**Lemma (4.6)** (Existence of étale projections)**.** *Let $f : X \to S$ be an algebraic space, locally of finite type, and $\alpha$ a relative cycle on $X/S$ with support $Z$. Then for any point $z \in Z$ there is an étale projection $(U,B,S,p,g)$ adapted to $Z$ and $u \in p^{-1}(z)$ such that $u \in p^{-1}(Z)|_{\mathrm{dom}/B}$.*

*Proof.* Replacing $X$ with an étale cover, we can assume that $X$ is a scheme. Let $r = \dim_z\bigl(X_{f(z)}\bigr)$. There is an open neighborhood $U \subseteq X$ of $z$ and a morphism $U \cap Z_r \to \mathbb{A}^r_S$ which is equidimensional of dimension zero [EGA$_{\mathrm{IV}}$, Prop. 13.3.1 b]. After shrinking $U$, we can assume that we have a morphism $U \to \mathbb{A}^r_S$ such that $(U \cap Z)_{\mathrm{dom}/\mathbb{A}^r_S} = (U \cap Z_r)$ in a neighborhood of $z$. The result then follows from Lemma (2.11). □

The following proposition shows that the support of a relative cycle behaves similarly as the support of a flat and finitely presented sheaf. One difference though is that the irreducible components of the support of a flat sheaf always are equidimensional [EGA$_{\mathrm{IV}}$, Prop. 12.1.1.5].

**Proposition (4.7).** *Let $X/S$ be locally of finite type. The support of a relative cycle $\alpha$ on $X/S$ is universally open. In particular, an equidimensional relative cycle is universally equidimensional and equality always holds in Proposition* (4.5)*.*

*Proof.* Let $\alpha$ be a relative cycle. It is enough to show that the support $Z_r$ of $\alpha_r$ is universally open over $S$ for every $r$. This follows from Lemma (4.6) and Proposition (2.13). □

*Remark* **(4.8).** The support of a single irreducible component of $\alpha$ need not be universally open. For example, if $S$ consists of two secant lines and $X = S$, then there is a relative zero-cycle on $X/S$ with support $X$ but the inclusion of one of the lines is not open. This is also illustrated in the following example.

**Example (4.9)** ([EGA$_{\mathrm{IV}}$, Rem. 14.4.10 (ii)])**.** Let $S$ be a regular quasi-projective surface and choose a closed point $s \in S$. Let $Z_1$ be the blow-up of $S$ in $s$ and let $Z_2 = \mathbb{P}^1_S$. Then $(Z_1)_s \cong (Z_2)_s \cong \mathbb{P}^1_s$. We let $Z = Z_1 \amalg_{\mathbb{P}^1_s} Z_2$ be the gluing of $Z_1$ and $Z_2$ along the common fiber. This is a scheme [Fer03, Thm. 5.4] with irreducible components $Z_1$ and $Z_2$.

Note that $Z_1 \to S$ does not satisfy (T) but that $Z \to S$ satisfies (T). It follows from Chevalley's theorem [EGA$_{\mathrm{IV}}$, Thm. 14.4.1] that $Z \to S$ is universally open but that $Z_1 \to S$ is not universally open. Later on, in Theorem (10.1), we will see that $Z/S$ determines a unique relative cycle on $Z/S$ with underlying cycle $[Z] = [Z_1] + [Z_2]$. Thus, this an example of a relative cycle for which the irreducible components are not equidimensional. This is a phenomenon which does not occur in flat families.

**Proposition (4.10)** (Étale descent). *Let $X/S$ be locally of finite type and let $p : U \to X$ be an étale morphism. Let $\pi_1$ and $\pi_2$ be the projections of $U \times_X U$ onto the first and second factors. Let $\beta$ be a relative cycle on $U/S$ such that $\pi_1^*\beta = \pi_2^*\beta$. Then there is a unique relative cycle $\alpha$ on $X/S$ with support contained in $p(U)$ such that $\beta = p^*\alpha$.*

*Proof.* Let $W \subseteq U$ be the support of $\beta$. Then $\pi_1^{-1}(W) = \pi_2^{-1}(W)$ and by étale descent, we obtain a locally closed retro-compact subset $Z \subseteq p(U)$. If $(V, B, T)$ is a projection adapted to $Z/S$, then $(U \times_X V, B, T)$ is a projection quasi-adapted to $W/S$. The relative zero-cycle $\alpha_{U \times_X V/B/T}$ then descends uniquely to a relative zero-cycle $\alpha_{V/B/T}$ by Proposition (2.18). □

**Proposition (4.11).** *Let $\alpha$ be a relative cycle on $X/S$. Let $(U, B, T, p, g)$ be a projection such that $B \to T$ factors through an étale morphism $h : T' \to T$. Then $(U, B, T', p, g \circ h)$ is a projection and $\alpha_{U/B/T'} = \alpha_{U/B/T}$.*

*Proof.* As $T' \to T$ is étale, $U \to X \times_S T$ is étale and $B \to T$ is smooth, it follows that $U \to X \times_S T'$ is étale and that $B \to T'$ is smooth. Thus $(U, B, T', p, g \circ h)$ is a projection. We also have a natural map of projections $(\mathrm{id}_U, \mathrm{id}_B, h) : (U/B/T') \to (U/B/T)$. The compatibility condition for this map is that

$$\mathcal{N}_{B/B \times_T T'} * \alpha_{U/B/T'} = \pi_2^* \alpha_{U/B/T} * \mathcal{N}_{U/U \times_T T'}$$

where the maps are given by the diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \times_T T' \arrow[r] \arrow[d] & U \arrow[d] \\
B \arrow[r, hook] \arrow[dr, "="'] & B \times_T T' \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & B \arrow[d] \\
& T' \arrow[r, "h"] & T
\end{tikzcd}$$

But as $B \to B \times_T T'$ is an open immersion, it is obvious that this is equivalent to $\alpha_{U/B/T'} = \alpha_{U/B/T}$. □

**Corollary (4.12).** *There is a one-to-one correspondence between relative zero-cycles as of Definition (2.3), and relative cycles of dimension zero, as of Definition (4.2). In this correspondence the support remains the same and $\alpha_{U/B/T} = \alpha_{U/B}$.*

*Proof.* As $\alpha_{U/B/T} = \alpha_{U/B/B}$ by Proposition (4.11), this correspondence is well-defined. Under the hypothesis that $\alpha_{U/B/T} = \alpha_{U/B/B}$, it is then enough to check the compatibility condition in the second definition for morphisms between projections of the form $(U_1, B_1, B_1) \to (U_2, B_2, B_2)$. This compatibility condition coincides with the compatibility condition between neighborhoods $(U_1, B_1) \to (U_2, B_2)$ in the first definition. □

**Definition (4.13).** We let $Cycl(X/S)$ be the set of relative cycles on $X/S$. We let $Cycl_{\mathrm{equi}}(X/S)$ (resp. $Cycl_r(X/S)$, resp. $Cycl^{\mathrm{prop}}(X/S)$, resp. $Cycl^{\mathrm{cl}}(X/S)$) be the subset consisting of relative cycles which are equidimensional (resp. are equidimensional of dimension $r$, resp. have proper support, resp. have closed support). We let $\mathrm{Chow}_r(X/S)$ and $\mathrm{Chow}(X/S)$ be the functors from $S$-schemes to sets given by

$$\mathrm{Chow}_r(X/S)(T) = Cycl_r^{\mathrm{prop}}(X \times_S T/T)$$
$$\mathrm{Chow}(X/S)(T) = Cycl_{\mathrm{equi}}^{\mathrm{prop}}(X \times_S T/T)$$

with the natural pull-back.

As before it follows that $Cycl(X/S)$, $\mathrm{Chow}(X/S)$, $\mathrm{Chow}_r(X/S)$, etc., are fppf-sheaves as $\Gamma^\star(U/B)$ is representable.

**Definition (4.14).** Let $X/S$ be locally of finite type. We say that a relative cycle $\alpha$ on $X/S$ is a *relative Weil divisor* if for every $s \in S$ and $z \in \mathrm{Supp}(\alpha)_s$ we have that $\mathrm{codim}_z(\mathrm{Supp}(\alpha)_s, X_s) = 1$.

**(4.15)** *Addition of cycles* — Let $\alpha$ and $\beta$ be relative cycles on $X/S$ with supports $Z_\alpha$ and $Z_\beta$. If $Z_\alpha$ and $Z_\beta$ are closed in $Z_\alpha \cup Z_\beta$, e.g., if $Z_\alpha$ and $Z_\beta$ are closed in $X$, then there is a relative cycle $\alpha + \beta$ on $X/S$ with support $Z_\alpha \cup Z_\beta$ defined by $(\alpha+\beta)_{U/B/T} = \alpha_{U/B/T} + \beta_{U/B/T}$ for any projection adapted to $Z_\alpha \cup Z_\beta$, cf. (2.7). This makes $Cycl^{\mathrm{cl}}(X/S)$ a commutative monoid.

## 5. Smooth projections

In this section, we show that in the definition of a relative cycle, given in the previous section, it is enough to consider smooth projections. That is, relative zero-cycles on every smooth projection satisfying the compatibility condition, determine a unique relative cycle. We then discuss variants of the definition of a relative cycle that are more well-behaved.

**Lemma (5.1).** *Let $S_0 \hookrightarrow S$ be a closed immersion. Let $X_0 \to S_0$ be smooth (resp. étale) and $x_0 \in X_0$. Then there is an open neighborhood $U_0 \subseteq X_0$ of $x_0$ and a smooth (resp. étale) scheme $U \to S$ such that $U_0 = U \times_S S_0$.*

*Proof.* Replacing $X_0$ with an open neighborhood of $x_0$, we can assume that there is an étale morphism $X_0 \to \mathbb{A}^n_{S_0}$. This lifts to an étale morphism $X \to \mathbb{A}^n_S$ such that $X_0 = X \times_S S_0$ [SGA$_1$, Exp. I, Prop. 8.1]. □

**Lemma (5.2).** *Let $S_0 \hookrightarrow S$ be a closed immersion. Let $X/S$ be a scheme and let $Y/S$ be smooth. Let $X_0 = X \times_S S_0$, let $x_0 \in X_0$ and let $f_0 : X_0 \to Y$ be a morphism. Then there exists an open neighborhood $U_0 \subseteq X_0$, an étale morphism $U \to X$ such that $U_0 = U \times_X X_0$, and a map $f : U \to Y$ which restricts to $(f_0)|_{U_0}$.*

*Proof.* Replacing $X$ and $Y$ with open neighborhoods, we can assume that $Y/S$ factors through an étale map $Y \to \mathbb{A}^n_S$. As $f_0$ lifts to $X \to \mathbb{A}^n_S$, we can replace $S$ with $\mathbb{A}^n_S$ and assume that $Y/S$ is étale. Let $V = X \times_S Y$ and $V_0 = V \times_X X_0 = X_0 \times_S Y$.

Then $V_0 \to X_0$ has an open section $s$. Any open subset $U \subseteq V$ restricting to $s(X_0)$ gives a map as in the lemma. $\square$

**Proposition (5.3).** *In the definition of relative cycles, Definition* (4.2)*, it is enough to consider projections* $(U, B, T)$ *such that* $T = \mathbb{A}^n_S$ *for some* $n$*. That is, given* $Z$ *as in the definition and relative cycles* $\alpha_{U/B/T}$ *on projections* $(U, B, T)$ *with* $T = \mathbb{A}^n_S$ *for some* $n$ *satisfying the compatibility condition, these data extends uniquely to a relative cycle.*

*Proof.* It is clear that in the definition of relative cycles, we can assume that $U$, $B$ and $T$ are affine. We can also assume that $S$ and $X$ are affine. Let $(U, B, T)$ be a projection. Then $T$ is the inverse limit of finitely presented affine $S$-schemes $T_\lambda$. As $U/X \times_S T$, and $B/T$ are of finite presentation, it follows that the projection descends to a projection $(U_\lambda, B_\lambda, T_\lambda)$ for sufficiently large $\lambda$. Similarly, every morphism of neighborhoods $(U_1, B_1, T) \to (U_2, B_2, T)$ descends to a morphism of finitely presented neighborhoods. In the definition of relative cycles, we can thus assume that all projections are finitely presented.

Let $(U, B, T)$ be a projection with $T$ a finitely presented affine $S$-scheme. There is then a closed immersion $T \hookrightarrow T_1 = \mathbb{A}^n_S$. Lemmas (5.1) and (5.2) shows that, locally on $U$ and $B$, there exists an étale morphism $U_1 \to X_1 \times_S T_1$, a smooth morphism $B_1 \to T_1$ and a morphism $U_1 \to B_1$, lifting $U \to X \times_S T$, $B \to T$ and $U \to B$ respectively.

To show that $\alpha_{U/B/T}$ is uniquely defined by smooth projections, we can assume that $B = \mathbb{A}^n_T$. Let $(U_1, B_1, T_1)$ and $(U_2, B_2, T_2)$ be two smooth liftings, i.e., $T_i = \mathbb{A}^{n_i}_S$, $B_i = \mathbb{A}^r_{T_i}$ and $(U_i, B_i, T_i) \times_{T_i} T = (U, B, T)$. Then $T \to T_2$ (resp. $B \to B_2$) factors non-canonically through $T \to T_1$ (resp. $B \to B_1$). Replacing $U_1$ with an étale cover, we can also arrange so that $U \to U_2$ factors through $U \to U_1$. Thus, if the smooth projections are compatible, then $\alpha_{U/B/T}$ is uniquely defined by them.

Finally, let us show that the compatibility condition for smooth projections imply the compatibility condition for arbitrary projections. As the $\alpha_{U/B/T}$'s are compatible with base change by assumption, it is enough to check the compatibility for $(U, B_1, T) \to (U, B_2, T)$ and $(U_1, B, T) \to (U_2, B, T)$. By Lemmas (5.1) and (5.2), these morphisms lift to morphisms of projections over $\mathbb{A}^n_S$. $\square$

**Corollary (5.4).** *Let* $Z \hookrightarrow X$ *be a locally closed subset, universally open over* $S$*, and assume that we are given relative zero-cycles* $\alpha_{U/B/S}$ *for every projection* $(U, B, S)$ *adapted to* $Z$*. Then there is at most one relative cycle inducing these relative zero-cycles.*

*Proof.* By Proposition (5.3) a relative cycle $\alpha$ is given by its smooth projections. By Corollary (B.3), a relative cycle $\alpha$ is determined by its étale projections. Finally if $(U, B, T)$ is an étale projection, then $\alpha_{U/B/T} = \alpha_{U/B/S}$ by Proposition (4.11). $\square$

A relative cycle on $X/S$ is expected to behave as if it is induced by an object living on $X$. Thus, the following condition is reasonable.

(*) For any smooth projection $(U,B,T)$ the relative zero-cycle $\alpha_{U/B/T}$ does not depend on $T$.

We will show that this is satisfied in many situations, cf. Proposition (9.17). I do not know if this condition always holds for a relative cycle but this seems unlikely. If not, then this condition should probably be imposed on relative cycles to get a well-behaved functor, cf. Section 16.

Moreover, it is also reasonable to require that for any pair of smooth morphisms $p : U \to X$ and $B \to S$ and a morphism $U \to B$ such that $U \to B$ is quasi-finite over $p^{-1}(Z)$, there is a relative zero-cycle $\alpha_{U/B}$ on $U/B$. Indeed, if smooth pull-back of relative cycles exists, then such relative zero-cycles $\alpha_{U/B}$ exist. This is the case if $S$ is reduced, cf. Section 14. I do not know if this follows in general from condition $(*)$.

Given a relative cycle $\alpha$ on $X/S$, it is also fair to require that there should be an infinitesimal neighborhood $Z$ of $\mathrm{Supp}(\alpha)$ such that $\alpha$ is the push-forward of a relative cycle on $Z/S$. If $\alpha$ is a relative zero-cycle, then there is the canonical choice $Z = \mathrm{Image}(\alpha)$. The following proposition gives sufficient and necessary conditions for the existence of an infinitesimal neighborhood $Z$ as above.

**Proposition (5.5).** *Let $\alpha$ be a relative cycle on $X/S$ with support $Z_0 \subseteq X$. Let $Z_0 \hookrightarrow Z$ be an infinitesimal neighborhood. Then $\alpha$ is the push-forward of a relative cycle on $Z$ if and only if*

(i) *For any smooth projection $(U,B,T,p)$ adapted to $Z_0$, the image of $\alpha_{U/B/T}$ is contained in $p^{-1}(Z)$.*
(ii) *For any smooth projection $(U,B,T,p)$ adapted to $Z_0$, the relative cycle $\alpha_{U/B/T}$ only depends upon $U|_{p^{-1}(Z)} \to B$ and $p|_Z$.*

*Proof.* The two conditions are clearly necessary. To show that they are sufficient it is enough to show that given a smooth projection $(U,B,T,p)$ of $Z/S$ adapted to $Z_0$, there is a smooth projection $(U',B,T,p')$ of $X/S$ adapted to $Z_0$ which restricts to the first projection over $Z$, and similarly for morphisms of projections. This follows from Lemmas (5.1) and (5.2). $\square$

## 6. Uniqueness and extension of relative cycles

**Proposition (6.1).** *Let $S$ be an irreducible normal scheme with generic point $\xi$ and $X/S$ locally of finite type. Let $Z \hookrightarrow X$ be a subscheme such that $Z/S$ is equidimensional of dimension zero, i.e., locally quasi-finite and such that $Z_{\mathrm{dom}/S} = Z$. Then any relative cycle on $X_\xi/\mathrm{Spec}(k(\xi))$ with support $Z_\xi$ extends uniquely to a relative cycle on $X/S$.*

*Proof.* If $g : T \to S$ is étale then $T$ is normal. As it is enough to consider étale neighborhoods $(U,T,p,g)$ in the definition of a relative non-proper cycle, we can thus assume that $Z/S$ is finite. Let $\alpha_\xi$ be a relative cycle on $X_\xi/\mathrm{Spec}(k(\xi))$ and let $W_\xi$ be its image, which is an infinitesimal neighborhood of $Z_\xi$. Let $W \hookrightarrow X$ be the closure of $W_\xi$. Then as $\Gamma^d(W/S) \to S$ is finite [**I**, Prop. 4.3.1] and $S$ is normal, it follows that the morphism $\alpha_\xi : \mathrm{Spec}(k(\xi)) \to \Gamma^d(W/S)$ extends to a section of $\Gamma^d(W/S) \to S$. □

**Corollary (6.2).** *Let $S$ be an irreducible normal scheme with generic point $\xi$ and $X/S$ locally of finite type. Let $Z \hookrightarrow X$ be a subscheme satisfying* (T)*. Then any relative cycle on $X_\xi/\mathrm{Spec}(\xi)$ with support $Z_\xi$ extends uniquely to a relative cycle on $X/S$.*

*Proof.* Follows from Proposition (6.1) as it is enough to consider smooth projections. □

Chevalley's criterion for universally open morphisms [$\mathrm{EGA_{IV}}$, Thm. 14.4.1] easily follows. Note that the proof given in *loc. cit.* only is valid if $X/S$ is locally of finite presentation [$\mathrm{EGA_{IV}}$, $\mathrm{Err_{IV}}$ 37].

**Corollary (6.3)** (Chevalley's theorem)**.** *Let $S$ be a geometrically unibranch scheme (e.g. a normal scheme) with a finite number of components and let $X \to S$ be locally of finite type satisfying* (T)*, i.e., such that every point $x \in X$ is contained in an irreducible component which is equidimensional over $S$ at $x$. Then $X \to S$ is universally open.*

*Proof.* Let $\widetilde{S} \to S$ be the normalization. As this is a universal homeomorphism, we can assume that $S$ is normal. We will now construct a canonical relative cycle $\alpha$ on $X/S$ with support $X$. The underlying cycle, cf. Section 8, of $\alpha$ is going to be $[X]$. Let $(U, B, T)$ be any smooth projection. Then $B$ is normal and $U/B$ is generically flat. We let $\alpha_{U/B/T}$ be the unique extension of $\mathcal{N}_{U_\xi/B_\xi}$ given by Proposition (6.1). The corollary then follows from Proposition (4.7). □

Note that the condition that $S$ has a finite number of components is essential. In fact, there are non-noetherian normal schemes such that the irreducible components are not open, e.g., the absolutely flat scheme associated to the affine line. The inclusion of such a component is a counter-example.

We have the following simple analog of the flatification by Raynaud and Gruson [RG71]:

**Proposition (6.4).** *Let $S$ be a scheme, $X/S$ locally of finite type and let $U \subseteq S$ be an open retro-compact subset. Let $Z \hookrightarrow X$ be a subscheme such that $Z/S$ is universally open. Let $\alpha_U$ be a relative cycle on $X|_U/U$ with support $Z|_U$. Let $S' \to S$ be the normalization of $S$ in $U$, i.e., the spectrum of the integral closure of $\mathcal{O}_S$ in the direct image of $\mathcal{O}_U$. Then $\alpha_U$ extends to a relative cycle on $X'/S'$.*

*Proof.* As the integral closure commutes with smooth morphisms, we can assume that $Z/S$ is zero-dimensional. Then reason as in the proof of Proposition (6.1). □

**Proposition (6.5).** *Let $S$ be a locally noetherian scheme, $X/S$ locally of finite type and let $U \subseteq S$ be an open subscheme. Let $Z \hookrightarrow X$ be a subscheme such that $Z/S$ satisfies* (T). *Let $\alpha_U$ be a relative cycle on $X|_U/U$ with support $Z|_U$.*

(i) *If $U$ contains all points of depth zero, then there is at most one relative cycle on $X/S$ extending $\alpha_U$.*
(ii) *If $U$ contains all points of depth at most one, then there is a unique relative cycle on $X/S$ extending $\alpha_U$.*

*Proof.* If $B \to S$ is flat and $U \subseteq S$ contains all points of depth zero (resp. at most one) then so does $B \times_S U \subseteq B$. As it is enough to consider smooth projections, we can thus assume that $Z/S$ is finite. Then $\alpha_U$ is a relative proper zero-cycle and we let $W \hookrightarrow X$ be its image. If $U$ contains all points of depth zero, then the morphism $U \to \Gamma^d(W/S)$ has at most one extension to $S$. If $U$ contains all points of depth one, then as $\Gamma^d(W/S) \to S$ is finite and in particular affine, it follows that the section $U \to \Gamma^d(W/S)$ extends to $S$. Indeed, if $j : U \hookrightarrow S$ is the inclusion, then $j_*\mathcal{O}_U = \mathcal{O}_S$. $\square$

We can make the extension property slightly more precise.

**Corollary (6.6).** *Let $S$ be a locally noetherian scheme, let $f : X \to S$ be locally of finite type and let $U \subseteq X$ be an open subscheme. Let $Z \hookrightarrow X$ be a subscheme such that $Z/S$ satisfies* (T). *Let $\alpha_U$ be a relative cycle on $U/S$ with support $Z|_U$.*

(i) *If $U$ contains all points $z \in Z$ such that* $\operatorname{depth} f(z) + \operatorname{codim}_z(Z_{f(z)}) = 0$, *then there is at most one relative cycle on $X/S$ with support $Z$ extending $\alpha_U$.*
(ii) *If $U$ contains all points $z \in Z$ such that* $\operatorname{depth} f(z) + \operatorname{codim}_z(Z_{f(z)}) \leq 1$, *then there is a unique relative cycle on $X/S$ with support $Z$ extending $\alpha_U$.*

*Proof.* This follows from Proposition (6.5) and the observation that if $h : B \to S$ is smooth, then the depth of a point $b \in B$ is the sum of the depth of $h(b)$ and the codimension of $b$ in its fiber $h^{-1}(h(b))$. $\square$

## 7. Flat families

In this section, we will define a relative cycle $\mathcal{N}_{\mathcal{F}/S}$ on $X/S$ for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ which is flat over $S$. If $(U, B, T, p)$ is a projection, then $p^*\mathcal{F}$ is not flat over $B$, but only of *finite Tor-dimension*, cf. Lemma (7.12). If for every point $s \in S$ of depth zero, $\mathcal{F}_s$ has no embedded components in codimension one, then $p^*\mathcal{F}/B$ is flat at every point of depth one and the existence of $(\mathcal{N}_{\mathcal{F}/S})_{U/B/T}$ follows from Proposition (6.5). In general, however, we need to associate a relative zero-cycle to a coherent sheaf of finite Tor-dimension. Similar constructions can be found in [GIT, Ch. 5, §3], [Fog69, §2] and [KM76]. To avoid complicated notions such as pseudo-coherence, we only use the notion of finite Tor-dimension for coherent modules over noetherian schemes.

**Definition (7.1)** ([SGA$_6$, Exp. I, Def. 5.2])**.** Let $(X, \mathcal{A})$ be a locally ringed space. An $\mathcal{A}$-module $\mathcal{F}$ on $X$ has *finite Tor-dimension* if $\mathcal{F}_x$ is an $\mathcal{A}_x$-module of finite Tor-dimension for every $x \in X$, i.e., if $\mathcal{F}_x$ admits a finite resolution of flat $\mathcal{A}_x$-modules. The Tor-dimension of $\mathcal{F}$ at $x$, denoted $\operatorname{Tor-dim}_x(\mathcal{F})$, is the length $n$ of a minimal flat resolution of $\mathcal{A}_x$-modules

$$0 \to \mathcal{P}_n \to \mathcal{P}_{n-1} \to \cdots \to \mathcal{P}_0 \to \mathcal{F}_x \to 0.$$

Note that $\mathcal{F}$ is flat at $x$ if and only if the Tor-dimension of $\mathcal{F}$ at $x$ is zero.

**Definition (7.2)** ([SGA$_6$, Exp. III, Def. 3.1])**.** Let $f : X \to S$ be a morphism of algebraic spaces and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. The module $\mathcal{F}$ has *finite Tor-dimension* over $S$ if $\mathcal{F}$ has finite Tor-dimension as a $f^{-1}\mathcal{O}_S$-module.

*Remark* **(7.3).** If $f : X \to S$ is affine and $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\mathcal{F}$ is of finite Tor-dimension over $S$ if and only if $f_*\mathcal{F}$ is of finite Tor-dimension.

**(7.4)** *Auslander-Buchsbaum formula* — Let $X$ be a locally noetherian scheme and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite Tor-dimension. Then $\operatorname{Tor-dim}_x(\mathcal{F}) + \operatorname{depth}_x(\mathcal{F}) = \operatorname{depth} x$ [AB57, Thm. 3.7]. In particular, $\mathcal{F}$ is flat, and hence free, over points of depth zero. If $\mathcal{F}$ is $(\mathrm{S_k})$, then $\mathcal{F}$ is flat over points of depth at most $k$.

**(7.5)** *Norms and traces* — Let $A$ be a ring, $B$ an $A$-algebra and $M$ a $B$-module which is locally free of rank $d$ as an $A$-module. Then the norm map of $M$, which defines $\mathcal{N}_M$, is given by

$$B \longrightarrow \operatorname{End}_A(M) \xrightarrow{\det} \operatorname{End}_A(\wedge^d M) \cong A$$

where the first homomorphism is the multiplication, and the second *map* takes an endomorphism $\varphi \in \operatorname{End}_A(M)$ onto the endomorphism $\wedge^d\varphi$ given by

$$x_1 \wedge \cdots \wedge x_d \mapsto \varphi(x_1) \wedge \cdots \wedge \varphi(x_d).$$

We also have a *trace* homomorphism given by a similar composition where the second map is the *homomorphism* which takes $\varphi$ onto the endomorphism

$$x_1 \wedge \cdots \wedge x_n \mapsto \varphi(x_1) \wedge x_2 \wedge \cdots \wedge x_n + x_1 \wedge \varphi(x_2) \wedge \cdots \wedge x_n + \cdots + x_1 \wedge x_2 \wedge \cdots \wedge \varphi(x_n).$$

Now assume that $M$ is not locally free but of finite Tor-dimension. Let $0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$ be a locally free resolution. If $\varphi \in \operatorname{End}_A(M)$ then, as the $P_i$'s are projective, there is a (non-unique) lifting of the endomorphism $\varphi$ to an endomorphism $\varphi_\bullet$ of the complex $P_\bullet$. The trace of $\varphi$ on $M$, can then be defined as the alternating sum $\sum_i (-1)^i \operatorname{tr}_{P_i}(\varphi_i)$. If $M$ is locally free, the resolution splits locally and it is clear that this definition of the trace of $M$ coincides with the previous definition. It thus follows that the trace of an arbitrary $M$ of finite Tor-dimension is independent of the resolution and the choice of lifting $\varphi_\bullet$. In fact, $M$ is free over every point in $\operatorname{Spec}(A)$ of depth zero.

Naïvely, we would define the norm of a module of finite Tor-dimension similarly, i.e., $\prod_i \mathrm{N}_{P_i}(\varphi_i)^{(-1)^i}$, but this does not make sense unless $\mathrm{N}_{P_i}(\varphi_i)$ is invertible for every odd $i$. The following easy lemma, similar to Gauss's Lemma, solves this.

**Lemma (7.6).** *Let $A \hookrightarrow A'$ be a ring extension. Let $p, q \in A[t]$ and $r \in A'[t]$ be monic polynomials. If $rp = q$ in $A'[t]$ then $r \in A[t]$.*

**Lemma (7.7).** *Let $A$ be a noetherian ring and let*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

*be an exact sequence of $A$-modules such that the $P_i$'s are free of finite ranks. Let $\varphi \in \mathrm{End}_A(M)$. Then there is a unique element, $\det(\varphi) \in A$, coinciding with the usual determinant of $\varphi$ at any point $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $M_{\mathfrak{p}}$ is free.*

*Proof.* First note that the uniqueness of $\det(\varphi)$ is clear as $M_{\mathfrak{p}}$ is free over any point $\mathfrak{p}$ of depth zero. Let $\varphi_i \in \mathrm{End}_A(P_i)$, be liftings of the endomorphism $\varphi$. Let $\varphi' \in \mathrm{End}_{A[t]} M[t]$ and $\varphi'_i \in \mathrm{End}_{A[t]} P_i[t]$ be defined by

$$\varphi' = \mathrm{id}_M \otimes t + \varphi \otimes \mathrm{id}_{A[t]}$$
$$\varphi'_i = \mathrm{id}_{P_i} \otimes t + \varphi_i \otimes \mathrm{id}_{A[t]}$$

where $t$ also denotes multiplication by $t$. Then $\det(\varphi'_i)$ — the characteristic polynomial of $\varphi_i$ — is a monic polynomial for all $i$. It is enough to show the existence of $\det(\varphi')$.

Let $\mathrm{Tot}(A)$ be the total ring of fractions of $A$, i.e., the localization in the set of all regular elements. Recall that $\mathrm{Tot}(A)$ is a semi-local ring such that every maximal ideal has depth zero. It follows that $M \otimes_A \mathrm{Tot}(A)$ is locally free of rank $d$, and hence free [Bou61, Ch. II, §2.3, Prop. 5]. Thus, there exists a regular element $f \in A$ such that $M_f$ is free.

Let $p = \prod_{2\nmid i} \det(\varphi'_i) \in A[t]$, $q = \prod_{2|i} \det(\varphi'_i) \in A[t]$ and $r = \det(\varphi'_f) \in A_f[t]$. Then $rp = q$ in $A_f[t]$ and hence $p \in A[t]$ by the lemma. The element $p(0) \in A$ is the determinant of $\varphi$. ☐

**Proposition (7.8).** *Let $S$ be a locally noetherian space and let $f : X \to S$ be a morphism of algebraic spaces, locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module such that $\mathrm{Supp}(\mathcal{F})$ is finite over $S$ and $\mathcal{F}$ has finite Tor-dimension over $S$. Then there is a unique proper relative zero-cycle $\mathcal{N}_{\mathcal{F}/S} : S \to \Gamma^\star(X/S)$ on $X/S$ such that for any point $s \in S$ of depth zero, the induced cycle $\left(\mathcal{N}_{\mathcal{F}/S}\right)_s$ is given by the norm $\mathcal{N}_{(f_*\mathcal{F})_s}$ of the free $\mathcal{O}_{S,s}$-module $(f_*\mathcal{F})_s$. In particular, the degree of $\mathcal{N}_{\mathcal{F}/S}$ at $s \in S$ is the rank of $\mathcal{F}$ over any generization of $s$ and the support of $\mathcal{N}_{\mathcal{F}/S}$ is the closure of the support of $\mathrm{Supp}(\mathcal{F})$ over the generic points. This construction commutes with cohomologically flat base change, i.e., base change $S' \to S$ such that $\mathrm{Tor}_i^S(\mathcal{O}_{S'}, \mathcal{F}) = 0$ for all $i > 0$.*

*Proof.* Let $\mathcal{I} = \mathrm{Ann}_{\mathcal{O}_X}(\mathcal{F})$ be the annihilator of $\mathcal{F}$ and let $j : Z \hookrightarrow X$ be the closed subscheme defined by $\mathcal{I}$. Then $Z \to S$ is finite and $\mathcal{F} = j_* j^* \mathcal{F}$. Replacing

$X$ with $Z$, we can thus assume that $f$ is finite. The norm

$$f_*\mathcal{O}_X \longrightarrow \operatorname{End}_{\mathcal{O}_S}(f_*\mathcal{F}) \xrightarrow{\det} \mathcal{O}_S$$

defines a multiplicative law and hence a proper relative zero-cycle $\mathcal{N}_{\mathcal{F}/S}$ as in the proposition. □

**Corollary (7.9).** *Let $S$ be locally noetherian and let $f : X \to S$ be a morphism locally of finite type. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite type such that $\operatorname{Supp}(\mathcal{F})$ is quasi-finite over $S$ and such that $\mathcal{F}$ has finite Tor-dimension over $S$. Then there is a unique relative zero-cycle $\mathcal{N}_{\mathcal{F}}$ on $X/S$ with support $\operatorname{Supp}(\mathcal{F})_{\mathrm{dom}/S}$ such that for any point $s \in S$ of depth zero, the induced cycle $(\mathcal{N}_{\mathcal{F}})_s$ is given by the norm $\mathcal{N}_{(f_*\mathcal{F})_s}$ of the free $\mathcal{O}_{S,s}$-module $(f_*\mathcal{F})_s$. This construction commutes with cohomologically flat base change.*

*Proof.* Proposition (7.8) gives a unique proper relative zero-cycle $\alpha_{U/T}$ on any étale neighborhood $(U, T, p, g)$ which thus determines the relative zero-cycle $\mathcal{N}_{\mathcal{F}}$ by Corollary (2.20). □

*Remark* **(7.10).** Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be coherent $\mathcal{O}_X$-modules of finite Tor-dimension over $S$ and with quasi-finite support over $S$. The following properties of the norm of a sheaf of finite Tor-dimension are easily verified.

(i) If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-sheaf, then $\mathcal{N}_{\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}} = \mathcal{N}_{\mathcal{F}}$.
(ii) If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is an exact sequence, then $\mathcal{N}_{\mathcal{G}} = \mathcal{N}_{\mathcal{F}} + \mathcal{N}_{\mathcal{H}}$. In particular, we have that $\mathcal{N}_{\mathcal{F}\oplus\mathcal{G}} = \mathcal{N}_{\mathcal{F}} + \mathcal{N}_{\mathcal{G}}$.

*Remark* **(7.11).** *Norms of perfect complexes* — There is an analog of Proposition (7.8) for certain perfect complexes. Note that not every perfect complex determines a relative cycle. Indeed, a necessary condition is that it is possible to define a relative cycle on depth zero points, i.e., that the alternating determinant is defined on depth zero points. This is also a sufficient condition by the proof of Lemma (7.7).

If $\mathcal{F}_\bullet$ is a perfect complex on $S$ such that the norm of $\mathcal{F}_\bullet$ is defined and $\bigoplus_i \mathrm{H}_i(S, \mathcal{F}_\bullet)$ is of finite Tor-dimension and zero in odd degree, then we have that the norms of $\mathcal{F}_\bullet$ and $\bigoplus_i \mathrm{H}_i(S, \mathcal{F}_\bullet)$ coincide. In particular, if $\mathcal{F}_\bullet$ is a perfect complex on $S$ such that at depth zero points, $\mathrm{H}_i(S, \mathcal{F}_\bullet)$ is zero for odd $i$ and locally free for even $i$, then the norm of $\mathcal{F}_\bullet$ is defined.

**Lemma (7.12)** ([GIT, Lem. 5.8]). *Let $h : B \to S$ be smooth and let $\varphi : X \to B$ be locally of finite type. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module which has finite Tor-dimension over $S$, then $\mathcal{F}$ has finite Tor-dimension over $B$.*

*Proof.* Consider the product $X \times_S B$. The first projection $\pi_1$ has a section $s = (\mathrm{id}_X, \varphi) : X \to X \times_S B$ which is a regular immersion. Thus, $\mathcal{O}_X$ has finite Tor-dimension over $X \times_S B$. The pull-back $\pi_1^*\mathcal{F}$ has finite Tor-dimension over $B$ and thus $\mathcal{F} = s^*\pi_1^*\mathcal{F}$ has finite Tor-dimension over $B$. □

**Proposition (7.13).** *Let $S$ be locally noetherian and let $f : X \to S$ be a smooth curve. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module of finite Tor-dimension over $S$ and such that $Z = \mathrm{Supp}(\mathcal{F})$ is quasi-finite over $S$. Then $\mathcal{N}_{\mathcal{F}/S} = \mathcal{N}_{\mathrm{Div}(\mathcal{F})/S}$ where $\mathrm{Div}(\mathcal{F})$ is the relative Cartier-divisor on $X$ defined by Mumford* [GIT, KM76].

*Proof.* Note that $\mathcal{F}$ has finite Tor-dimension as an $\mathcal{O}_X$-module by Lemma (7.12) and thus $\mathrm{Div}(\mathcal{F})$ is defined. The support of both $\mathcal{N}_{\mathcal{F}/S}$ and $\mathcal{N}_{\mathrm{Div}(\mathcal{F})/S}$ is $Z_{\mathrm{dom}/S}$. By Proposition (6.5) it is enough to show the equality on depth zero points. Taking an étale neighborhood, we can thus assume that $Z/S$ is finite and that $f_*\mathcal{F}$ is a *free* $\mathcal{O}_S$-module. The equality now follows from [Del73, Prop. 6.3.11.1]. $\square$

**Theorem (7.14).** *Let $f : X \to S$ be a morphism locally of finite presentation and let $\mathcal{F}$ be a finitely presented $\mathcal{O}_X$-module which is flat over $S$. Then there is a canonical relative cycle, denoted $\mathcal{N}_{\mathcal{F}}$ on $X/S$ with support* $\mathrm{Supp}(\mathcal{F})$*. This construction commutes with arbitrary base change. If $Z \hookrightarrow X$ is a subscheme such that $Z$ is flat and of finite presentation over $S$, we let $\mathcal{N}_Z = \mathcal{N}_{\mathcal{O}_Z}$.*

*Proof.* The question is local so we can assume that $X$ and $S$ are affine. By a limit argument, we can then assume that $S$ is noetherian. First note that the support $Z$ of $\mathcal{F}$ is universally open [$\mathrm{EGA}_{\mathrm{IV}}$, 2.4.6]. Let $(U, B, T)$ be a projection adapted to $Z$. Then $\mathcal{F}$ is of finite Tor-dimension over $B$ and we let $(\mathcal{N}_{\mathcal{F}})_{U/B/T} = \mathcal{N}_{\mathcal{F}/B}$. This defines a relative cycle. Note that any base change $B \times_S S' \to B$ is cohomologically flat with respect to $\mathcal{F}$ over $B$ as $\mathcal{F}$ is flat over $S$. $\square$

Note that $\mathcal{N}_{\mathcal{F}}$ is defined for sheaves $\mathcal{F}$ with non-proper and non-equidimensional support. We do not even require that $X/S$ is separated. For representability, we need families of cycles to be equidimensional. However, even if $\mathcal{F}$ is a sheaf whose support is not equidimensional, then we have the equidimensional relative cycle $(\mathcal{N}_{\mathcal{F}})_r$. If $\mathcal{F}$ has proper support with fibers of dimension at most $r$, then $(\mathcal{N}_{\mathcal{F}})_r$ is a proper relative cycle of dimension $r$. In particular, we obtain the following morphism:

**Corollary (7.15).** *There is a canonical morphism from the functor* $\mathrm{Quot}_r(\mathcal{G}/X/S)$ *to the functor* $\mathrm{Chow}_r(X/S)$ *given by $\mathcal{F} \mapsto (\mathcal{N}_{\mathcal{F}})_r$. Similarly, there is a canonical morphism from the functor* $\mathrm{Hilb}_r(X/S)$ *to the functor* $\mathrm{Chow}_r(X/S)$ *given by $Z \mapsto (\mathcal{N}_Z)_r$. Here* $\mathrm{Hilb}_r(X/S)$ *is the Hilbert functor parameterizing subschemes $Z$ which are proper and of dimension $r$ but not necessary equidimensional, and* $\mathrm{Chow}_r(X/S)$ *is the Chow functor parameterizing equidimensional proper relative cycles of dimension $r$.*

Later on, we will see that there also are morphisms from the Hilbert stack and from the Kontsevich space of stable maps to the Chow functor. In particular, we obtain a morphism from the stack of Branch varieties [AK06] and from the space of Cohen-Macaulay curves [Høn04] to the Chow functor. This is discussed in Section 13.

## 8. The underlying cycle

In this section we will assign to any relative cycle $\alpha$ on $X/S$, an ordinary cycle, denoted $\mathrm{cycl}(\alpha)$. The support of $\mathrm{cycl}(\alpha)$ coincides with the support of $\alpha$. As the support is universally open, the only thing that we need to define is the multiplicities of the components over a generic point of $S$.

**Proposition-Definition (8.1).** *Let $S = \mathrm{Spec}(k)$ be the spectrum of a field and $\alpha$ be a relative cycle on $X/S$. Let $x \in X$ be a point which is generic in* $\mathrm{Supp}(\alpha)$*. Then there is a unique number* $\mathrm{mult}_x(\alpha)$*, the* multiplicity *of $\alpha$ at $x$, such that for any projection $(U, B, T, p, g)$ and $u \in U$ above $x \in X$ and $t \in T$ such that $k(t)/k(s)$ is* separable*, we have that* $\mathrm{mult}_u(\alpha_{U/B/T}) = \mathrm{mult}_x(\alpha)$*, cf. Definition* (1.10)*. The* geometric multiplicity *at $x$ is the product of the multiplicity at $x$ and the radical multiplicity of* $k(x)/k(s)$ [EGA$_{\mathrm{IV}}$, Def. 4.7.4]*. The geometric multiplicity is constant under arbitrary base change.*

*Proof.* We first observe that if $(U, B, T)$ is a projection, $T' \to T$ is an arbitrary morphism and $u' \in U' = U \times_T T'$ a point above $u$, then the geometric multiplicities of $\alpha_{U/B/T}$ at $u$ and $\alpha_{U'/B'/T'}$ at $u'$ coincide. This is Proposition (1.13) (ii). Thus, $\mathrm{mult}_{u'}(\alpha_{U'/B'/T'}) = \mathrm{mult}_u(\alpha_{U/B/T})r$ where $r$ is the length of $\mathrm{Spec}(k(u) \otimes_{k(t)} k(t'))$ at $u'$ [EGA$_{\mathrm{IV}}$, Prop. 4.7.3]. It follows that the multiplicites at $u'$ and $u$, multiplied with the radical multiplicity of $k(u')/k(t')$ and $k(u)/k(t)$ respectively, coincide.

It is thus enough to show that if $\overline{k}$ is an algebraically closed field, $(U_1, B_1, \mathrm{Spec}(\overline{k}))$ and $(U_2, B_2, \mathrm{Spec}(\overline{k}))$ are two projections and $u_1 \in U_2$ and $u_2 \in U_2$ are two points above $x$, then the multiplicites of $\alpha_{U_1/B_1/\overline{k}}$ at $u_1$ and $\alpha_{U_2/B_2/\overline{k}}$ at $u_2$ coincide. As the multiplicity is constant under pull-back by étale morphisms $U' \to U_i$ by Proposition (1.13) (iv), we can replace the $U_i$'s with $U_1 \times_X U_2$ and the $u_i$'s with $(u_1, u_2)$ and hence assume that $X = U_1 = U_2$ and $x = u$. Taking étale projections $B_1 \to \mathbb{A}^r$ and $B_2 \to \mathbb{A}^r$, which is possible locally around the images of $x$ and using Proposition (1.13) (vi) we can assume that $B_1 = B_2 = \mathbb{A}^r$. Taking an open neighborhood of $x$, we can assume that $Z = \mathrm{Supp}(\alpha)$ is smooth and irreducible.

Let $\varphi_1$ and $\varphi_2$ be the two projections $X \to \mathbb{A}^r$. It is enough to show that the multiplicity of $\alpha_{\varphi_1}$ at $x$ coincides with the multiplicity of $\alpha_{\varphi_2}$ for a particular choice of $\varphi_2$. Taking a generic projection, we can thus assume that $\varphi_2|_Z$ is étale. The morphisms $\varphi_1$ and $\varphi_2$ can be put into a single projection

$$\varphi : X \times_{\overline{k}} T \to \mathbb{A}^r \times_{\overline{k}} T$$

over $T = \mathbb{A}^1_{\overline{k}}$ such that $\varphi_1 = \varphi|_{t=0}$ and $\varphi_2 = \varphi|_{t=1}$. Let $U \subseteq X \times T$ be the open subset where $\varphi|_{Z \times T}$ is quasi-finite. This subset contains $X \times \{0\}$ and $X \times \{1\}$. As $Z \times T \to T$ is Cohen-Macaulay it follows that $\varphi|_{Z \times T}$ is flat over $U$. Moreover, as $\varphi_2|_Z$ is étale it follows that $\varphi|_{Z \times T}$ is generically étale. It then readily follows from [**II**, Prop. 8.6] that the (non-proper) relative zero-cycle $\alpha_\varphi$ is of the form $m \cdot \mathcal{N}_{Z \times T/\mathbb{A}^r \times T}$ for some positive integer $m$. We thus have that $\alpha_{\varphi_i} = m\mathcal{N}_{Z/_{\varphi_i}\mathbb{A}^r}$ for $i = 1, 2$. It follows that $\mathrm{mult}_x \alpha_{\varphi_1} = \mathrm{mult}_x \alpha_{\varphi_2} = m$. $\square$

**Definition (8.2).** Let $S$ be arbitrary and let $\alpha$ be a relative cycle on $X/S$ with support $Z$. The *underlying cycle* of $\alpha$ is the effective cycle $\operatorname{cycl}(\alpha)$ with $\mathbb{Q}$-coefficients defined by

$$\operatorname{cycl}(\alpha) = \sum_{\substack{s \in S_{\max} \\ x \in (Z_s)_{\max}}} \operatorname{mult}_x(\alpha_s) \left[ \overline{\{x\}} \right].$$

Here $\overline{\{x\}}$ denotes the closure of $x$ in $\operatorname{Supp}(\alpha)$ as a reduced subscheme.

*Remark* **(8.3).** It follows from Proposition-Definition (8.1) that if $(U, B, T, p)$ is a smooth projection, then $p^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(\alpha_{U/B/T})$.

The following definition generalizes [**II**, Def. 8.1].

**Definition (8.4).** Let $K/k$ be a finitely generated field extension. The *inseparable degree*, or radical multiplicity [EGA$_{\mathrm{IV}}$, Def. 4.7.4], is the maximum length of $K \otimes_k k'$ where $k'/k$ is an inseparable extension. The *exponent* of $K/k$ is the smallest integer $e$ such that $K^e k/k$ is separable. The *inseparable discrepancy* is the quotient of the inseparable degree and the exponent.

If $K/k$ is a finitely generated field extension and $k' = k(x_1, x_2, \dots, x_r) \subseteq K$ is a transcendence basis, then the exponent of $K/k'$ is a multiple of the exponent of $K/k$. Moreover, there is a transcendence basis such that the exponent of $K/k'$ equals the exponent of $K/k$, e.g., take $k'$ as a separating transcendence basis of $K^e k/k$.

**Definition (8.5).** Let $S$ be a scheme and let $X/S$ be locally of finite type. A cycle $\mathcal{Z}$ on $X$ with $\mathbb{Q}$-coefficients is *quasi-integral* if the multiplicity of every irreducible component $Z_i$ of $\mathcal{Z}$ becomes an integer after multiplying it with the inseparable discrepancy of $k(Z_i)/k(S_i)$. Here $S_i$ denotes the image of $Z_i$ in $S$.

**Theorem (8.6).** *Let $S = \operatorname{Spec}(k)$ be the spectrum of a field. Then there is a one-to-one correspondence between relative cycles on $X/S$ and effective cycles on $X$ with quasi-integral coefficients. This correspondence is given by associating the underlying cycle to a relative cycle.*

*Proof.* It is clear from [**II**, Prop. 8.6] that every cycle comes from at most one relative cycle. If $\alpha$ is a family on $X/S$ then $\alpha$ has quasi-integral coefficients. In fact, let $Z$ be an irreducible component of $\operatorname{Supp}(\alpha)$ and let $e_Z$ be the exponent of $K(Z)/k$. Then $K(Z)^{e_Z}/k$ is separable and there is a separating transcendence basis $t_1, t_2, \dots, t_r$. The homomorphism $k[t_1, t_2, \dots, t_r] \to K(Z)^{e_Z} \to K(Z)$ extends to a morphism $U \to \mathbb{A}^r_k$ for some open subset $U \subseteq Z$. The inseparable discrepancy of $K(Z)/k$ coincides with the inseparable discrepancy of $K(Z)/K(t_1, t_2, \dots, t_r)$ and thus it follows from [**II**, Prop. 8.11] that the multiplicity of $\alpha$ at $Z$ is quasi-integral.

Conversely, let us show that the quasi-integral cycle $\frac{1}{e_Z}[Z]$ is the underlying cycle of a relative cycle. We can assume that $X = Z$. Let $(U, B, \mathbb{A}^n_k, p, g)$ be a smooth projection adapted to $X$. We want to construct a canonical relative zero-cycle $\alpha_{U/B/\mathbb{A}^n_k}$ on $U/B$ with underlying cycle $\frac{1}{e_Z}[U]$. As $B$ is normal (even regular), it is enough to construct this canonical relative zero-cycle over a generic point of $B$ by Theorem (6.1). We can thus assume that $U$ and $B$ are irreducible. The inseparable discrepancy of $k(U)/k(B)$ is a multiple of the inseparable discrepancy of $k(X)/k$ and the existence of the relative cycle follows from [**II**, Prop. 8.11]. □

**Corollary (8.7).** *Let $S$ be a reduced scheme. Then there is an injective map*

$$\mathit{Cycl}(X/S) \to \{\textit{quasi-integral effective cycles on } X\}$$

*taking a relative cycle $\alpha$ on $X/S$ to its underlying cycle.*

**Corollary (8.8).** *Let $S$ be a* reduced *scheme and let $\alpha$ be a relative cycle on $X/S$ with support $Z$. Then $\alpha$ satisfies condition $(*)$ of Section 5 and $\alpha$ is the push-forward of a relative cycle on $Z/S$.*

*Proof.* If $(U, B, T, p)$ is a smooth projection, then $p^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(\alpha_{U/B/T})$. This shows condition $(*)$, i.e., that $\alpha_{U/B/T}$ does not depend on the morphisms $B \to T$ and $T \to S$. The last statement follows from Proposition (5.5). □

**Lemma (8.9).** *Let $S = \operatorname{Spec}(k)$ be the spectrum of a field and let $\mathcal{Z}$ be an effective cycle with $\mathbb{Q}$-coefficients on $X/S$. Then $\mathcal{Z}$ is quasi-integral if and only if $k$ is the intersection of all inseparable field extensions $k'/k$ such that $\mathcal{Z}_{k'}$ has integral coefficients.*

**Proposition (8.10).** *Let $X/S$ be a quasi-projective scheme with a given embedding $X \hookrightarrow \mathbb{P}(\mathcal{E})$ where $\mathcal{E}$ is a locally free $\mathcal{O}_S$-sheaf. Then there is functorial bijection between $k$-points of* $\operatorname{Chow}(X/S)$ *and $k$-points of* $\operatorname{ChowVar}(X \hookrightarrow \mathbb{P}(\mathcal{E}))$.

*Proof.* This follows from Lemma (8.9) and [Kol96, Thm. 4.5]. □

**Definition (8.11).** Let $\alpha$ be a relative cycle on $f : X \to S$. We say that $\alpha$ is *multiplicity-free* at a point $x \in X$ if the geometric multiplicity of $\alpha_{f(x)}$ is one at the generic points of the irreducible components of $\operatorname{Supp}(\alpha)_{f(x)}$ containing $x$. We say that $\alpha$ is *normal* (resp. *smooth*) at $x$ if $\alpha$ is multiplicity-free and equidimensional at $x$ and $\operatorname{Supp}(\alpha_{f(x)}) = \big(\operatorname{Supp}(\alpha)_{f(x)}\big)_{\mathrm{red}}$ is geometrically normal (resp. smooth) at $x$ over $k(f(x))$.

The requirement that $\alpha$ is equidimensional at $x$ is explained by the following example:

**Example (8.12).** Let $S = \operatorname{Spec}(k[t])$ and $X = \operatorname{Spec}(k[t, x, y]/x(y, x - t))$. Then $X$ is the union of a plane and a line meeting in the origin. The natural morphism $X \to S$ is smooth outside the origin. The special fiber $X_0$ is an affine line with an embedded point. The corresponding relative cycle $\mathcal{N}_{X/S}$ has underlying cycle $[X]$ and special fiber $[X_0] = [(X_0)_{\mathrm{red}}]$ which is smooth.

Note that the fact that $\alpha$ is multiplicity-free at $x \in X$, does *not* imply that $\operatorname{Supp}(\alpha)$ is reduced at $x$ in its fiber. However, if $f : Z \to S$ is flat and $\alpha = \mathcal{N}_{Z/S}$, then $\alpha$ is multiplicity-free (resp. normal, resp. smooth) at $z \in Z$ if and only if $Z$ is geometrically $(\mathrm{R}_0)$ (resp. geometrically normal, resp. smooth) at $z$ in $Z_{f(z)}$.

## 9. Representable relative cycles

We have showed that if $S$ is reduced, then any relative cycle on $X/S$ is represented by an ordinary cycle on $X/S$, cf. Corollary (8.7). In this section, we will show that smooth relative cycles correspond to subschemes which are smooth over $S$ and that if $X/S$ is smooth then relative Weil divisors on $X/S$ correspond to relative Cartier divisors on $X/S$. Unfortunately, these result are so far only proven when either $S$ is reduced or $S$ is of characteristic zero. I conjecture that these results hold in general.

It then follows (assuming that $S$ is reduced or that $S$ has characteristic zero), that multiplicity-free relative cycles and relative Weil divisors on $(\mathrm{R}_1)$-schemes are represented by unique subschemes which are flat in relative codimension zero. When $\alpha$ is a relative cycle on $X/S$ such that either $S$ is reduced, $\alpha$ is multiplicity-free or $\alpha$ is a relative Weil divisor on a $(\mathrm{R}_1)$-scheme (cases (A1)–(A3) in the introduction), $\alpha$ has several nice properties. In the following sections, these three cases are discussed in more detail.

**Proposition (9.1).** *Let $f : X \to S$ be an algebraic space, locally of finite type and let $\alpha$ be a relative cycle on $X/S$ with support $Z$. The set of points $z \in Z$ such that $\alpha$ is multiplicity-free at $z$ is open.*

*Proof.* Let $z \in Z$ be a point at which $\alpha$ is multiplicity-free and let $s = f(z)$. After replacing $X$ with an open neighborhood of $z$, we can assume that every irreducible component of $Z_s$ contains $z$. It is enough to show that $\alpha_r$ is multiplicity-free in a neighborhood of $z$ for every $r$. Thus we can assume that $Z$ is equidimensional of dimension $r$ but it is now possible that $z \notin Z$.

After restricting $X$ and $S$ further, we can assume that there is an embedding $X \hookrightarrow \mathbb{A}^n_S$. There is then a projection $\pi_s : \mathbb{A}^n_s \to \mathbb{A}^r_s$ such that $\pi_s|_{Z_s}$ is quasi-finite in a neighborhood of $z$ and generically étale. After restricting $S$, we can assume that this projection extends to a projection $\pi : \mathbb{A}^n_S \to \mathbb{A}^r_S$. Choose étale morphisms $B \to \mathbb{A}^r_S$ and $U \to \mathbb{A}^n_S$ such that $(U, B, S, p, g)$ is a projection adapted to $Z$ with $z \in p(U)$. Then $\alpha_{U/B/S}$ is non-degenerate at the generic points of $B_s$. The non-degeneracy locus $B_{\mathrm{nondeg}}$ of the proper relative zero-cycle $\alpha_{U/B/S}$ is open. As the fibers of $B \to S$ are irreducible, it follows that $\alpha$ is multiplicity-free over the image of $B_{\mathrm{nondeg}}$ in $S$. This is an open subset as $B \to S$ is open. □

**Proposition (9.2).** *Let $f : X \to S$ be an algebraic space, locally of finite type and let $\alpha$ be a relative cycle on $X/S$ with support $Z_0$. Let $x \in X$ be a smooth point of $\alpha$ and let $s = f(x)$. Then there is a smooth projection $(U, B, S, p)$, and a point $u \in U$ over $x$ such that $p^{-1}((Z_0)_s)_{\mathrm{red}} \to B_s$ is étale at $x$. If $(U, B, S, p)$ is any such projection with $u \in U$ above $x$ then, in a neighborhood of $U$, there exists a closed subscheme $Z \hookrightarrow U$ which is smooth over $S$ and étale over $B$ such that*

*$\alpha_{U/B/S} = \mathcal{N}_{Z/B}$. In particular, the set of points $x \in X$ such that $\alpha$ is smooth at $x$ is open.*

*Proof.* Let $x \in \mathrm{Supp}(\alpha)$ be a point at which $\alpha$ is smooth and let $s$ be its image in $S$. Let $Z_0 = \mathrm{Supp}(\alpha) \hookrightarrow X$. Then $((Z_0)_s)_{\mathrm{red}} \to \mathrm{Spec}(k(s))$ is smooth at $x$. Thus in a neighborhood of $x$ there is a factorization $((Z_0)_s)_{\mathrm{red}} \to \mathbb{A}^r_{k(s)} \to \mathrm{Spec}(k(s))$ such that $((Z_0)_s)_{\mathrm{red}} \to \mathbb{A}^r_{k(s)}$ is étale at $x$. As $Z_0 \to S$ is equidimensional at $x$, this factorization lifts to a neighborhood $U \subseteq X$ of $x$ such that $U \cap Z_0 \to \mathbb{A}^r_S$ is quasi-finite and dominant. We thus have a quasi-adapted projection and after étale localization, we obtain an adapted projection.

Now let $(U, B, S, p)$ be any smooth projection such that $p^{-1}((Z_0)_s)_{\mathrm{red}}$ is étale over $B_s$. Then passing to the fiber at $s$ we obtain a family $\alpha_{U_s/B_s/s}$ which is étale, i.e., non-degenerate, at $u$. Thus, so is $\alpha_{U/B/S}$ in a neighborhood of $u$ and the proposition follows with $Z = \mathrm{Image}(\alpha_{U/B/S})$. □

**Corollary (9.3).** *Let $f : X \to S$ be an algebraic space, locally of finite type and let $\alpha$ be a relative cycle on $X/S$. Then* $\mathrm{Supp}(\alpha) \to S_{\mathrm{red}}$ *is smooth at smooth points of $\alpha$.*

*Proof.* We can assume that $S$ is reduced. Then for any smooth projection $(U, B, S)$ we have that $B$ is reduced. It follows that $\mathrm{Image}(\alpha_{U/B/S}) = \mathrm{Supp}(\alpha_{U/B/S})$ and hence $\mathrm{Supp}(\alpha) \to S_{\mathrm{red}}$ is smooth by Proposition (9.2). □

Proposition (9.2) states that locally at $x \in X$ there is a *subscheme* $Z \hookrightarrow X$ such that $Z \to S$ is smooth and $\mathcal{N}_Z$ is equal to $\alpha$ under a certain projection. However, it does *not* follow trivially that this subscheme is independent on the choice of projection, except when $S$ is reduced. At the moment, I can only show that $Z$ is independent on the choice of projection in characteristic zero. We begin with two lemmas valid in arbitrary characteristic.

**Lemma (9.4).** *Let $S$ be a scheme, let $X \to S$ be flat and locally of finite presentation and let $G \to S$ be proper and smooth with geometrically connected fibers. Let $S_0 = S_{\mathrm{red}}$, $G_0 = G \times_S S_0$ and $X_0 = X \times_S S_0$. Let $Z_0 \hookrightarrow X_0$ be a subscheme which is flat and locally of finite presentation over $S_0$. Let $W_0 = Z_0 \times_{S_0} G_0$ and let $W \hookrightarrow X \times_S G$ be a subscheme such that $W \times_S S_0 = W_0$. Further, assume that $W \to G$ is flat and finitely presented over a schematically dense open retro-compact subset $U \subseteq W$ which contains all points of relative codimension one over $Z_0$. Then there exists a unique subscheme $Z \hookrightarrow X$, flat and locally of finite presentation over $S$, such that $Z_0 = Z \times_S S_0$ and $W = Z \times_S G$.*

Note that *a priori* $W \to G$ is only flat over $U$ but that *a posteriori* it follows that $W \to G$ is flat. The lemma thus essentially states that all deformations of $W_0$ come from deformations of $Z_0$.

*Proof.* The question is local on $X$ and $S$ and we can thus assume that $X$ and $S$ are affine and that $W$ and $Z_0$ are closed subschemes. By a limit argument, we can also assume that $S$ is noetherian. By effective descent of closed subschemes for the smooth morphism $X \times_S G \to X$, the existence of a $Z$ such that $W =$

$Z \times_S G$ is equivalent to the condition that $\pi_1^{-1}W = \pi_2^{-1}W$ where $\pi_1$ and $\pi_2$ are the two projections $X \times_S G \times_S G \to X \times_S G$. This can be checked on infinitesimal neighborhoods of depth zero points on $W$, and hence on infinitesimal neighborhoods of depth zero points on $S$. We can thus assume that $S$ is the spectrum of a local artinian ring $A$ with maximal ideal $\mathfrak{m}$ and residue field $k$.

We will show the lemma by induction on the integer $n$ such that $\mathfrak{m}^n = 0$. If $n = 1$, then there is nothing to prove. If $n > 1$, let $A_1 = A/\mathfrak{m}^{n-1}$ and let $J = \ker(A \to A_1)$ so that $J\mathfrak{m} = 0$. Then $J$ is a $k$-module. Let $S_1 = \operatorname{Spec}(A_1)$ and let $X_1 = X \times_S S_1$, $G_1 = G \times_S S_1$ and $W_1 = W \times_S S_1$. Then by induction there is a subscheme $Z_1 \hookrightarrow X_1$, flat and finitely presented over $S_1$, such that $W_1 = Z_1 \times_{S_1} G_1$. Let $\mathcal{I}_i$ be the ideal sheaves defining $Z_i \hookrightarrow X_i$, for $i = 0, 1$, and let $p : U \times_G G_1 \to Z_1$ be the composition of the open immersion $j : U \times_G G_1 \to W_1$ and the projection $\pi : W_1 = Z_1 \times_{S_1} G_1 \to Z_1$.

By the deformation theory of Hilbert schemes, cf. [FGA, No. 221, p. 21] or [Kol96, I.2], the obstruction to extend the flat family $W_1|_U = p^{-1}Z_1$ over $S_1$ to a flat family over $S$ is an element

$$c_J(W_1|_U) \in \operatorname{Ext}^1_{p^{-1}X_1}\bigl(p^*\mathcal{I}_1, \mathcal{O}_{W_0|_U} \otimes_k J\bigr) = \operatorname{Ext}^1_{X_1}\bigl(\mathcal{I}_1, p_*p^*(\mathcal{O}_{Z_0}) \otimes_k J\bigr).$$

As such an extension exists, namely the deformation $W|_U \to S$, the obstruction $c_J(W_1|_U)$ is zero. Moreover, $W|_U$ corresponds (non-canonically) to an element in

$$\operatorname{Hom}_{p^{-1}X_0}\bigl(p^*\mathcal{I}_0, \mathcal{O}_{W_0|_U} \otimes_k J\bigr) = \operatorname{Hom}_{X_0}\bigl(\mathcal{I}_0, p_*p^*(\mathcal{O}_{Z_0}) \otimes_k J\bigr).$$

Now, as $G \to S$ is proper and smooth, we have that $\pi$ is cohomologically flat in dimension zero [$\text{EGA}_{\text{III}}$, Prop. 7.8.6] and as $G \to S$ has geometrically connected fibers it thus follows that $\pi_*\mathcal{O}_{W_0} = \mathcal{O}_{Z_0}$. As the open immersion $j$ contains all points of depth one of $W_0$, it follows that $j_*j^*\mathcal{O}_{W_0} = \mathcal{O}_{W_0}$ and hence $p_*p^*\mathcal{O}_{Z_0} = \mathcal{O}_{Z_0}$. It follows that the obstruction

$$c_J(Z_1) \in \operatorname{Ext}^1_{X_1}\bigl(\mathcal{I}_1, \mathcal{O}_{Z_0} \otimes_k J\bigr)$$

is zero and that the deformation $W \to S$ of $W_1 \to S_1$ is the pull-back of a deformation $Z \to S$ of $Z_1 \to S_1$. $\square$

We the need the following construction of Angéniol and El Zein [AEZ78, §I].

**(9.5)** *Grassmannians of projections* — Let $S$ be a scheme, let $\mathcal{E} = \mathcal{O}_S^n$ be a free sheaf of rank $n$ and let $X = \mathbb{A}^n_S = \operatorname{Spec}_S(\mathcal{E}^\vee)$. Let $\mathbb{G} = \mathbb{G}(r,n) = \mathbb{G}_r(\mathcal{E})$ be the grassmannian parameterizing quotients $\mathcal{E} \twoheadrightarrow \mathcal{F}$ such that $\mathcal{F}$ is locally free of rank $r$ [$\text{EGA}_{\text{I}}$, 9.7]. Let $\pi : \mathbb{G} \to S$ be the structure morphism and let $\pi^*\mathcal{E} \twoheadrightarrow \mathcal{F}$ be the universal quotient. We then let $\mathbb{B} = \operatorname{Spec}_{\mathbb{G}}(\mathcal{F}^\vee)$. The morphism $\mathbb{B} \to \mathbb{G}$ is a vector bundle of rank $r$. The morphism $\mathbb{A}^n_{\mathbb{G}} = \operatorname{Spec}_{\mathbb{G}}(\mathcal{E}^\vee) \to \operatorname{Spec}_{\mathbb{G}}(\mathcal{F}^\vee) = \mathbb{B}$ is the *universal projection*.

Let $Z \hookrightarrow \mathbb{A}^n_S$ be a closed subset, equidimensional of dimension $r$ over $S$. Let $U \subseteq Z \times_S \mathbb{G}$ be the open subset over which $Z \times_S \mathbb{G} \hookrightarrow \mathbb{A}^n_{\mathbb{G}} \to \mathbb{B}$ is quasi-finite. We say that $Z \hookrightarrow \mathbb{A}^n_S$ has property (P$'$) if $U \subseteq Z \times_S \mathbb{G}$ contains all points of relative codimension at most one over $Z$.

**Lemma (9.6).** *Let $S$ be an affine scheme, let $X \to S$ be a scheme with a closed immersion $X \hookrightarrow \mathbb{A}^n_S$ and let $Z_0 \hookrightarrow X \times_S S_{\mathrm{red}}$ be a closed subscheme such that $Z_0 \to S_{\mathrm{red}}$ is a finitely presented morphism. Then there exists, Zariski-locally on $X$, a closed immersion $X \hookrightarrow \mathbb{A}^{n+m}_S$ such that the projection onto the first $n$ factors is the original embedding of $X$ in $\mathbb{A}^n_S$ and such that $Z_0 \hookrightarrow \mathbb{A}^{n+m}_S$ has property* (P′).

*Proof.* By a limit argument, there exists a noetherian scheme $S_\alpha$, an affine morphism $S \to S_\alpha$ and a morphism of finite type $Z_\alpha \to (S_\alpha)_{\mathrm{red}}$ such that $Z_0 \to S_{\mathrm{red}}$ is the pull-back of $Z_\alpha \to (S_\alpha)_{\mathrm{red}}$ along $S_{\mathrm{red}} \to (S_\alpha)_{\mathrm{red}}$. By [AEZ78, Lem. I.3], every point $z \in Z_\alpha$ admits an open neighborhood $V_\alpha$ and a closed immersion $V_\alpha \hookrightarrow \mathbb{A}^m_{S_\alpha}$ satisfying (P′). If $V = Z_0 \times_{Z_\alpha} V_\alpha$, then the corresponding immersion $V \hookrightarrow \mathbb{A}^m_S$ also satisfies (P′). After replacing $X$ with an open neighborhood of $z$, we can lift this immersion to a morphism $X \to \mathbb{A}^m_S$. We thus obtain an immersion $X \hookrightarrow \mathbb{A}^{n+m}_S$. By *loc. cit.* the immersion $Z_0 \hookrightarrow X \hookrightarrow \mathbb{A}^{n+m}_S$ satisfies (P′). □

**Proposition (9.7).** *Let $S$ be purely of* characteristic zero *and let $X \to S$ be locally of finite type. Let $\alpha$ be a smooth relative cycle on $X/S$. Let $(X, B_1, S)$ and $(X, B_2, S)$ be two projections quasi-adapted to* $\mathrm{Supp}(\alpha)$. *Assume that there exists a locally closed subscheme $Z \hookrightarrow X$, such that $Z \to B_1$ is étale and such that $\alpha_{X/B_1/S} = \mathcal{N}_{Z/B_1}$. Then $\alpha_{X/B_2/S} = \mathcal{N}_{Z/B_2}$.*

*Proof.* Let $Z_0 \hookrightarrow X \times_S S_{\mathrm{red}}$ be the support of $\alpha$. This is smooth over $S_{\mathrm{red}}$ by Corollary (9.3). The question is local on $X$ and $S$ and can be checked at neighborhoods of the generic points of $Z_0$. Taking étale projection $B_i \to \mathbb{A}^r_S$, we can assume that $B_i = \mathbb{A}^r_S$. Locally on $X$ there is then a closed immersion $X \hookrightarrow \mathbb{A}^n_S$ such that the two projections $(X, \mathbb{A}^r_S, S)$ lifts to linear projections $\mathbb{A}^n_S \to \mathbb{A}^r_S$. We then take a closed immersion $X \hookrightarrow \mathbb{A}^{n+m}_S$ as in Lemma (9.6). We thus have a grassmannian $\mathbb{G} \to S$ and a projection $(X \times_S \mathbb{G}, \mathbb{B}, \mathbb{G})$ which is quasi-adapted to $Z_0$ over an open subscheme $U \subseteq X \times_S \mathbb{G}$ containing all points of relative codimension at most one over $X$. Furthermore, the two projections $X \to \mathbb{A}^r_S$ that we started with, appear as two of the fibers of the grassmannian family.

As the family of one of these fibers is non-degenerate, it follows that $\alpha_{U/\mathbb{B}/\mathbb{G}}$ is generically non-degenerate. It follows that there exists a closed subscheme $W \hookrightarrow X \times_S \mathbb{G}$ and an open subset $V \subseteq U \subseteq X \times_S \mathbb{G}$ such that $W|_V \subseteq W$ is schematically dense and $W|_V \to \mathbb{B}$ is étale. Furthermore, as $S$ is of *characteristic zero*, it follows that $V$ contains all points lying over a generic point of $Z_0$. Indeed, any quasi-finite morphism between regular schemes is generically étale in characteristic zero.

Replacing $X$ with an open neighborhood of any generic point of $Z_0$ we can thus assume that $W|_U$ is smooth over $\mathbb{G}$. It follows from Lemma (9.4) that $W = Z \times_S \mathbb{G}$ for a unique subscheme $Z \hookrightarrow X$ with support $Z_0$. □

If $S$ is noetherian and such that the residue field of every points of depth zero has characteristic zero, then Proposition (9.7) is still true, as can be seen from the proof of Lemma (9.4). I do not know if the proposition is false in positive characteristic.

**Theorem (9.8).** *Let $S$ be a scheme purely of characteristic zero and let $X/S$ be locally of finite type. Let $\alpha$ be a smooth relative cycle on $X/S$. Then there is a unique subscheme $Z \hookrightarrow X$ which is smooth over $S$ such that $\alpha = \mathcal{N}_Z$.*

*Proof.* Let $x \in Z$ and let $(U, B, S, p, g, \varphi)$ be a projection with a lifting $u \in U$ of $x$ as in Proposition (9.2) such that $\alpha_{U/B} = \mathcal{N}_W$ for a subscheme $W \hookrightarrow U$ which is smooth over $S$. We apply Proposition (9.7) with $(U \times_X U, B, S)$ and the two morphisms $\varphi_i : U \times_X U \to B$ given by the compositions of the projections and the morphism $\varphi : U \to B$. By étale descent, it then follows that there exists a subscheme $Z \hookrightarrow X$ which is smooth over $S$ such that $W = Z \times_X U$.

Now let $(U', B', T', p')$ be an arbitrary projection. We will show that $\alpha_{U'/B'/T'} = \mathcal{N}_{p'^{-1}(Z)/B'}$, and it suffices to show this equality étale-locally on $U'$. This follows from Proposition (9.7) applied on the two projections $(U' \times_X U, B', T', p' \circ \pi_1)$ and $(U' \times_X U, B \times_T T', T', p \circ \pi_2)$, ☐

**Corollary (9.9).** *Let $S$ be a scheme purely of characteristic zero and let $X/S$ be locally of finite type. Let $\alpha$ be a relative cycle on $X/S$ which is multiplicity-free. There is a unique subscheme $Z \hookrightarrow X$ which has support* $\mathrm{Supp}(\alpha)$ *and a fiberwise dense open subset $U \subseteq Z$, containing all associated points, such that $U \to S$ is smooth and such that $\mathcal{N}_{U/S} = \alpha|_U$. Moreover $\alpha$ is uniquely determined by $Z$. If $S' \to S$ is an arbitrary morphism, then the unique subscheme corresponding to $\alpha \times_S S'$ is the closure of $U \times_S S'$ in $Z \times_S S'$.*

*Proof.* Let $Z_0$ be the support of $\alpha$. By Proposition (9.2), the subset $U_0 \subseteq Z_0$ of points where $\alpha$ is smooth is open. As $\alpha$ is multiplicity-free, this subset contains all points which are generic in their fibers, i.e., $U_0 \subseteq Z_0$ is fiberwise dense. Let $V \subseteq X$ be any open subset restricting to $U_0$. It then follows from Theorem (9.8) that $\alpha|_V = \mathcal{N}_{Z_V}$ for a unique subscheme $Z_V \hookrightarrow V$ which is smooth over $S$. This extends uniquely to a locally closed subscheme $Z \hookrightarrow X$ such that $Z|_V = Z_V$ is schematically dense in $Z$. ☐

**Corollary (9.10).** *Let $S$ be a scheme purely of characteristic zero and let $X/S$ be locally of finite presentation. The morphism* $\mathrm{Hilb}_r^{\mathrm{red}}(X/S) \to \mathrm{Chow}_r^{\mathrm{red}}(X/S)$ *from the Hilbert functor parameterizing equidimensional and reduced subschemes of dimension $r$ to the Chow functor parameterizing equidimensional and multiplicity-free families of cycles of dimension $r$ is a monomorphism.*

*Remark* **(9.11).** If $X/S$ is quasi-projective, it is not difficult to show that the above morphism is an immersion when restricted to a component $\mathrm{Hilb}_P^{\mathrm{red}}(X/S)$ where $P$ is a polynomial of degree $r$. This also follows from the representability of $\mathrm{Hilb}_r(X/S)$ and $\mathrm{Chow}_r(X/S)$ for a projective scheme $X/S$ as it then follows that $\mathrm{Hilb}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is proper.

**Proposition (9.12).** *Let $f : X \to S$ be an algebraic space and let $\alpha$ be a relative cycle on $X/S$. Let $x \in X$ be a point such that $\alpha$ is a relative Weil divisor at $x$ and $f$ is smooth at $x$. Then there is a projection $(U, B, S, p, g)$ quasi-adapted to* $\mathrm{Supp}(\alpha)$, *such that $p^{-1}(x)$ is non-empty and such that $U \to B$ is smooth. Furthermore, for any such projection, we have that $\alpha_{U/B/S} = \mathcal{N}_{Z/B}$ for a unique subscheme $Z \hookrightarrow U$ flat over $B$.*

*Proof.* The existence of the projection follows from an argument similar as in Proposition (9.2). The existence of $Z$ follows from (2.9) as $U/B$ is a smooth curve. □

**Corollary (9.13).** *Let $f : X \to S$ be smooth and let $\alpha$ be a relative Weil divisor on $X/S$. Then $\operatorname{Supp}(\alpha) \to S_{\mathrm{red}}$ is flat.*

As before we would like to show that $Z$ is independent upon the choice of smooth projection but this is only accomplished in characteristic zero.

**Proposition (9.14).** *Let $S$ be a scheme purely of* characteristic zero. *Let $X/S$ be smooth and let $\alpha$ be a relative Weil divisor on $X/S$. Let $(X, B_1, S)$ and $(X, B_2, S)$ be two projections quasi-adapted to* $\operatorname{Supp}(\alpha)$. *Assume that $X \to B_1$ is smooth, such that $\alpha_{X/B_1/S} = \mathcal{N}_{W/B_1}$ for a locally closed subscheme $W \hookrightarrow X$, flat over $B_1$. Then $\alpha_{X/B_2/S} = \mathcal{N}_{W/B_2}$.*

*Proof.* Similar as Proposition (9.7) using (9.12). □

**Theorem (9.15).** *Let $S$ be a scheme purely of* characteristic zero. *Let $X/S$ be smooth and let $\alpha$ be a relative Weil divisor on $X/S$. Then there is a unique subscheme $Z \hookrightarrow X$ which is flat with Cohen-Macaulay fibers over $S$ such that $\alpha = \mathcal{N}_Z$, i.e., $Z$ is a relative Cartier divisor.*

*Proof.* Follows from Proposition (9.14) exactly as Theorem (9.8) follows from Proposition (9.7). □

**Corollary (9.16).** *Let $S$ be a scheme purely of* characteristic zero. *Let $X \to S$ be locally of finite type and smooth at points of relative codimension at most one, e.g., $X \to S$ flat with* $(\mathrm{R}_1)$*-fibers. Let $\alpha$ be a relative Weil-divisor on $X/S$. Then there is a unique subscheme $Z \hookrightarrow X$ which has support* $\operatorname{Supp}(\alpha)$ *and a fiberwise dense open subset $U \subseteq Z$, containing all associated points, such that $U \to S$ is a relative Cartier divisor and such that $\mathcal{N}_{U/S} = \alpha|_U$. The relative Weil divisor $\alpha$ is uniquely determined by $Z$. If $S' \to S$ is an arbitrary morphism, then the unique subscheme corresponding to $\alpha \times_S S'$ is the closure of $U \times_S S'$ in $Z \times_S S'$.*

**Proposition (9.17).** *Let $X/S$ be locally of finite type and let $\alpha$ be a relative cycle on $X/S$. Assume that one of the following conditions are satisfied:*

(i) *$S$ is reduced.*
(ii) *$\alpha$ is multiplicity-free and $S$ is of characteristic zero.*
(iii) *$X/S$ is smooth in relative codimension one, $\alpha$ is a relative Weil divisor and $S$ is of characteristic zero.*

*Then there is a locally closed subscheme $Z \hookrightarrow X$, such that $|Z| = \operatorname{Supp}(\alpha)$ and such that $\alpha$ is the push-forward of a relative cycle on $Z/S$. The relative cycle $\alpha$ satisfies condition $(*)$ of Section 5.*

*Proof.* These assertions follow from Corollaries (8.7), (9.9) and (9.16). In fact, let $Z$ be the representing subscheme in the last two cases and the support of $\alpha$ in the first case. Then for any smooth projection $(U, B, T, p)$, the relative cycle $\alpha_{U/B/T}$ is determined by $p^{-1}(Z)$ (resp. $p^{-1}\operatorname{cycl}(\alpha)$ in the first case) and hence do not depend on $T$. This is condition $(*)$. Proposition (5.5) shows that $\alpha$ is the push-forward of a cycle on $Z/S$. □

## 10. Families over reduced parameter schemes

Let $X$ be locally of finite type over a *reduced* scheme $S$. We describe the subset of effective cycles with $\mathbb{Q}$-coefficients which corresponds to the set of relative cycles on $X/S$, cf. Corollary (8.7). When $S$ is semi-normal and of characteristic zero, we obtain the descriptions of Kollár [Kol96] and Suslin-Voevodsky [SV00]. When $S$ is semi-normal and of positive characteristic, then the description is slightly different as Kollár does not include cycles with quasi-integral coefficients. This is a minor difference though, as Kollár has characterized the quasi-integral cycles. Suslin and Voevodsky work either with integral coefficients or with arbitrary rational coefficients. We also show that the fibers of a relative cycle can be computed via Samuel multiplicities of its underlying cycle.

**Theorem (10.1).** *Let $S$ be* normal *with a finite number of irreducible components. Then there is a one-to-one correspondence between relative cycles on $X/S$ and effective cycles on $X$ with quasi-integral coefficients and universal open support.*

*Proof.* This follows from Theorem (8.6) and Corollary (6.2). □

**Corollary (10.2).** *Let $S$ be normal with a finite number of irreducible components. Then the commutative monoid $Cycl_r^{\mathrm{cl}}(X/S)$ of $r$-dimensional cycles with closed support is freely generated by cycles of the form $(1/p^\delta)[Z]$ where $Z$ is an irreducible and reduced closed subscheme of $X$ which is equidimensional of dimension $r$ over $S$, and $\delta$ is the inseparable discrepancy of $k(Z)/k(S)$.*

**Definition (10.3).** Let $X/S$ be locally of finite type, let $f : S' \to S$ be a morphism and let $\mathcal{Z} = \sum_i m_i[Z_i]$ be a cycle on $X$ such that every irreducible component $Z_i$ dominates an irreducible component of $S$. The *pull-back* of $\mathcal{Z}$ along $f$ is the cycle $f^*\mathcal{Z} = \mathcal{Z} \times_S S' = \sum_i m_i[f^{-1}(Z_i)_{\mathrm{dom}/S'}]$.

The pull-back of a relative cycle *does not* correspond to taking the pull-back of the underlying cycle. This is because the underlying cycle need not be flat. Also, the pull-back of a cycle is not functorial as we forget all non-dominating and embedded components.

**Proposition (10.4).** *Let $S$ be reduced and let $\alpha$ be a relative cycle on $X/S$. Assume that* $\operatorname{cycl}(\alpha) = \mathcal{Z} = \sum_i m_i[Z_i]$ *where the $Z_i$'s are subschemes of $X$,* flat *and finitely presented over $S$, but not necessarily reduced or irreducible, and the $m_i$'s* *are rational numbers. Let $S'$ be reduced and let $S' \to S$ be any morphism. Then*

$$\operatorname{cycl}(\alpha \times_S S') = \mathcal{Z} \times_S S' = \sum_i m_i [Z_i \times_S S'].$$

*Proof.* The question is local on $X$ and $S$ and thus we can assume that $X$ and $S$ are quasi-compact. Let $q$ be an integer clearing the denominators of the $m_i$'s. As addition of cycles commutes with pull-back it is enough to show that $\operatorname{cycl}(q\alpha) \times_S S' = \sum_i qm_i[Z_i \times_S S']$ and we can thus assume that the $m_i$'s are integers. Then $\alpha = \sum_i m_i \mathcal{N}_{Z_i}$ and it follows that $\alpha \times_S S' = \sum_i m_i(\mathcal{N}_{Z_i} \times_S S') = \sum_i m_i \mathcal{N}_{Z_i \times_S S'}$. □

**Corollary (10.5).** *Let $S$ be a smooth curve, i.e., a noetherian regular scheme of dimension one, or the spectrum of a valuation ring. Let $\alpha$ be a relative cycle on $X/S$. Then for any point $s \in S$ we have that* $\operatorname{cycl}(\alpha_s) = \operatorname{cycl}(\alpha)_s$.

*Proof.* Follows from the previous proposition as any irreducible and reduced subscheme $Z$ of $X$ dominating $S$ is flat over $S$. In fact, $S$ is a Prüfer scheme, i.e., every finitely generated ideal of $\mathcal{O}_S$ is locally free. □

**Definition (10.6)** ([SV00, 3.1.1])**.** Let $S$ be a scheme, let $k$ be a field and let $s : \operatorname{Spec}(k) \to S$ be a point. A fat point over $s$ is a triple $(s_0, s_1, V)$ where $V$ is a valuation ring and $s_0 : \operatorname{Spec}(k) \to \operatorname{Spec}(V)$ and $s_1 : \operatorname{Spec}(V) \to S$ are morphisms such that

(i) $s = s_1 \circ s_0$.
(ii) The image of $s_0$ is the closed point of $\operatorname{Spec}(V)$.
(iii) The image under $s_1$ of the generic point of $\operatorname{Spec}(V)$ is a generic point of $S$.

*Remark* **(10.7).** For every point $s \in S$ and generization $\xi \in S_{\max}$, there is a field extension $k/k(s)$ and a fat point $(s_0, s_1, V)$ over $s : \operatorname{Spec}(k) \to S$ such that the image of the generic point by $s_1$ is $\xi$ [EGA$_{\text{II}}$, Prop. 7.1.4]. If $S$ is locally noetherian, then there is a fat point with $V$ a discrete valuation ring [EGA$_{\text{II}}$, Prop. 7.1.7].

**Proposition (10.8).** *Let $S$ be reduced and let $\alpha$ be a relative cycle on $X/S$. Let $s : \operatorname{Spec}(k) \to S$ be a point of $S$ and $(s_0, s_1, V)$ a fat point over $s$. Then*

$$\operatorname{cycl}(s^*\alpha) = s_0^*\big(s_1^* \operatorname{cycl}(\alpha)\big).$$

*Proof.* As $s_1$ is flat over the generic point, it is clear that $\operatorname{cycl}(s_1^*\alpha) = s_1^* \operatorname{cycl}(\alpha)$. The result thus follows from Corollary (10.5). □

The pull-back $s_0^* s_1^*$ can be interpreted as taking the limit fiber over $s$ along a general curve through $s$.

**Definition (10.9).** Let $S$ be reduced, let $X/S$ be an algebraic space, locally of finite type and let $\mathcal{Z}$ be a cycle on $X$. We say that $\mathcal{Z}$ satisfies the *limit cycle condition* if for every point $s : \operatorname{Spec}(k) \to S$, the pull-back $s_0^* s_1^* \mathcal{Z}$ is independent on the choice of fat point $(s_0, s_1, V)$ over $s$. When $\mathcal{Z}$ satisfies the limit cycle condition, then we let $s^{[-1]}(\mathcal{Z})$ denote the pull-back $s_0^*s_1^*\mathcal{Z}$ for any choice of fat point over $s$, under the assumption that there exists a fat point over $s$.

**Proposition (10.10).** *Let $S$ be reduced, let $X/S$ be locally of finite type and let $\mathcal{Z}$ be a cycle on $X$ flat over $S$, i.e., $\mathcal{Z} = \sum_i m_i[Z_i]$ where the $Z_i$'s are flat over $S$. Then $\mathcal{Z}$ satisfies the limit cycle condition and $s^{[-1]}(\mathcal{Z}) = \mathcal{Z}_s$.*

*Proof.* Trivial, as the pull-back of a flat cycle is functorial. □

**Corollary (10.11).** *Let $X/S$ be locally of finite type, and let $\mathcal{Z}$ be a cycle on $X$. Let $f : S' \to S$ be a proper morphism such that $\mathcal{Z}' = f^*\mathcal{Z}$ is flat over $S'$. Then $\mathcal{Z}$ satisfies the limit cycle condition if and only if for any point $s : \operatorname{Spec}(k) \to S$ the cycle $\mathcal{Z}'_{s'}$ is independent on the choice of a lifting $s' : \operatorname{Spec}(k) \to S'$ of $s$. If this is the case, then $s^{[-1]}\mathcal{Z} = \mathcal{Z}'_{s'}$ for any such lifting.*

*Proof.* Follows easily from the valuative criterion for proper morphisms and the previous proposition. □

If $S$ is reduced and noetherian and $X/S$ is of finite type, then there exists a proper morphism $S' \to S$ which flatifies $\mathcal{Z}$. In fact, under these hypotheses there is an open dense subset $U \subseteq S$ such that $\mathcal{Z}$ is flat over $U$ [$\text{EGA}_{\text{IV}}$, Cor. 11.3.2]. If $\operatorname{Supp}(\mathcal{Z})$ is proper over $S$, the existence of $S' \to S$ then follows from the existence of the Hilbert scheme $\operatorname{Hilb}(\operatorname{Supp}(\mathcal{Z})/S)$. In the non-proper case, this is Raynaud and Gruson's flatification theorem [RG71].

**Lemma (10.12).** *Let $S$ be reduced, let $X/S$ be locally of finite type and let $\mathcal{Z}$ be a cycle on $X$ satisfying the limit cycle condition and such that any component of $\mathcal{Z}$ dominates a component of $S$. Then for any point $s \in S$, the support of $s^{[-1]}\mathcal{Z}$ equals the support of* $\operatorname{Supp}(\mathcal{Z})_s$. *Also, the support of $\mathcal{Z}$ satisfies condition* (T) *universally.*

*Proof.* Let $Z = \operatorname{Supp}(\mathcal{Z})$. Let $z \in Z$ be a point and choose a generization $\eta \in Z_{\max}$. Let $s \in S$ and $\xi \in S_{\max}$ be the images of $z$ and $\eta$. Choose a valuation ring $V$ and a morphism $\operatorname{Spec}(V) \to X$ such that the closed point $v_0$ is mapped onto $z$ and the generic point $v_1$ is mapped onto $\eta$. Let $s_1 : \operatorname{Spec}(V) \to S$ be the composition of $\operatorname{Spec}(V) \to X$ and $X \to S$. Let $k/k(v_0)$ be an extension such that $k$ is algebraically closed and let $s_0 : \operatorname{Spec}(k) \to \operatorname{Spec}(k(v_0)) \hookrightarrow \operatorname{Spec}(V)$ be the corresponding morphism. Then $X \times_S \operatorname{Spec}(V) \to \operatorname{Spec}(V)$ has a section mapping $v_0$ onto the $k(v_0)$-point $(z, v_0)$. It follows that $(z, v_0)$ is in the support of $s_1^*\mathcal{Z}$ and hence that $(z, s_1 \circ s_0)$ is in the support of $s_0^*s_1^*\mathcal{Z}$. For any $\psi \in \operatorname{Aut}_{k(s)}(k)$ we have by assumption that $s_0^*s_1^*\mathcal{Z} = (s_0 \circ \psi)^*s_1^*\mathcal{Z}$. Thus any closed point in $(s_1 \circ s_0)^{-1}Z$ above $z$ is contained in the support of $s_0^*s_1^*\mathcal{Z}$. It follows that $s_0^*s_1^*\mathcal{Z}$ contains the whole fiber above $z$. Thus $\operatorname{Supp}(s^{[-1]}\mathcal{Z}) = |Z_s|$.

In particular, $\operatorname{Supp}(s_1^*\mathcal{Z}) = |s_1^{-1}Z|$ for any valuation ring $V$ and morphism $s_1 : \operatorname{Spec}(V) \to S$. It follows from Proposition (3.6) that $Z/S$ satisfies (T) universally. □

We denote by $k^{\mathrm{perf}} = k^{p^{-\infty}}$ the perfect closure of $k$, where $p$ is the characteristic of $k$.

**Proposition (10.13).** *Let $S$ be reduced, let $X/S$ be locally of finite type and let $\mathcal{Z}$ be a cycle on $X$ satisfying the limit cycle condition and such that any component of $\mathcal{Z}$ dominates a component of $S$. For any point $s \in S$ there is a unique cycle $s^{[-1]}\mathcal{Z}$ on $X \times_S \operatorname{Spec}(k(s))$ such that for any field extension $k/k(s)$ and fat point $(s_0, s_1, V)$ over $\operatorname{Spec}(k) \to \operatorname{Spec}(k(s)) \hookrightarrow S$, the cycle $(s_0^* s_1^* \mathcal{Z})$ coincides with $s^{[-1]}\mathcal{Z} \times_{k(s)} \operatorname{Spec}(k)$.*

*Proof.* From the previous lemma, it follows that the support of $s^{[-1]}\mathcal{Z}$ should be $\operatorname{Supp}(\mathcal{Z})_s = \operatorname{Supp}(\mathcal{Z}) \times_S \operatorname{Spec}(k(s)^{\mathrm{perf}})$. Thus, it is enough to assign multiplicities for the irreducible components of $\operatorname{Supp}(\mathcal{Z})_s$. If $W \subseteq \operatorname{Supp}(\mathcal{Z})_s$ is an irreducible component, then there is a finite separable and normal field extension $k/k(s)^{\mathrm{perf}}$ such that the irreducible components of $W_k$ are geometrically irreducible [$\mathrm{EGA_{IV}}$, Cor. 4.5.11]. It then follows from the limit cycle condition, and the action of $Gal(k/k(s)^{\mathrm{perf}})$ on any algebraically closed extension of $k$, that the multiplicities of the irreducible components of $W_k$ are all equal. The multiplicity of $s^{[-1]}\mathcal{Z}$ at $W$ is then this common value divided by the inseparable degree of $k(W)/k(s)$. $\square$

Recall that a morphism $f : X \to Y$ is *integral* if $f$ is affine and $f_*\mathcal{O}_X$ is integral over $\mathcal{O}_Y$. A morphism $f : X \to Y$ is a *universal homeomorphism* if $f' : X' \to Y'$ is a homeomorphism for any base change $Y' \to Y$. A morphism of *schemes* $f : X \to Y$ is a universal homeomorphism if and only if $f$ is integral, universally injective and surjective [$\mathrm{EGA_{IV}}$, Cor. 18.12.11]. The same holds for a *locally separated* morphism of algebraic spaces [Ryd08b, Cor. 4.22]. We recall the following definitions, cf. [AB69, Tra70, Swa80, Man80, Yan83, Kol96, Ryd08b].

**Definition (10.14).** A morphism $f : X \to Y$ is weakly subintegral (resp. subintegral) if it is a separated universal homeomorphism (resp. a separated universal homeomorphism with trivial residue field extensions). A reduced algebraic space $X$ is *weakly normal* (resp. semi-normal) if every birational weakly subintegral (resp. subintegral) morphism $X' \to X$, from a reduced space $X'$, is an isomorphism.

Let $f : X \to Y$ be a morphism. Consider the set of factorizations $X \to Y' \to Y$ of $f$ such that $X \to Y'$ is schematically dominant and $g : Y' \to Y$ is subintegral (resp. weakly subintegral). We have corresponding homomorphisms $\mathcal{O}_Y \to g_*\mathcal{O}_{Y'} \hookrightarrow f_*\mathcal{O}_X$ and as $g$ is affine, the set of such factorizations is partially ordered with $Y'_1 \geq Y'_2$ if and only if there exists a morphism $Y'_1 \to Y'_2$ or equivalently if and only if $(g_2)_*\mathcal{O}_{Y'_2} \subseteq (g_1)_*\mathcal{O}_{Y'_1}$. The *subintegral closure*, or semi-normalization, $Y^{X/\mathrm{sn}} \to Y$ (resp. *weak subintegral closure* or weak normalization $Y^{X/\mathrm{wn}} \to Y$) of $f$ is the maximal element in this set.

If $X$ is an algebraic space with a finite number of irreducible components, then the semi-normalization $X^{\mathrm{sn}}$ (resp. weak normalization $X^{\mathrm{wn}}$) is the subintegral closure (resp. weak subintegral closure) of $X$ with respect to the normalization

$\widetilde{X} \to X$. As (weakly) subintegral morphisms are integral, it follows that $X^{\mathrm{sn}}$ is semi-normal and that $X^{\mathrm{wn}}$ is weakly normal.

The following proposition is a special form of "$h$-descent". In general, if $S' \to S$ is universal subtrusive of finite presentation (e.g. faithfully flat or proper and surjective) and $X$ is a scheme, then the sequence

$$\mathrm{Hom}(S, X) \longrightarrow \mathrm{Hom}(S', X) \rightrightarrows \mathrm{Hom}\big((S' \times_S S')_{\mathrm{red}}, X\big)$$

is exact if $S$ is absolutely weakly normal [Voe96, Ryd08b].

**Proposition (10.15).** *Let $S$ be a reduced scheme, $X/S$ an algebraic space, locally of finite type and let $p : S' \to S$ be an integral surjective morphism of reduced schemes. Let $S'' = (S' \times_S S')_{\mathrm{red}}$ and denote the two projections by $\pi_1$ and $\pi_2$. Let $\alpha'$ be a relative cycle on $X'/S'$ such that $\pi_1^*\alpha' = \pi_2^*\alpha'$. Assume that either of the following conditions is satisfied.*

(i) *$S$ is weakly subintegrally closed in $S'$,*
(ii) *$S$ is subintegrally closed in $S'$ and for any $s \in S$, there exists a relative cycle $\alpha_s$ on $X_s/\mathrm{Spec}(k(s))$ such that $\alpha_s \times_s p^{-1}(s) = \alpha' \times_{S'} p^{-1}(s)$.*

*Then there exists a unique relative cycle $\alpha$ on $X/S$ such that $\alpha' = p^*\alpha$.*

*Proof.* Let $Z' = \mathrm{Supp}(\alpha') \hookrightarrow X'$ and let $Z \hookrightarrow X$ be the image of $Z'$. As $X' \to X$ is universally closed and $Z' = p^{-1}(Z)$, it follows that $Z$ is a locally closed subset of $X$. As the support commutes with arbitrary base change, we have that $\pi_1^{-1}(Z') = \pi_2^{-1}(Z')$ and hence that $Z' = p^{-1}(Z)$. The support of $\alpha$, if it exists, is $Z$.

As the (weak) subintegral closure and the reduction commutes with smooth base change [Ryd08b, App. B] we can take a smooth projection adapted to $Z$ and assume that $Z \to S$ is finite. Then $\alpha'$ corresponds to a morphism $\alpha' : S' \to \Gamma^\star(X/S)$ such that $\alpha' \circ \pi_1 = \alpha' \circ \pi_2$. Moreover, as $S'$ is reduced, it follows that $\alpha'$ factors through $\Gamma^\star(Z'/S')$ and hence through $\Gamma^\star(Z/S)$. Note that $\Gamma^\star(Z/S)$ is finite, and in particular affine, over $S$.

Let $W$ be the image of $\alpha' : S' \to \Gamma^\star(Z/S)$ and consider the factorization $S' \to W \to S$. As $\alpha \circ \pi_1 = \alpha' \circ \pi_2$ we obtain a bijective section $\alpha : S \to W$ of *sets* such that $\alpha' = \alpha \circ p$. As $S' \to S$ is *submersive*, i.e., $S$ is equipped with the quotient topology, this section is *continuous* and it follows that $W \to S$ is *weakly subintegral*, i.e., a universal homeomorphism. If $\alpha'_s$ lifts to a morphism $\alpha_s : k(s) \to W$ for every $s \in S$, then $W \to S$ is *subintegral.* Thus $W = S$ under either of the two conditions and $\alpha'$ lifts to a *morphism* $\alpha : S = W \hookrightarrow \Gamma^\star(Z/S)$. $\square$

**Theorem (10.16).** *Let $S$ be* weakly normal *with a finite number of components. Then there is a one-to-one correspondence between relative cycles $\alpha$ on $X/S$ and effective cycles $\mathcal{Z}$ on $X$ such that:*

(i) *Every irreducible component of $\mathcal{Z}$ dominates an irreducible component of $S$.*
(ii) *$\mathcal{Z}$ satisfies the limit cycle condition.*

(iii) *$\mathcal{Z}$ has quasi-integral coefficients, i.e., for any* generic *point $s \in S_{\max}$, the cycle $\mathcal{Z}_s$ has quasi-integral coefficients.*

*Proof.* If $\alpha$ is a relative cycle then $\operatorname{cycl}(\alpha)$ satisfies the three conditions. Indeed, the first follows by definition, the second follows from Proposition (10.8) and the third from Theorem (8.6).

Conversely, assume that we are given a cycle $\mathcal{Z}$ satisfying the three conditions. Let $S' \to S$ be the normalization. Then by Theorem (10.1) we have that $\mathcal{Z} \times_S S' = \operatorname{cycl}(\alpha')$ for a unique relative cycle $\alpha'$ on $X'/S'$. Let $S'' = (S' \times_S S')_{\mathrm{red}}$ and denote the two projections with $\pi_1$ and $\pi_2$. Then $\pi_1^* \alpha' = \pi_2^* \alpha'$. In fact, for any point $s'' \in S''$ we have that $(\pi_1^* \alpha')_{s''} = (\pi_2^* \alpha')_{s''}$ as their underlying cycles coincide with $s''^{[-1]}\mathcal{Z}$. The theorem then follows by $h$-descent, cf. Proposition (10.15). □

**Theorem (10.17).** *Let $S$ be* semi-normal *with a finite number of components. Then there is a one-to-one correspondence between relative cycles $\alpha$ on $X/S$ and effective cycles $\mathcal{Z}$ on $X$ such that:*

(i) *Every irreducible component of $\mathcal{Z}$ dominates an irreducible component of $S$.*
(ii) *$\mathcal{Z}$ satisfies the limit cycle condition.*
(iii) *For every $s \in S$, the cycle $s^{[-1]}\mathcal{Z}$ has quasi-integral coefficients.*

*In particular, a relative cycle such that its underlying cycle has* integral *coefficients, is a well defined family of cycles satisfying the Chow-field condition in the terminology of Kollár* [Kol96, Defs. I.3.10, I.4.7].

*Proof.* Reason as in the proof of Theorem (10.16). □

**Corollary (10.18).** *Let $S$ be a* semi-normal *scheme over* $\operatorname{Spec}(\mathbb{Q})$ *with a finite number of components. Then there is a one-to-one correspondence between relative cycles $\alpha$ on $X/S$ and effective cycles $\mathcal{Z}$ on $X$ such that:*

(i) *Every irreducible component of $\mathcal{Z}$ dominates an irreducible component of $S$.*
(ii) *$\mathcal{Z}$ satisfies the limit cycle condition.*

*In particular, under this hypothesis on $S$, a relative cycle corresponds to a relative effective cycle in the terminology of Suslin and Voevodsky* [SV00, Def. 3.1.3].

**Corollary (10.19)** ([Bar75, Ch. II, §3])**.** *Let $S$ be semi-normal and let $\mathcal{Z}$ be a cycle on $X/S$. Then there is a one-to-one correspondence between relative cycles $\alpha$ on $X/S$ and effective cycles $\mathcal{Z}$ on $X$ such that:*

(i) *The support of $\mathcal{Z}$ satisfies* (T).
(ii) *There is a smooth projection $(U, B, S, p)$ such that* $\operatorname{Supp}(\alpha) \subseteq p(U)$ *and such that $p^*(\mathcal{Z})$ satisfies the limit cycle condition over $B$.*
(iii) *For every $s \in S$, the cycle $s^{[-1]}\mathcal{Z}$ has quasi-integral coefficients.*

*Proof.* This follows from the observation that the limit cycle condition on $X/S$ is equivalent to the limit cycle condition on $U/B$. □

Finally, we define the pull-back of a relative cycle using intersection theory.

**Definition (10.20)** ([Ful98, Ex. 4.3.4])**.** Let $W \hookrightarrow Z$ be a closed subscheme with irreducible components $\{W_i\}$. The multiplicity of $Z$ along $W$ at $W_i$, denoted $(e_W Z)_{W_i}$, is the Samuel multiplicity of the primary ideal determined by $W$ in the local ring $\mathcal{O}_{Z,w_i}$ where $w_i$ is a generic point of $W_i$.

If $[Z] = \sum_j m_j [Z_j]$ then $(e_W Z)_{W_i} = \sum_j m_j (e_{W \cap Z_j} Z_j)_{W_i}$ [Ful98, Lem. 4.2]. This motivates the following definition:

**Definition (10.21).** Let $S$ be reduced with a finite number of irreducible components and let $X/S$ be locally of finite type. Let $Z \hookrightarrow X$ be an irreducible locally closed subscheme and let $s \in S$. We denote by $[Z]_s$ the cycle

$$\sum_V \frac{(e_{Z_s} Z)_V}{e_s S} [V]$$

where the sum is taken over the irreducible components of $Z_s$. We extend this definition linearly to cycles on $X$.

We have the following generalization of [SV00, Thm. 3.5.8]:

**Theorem (10.22).** *Let $S$ be a reduced scheme with a finite number of irreducible components and let $\alpha$ be a relative cycle on $f : X \to S$ with underlying cycle $\mathcal{Z} = \mathrm{cycl}(\alpha)$. Then for any point $s \in S$ we have that* $\mathrm{cycl}(\alpha_s) = [\mathrm{cycl}(\alpha)]_s$.

*Proof.* Let $Z = \mathrm{Supp}(\alpha)$. Let $V \hookrightarrow Z_s$ be an irreducible component with generic point $v$. Let $(U, B, S, p)$ be a smooth projection adapted to $Z$ such that there exists a point $v' \in U$ above $v$ such that $v'$ is the only point of $p^{-1}(Z)$ in its fiber over $B$. Let $V' \hookrightarrow p^{-1}(Z_s)$ be the corresponding irreducible component. Let $W \hookrightarrow B_s$ be the image of $V'$ — this is a connected component of $B_s$ — and let $w$ be its generic point.

Then since $p : U \to X$ and $B \to S$ are smooth it follows from [SV00, Lem. 3.5.2] that $e_W B = e_s S$ and

$$\big(e_{p^{-1}(Z_s)} p^* \mathcal{Z}\big)|_{V'} = (e_{Z_s} \mathcal{Z})_V.$$

Thus, if we show that $e_{p^{-1}(Z_s)}(p^* \mathcal{Z})|_{V'}/e_W B$ is the multiplicity of $\alpha_{U/B/S}$ at $v'$, the result follows. Replacing $X$, $S$ and $s$ with $U$, $B$ and $w$, we can thus assume that $\alpha$ is a proper relative zero-cycle such that $Z_s$ consists of a single (non-reduced) point $z$.

Let $S_j$ be an irreducible component of $S$ and let $\mathcal{Z}_j = \mathcal{Z}|_{S_j} = \sum_i m_i [Z_i]$ be the pull-back of the cycle to $S_j$. Then

$$e_{(Z_i)_s} Z_i = \frac{e_s f_* [Z_i]}{\deg\big(k(z)/k(s)\big)} = e_s S_j \frac{\deg\big(k(Z_i)/k(S_j)\big)}{\deg\big(k(z)/k(s)\big)},$$

cf. [SV00, Lem. 3.5.3]. Thus

$$e_{Z_s} \mathcal{Z}_j = e_s S_j \frac{\deg(\alpha)}{\deg\big(k(z)/k(s)\big)} = e_s S_j \,\mathrm{mult}_z(\alpha)$$

and the theorem follows. □

**Corollary (10.23).** *Let $S$ be a smooth scheme and let $\alpha$ be an equidimensional relative cycle on $X/S$. Then the pull-back of $\alpha$ coincides with the pull-back of* $\mathrm{cycl}(\alpha)$ *given by intersection theory. That is,* $\mathrm{cycl}(\alpha_s) = \mathrm{cycl}(\alpha)_s$ *where the right-hand side is the cycle (not the rational equivalence class) defined in* [Ful98, §10.1].

*Proof.* As $S$ is smooth, $e_s S = 1$, and thus the corollary follows from the theorem and [Ful98, 10.1.1]. □

## 11. Multiplicity-free relative cycles and relative Weil divisors

In Section 9 we saw that multiplicity-free relative cycles and relative Weil divisors on ($\mathrm{R}_1$)-schemes are given by unique subschemes which are flat over a fiberwise dense open subset. Conversely, we would like to characterize the subschemes, fiberwise generically flat, which correspond to such relative cycles. This is not accomplished in general. We only mention the simple cases in which such correspondences are known.

Note that under these correspondences, the pull-back of a relative cycle corresponds to the ordinary pull-back of the corresponding subscheme after removing embedded components of relative codimension at least one.

Let $X$ be a locally noetherian scheme. We recall that $X$ is ($\mathrm{R_n}$) if $X$ is regular at every point of codimenson $n$ and ($\mathrm{S_n}$) if every point of depth $d \leq n$ has codimension $d$. In particular, $X$ is ($\mathrm{R}_0$) if it is reduced at every generic point and ($\mathrm{S}_1$) if it has no embedded components. A scheme is ($\mathrm{S}_2$) if it is ($\mathrm{S}_1$) and every point of depth 1 has codimension 1. Serre's condition states that $X$ is normal if and only if $X$ is ($\mathrm{R}_1$) and ($\mathrm{S}_2$).

Recall that a morphism $f$, locally of finite type, is reduced (resp. normal, resp. ($\mathrm{R_n}$), resp. ($\mathrm{S_n}$)) if it is flat and its geometric fibers are reduced (resp. normal, etc.) [$\mathrm{EGA_{IV}}$, Def. 6.8.1]. We say that a relative cycle $\alpha$ on $X/S$ is ($\mathrm{R_n}$), if $\alpha_s$ is multiplicity-free and $\mathrm{Supp}(\alpha_s)$ is geometrically ($\mathrm{R_n}$) for every $s \in S$.

**Definition (11.1).** Let $Z \to S$ be locally of finite type. We say that $Z \to S$ is *$n$-flat* (resp. *$n$-smooth*) if there exists a schematically dense open subset $U \subseteq Z$, containing all points of relative codimension at most $n$, such that $U \to S$ is flat, ($\mathrm{S}_1$) and locally of finite presentation (resp. smooth).

The condition that $U$ is schematically dense is equivalent to demanding that all (weakly) associated points of $Z$ have relative codimension zero. Indeed, by flatness and the ($\mathrm{S}_1$)-condition, any associated point in $U$ has relative codimension zero.

*Remark* **(11.2).** If $Z \to S$ is 0-flat, then $Z \to S$ satisfies the condition (T) universally, i.e., $Z' \to S'$ satisfies (T) after any base change $S' \to S$. In particular, if $Z \to S$ is 0-flat and equidimensional, then $Z \to S$ is universally equidimensional. If in addition $Z \to S$ is locally of finite presentation or $S$ has a finite number of components, then $Z \to S$ is universally open. This follows from [$\mathrm{EGA_{IV}}$, Cor. 1.10.14] and Corollary (6.3).

Conceptually, an $n$-flat morphism is a family of $(\mathrm{S}_1)$-schemes, i.e., schemes without embedded components. Of course, the ordinary fibers are not necessarily $(\mathrm{S}_1)$-schemes but this is taken care of by the following definition

**Definition (11.3).** If $g : S' \to S$ is a morphism and $Z \to S$ is 0-flat, then we let $g^*_{\mathrm{emb}}(Z)$ be the closure of $U \times_S S'$ in $Z \times_S S'$ for some $U \subseteq Z$ as in the definition of 0-flat.

Note that since $U \times_S S' \subseteq Z \times_S S'$ is dense, $g^*_{\mathrm{emb}}(Z)$ has the same support as the usual pull-back. Also, $g^*_{\mathrm{emb}}(Z)$ can be described as removing all embedded components of relative codimension at least one. In particular, $g^*_{\mathrm{emb}}(Z)$ does not depend upon the choice of $U$. If $Z$ is $n$-flat (resp. $n$-smooth) then $g^*_{\mathrm{emb}}(Z)$ is $n$-flat (resp. $n$-smooth).

*Remark* **(11.4).** Let $X/S$ be locally of finite presentation. If $Z \hookrightarrow X$ is a subspace and $Z \to S$ is 0-flat with $U \subseteq Z$ as in the definition of 0-flat, then $Z/S$ defines a relative cycle $\mathcal{N}_{U/S}$ on $X/S$. By Corollary (6.6), this relative cycle has at most one extension to $Z$. If $Z \to S$ is 1-flat or $S$ is reduced, then the same corollary (together with a limit argument in the non-noetherian case) shows that such an extension exists. We will denote this extension by $\mathcal{N}_{Z/S}$.

**Theorem (11.5).** *Let $X/S$ be locally of finite type. There is a one-to-one correspondence between multiplicity-free relative cycles on $X/S$ and subschemes $Z \hookrightarrow X$ such that $Z \to S$ is* 0*-smooth and $\mathcal{N}_U$ extends to a cycle on $Z$. Under this correspondence, the pull-back of a relative cycle corresponds to the pull-back of a* 0*-smooth morphism as defined in Definition* (11.3)*. In particular, we have the following correspondences:*

- (i) *If $S$ is* reduced*, there is a one-to-one correspondence between multiplicity-free relative cycles on $X/S$ and subschemes $Z \hookrightarrow X$ such that $Z/S$ is* 0*-smooth.*
- (ii) *For arbitrary $S$, and $n \geq 1$, there is a one-to-one correspondence between relative* $(\mathrm{R_n})$*-cycles on $X/S$ and subschemes $Z \hookrightarrow X$ such that $Z \to S$ is $n$-smooth.*

*Proof.* As 0-smooth morphisms satisfies condition (T), the first correspondence follows from Corollary (9.9). The last two correspondences follows from Remark (11.4) and Theorem (9.8). ☐

Let $X/S$ be flat and locally of finite presentation. An effective *relative Cartier divisor* on $X/S$ is an immersion $Z \hookrightarrow X$ which is transversally regular relative to $S$ of codimension one [$\mathrm{EGA_{IV}}$, 21.15.3.3]. By definition, this means that $Z/S$ is flat and that $Z \hookrightarrow X$ is a Cartier divisor. Equivalently, $Z/S$ is flat and $Z_s \hookrightarrow X_s$ is a Cartier divisor for every $s \in S$ [$\mathrm{EGA_{IV}}$, Prop. 19.2.4].

**Definition (11.6).** Let $X/S$ be $(n+1)$-flat. We say that a subscheme $Z \hookrightarrow X$ is $n$-Cartier if $Z|_U \hookrightarrow X|_U$ is a relative Cartier divisor for some open subset $U \subseteq X$ containing all point of relative codimension $n+1$.

By definition, if $Z \hookrightarrow X$ is $n$-Cartier, then $Z/S$ is $n$-flat. An $n$-flat subscheme $Z \hookrightarrow X$ is $n$-Cartier if and only if $Z_s \hookrightarrow X_s$ is $n$-Cartier.

**Theorem (11.7).** *Let $X/S$ be locally of finite type and 1-smooth. There is a one-to-one correspondence between relative Weil divisors on $X/S$ and subschemes $Z \hookrightarrow X$ which are 0-Cartier and such that $\mathcal{N}_U$ extends to $Z$. The pull-back of relative Weil divisor corresponds to the pull-back of 0-flat morphisms. We also have the following correspondences.*

(i) *If $S$ is* reduced*, then there is a one-to-one correspondence between relative Weil-divisors on $X/S$ and subschemes $Z \hookrightarrow X$ which are 0-Cartier.*
(ii) *If $X/S$ is $(n+1)$-smooth, for some $n \geq 1$, then there is a one-to-one correspondence between relative Weil divisors on $X/S$ and subschemes $Z \hookrightarrow X$ which are $n$-Cartier.*

*Proof.* Follows from Theorem (9.15) as in the proof of Theorem (11.5). □

**Corollary (11.8).** *Let $X/S$ be smooth of dimension $r+1$. Then* $\mathrm{Chow}_r(X/S)$ *is isomorphic to the functor* $\mathrm{Div}(X/S)$ *parameterizing relative Cartier divisors on $X/S$.*

When $X/S$ is smooth of relative dimension $r+1$, then the morphism

$$\mathrm{Hilb}_{r-1}(X/S) \to \mathrm{Chow}_{r-1}(X/S) \cong \mathrm{Div}(X/S),$$

taking a proper family of subschemes of dimension $r-1$ to the corresponding equidimensional relative cycle, can be described as follows. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$ with support of dimension $r-1$ such that $\mathcal{F}$ is flat over $S$. Then $\mathcal{F}$ has finite Tor-dimension over $X$ by Lemma (7.12). The *determinant* of $\mathcal{F}$, denoted $\det(\mathcal{F})$ is the alternating determinant of a locally free resolution of $\mathcal{F}$ [GIT, Ch. 5, §3], [Fog69, §2], [KM76]. This is a locally free sheaf on $X$ and there is a section of $\det(\mathcal{F})$ which is unique up to a unit in $\mathcal{O}_X$. This determines an effective Cartier divisor on $X$ and the corresponding relative cycle coincides with $\mathcal{N}_{\mathcal{F}}$ by Proposition (7.13). The morphism $\mathrm{Hilb}_{r-1}(X/S) \to \mathrm{Div}(X/S)$ was used by Fogarty to study the Hilbert scheme of a smooth surface [Fog68].

In [Fog71], Fogarty considers families of Weil divisors on a projective $(\mathrm{R}_1)$-scheme $X/k$ which is equidimensional of dimension $r$. He then defines a relative Weil-divisor on $X \times_k S/S$ as a subscheme $Z \hookrightarrow X \times S$ which is Cartier over the smooth locus of $X$. Thus, when either $S$ is reduced or $X/k$ is $(\mathrm{R}_2)$, Fogarty's definition agrees with our definition. Fogarty then shows [Fog71, Prop. 4.4] that the classical Chow construction, reviewed in Section 17, extends to give a morphism $\mathrm{Chow}_{r-1,d}(X) \to \mathrm{Div}_d(G)$ under one of the following conditions.

(i) $S$ is normal.
(ii) $X/k$ is $(\mathrm{R}_2)$ and $(\mathrm{S}_2)$.

The results of Section 17 shows that such a morphism exists if either $S$ is reduced or $X/k$ is $(\mathrm{R}_2)$. Conjecturally, this morphism exists without any assumptions on $S$ and $X$, but then the elements of $\mathrm{Chow}_{r-1}(X)(S)$ are not represented by subschemes of $X \times S$.

Fogarty also shows [Fog71, §5], assuming that $S$ is reduced or $X/k$ is $(\mathrm{R}_2)$, that the morphism $\mathrm{Chow}_{r-1,d}(X)(S) \to \mathrm{Div}_d(G)(S)$ is injective. Finally [Fog71, §6] he shows that the normalization of $\mathrm{Chow}_{r-1,d}(X)$ is representable (this is simply the normalization of the classical Chow variety) and that if $X/k$ is $(\mathrm{R}_2)$, then $\mathrm{Chow}_{r-1,d}(X)_{\mathrm{red}}$ is representable (i.e., the classical Chow variety is independent of the embedding in this case).

## 12. Relative normal cycles

In this section we prove a generalized version of Hironaka's lemma. The standard version of Hironaka's lemma is that if $S$ is the spectrum of a discrete valuation ring and $X \to S$ is an equidimensional morphism such that the generic fiber is normal, the special fiber is generically reduced and the reduction of the special fiber is normal, then the special fiber is normal. In the terminology of the previous section, Grothendieck and Seydi's [GS71] generalization of Hironaka's lemma states that if $S$ is *reduced* and $X \to S$ is 0-smooth and equidimensional and such that the reduction of any fiber is normal, then $X \to S$ is normal.

The version of Hironaka's lemma that we will prove states that for arbitrary $S$, any 1-smooth equidimensional morphism $X \to S$ such that the reduction of its fibers are normal, is normal.

**Lemma (12.1).** *Let $S$ be a locally noetherian scheme and let $X$ be a locally noetherian $S$-scheme. Let $X' = \mathcal{H}^0_{X/Z^{(2)}}(\mathcal{O}_X)$ be the $Z^{(2)}$-closure of $X$* [EGA$_{\mathrm{IV}}$, 5.10.16]*.*

(i) *If $X_{\mathrm{red}}$ is $(\mathrm{S}_2)$, then $X' \to X$ is finite and $X'_{\mathrm{red}} = X_{\mathrm{red}}$. In particular $X'_{\mathrm{red}}$ is $(\mathrm{S}_2)$.*

(ii) *If $X$ is $(\mathrm{S}_1)$ and $X' \times_S S_{\mathrm{red}}$ is reduced then $X' = X$. In particular, $X$ is $(\mathrm{S}_2)$ and $X \times_S S_{\mathrm{red}}$ is reduced.*

*Proof.* The question is local on $S$ and $X$ and we can thus assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$. As taking reduced rings commutes with direct limits [EGA$_{\mathrm{IV}}$, Cor. 5.13.2], it follows from the definition of the $Z^{(2)}$-closure that it commutes with the reduction. In particular, if $X_{\mathrm{red}}$ is $(\mathrm{S}_2)$ then $X'_{\mathrm{red}} = X_{\mathrm{red}}$. By [EGA$_{\mathrm{IV}}$, Prop. 5.11.1], it follows that the $Z^{(2)}$-closure of $X$ is finite if and only if the $Z^{(2)}$-closure of $X_{\mathrm{red}}$ is finite. As the last closure is trivial, it follows that $X' \to X$ is finite.

Now assume that $X$ is $(\mathrm{S}_1)$ and $X'_{\mathrm{red}} = X' \times_S S_{\mathrm{red}}$. Then $B \to B'$ is injective and $B/\mathfrak{N}_B \to B'/\mathfrak{N}_A B'$ is an isomorphism. Thus $B' = B + \mathfrak{N}_B B'$ and it follows by Nakayama's Lemma that $B' = B$. $\square$

**Lemma (12.2).** *Let $S$ be a local artinian scheme and let $X$ be a locally noetherian $S$-scheme. Let $S_1 \hookrightarrow S$ be a small nil-immersion, i.e., $\ker(\mathcal{O}_S \to \mathcal{O}_{S_1})\mathfrak{N}_{\mathcal{O}_S} = 0$.*

*Assume that $X$ is* $(\mathrm{S}_2)$ *and that $X \to S$ is flat with* $(\mathrm{S}_1)$*-fibers at every point $x \in X$ of codimension at most* $1$*. Then $X \times_S S_1$ is* $(\mathrm{S}_1)$*, i.e., has no embedded components.*

*Proof.* The question is local on $S$ and $X$ and we can thus assume that $S = \mathrm{Spec}(A)$, $S_1 = \mathrm{Spec}(A_1)$ and $X = \mathrm{Spec}(B)$. Let $I = \ker(A \to A_1)$, let $\mathfrak{N}_A$ be the nilradical of $A$ and let $k = A/\mathfrak{N}_A$. Then $I\mathfrak{N}_A = 0$ by hypothesis and this makes $I$ a $k$-module. Now let $u \in B$ such that there exists a non-zero divisor $f \in B$ with $uf \in IB$. To show that $B/IB$ is $(\mathrm{S}_1)$ it is enough to show that $u \in IB$.

Let $\epsilon_1, \epsilon_2, \dots, \epsilon_n \in I$ be a $k$-basis of $I$ and let $uf = \sum_i \epsilon_i b_i$ where $b_i \in B$. As $f$ is regular and $B$ is $(\mathrm{S}_2)$ we have that $B/f$ is $(\mathrm{S}_1)$ [EGA$_{\mathrm{IV}}$, Cor. 5.7.6]. As $X \times_S S_{\mathrm{red}}$ has no embedded components in codimension one, it follows that the image of $f$ in $B/\mathfrak{N}_A B$ is regular in codimension one. Thus, $B/fB$ is flat in codimension zero as $\mathrm{Tor}_1(B/fB, A/\mathfrak{N}_A) = 0$ at points of codimension zero on $\mathrm{Spec}(B/fB)$.

Let $C = \mathrm{Tot}(B/f)$ be the total fraction ring of $B/f$. This is a zero-dimensional ring which is flat over $A$ and $B/f \hookrightarrow C$. By the infinitesimal criterion of flatness, we have that the images of the $b_i$'s in $C$ are in $\mathfrak{N}_A C$. As $B/f \hookrightarrow C$ is faithfully flat, it follows that the images of the $b_i$'s in $B/f$ are in $\mathfrak{N}_A(B/f)$, i.e., $b_i \in (f + \mathfrak{N}_A B)$. Thus $uf = \sum_i \epsilon_i f b_i'$ where $b_i' \in B$. As $f$ is regular, it follows that $u = \sum_i \epsilon_i b_i' \in IB$. $\square$

**Proposition (12.3).** *Let $S$ be a local artinian scheme and let $X$ be a locally noetherian $S$-scheme. Assume that $X$ is* $(\mathrm{S}_1)$*, that $X_{\mathrm{red}}$ is* $(\mathrm{S}_2)$*, and that $X \to S$ is flat with reduced fibers at every point $x \in X$ of codimension at most* $1$*. Then $X \times_S S_{\mathrm{red}}$ is reduced and hence* $(\mathrm{S}_2)$*.*

*Proof.* Let $n$ be such that $\mathfrak{N}_S^n = 0$. We will show that $X \times_S S_{\mathrm{red}}$ is reduced by induction on $n$. If $n = 0$, then $X$ is $(\mathrm{R}_0)$ and $(\mathrm{S}_1)$ and hence reduced.

Let $S_1 = \mathrm{Spec}(\mathcal{O}_S/\mathfrak{N}_S^{n-1})$. Let $X'$ be the $Z^{(2)}$-closure of $X$. Then $X' \to X$ is an isomorphism in codimension 1. By Lemma (12.1) (i) we have that $X'_{\mathrm{red}} = X_{\mathrm{red}}$ and $X' \to X$ is finite. In particular, $X'$ is noetherian and $(\mathrm{S}_2)$. Thus $X' \to S$ satisfies the conditions of Lemma (12.2) and it follows that $X' \times_S S_1$ is $(\mathrm{S}_1)$. By induction, it follows that $X' \times_S S_{\mathrm{red}}$ is reduced. We now have that $X' = X$ by Lemma (12.1) (ii) and thus that $X \times_S S_{\mathrm{red}}$ is reduced. $\square$

**Corollary (12.4).** *Let $S$ be a local artinian scheme and let $X$ be a locally noetherian $S$-scheme. Assume that $X$ is* $(\mathrm{S}_1)$ *and that $X \to S$ is flat with reduced fibers at every point $x \in X$ of codimension at most* $1$*. Then $X \to S$ is flat with* $(\mathrm{R}_0)$+$(\mathrm{S}_2)$*-fibers at all points at which $X_{\mathrm{red}}$ is* $(\mathrm{S}_2)$*. This locus is open in $X$.*

*Proof.* We can assume that $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine. Let $X_{\max} = \{x_1, x_2, \dots, x_n\}$ be the generic points of $X$. Let $Z = \coprod_i \mathrm{Spec}(\mathcal{O}_{X,x_i}) = \mathrm{Spec}(C)$ and let $f : Z \hookrightarrow X$ be the canonical inclusion. Then $f$ is universally schematically dominant relative to $S$ by [EGA$_{\mathrm{IV}}$, Thm. 11.10.9] and Proposition (12.3). This means that $B \hookrightarrow C$ remains injective after tensoring with any $A$-algebra $A'$. As $C$ is flat, we have the long exact sequence

$$0 \to \mathrm{Tor}_1^A(C/B, A') \to B \otimes_A A' \to C \otimes_A A' \to C/B \otimes_A A' \to 0$$

and it follows that $C/B$ is flat. Thus $\mathrm{Tor}_2^A(C/B, A') = 0$ and it follows that $\mathrm{Tor}_1^A(B, A') = 0$ and hence $B$ is flat as well. □

I recently became aware that Kollár [Kol95, Thm. 10], cf. Theorem (12.7), implies a stronger version of Corollary (12.4). When $S$ is artinian, he shows the following. If $X$ is $(\mathrm{S}_1)$ and $X \to S$ is flat with $(\mathrm{S}_1)$-fibers at every point $x \in X$ of codimension at most 1, and $(X \times_S S_{\mathrm{red}})_{\mathrm{dom}/S}$ is $(\mathrm{S}_2)$, then $X \to S$ is flat with $(\mathrm{S}_2)$-fibers. It is not difficult to modify the proofs above to obtain this result.

We now have the following generalization of a theorem of Grothendieck and Seydi [GS71, Thm. II 1]. In *loc. cit.*, only the case where $S$ is reduced is treated.

**Theorem (12.5)** (Generalized Hironaka's lemma)**.** *Let $S$ be a locally noetherian scheme. Let $f : X \to S$ be locally of finite type and* 0*-smooth. Let $x \in X$ such that $(X_{f(x)})_{\mathrm{red}}$ is geometrically normal at $x$. Assume that $f$ is (locally) equidimensional and that either one of the following conditions hold:*

(i) *$S$ is reduced and excellent.*
(ii) *$f$ is* 1*-smooth.*

*Then $f$ is normal, i.e., flat with geometrically normal fibers, in a neighborhood of $x$.*

*Proof.* Let $U \subseteq X$ be the open subset of $f$ such that $f|_U$ is smooth. It is by [GS71, Prop. I 1.0] enough to show that $U \subseteq X$ is universally schematically dominant with respect to $S$. Moreover, it is by [GS71, Thm. I 2] enough to show this when $S$ is the spectrum of either a local artinian ring or a discrete valuation ring, and if $S$ is reduced and excellent only the second case is required. Note that since $f$ is 0-flat, it is universally equidimensional.

We can thus assume that either $S$ is the spectrum of a DVR or that $S$ is local artinian and $f$ is 1-smooth. The first case is the usual Hironaka lemma [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 5.12.8]. The second case is Corollary (12.4). □

*Remark* **(12.6).** The excellency condition in (i) is not necessary as follows by a limit argument. Similarly, the theorem is valid without the noetherian assumption if we assume that $f : X \to S$ is locally of finite presentation.

If $f$ is normal at $x$ then $f$ is 1-smooth at $x$. Hence the theorem shows that condition (i) implies condition (ii). Under assumption (i) of the theorem, the hypothesis that $f$ is equidimensional is necessary as shown by Example (8.12). The hypothesis that $f$ is equidimensional is not needed in (ii). In fact, the following theorem is a special case of Kollár's theorem [Kol95, Thm. 10].

**Theorem (12.7).** *Let $S$ be a locally noetherian scheme. Let $f : X \to S$ be locally of finite type and* 1*-smooth. Let $x \in X$ such that $X_{f(x)}$ is $(\mathrm{S}_2)$ at $x$ after removing embedded components. Then $f$ is $(\mathrm{S}_2)$, i.e., flat with geometrically $(\mathrm{S}_2)$-fibers, in a neighborhood of $x$.*

**Theorem (12.8).** *Let $X/S$ be locally of finite presentation and let $\alpha$ be a relative cycle on $X/S$ which is multiplicity-free. The subset of points $Z_{\mathrm{norm}}$ at which $\alpha$ is* *normal, is open. The morphism $Z \to S$ is normal over $Z_{\mathrm{norm}}$, i.e. flat, locally of finite presentation and with geometrically normal fibers.*

*Proof.* Follows by Theorem (12.5). □

**Corollary (12.9).** *The functor* $\mathrm{Hilb}_r^{\mathrm{equi}}(X) \to \mathrm{Chow}_r(X)$ *induces an isomorphism between normal families of subschemes and normal families of cycles.*

**Theorem (12.10).** *Let $X/S$ be locally of finite presentation and 2-smooth. Let $\alpha$ be a relative Weil divisor on $X/S$ represented by the subscheme $Z \hookrightarrow X$. Let $z \in Z$ be a point over $s \in S$. If $(Z_s)_{\mathrm{emb}}$ is* $(\mathrm{S}_2)$ *at $z$, then $Z \to S$ is flat over $z$. In particular, a relative Weil divisor, parameterizing Weil divisors which are* $(\mathrm{S}_2)$*, is flat.*

*Proof.* Follows by Kollár's Theorem (12.7). □

**Corollary (12.11).** *Let $X/S$ be flat of relative dimension $r+1$ with* $(\mathrm{R}_2)$*-fibers. The functor* $\mathrm{Hilb}_r(X) \to \mathrm{Chow}_r(X)$ *induces an isomorphism between families of Cartier divisors which are* $(\mathrm{S}_2)$ *and families of Weil divisors which are* $(\mathrm{S}_2)$*.*

## 13. Push-forward

In this section we first define the push-forward of a (closed) relative cycle along a *finite* morphism. This definition then extends to the push-forward along a *proper* morphism, assuming that either the morphism is generically finite, i.e., that no components are collapsed under the push-forward, or that the relative cycle is represented by a flat subscheme in relative codimension one over depth zero points, e.g., the cases (A1) and (B1)–(B9) in the introduction. In particular, the proper push-forward is defined when the parameter scheme is reduced (A1) or when the relative cycle has $(\mathrm{R}_1)$-fibers (B7).

**Definition (13.1).** Let $f : X \to Y$ be a morphism locally of finite type. We say that $f$ is proper onto its image if $f(X)$ is locally closed and $f|_{f(X)}$ is proper.

A proper morphism is proper onto its image. A morphism which is proper at each point of $f(X)$ is proper onto its image [$\mathrm{EGA_{IV}}$, Cor. 15.7.6] (at least if $Y$ is locally noetherian).

**Definition (13.2).** Let $f : X \to Y$ be quasi-finite and let $\alpha$ be a relative cycle on $X/S$ with support $Z$. Assume that $f|_Z$ is proper onto its image, e.g., that $Z$ is closed and $f$ is proper or that $Z/S$ is proper and $Y/S$ is separated. We let $f_*\alpha$ be the relative cycle on $Y/S$ with support $f(Z)$ such that for any projection $(U,B,T,p,g)$ of $Y/S$ adapted to $f(Z)$ we have that $(f_*\alpha)_{U/B/T} = (\pi_1)_*\alpha_{U\times_X Y/B/T}$. Here $\pi_1 : U \times_X Y \to U$ is the projection on the first factor.

It is easily verified that $f_* \operatorname{cycl}(\alpha) = \operatorname{cycl}(f_*\alpha)$. The addition of two cycles $\alpha$ and $\beta$ is the push-forward of $\alpha \amalg \beta$ along the morphism $X \amalg X \to X$.

**(13.3)** *Hilbert stack* — Let $X/S$ be locally of finite presentation. The *Hilbert stack* $\mathscr{H}(X/S)$, parameterizes proper flat families $p : Z \to T$ equipped with a morphism $q : Z \to X$ such that $(q, p) : Z \to X \times_S T$ is quasi-finite. Even if $X/S$ is proper, this stack is very non-separated and does not have finite automorphism groups. If $X/S$ is separated, then the Hilbert stack is algebraic [Lie06]. It is also expected that the Hilbert stack of a non-separated scheme is algebraic [Art74, App.], in contrast to the Hilbert functor of a non-separated scheme which is not representable. Indeed, the algebraicity for zero-dimensional families is shown in [Ryd08a].

**Proposition (13.4).** *Let $X/S$ be* separated *and locally of finite presentation. There is a morphism from the Hilbert stack $\mathscr{H}_r(X/S)$, parameterizing $r$-dimensional proper flat families, to the Chow functor* $\mathrm{Chow}_r(X/S)$. *This morphism takes a family $Z/T$ to the relative cycle $(q, p)_*(\mathcal{N}_{Z/T})_r$ on $X \times_S T/T$.*

*Remark* **(13.5).** *Branchvarieties* — Let $X/S$ be separated. The *stack of branchvarieties* of Alexeev and Knutson [AK06] is the substack of the Hilbert stack parameterizing proper and flat morphisms $p : Z \to T$ together with a morphism $q : Z \to X$ such that $(q, p)$ is quasi-finite and $p$ has *geometrically reduced fibers*. This stack is *proper* and has finite stabilizers but it is not Deligne-Mumford in positive characteristic. The open substack such that $(q, p)$ is a closed immersion coincides with the open subset of the Hilbert scheme parameterizing reduced families. In particular, the morphism $\mathrm{Branch}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is an isomorphism over normal embedded families, Corollary (12.9), and a monomorphism over reduced embedded families, Corollary (9.10). The morphism $\mathrm{Branch}_r(X/S) \to \mathrm{Chow}_r(X/S)$ is injective over the open locus parameterizing normal families $Z \to T$ such that $Z \to X \times_S T$ is birational onto its image. I do not know if this is a monomorphism but it seems likely.

*Remark* **(13.6).** *Cohen-Macaulay curves* — The space of Cohen-Macaulay curves [Høn05], is the open subset of the Hilbert stack parameterizing Cohen-Macaulay curves $Z \to T$ together with a morphism $Z \to X \times_S T$ which is birational onto its image. This is a proper algebraic space.

There is thus a plethora of moduli spaces which all maps into the Chow functor. This also includes the stack of stable maps as we will see in Corollary (13.11). To show this, we first need to define the push-forward of a relative cycle along a *proper* morphism. For simplicity we only define the push-forward for relative cycles with equidimensional support.

**Definition (13.7).** Let $X/S$ and $Y/S$ be algebraic spaces locally of finite type over $S$. Let $f : X \to Y$ be a morphism and let $Z \subseteq X$ be a locally closed subset such that $Z/S$ is equidimensional and $f|_Z$ is proper onto its image. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. We let $f_*(Z) \subseteq f(Z)$ be the closure of $U$.

*Remark* **(13.8).** If $f|_Z : Z \to Y$ is quasi-finite at the generic points of $Z$, then $f_*(Z) = f(Z)$.

**Lemma (13.9).** *Let $f : X \to Y$ be a morphism and let $Z \subseteq X$ be a locally closed subset such that $Z/S$ is equidimensional and such that $f|_Z$ is proper onto its image. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. Then $f_*(Z)$ is equidimensional over $S$ and $U \subseteq f_*(Z)$ is fiberwise dense. In particular, $f_*(Z)$ commutes with base change, i.e., for any morphism $g : S' \to S$ we have that*

$$g^{-1}(f_*(Z)) = f'_*(g^{-1}(Z))$$

*Proof.* Let $s \in S$ and let $y \in f_*(Z)_s$ be a generic point and let $r$ be the dimension of $f_*(Z)_s$ at $y$. Let $W \subseteq f_*(Z)$ be an irreducible component containing $y$ and let $V \subseteq Z$ be an irreducible component mapping onto $W$ such that $V \to W$ is quasi-finite. Then as $V$ is equidimensional, it follows that $W$ is equidimensional at $y$ and that $V \to W$ is quasi-finite over $y$. This shows that $y \in U$. $\square$

**Theorem (13.10).** *Let $S$ be locally noetherian, let $X/S$ and $Y/S$ be locally of finite type, let $f : X \to Y$ be a morphism and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module which is flat over $S$. Let $Z = \operatorname{Supp}(\mathcal{F})$ and assume that $f|_Z$ is proper onto its image and that $Z/S$ is equidimensional. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. Let $V = f^{-1}(U)$. Then the relative cycle $f_*\big(\mathcal{N}_{\mathcal{F}|_V}\big)$ on $U/S$ extends uniquely to a relative cycle on $Y/S$ with support $f_*(Z)$. This cycle is denoted by $f_*\mathcal{N}_{\mathcal{F}/S}$.*

*Proof.* Replacing $X$ with the closed subscheme defined by $\operatorname{Ann}_{\mathcal{O}_X}(\mathcal{F})$, and $Y$ with its image, we can assume that $f$ is proper.

First assume that $f$ is not only proper but also projective. Let $\mathcal{L}$ be an invertible sheaf on $X$ which is $f$-ample. Then for sufficiently large $n$, we have that $\mathrm{R}^i f_*(\mathcal{F} \otimes \mathcal{L}^n) = 0$ for all $i > 0$ and that $\mathcal{G} = f_*(\mathcal{F} \otimes \mathcal{L}^n)$ is coherent and flat over $S$. As $f_*(\mathcal{F} \otimes \mathcal{L}^n)|_U$ and $f_*(\mathcal{F})|_U$ are locally isomorphic, it follows that $\mathcal{N}_{\mathcal{G}}$ is an extension of $f_*(\mathcal{N}_{\mathcal{F}|_V})$.

In general, $\mathbf{R}f_*(\mathcal{F})$ is a perfect complex relative to $S$ [$\mathrm{SGA}_6$, Exp. III, Prop. 4.8] and $\mathcal{N}_{\mathbf{R}f_*(\mathcal{F})/S}$ is the required extension, cf. Remark (7.11). $\square$

**Corollary (13.11).** *Let $X/S$ be separated and locally of finite presentation. For any genus $g$, number of marked points $n$ and homology class $\beta$, there is a functor from the Kontsevich space $\overline{\mathscr{M}}_{g,n}(X/S, \beta)$ of stable maps into $X$ to the Chow functor $\mathrm{Chow}_1(X/S)$ taking a stable curve onto its image cycle.*

**Theorem (13.12).** *Let $f : X \to Y$ be a morphism and let $\alpha$ be a relative cycle on $X/S$ with equidimensional support $Z$ such that $f|_Z$ is proper onto its image. Let $U \subseteq f(Z)$ be the open subset over which $Z \to f(Z)$ is quasi-finite. Then there is at most one extension of $f_*(\alpha|_{f^{-1}(U)})$ to $f_*(Z)$. When such an extension exists, we denote it by $f_*\alpha$. An extension exists if one of the following conditions is satisfied:*

(1) *$Z \to f(Z)$ is quasi-finite at points $y \in f_*(Z)$ such that $y$ has codimension one in a fiber over a point of depth zero in $S$.*

(1a) *$f|_Z$ is generically finite, i.e., $f_*(Z) = f(Z)$.*

(2) *There is an open subset $V \subseteq Z$ containing all points $x \in Z$ of relative codimension one over points of depth zero of $S$, such that $\alpha|_V = \mathcal{N}_{V_1/S}$ where $V_1 \to S$ is flat and finitely presented.*

(2a) *$S$ is reduced.*

(2b) *$\alpha$ has* $(\mathrm{R}_1)$*-fibers.*

(2c) *$X \to S$ is 2-smooth, e.g.* $(\mathrm{R}_2)$*, and $\alpha$ is a relative Weil divisor.*

*Proof.* Note that (1a) is a special case of (1) and that (2a)–(2c) are special cases of (2). By Lemma (13.9), the open subset $U \subseteq f(Z)$ contains all points of $f_*(Z)$ which are generic in their fibers over $S$. By Corollary (6.6) there is thus at most one extension and an extension to $f_*(Z)$ exists if an extension to all points of $f_*(Z)$ which are of codimension one in its fiber over a point $s \in S$ of depth zero exist. In (1) all such points are already in $U$ and in (2) an extension exists by Theorem (13.10). $\square$

Conditions (2a)–(2c) contain the cases (A1) and (B1)–(B9) of the introduction. It is likely that $f_*\alpha$ always is defined, i.e., that $f_*(\alpha|_{f^{-1}U})$ always extends to $f_*(Z)$.

## 14. Flat pull-back and products of cycles

Let $S$ be a locally noetherian scheme, let $X/S$ be locally of finite type and let $\alpha$ be a relative cycle on $X/S$ with support $Z$. Let $f : Y \to X$ be a flat morphism, locally of finite presentation. We would like to define the pull-back $f^*\alpha$ of $\alpha$ as a relative cycle on $Y/S$. The pull-back should satisfy the following two conditions

(P1) $f^* \operatorname{cycl}(\alpha) = \operatorname{cycl}(f^*\alpha)$.

(P2) If $\mathcal{F}$ is a coherent sheaf on $X$ which is flat over $S$, then $f^*\mathcal{N}_{\mathcal{F}/S} = \mathcal{N}_{f^*\mathcal{F}/S}$.

Note that $f^* \operatorname{cycl}(\mathcal{N}_{\mathcal{F}/S}) = \operatorname{cycl}(\mathcal{N}_{f^*\mathcal{F}/S})$ so these two conditions are compatible. When one of the following conditions holds

(A1) $S$ is reduced.

(A2) $\alpha$ is multiplicity-free.

(A3) $\alpha$ is a relative Weil divisor and $X/S$ is 1-smooth.

then there is at most one relative cycle $f^* \operatorname{cycl}(\alpha)$ satisfying (P1)–(P2) by Corollary (6.6) and the results of Sections 8–9. Similarly, we obtain the following result from Corollary (6.6).

**Proposition (14.1).** *Let $S$ be arbitrary and let $\alpha$ be a relative cycle on $X/S$ with support $Z$. Let $f : Y \to X$ be a flat morphism. Assume that there exists an open subset $U \subseteq Z$ containing all points $z \in Z$ over $s \in S$ with* $\operatorname{codim}_z Z_s + \operatorname{depth}_s S \leq 1$*, such that $\alpha$ is represented by a flat subscheme or cycle over $U$, cf.* (B1), (B3)–(B9) *in the introduction. Then there is a unique cycle* $f^* \operatorname{cycl}(\alpha)$ *satisfying* (P1)–(P2).

The proposition is also valid when $S$ is semi-normal and $\alpha$ is arbitrary, i.e., in the case (B2). This follows from Corollary (10.19) and the following discussion.

Let us now discuss the general case. Locally on $Y$ there exists a factorization $Y \to U \to X$ of $f$ such that the first morphism is quasi-finite and the second morphism is smooth. By Lemma (7.12), $Y \to U$ is of finite Tor-dimension. Locally, $U \to X$ factors through an étale morphism $U \to \mathbb{A}^n_X$. If $(U, B, \mathbb{A}^n_S)$ is a projection adapted to $Z/S$ we then define

$$(f^*\alpha)_{Y/B/S} = \alpha_{U/B/\mathbb{A}^n_S} \circ \mathcal{N}_{Y/U}.$$

If $\alpha$ satisfies condition $(*)$ of Section 5, then $(f^*\alpha)_{Y/B/S}$ does not depend upon the morphism $B \to \mathbb{A}^n_S$ and is thus well-defined. Now, the problem is that a general smooth projection $(Y, B, S)$ adapted to $f^{-1}(Z)$ does not admit such a factorization.

If $S$ is of *characteristic zero* and $f$ is *smooth*, then any smooth projection $(Y, B, S)$ generically admits such a factorization. Indeed, let $n$ be the relative dimension of $Y \to X$ and let $B \to \mathbb{A}^{r+n}_S$ be an étale morphism. Then the induced morphism $Y \to X \times_S \mathbb{A}^{r+n}_S$ is quasi-finite. Thus, there is a projection $\mathbb{A}^{r+n}_S \to \mathbb{A}^n_S$ such that $Y \to X \times_S \mathbb{A}^n_S$ is quasi-finite and hence generically étale fiberwise over $X$.

Now assume as before that $f$ is smooth but let $S$ be arbitrary. Then there exists smooth projections $(Y, B, S)$ such that $Y \to X \times_S \mathbb{A}^n_S$ cannot be chosen so that it is generically étale. For example, let $k$ be a field of characteristic $p$, $X = S = \operatorname{Spec}(k)$, $Y = \operatorname{Spec}(k[t])$ and $B = \operatorname{Spec}(k[t^p])$. For a *generic* choice of $(Y, B, S)$ we can however find a factorization $Y \to X \times_S \mathbb{A}^n_S$ which is generically étale fiberwise over $X$.

**Theorem (14.2).** *Let $S$ be an arbitrary scheme, let $\alpha$ be a relative cycle on $X/S$ with support $Z$ and let $f : Y \to X$ be a* smooth *morphism. Assume that $\alpha$ is represented by a flat subscheme or a flat cycle over an open subset $U \subseteq Z$ containing all points of relative codimension at most one over points of depth zero in $S$. This is the case if $S$ is reduced or if $\alpha$ is as in* (B1)–(B9) *of the introduction. Then there is a unique relative cycle $f^*\alpha$ on $Y/S$ satisfying* (P1)–(P2). *Furthermore, for every commutative diagram*

$$\begin{tikzcd}
U \arrow[r, "p"] \arrow[d] & Y \arrow[r, "f"] & X \arrow[dd] \\
B \arrow[d] & & \\
T' \arrow[r, "g'"] & T \arrow[r, "g"] & S
\end{tikzcd}$$

*with $p, g, g'$ smooth, $U \to Y \times_S T'$ étale and $U \to X \times_S T$ étale, we have that*

$$(f^*\alpha)_{U/B/T'} = \alpha_{U/B/T}.$$

*Proof.* First note that since $\alpha$ is represented by a flat subscheme or flat cycle in relative codimension zero over depth zero points, $\alpha$ satisfies condition $(*)$. Let $(U, B, T', p, g'')$ be a smooth projection of $Y/S$. As discussed above, there is then a factorization of $g'' : T' \to S$ into smooth morphisms $g' : T' \to T$ and $g$ :

$T \to S$ such that $U \to X \times_S T$ is quasi-finite (but not necessarily generically étale in characteristic $p$). Picking a generic smooth projection and placing the two projections in a family, we obtain a smooth projection and morphisms as above such that $U \to X \times_S T$ is generically étale, fiberwise over $X$, and such that the original projection is obtained as a pull-back of this family.

As $\alpha$ is represented by a flat subscheme or flat cycle in relative codimension at most one over depth zero points, it follows that the common definition of $f^*\alpha$ at points of relative codimension zero over depth zero points extends. □

Similarly, we would like to define *products* of cycles, i.e., if $\alpha$ is a relative cycle on $X/S$ and $\beta$ is a relative cycle on $Y/S$ we would like to define $\alpha \times \beta$ on $X \times_S Y/S$. This relative cycle should satisfy obvious conditions such as $\mathrm{cycl}(\alpha \times \beta) = \mathrm{cycl}(\alpha) \times \mathrm{cycl}(\beta)$ and $\mathrm{cycl}(\mathcal{N}_\mathcal{F} \times \mathcal{N}_\mathcal{G}) = \mathrm{cycl}(\mathcal{N}_{\mathcal{F}\otimes\mathcal{G}})$. When $\alpha$ and $\beta$ are as in (A1)–(A3) then there is at most one such product cycle and when $\alpha$ and $\beta$ are as (B1)–(B9) there exists a product cycle, cf. Proposition (14.1). I do not know if it is possible to employ similar methods as in Theorem (14.2) to show the existence of a product cycle when $S$ is reduced.

## 15. Projections and intersections

**Proposition (15.1)** (Projection)**.** *Let $\alpha$ be a relative cycle on $X/S$, let $Y \to S$ be smooth and let $X \to Y$ be a morphism such that* $\mathrm{Supp}(\alpha)|_{\mathrm{dom}/Y} \to Y$ *satisfies* (T)*. Then there is an induced relative cycle $\alpha'$ on $X/Y$ such that for any projection $(U, B, \mathbb{A}^r_Y)$ of $X/Y$ adapted to* $\mathrm{Supp}(\alpha')$ *we have that* $\alpha'_{U/B/\mathbb{A}^r_Y} = \alpha_{U/B/\mathbb{A}^r_S}$.

*Proof.* This follows from the fact, Proposition (5.3), that it is enough to consider projections of the form $(U, B, \mathbb{A}^r_Y)$ to define $\alpha'$. □

**Definition (15.2).** Let $\alpha$ be a relative cycle on $X/S$ with support $Z$. Let $\mathcal{L}$ be a invertible sheaf on $X/S$ and let $f \in \Gamma(X, \mathcal{L})$ be a global section. Assume that the closed subscheme $V(f)$ defined by $f$ intersects $Z$ properly in every fiber, i.e., that $V(f)_s$ does not contain an irreducible component of $Z_s$ for every $s \in S$. Locally on $X$, the section $f$ induces a projection $(X, \mathbb{A}^1, S)$ and hence a relative cycle $\alpha'$ on $X/\mathbb{A}^1$. We let $\mathcal{L}_f \cap \alpha$ be the relative cycle with support $Z \cap V(f)$ defined locally on $X$ as the pull-back of $\alpha'$ along the zero-section of $\mathbb{A}^1 \to S$.

In particular, if $D$ is a relative Cartier divisor on $X/S$ which intersects $Z$ properly, then we let $D \cap \alpha = \mathcal{O}(D)_f \cap \alpha$ where $f$ is the section given as the dual of $\mathcal{I}_D = \mathcal{O}(-D) \hookrightarrow \mathcal{O}_X$.

If $f_1, f_2, f_3, \dots, f_n$ is a sequence of sections of $\mathcal{O}_X$ such that $V(f_i)$ intersects $V(f_{i-1}) \cap \dots \cap V(f_1) \cap Z$ properly for $i = 1, 2, \dots, n$, then $V(f_n) \cap V(f_{n-1}) \cap \dots \cap V(f_1) \cap \alpha$ is defined. It is clear that this relative cycle, which we denote by $V(f_1, f_2, \dots, f_n) \cap \alpha$, does not depend upon the ordering of the $f_i$'s. On the other hand, if $g_1, g_2, \dots, g_n$ is another sequence such that the relative cycle $V(g_1, g_2, \dots, g_n) \cap \alpha$ exists and $(f_1, f_2, \dots, f_n) = (g_1, g_2, \dots, g_n)$ as ideals, then it is not clear that the corresponding cycles coincide.

Assume that $V(f_1, f_2, \dots, f_n) \cap \alpha$ only depends on the ideal $(f_1, f_2, \dots, f_n)$ in general. If $Y \hookrightarrow X$ is a regular immersion intersecting $Z = \operatorname{Supp}(\alpha)$ properly, we can then define a relative cycle $Y \cap \alpha$ locally using any regular sequence defining $Y$. Under this assumption, we can now define proper intersections of relative cycles on smooth schemes:

**Definition (15.3).** Let $X/S$ be smooth of relative dimension $n$ and let $\alpha, \beta$ be relative cycles on $X/S$, equidimensional of dimensions $r$ and $s$ respectively. Assume that $\operatorname{Supp}(\alpha)$ and $\operatorname{Supp}(\beta)$ intersect properly in each fiber, i.e., that $\operatorname{Supp}(\alpha) \cap \operatorname{Supp}(\beta)$ is equidimensional of dimension $r + s - n$. Then $\operatorname{Supp}(\alpha) \cap \operatorname{Supp}(\beta) = \Delta_{X/S} \cap \bigl(\operatorname{Supp}(\alpha) \times \operatorname{Supp}(\beta)\bigr)$ and the latter intersection is proper in each fiber. We let $\alpha \cap \beta = \Delta_X \cap (\alpha \times \beta)$ when the relative cycle $(\alpha \times \beta)$ is defined, cf. Section 14.

## 16. Relative fundamental classes of relative cycles

We briefly indicate the construction of relative fundamental classes and the relation with Angéniol's functor. Throughout this section, $S$ is a locally noetherian scheme over $\operatorname{Spec}(\mathbb{Q})$ and $X/S$ is of finite type and *separated*.

**Theorem (16.1).** *Let $\alpha$ be a relative cycle on $X/S$ which is equidimensional of dimension $r$. Then there exists an infinitesimal neighborhood $j : Z \hookrightarrow X$ of* $\operatorname{Image}(\alpha) \hookrightarrow X$ *such that $\alpha$ is the push-forward of a relative cycle on $Z$ along $j$. Moreover, there is a class $c_\alpha \in \operatorname{Ext}_Z^{-r}\bigl(\Omega^r_{Z/S}, \mathcal{D}^\bullet_{Z/S}\bigr)$, the* relative fundamental class *of $\alpha$, such that for any projection $(U, B, T, p, g)$ the composition of the canonical homomorphism $h_* h^* \Omega^r_{B/S} \to h_* \Omega^r_{Z/S}$ and the trace* $\operatorname{tr}(c_\alpha) : h_* \Omega^r_{Z/S} \to \Omega^r_{B/S}$, *coincides with the trace $h_* \mathcal{O}_{p^{-1}(Z)} \to \mathcal{O}_B$ induced by $\alpha_{U/B/T}$ after tensoring with $\Omega^r_{B/S}$. Here $h$ denotes the morphism $p^{-1}(Z) \to U \to B$.*

*Proof.* This can be proved using Bott's theorem on grassmannians almost exactly as in [AEZ78, §III]. We indicate the steps.

Note that $Z$ is not unique, but if we have obtained $Z$ and $c_\alpha$ on an open cover, then we can take a common infinitesimal neighborhood of the $Z$'s and on this neighborhood the $c_\alpha$'s glue. We can thus assume that $X$ and $S$ are affine.

Let $Z_0 = \operatorname{Supp}(\alpha)$ and take an embedding $X \hookrightarrow \mathbb{A}^n$ as in Lemma (9.6) and consider the corresponding universal projection $(\mathbb{A}^n_{\mathbb{G}}, \mathbb{B}, \mathbb{G}, p)$. To simplify the presentation, we will now let $X = \mathbb{A}^n$. Let $U \subseteq Z_0 \times_S \mathbb{G}$ be the open subset over which $Z_0 \times_S \mathbb{G} \hookrightarrow \mathbb{A}^n_{\mathbb{G}} \to \mathbb{B}$ is quasi-finite. The subset $U$ then contains all points of relative codimension at most one over $Z_0$ by Lemma (9.6).

Let $Z \hookrightarrow \mathbb{A}^n_S$ be the image of $\operatorname{Image}(\alpha_{U/\mathbb{B}/\mathbb{G}})$ along $p$, and denote the inclusion with $i$. Let $h : Z \times_S \mathbb{G} \to \mathbb{B}$ be the corresponding morphism, let $Z' = U \cap Z \times_S \mathbb{G}$ and denote the open immersion $Z' \hookrightarrow Z \times_S \mathbb{G}$ with $j$. On $Z$ we have the sheaf $i^*(\Omega^1_{\mathbb{A}^n_S})^\vee$ which is free of rank $n$. Thus, we have that $Z \times_S \mathbb{G} = \mathbb{G}_r(i^*(\Omega^1_{\mathbb{A}^n_S})^\vee)$. Let $\mathcal{H}$ be the universal quotient sheaf on $Z \times_S \mathbb{G}$. It is then readily verified that there is a natural isomorphism $\mathcal{H} \cong h^*(\Omega^1_{\mathbb{B}/\mathbb{G}})^\vee$ [AEZ78, §I.3].

Let $\mathcal{W} = \mathcal{E}xt^{-r}(\mathcal{O}_Z, \mathcal{D}^\bullet_{Z/S})$. The relative zero-cycle $\alpha_{U/\mathbb{B}/\mathbb{G}}$ induces a global section of

$$\begin{aligned}\mathcal{E}xt^{-r}_{Z'}\big(j^*h^*(\Omega^r_{\mathbb{B}/\mathbb{G}}), \mathcal{D}^\bullet_{Z'/\mathbb{G}}\big) &= \mathcal{E}xt^{-r}_{Z'}\big(j^*(\wedge^r\mathcal{H}^\vee), p^*\mathcal{D}^\bullet_{Z/S}\big)\\ &= j^*(\wedge^r\mathcal{H}) \otimes_{\mathcal{O}_{Z'}} p^*\mathcal{W}\end{aligned}$$

by (2.27). Bott's theorem [AEZ78, Cor. I.2] shows that the canonical homomorphism

$$\wedge^r i^*(\Omega^1_{\mathbb{A}^n_S})^\vee \otimes \mathcal{W} \to p_*\big(j^*(\wedge^r\mathcal{H}) \otimes p^*\mathcal{W}\big)$$

is an isomorphism. We thus obtain a global section of

$$i^*(\Omega^r_{\mathbb{A}^n_S})^\vee \otimes \mathcal{W} = \mathcal{E}xt^{-r}\big(i^*\Omega^r_{\mathbb{A}^n_S}, \mathcal{D}^\bullet_{Z/S}\big)$$

and this is the relative fundamental class of $\alpha$ as it can be shown that this factors through $\Omega^r_{Z/S}$. What remains is to show that for any projection $(U, B, S)$ the trace $c_{U/B/S}$ is the trace of the zero-cycle $\alpha_{U/B/S}$. This is done almost exactly as in [AEZ78, §III]. □

Let $X/S$ be smooth of relative dimension $n$ and let $Z \hookrightarrow X$ be a closed subset which is equidimensional of dimension $r$ over $S$. Let $c$ be a class in $\mathrm{H}^{n-r}_Z(X, \Omega^{n-r}_{X/S})$. This class lifts to a class in $\mathrm{Ext}^{-r}_X((j_m)_*\mathcal{O}_{Z_m}, \Omega^{n-r}_{X/S})$ for some infinitesimal neighborhood $j_m : Z_m \hookrightarrow X$ of $Z$.

If $(U, B, T, p, g, \varphi)$ is any smooth projection adapted to $Z$, there is an induced trace homomorphism $\mathrm{tr}(c) : \varphi_*(j_m)_*\Omega^r_{p^{-1}(Z_m)/T} \to \Omega^r_{B/T}$ which induces a homomorphism $\mathrm{tr}(c) : \varphi_*(j_m)_*\mathcal{O}_{Z_m} \to \mathcal{O}_B$.

Now if $c$ is a *Chow class* [Ang80, Def. 4.1.2] then $\mathrm{tr}(c)$ is the trace corresponding to a relative zero-cycle $c_{U/B/T}$ on $U$ with image contained in $Z_m$ by [Ang80, Prop. 2.3.5 and Thm. 1.5.3]. These zero-cycles define a relative cycle on $X/S$.

**Theorem (16.2).** *The morphism from Angéniol's Chow-space* $\mathrm{Ang}_r(X/S)(T)$ *to the Chow functor* $\mathrm{Chow}_r(X/S)(T)$ *taking a Chow class onto the corresponding relative cycle is a* monomorphism. *When $T$ is reduced, or when restricted to multiplicity-free cycles or relative Weil-divisors, this morphism is an isomorphism.*

*Proof.* Let $c$ be a Chow class. Then $c$ is determined by the induced relative zero-cycles $c_{U/B/T}$. In fact, $c$ is determined by $c_{U/\mathbb{B}/\mathbb{G}}$ for a universal projection as in the proof of Theorem (16.1).

Let $\alpha$ be a relative cycle on $X/S$ and let $(U, B, S, p, g, \varphi)$ be a smooth projection. The corresponding class $c$ is a Chow class if the trace homomorphism $\varphi_*(j_m)_*\Omega^r_{Z_m/S} \to \Omega^r_{B/S}$ satisfies the conditions of [Ang80, Thm. 4.1.1]. These conditions can be checked on depth zero points of $B$.

In the three special cases listed, $\alpha_{U/B/S}$ is represented by a flat subscheme or a flat cycle on a schematically dense open subset of $B$. That $c$ is a Chow class then follows from [Ang80, Prop. 7.1.1]. □

**Corollary (16.3).** *Let $X$ be a quasi-projective scheme over $\mathbb{C}$. Then the reduction of the Chow functor* $\mathrm{Chow}_r(X)$ *is represented by the Chow variety* $\mathrm{ChowVar}_{r,d}(X)$.

*Proof.* By the theorem, we have that $\mathrm{Chow}_r(X)_{\mathrm{red}} = \mathrm{Ang}_r(X)_{\mathrm{red}}$ and the latter space coincides with the Barlet space [Ang80, Thm. 6.1.1]. As the Barlet space coincides with the Chow variety [Bar75, Ch. IV, Cor. of Thm. 7] the result follows. □

## 17. The classical Chow embedding and representability

In this section, we briefly review the classical construction of the Chow variety, cf. [CW37, Sam55, GKZ94, Kol96], and the extension of this construction to arbitrary relative cycles.

**(17.1)** *The incidence correspondence* — Let $S$ be a scheme and $\mathcal{E}$ a locally free sheaf on $\mathcal{O}_S$ of rank $N+1$. There is a natural commutative square

$$\begin{tikzcd} \mathbb{P}(\mathcal{E}) \arrow[d, "S"'] & \mathbb{F}_{1,N-r}(\mathcal{E}) \arrow[l, "p"'] \arrow[d, "q"] \\ S & \mathbb{G}_{N-r}(\mathcal{E}) \arrow[l] \end{tikzcd}$$

where $G = \mathbb{G}_{N-r}(\mathcal{E})$ is the grassmannian parameterizing linear subvarieties of codimension $r+1$ in $\mathbb{P}(\mathcal{E})$ and $I = \mathbb{F}_{1,N-r}(\mathcal{E})$ is the flag variety parameterizing linear subvarieties of codimension $r+1$ with a marked point. The morphisms $p$ and $q$ are grassmannian fibrations and in particular smooth.

If $Z \hookrightarrow \mathbb{P}(\mathcal{E})$ is equidimensional of dimension $r$, then $p^{-1}(Z)$ is equidimensional of dimension $r+(N-r-1)(r+1) = (N-r)(r+1)-1$. It is easily seen that $q|_{p^{-1}(Z)}$ is generically finite, fiberwise over $S$, and thus $\mathrm{CH}(Z) := q(p^{-1}(Z))$ is a hypersurface in $\mathbb{G}$. If $S = \mathrm{Spec}(k)$, and $Z$ has degree $d$, then $\mathrm{CH}(Z)$ has degree $d$ with respect to the Plücker embedding of $G$. Note that if $S = \mathrm{Spec}(k)$, then $\mathrm{CH}(Z)$ is a Cartier divisor on $\mathbb{G}$.

*Remark* **(17.2).** A variant of the incidence correspondence is often used, cf. [CW37, GIT, Kol96]. Instead of grassmannians and flag varieties, we take $G$ as the multi-projective space $\mathbb{P}(\mathcal{E}^\vee)^{r+1}$ and $I$ as the subscheme of $\mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}^\vee)^{r+1}$ given as the intersection of the $r+1$ universal hyperplanes. Then $q(p^{-1}(Z))$ becomes a hypersurface of multi-degree $d, d, \dots, d$ in $G$.

**(17.3)** *The Chow variety* — Using Chevalley's theorem on the semi-continuity of the fiber dimension, it is easily seen that there is a closed subset of $\mathrm{Div}^d(G/S)$ corresponding to cycles on $\mathbb{P}(\mathcal{E})$ of dimension $r$ and degree $d$ [Kol96, I.3.25.1].

**Definition (17.4).** Let $X \hookrightarrow \mathbb{P}(\mathcal{E})$ be a subscheme and let $\alpha$ be a proper equidimensional relative cycle of dimension $r$ on $X/S$ such that the smooth pull-back is defined, cf. (14.2), then we let $\mathrm{CH}(\alpha) = q_* p^*(\alpha)$.

Note that $q$ is generically finite over $p^{-1}(\mathrm{Supp}(\alpha))$ so the existence of $q_*$ follows by Theorem (13.12). As $\mathrm{CH}(\alpha)$ is a relative Weil divisor and $G/S$ is smooth, we obtain by Theorem (9.15) a morphism

$$\mathrm{CH} : \mathrm{Chow}_{r,d}(X/S) \to \mathrm{Div}^d(G/S).$$

If $\alpha$ is a relative cycle on $X/S$, then $\mathrm{cycl}(\mathrm{CH}(\alpha)) = q_*p^*(\mathrm{cycl}(\alpha))$ so this morphism extends the usual map of cycles. If $Z \hookrightarrow X$ is a closed subscheme which is flat and proper over $S$, then

$$\mathrm{CH}\left(\mathcal{N}_{Z/S}\right) = \mathrm{CH}\left(\mathcal{N}_{q_*(p^{-1}(Z)\otimes\mathcal{L})}\right)$$

for some sufficiently $q$-ample sheaf $\mathcal{L}$ on $I$. That the corresponding Cartier divisor coincides with the divisor constructed by Mumford [GIT, Ch. 5, §3] with the Div-construction follows from Proposition (7.13).

**Proposition (17.5).** *Let $X/S$ be a quasi-projective scheme with a projective embedding morphism $X \hookrightarrow \mathbb{P}(\mathcal{E})$. Let $\alpha$ be a relative cycle on $X/S$, equidimensional of dimension $r$. Assume that one of the following holds:*

- (i) *$S$ is reduced.*
- (ii) *$X/S$ is of relative dimension $r+1$ and 1-smooth.*
- (iii) *$\alpha$ is multiplicity-free.*

*Then $\alpha$ can be recovered from* $\mathrm{CH}(\alpha)$.

*Proof.* If $S$ is reduced, it is enough to show that $\alpha_s$ can be recovered for any generic point $s$. We can thus assume that $S$ is the spectrum of a algebraically closed field. As the CH-morphism is additive, we can also assume that $\alpha$ corresponds to an irreducible variety $V$. Then $V = X \setminus p(q^{-1}(G \setminus \mathrm{CH}(V)))$.

Under the hypothesis in (ii) and (iii), $\alpha$ is represented by a subscheme $Z \hookrightarrow X$ which is either a relative Cartier divisor or smooth over $S$ on a schematically dense subset $U$ of $Z$. To show that $\alpha$ can be recovered from $\mathrm{CH}(\alpha)$ it is enough to construct $Z|_U$. This can be done as in [Fog71, §5]. □

*Questions* **(17.6).** In characteristic zero, we have that

$$\mathrm{Chow}_{r,d}(X/S)_{\mathrm{red}} = \mathrm{Ang}_{r,d}(X/S)_{\mathrm{red}} = \mathrm{ChowVar}_{r,d}(X/S)$$

and thus the morphism $\mathrm{Chow}_{r,d}(X/S)_{\mathrm{red}} \to \mathrm{Div}^d(G)$ is an immersion. This leads to the following questions:

- Is $\mathrm{Ang}_{r,d}(X/S) \to \mathrm{Div}^d(G)$ an immersion?
- In positive characteristic, is $\mathrm{Chow}_{r,d}(X/S) \to \mathrm{Div}^d(G)$ an immersion for sufficiently ample embeddings $X \hookrightarrow \mathbb{P}(\mathcal{E})$?
- In positive characteristic, is $\mathrm{Chow}_{r,d}(X/S)_{\mathrm{red}} \to \mathrm{Div}^d(G)$ an immersion for sufficiently ample embeddings $X \hookrightarrow \mathbb{P}(\mathcal{E})$?

## Appendix A. Duality and fundamental classes

Let $f : X \to S$ be a morphism of schemes. We assume that $S$ is noetherian and that $f$ is separated and of finite type. Then $f$ admits a compactification, i.e. there is a proper morphism $\overline{X} \to S$ and a schematically dominant immersion $X \hookrightarrow \overline{X}$ of $S$-schemes. This is a famous theorem by Nagata [Nag62, Lüt93]. Nagata's compactification result has been generalized by Raoult to algebraic spaces when either $X$ is normal or $S$ is the spectrum of a field [Rao71, Rao74] but we do not need this.

Using that separated, finite type morphisms are compactifiable, one constructs a pseudo-functor !, the twisted (or extraordinary) inverse image, from the category of noetherian schemes and finite type separated morphisms to the corresponding derived category. If $f : X \to S$ is a finite type separated morphism of noetherian schemes then $f^!(\mathcal{O}_S) = \mathcal{D}^\bullet_{X/S}$ is the *relative dualizing complex* constructed by Deligne [Har66, App. by Deligne]. If $g : U \to X$ is an étale morphism then $g^! = g^*$ and if $f : X \to S$ is a *proper* morphism, then $f^!$ is a right adjoint to $f_*$ (in the derived category). If $f$ is a finite type separated morphism of finite Tor-dimension, then $f^!(\mathcal{F}) = f^*(\mathcal{F}) \overset{\mathbf{L}}{\otimes}_{\mathcal{O}_X} \mathcal{D}^\bullet_{X/S}$. If $f : X \to S$ is smooth of relative dimension $r$, then $\mathcal{D}^\bullet_{X/S} = \Omega^r_{X/S}[r]$.

As $g^! = g^*$ for étale morphisms, we can extend the definition of ! to the category of noetherian algebraic spaces with finite type separated morphisms. We will use the following duality theorem:

**Theorem (A.1)** ([Har66, Ch. III, Thm. 6.7]). *Let $f : X \to Y$ be a finite morphism of noetherian schemes. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be complexes of sheaves on $X$ and $Y$ respectively. Then there is a quasi-isomorphism*

$$f_*\mathbf{R}\mathcal{E}xt_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \to \mathbf{R}\mathcal{E}xt_Y(f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet).$$

*In particular, we have that*

$$\operatorname{Ext}^n_X(\mathcal{F}^\bullet, f^!\mathcal{G}^\bullet) \to \operatorname{Ext}^n_Y(f_*\mathcal{F}^\bullet, \mathcal{G}^\bullet)$$

*for every integer $n$.*

We briefly recall some of the main results of [EZ78]. Let $k$ be a field. If $X/k$ is smooth of dimension $r$, then the *fundamental class* of $X/k$ is the canonical class

$$c_X \in \operatorname{Ext}^{-r}_X(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}) = \operatorname{Hom}_X(\Omega^r_{X/k}, \Omega^r_{X/k})$$

given by the identity. If $X/k$ is geometrically reduced, then there is a unique class

$$c_X \in \operatorname{Ext}^{-r}_X(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}),$$

the fundamental class of $X/k$, such that over the smooth locus $U \subseteq X$, the pull-back $c_X|_U = c_U$ coincides with the class defined above. The uniqueness of $c_X$ follows by Corollary (A.4) below. The existence of the fundamental class $c_X$ for arbitrary $X/k$ is shown by El Zein [EZ78, Ch. III, Thm.]. When $X/k$ is not geometrically reduced, $c_X$ is uniquely determined as follows. If the irreducible components of $X$ are $X_i$, then

$$c_X = \sum_i m_i c_{X_i}$$

where $m_i$ is the multiplicity of $X_i$, i.e., the length of the local ring of $\mathcal{O}_X$ at the generic point of $X_i$, cf. Remark (A.8). If $K/k$ is a perfect extension of $k$, then

$$\operatorname{Ext}_X^{-r}(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}) \to \operatorname{Ext}^{-r}_{X_K/K}(\Omega^r_{X_K/K}, \mathcal{D}^\bullet_{X_K/K})$$

is injective and the image of $c_X$ is $c_{X_K}$ [EZ78, Ch. III, No. 4, Prop.]. Note that when $k$ is of characteristic $p > 0$, then $c_X$ is zero at every irreducible component $X_i$ where $p$ divides the geometric multiplicity, i.e. the multiplicity of $(X_i)_K$.

Assume that $X$ can be smoothly embeddable, i.e., that there exists a closed immersion $j : X \hookrightarrow Y$ into a smooth scheme $Y/k$ of pure dimension $n$. Then we define the *algebraic de Rham homology* of $X$ by

$$\mathrm{H}^{\mathrm{dR}}_q(X) = \mathbf{H}^{2n-q}_X(Y, \Omega^\bullet_{Y/k}),$$

the hypercohomology, with supports in $X$, of the algebraic de Rham complex on $Y$. If $k$ has characteristic zero, then this homology group is independent on the choice of smooth embedding [Har75, Ch. II, Thm. 3.2].

We have a canonical homomorphism

$$\begin{aligned}\operatorname{Ext}_X^{-r}(\Omega^r_{X/k}, \mathcal{D}^\bullet_{X/k}) &\to \operatorname{Ext}_X^{-r}(j^*\Omega^r_{Y/k}, j^!\Omega^n_{Y/k}[n])\\ &\cong \operatorname{Ext}_Y^{-r}(j_*j^*\Omega^r_{Y/k}, \Omega^n_{Y/k}[n])\\ &\cong \operatorname{Ext}_Y^{n-r}(j_*\mathcal{O}_X, \Omega^{n-r}_{Y/k})\\ &\to \mathrm{H}^{n-r}_X(Y, \Omega^{n-r}_{Y/k}).\end{aligned}$$

By abuse of notation, we also denote the image of the fundamental class $c_X$ in $\mathrm{H}^{n-r}_X(Y, \Omega^{n-r}_{Y/k})$ by $c_X$. El Zein [EZ78, Ch. III, Thm.] shows that $c_X$ is in the kernel of the differential

$$d' : \mathrm{H}^{n-r}_X(Y, \Omega^{n-r}_{Y/k}) \to \mathrm{H}^{n-r}_X(Y, \Omega^{n-r+1}_{Y/k}).$$

Thus $c_X$ is the image of an element in the hypercohomology

$$\mathbf{H}^{n-r}_X(Y, \Omega^{n-r}_{Y/k} \to \Omega^{n-r+1}_{Y/k} \to \dots)$$

which we also denote by $c_X$. Finally, we have the image of this element in the algebraic de Rham homology:

$$\mathbf{H}^{2(n-r)}_X(Y, \Omega^\bullet_{Y/k}) = \mathrm{H}^{\mathrm{dR}}_{2r}(X).$$

In characteristic zero, this class coincides with the homology class $\eta(X)$ defined by Hartshorne [Har75, Ch. II, 7.6]. This is not proved by El Zein but not difficult to show. In fact, as $c_X = \sum_i m_i c_{X_i}$ where $X_i$ are the irreducible components of $X$ and $m_i$ their multiplicities, we can assume that $X$ is integral. Then $\mathrm{H}^{\mathrm{dR}}_{2r}(X) \cong \mathrm{H}^{\mathrm{dR}}_{2r}(X \setminus X_{\mathrm{sing}})$ and we can thus assume that $X$ is smooth. Then with the choice $X = Y$, we have that $c_X$ is the identity homomorphism $\Omega^r_{X/S} \to \Omega^r_{X/S}$.

In this paper, we are mostly interested in the relative case. Let $X/S$ be a scheme of relative dimension $r$. A relative fundamental class of $X/S$ will be a class in $\operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$ satisfying certain properties as stated below. The construction of this class is local:

**Lemma (A.2)** ([AEZ78, Lem. II.1])**.** *Let $S$ be a noetherian scheme and let $X \to S$ be equidimensional of relative dimension $r$. Then*

$$\operatorname{Ext}^{-r}(\mathcal{F}, \mathcal{D}^\bullet_{X/S}) = \Gamma\big(X, \mathcal{E}xt^{-r}(\mathcal{F}, \mathcal{D}^\bullet_{X/S})\big)$$

*for every $\mathcal{O}_X$-module $\mathcal{F}$.*

We will use the following duality isomorphism which is a special case of Theorem (A.1):

**Proposition (A.3)** ([EZ78, Ch. IV, Prop. 2])**.** *Let $X \to S$ be equidimensional of relative dimension $r$ and let $Y \to S$ be a smooth morphism of relative dimension $r$. Let $f : X \to Y$ be a finite $S$-morphism. Then there is a canonical isomorphism*

$$T_f \,:\, \operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S}) \xrightarrow{\cong} \operatorname{Hom}(f_*\Omega^r_{X/S}, \Omega^r_{Y/S}).$$

**Corollary (A.4)** ([EZ78, Ch. IV, Prop. 4])**.** *Let $X \to S$ be equidimensional of relative dimension $r$ and let $Y \to S$ be a smooth morphism of relative dimension $r$. Let $f : X \to Y$ be a finite $S$-morphism. Let $U \subseteq Y$ be a schematically dense open subset. Then the canonical homomorphism*

$$\operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S}) \to \operatorname{Ext}^{-r}(\Omega^r_{f^{-1}(U)/S}, \mathcal{D}^\bullet_{f^{-1}(U)/S})$$

*is injective.*

Recall that if $f : X \to Y$ is a finite and flat morphism, then $f_*\mathcal{O}_X$ is locally free and there is a trace homomorphism $f_*\mathcal{O}_X \to \mathcal{O}_Y$. By tensoring with $\Omega^r_{Y/S}$ we obtain the homomorphism

$$\operatorname{tr}(f) \,:\, f_*f^*\Omega^r_{Y/S} \to \Omega^r_{Y/S}.$$

**Definition (A.5)** ([EZ78, Ch. IV, Def. 2])**.** Let $S$ be reduced and let $X \to S$ be equidimensional of relative dimension $r$. We say that a class $c \in \operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$ satisfies the *property of the trace*, if for every open subset $U \subseteq X$, every smooth morphism $Y \to S$ of dimension $r$ and every finite and flat morphism $f : U \to Y$, we have that the composition of $f_*f^*\Omega^r_{Y/S} \to f_*\Omega^r_{X/S}$ and $T_f(c)$ is the trace $\operatorname{tr}(f)$.

**Proposition (A.6)** ([AEZ78, Prop. II.3.1])**.** *Let $S$ be reduced and let $X \to S$ be equidimensional of relative dimension $r$. There is at most one class*

$$c \in \operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$$

*satisfying the property of the trace.*

*Proof.* The question is local on $X$ by Lemma (A.2). We can thus assume that there is a closed immersion $j : X \to \mathbb{A}^n$. Let $\varphi : \mathbb{A}^n \to \mathbb{A}^r$ be a linear morphism such that the composition $f : X \to \mathbb{A}^r$ is generically finite. Note that as $S$ is reduced, we have that $f$ is generically flat. The property of the trace determines $c$ over the image of $j^*\varphi^*\Omega^r_{\mathbb{A}^r/S} \to j^*\Omega^r_{\mathbb{A}^n/S} \to \Omega^r_{X/S}$. Let $x$ be a generic point of $X$, the images of $\varphi^*\Omega^r_{\mathbb{A}^r/S} \to \Omega^r_{\mathbb{A}^n/S}$ for every $\varphi$ such that $j \circ \varphi$ is quasi-finite at $x$, generates $\Omega^r_{\mathbb{A}^n/S}$ in a neighborhood of $x$. For details, see [AEZ78, *loc. cit.*]. □

**Definition (A.7).** Let $S$ be reduced and let $X \to S$ be equidimensional of relative dimension $r$. The unique class $c_{X/S} \in \operatorname{Ext}^{-r}(\Omega^r_{X/S}, \mathcal{D}^\bullet_{X/S})$ satisfying the property of the trace, if it exists, is called the relative fundamental class of $X/S$.

The fundamental class $c_X$ for a scheme $X/k$ discussed above is the relative fundamental class $c_{X/k}$, cf. [EZ78, Ch. III, Cor.].

*Remark* **(A.8).** Let $S$ be reduced and let $X \to S$ be equidimensional of relative dimension $r$. If $X$ has irreducible components $X_i$ with multiplicities $m_i$ then it follows that $c_X = \sum_i m_i c_{X_i}$. In fact, if $f : X \to Y$ is a finite morphism, then $\operatorname{tr}(f) = \sum_i m_i \operatorname{tr}(f|_{X_i})$ at the generic points of $Y$ where all involved maps are flat.

The relative fundamental class exists in the following cases:

(i) $S$ normal and $X/S$ equidimensional of dimension $r$ [EZ78, Ch. IV, No. 3].
(ii) $S$ reduced and $X/S$ flat [EZ78, Ch. IV, No. 4].

If $S$ is not reduced things are slightly more complicated. We assume that $X/S$ is flat or at least of finite Tor-dimension. If $S$ is without embedded components, then the property of the trace as stated in Definition (A.5) is enough to ensure uniqueness. In fact, if $U \subseteq X$ is an open subset, $Y/S$ is smooth of relative dimension $r$, and $f : U \to Y$ is a quasi-finite morphism, then $f$ is generically flat and finite. In general, the property of the trace should be generalized to include morphisms $f : U \to Y$ which are finite and of finite Tor-dimension but not necessarily flat. The trace of such a morphism is defined by the alternating sum of the traces of a flat resolution, cf. [AEZ78, Ch. II]. The main result of [AEZ78] is that for any locally noetherian scheme $S$ of characteristic zero and $X/S$ of finite type and finite Tor-dimension, there exists a relative fundamental class of $X/S$.

## Appendix B. Schematically dominant families

Recall that a family of morphisms $u_\lambda : Z_\lambda \to X$ is schematically dominant if the intersection of the kernels of $\mathcal{O}_X \to (u_\lambda)_*\mathcal{O}_{Z_\lambda}$ is zero [EGA$_{\mathrm{IV}}$, 11.10]. The important fact is that a morphism from $X$ is determined on $\{Z_\lambda\}$, i.e., if $Y$ is a separated scheme, then

$$\operatorname{Hom}(X, Y) \to \prod_\lambda \operatorname{Hom}(Z_\lambda, Y)$$

is injective. In this section we show that if $X \to Y$ is a smooth morphism, then the family of all subschemes $Z_\lambda \hookrightarrow X$ which are étale over $Y$, is schematically dominant.

**Lemma (B.1).** *Let $S$ and $X$ be affine schemes and $X \to S$ a smooth morphism. Let $f \in \Gamma(X)$. Then there exists a locally closed subscheme $j : Z \hookrightarrow X$ such that $Z \hookrightarrow X \to S$ is étale and $j^*(f) \in \Gamma(Z)$ is non-zero.*

*Proof.* Let $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$. By a standard limit argument, we can assume that $A$ is noetherian. Let $x \in X$ be a point such that $f$ is not zero in $\mathcal{O}_{X,x}$. Let $s \in S$ be the image of $x$. Let $\mathfrak{p} \subseteq A$ be the prime ideal corresponding to $s$.

By Krull's intersection theorem there is an integer $n \geq 0$ such that $f \in \mathfrak{p}^n B_\mathfrak{p}$ but $f \notin \mathfrak{p}^{n+1} B_\mathfrak{p}$. Consider the $k(\mathfrak{p})$-module $M = \mathfrak{p}^n A_\mathfrak{p}/\mathfrak{p}^{n+1} A_\mathfrak{p} \otimes_{k(\mathfrak{p})} B \otimes_A k(\mathfrak{p})$. By flatness, this is the submodule $\mathfrak{p}^n B_\mathfrak{p}/\mathfrak{p}^{n+1} B_\mathfrak{p}$ of $B_\mathfrak{p}/\mathfrak{p}^{n+1} B_\mathfrak{p}$. Choose a basis for $\mathfrak{p}^n A_\mathfrak{p}/\mathfrak{p}^{n+1} A_\mathfrak{p}$ and let $g_1, g_2, \dots, g_k \in B \otimes_A k(\mathfrak{p})$ be the coefficients in this basis of the image of $f$ in $M$. As $f$ is not zero in $M$, there is at least one non-zero $g_i$ and we let $g = g_i$.

Let $U \subseteq X$ be an open subset such that $U \cap X_s = (X_s)_g$. Choose a closed point $x \in (X_s)_g$ such that $k(s) \to k(x)$ is separable. There is then by [EGA$_{\mathrm{IV}}$, Cor. 17.16.3] a locally closed affine subscheme $Z = \operatorname{Spec}(C) \hookrightarrow U$, containing the point $x$, such that $Z \to X \to S$ is étale. In particular we have that $g$ is invertible in $C \otimes_A k(\mathfrak{p})$. It follows that the image of $f$ in $\mathfrak{p}^n C_\mathfrak{p}/\mathfrak{p}^{n+1} C_\mathfrak{p}$ is non-zero. As $C$ is a flat $A$-algebra, this implies that the image of $f$ in $C_\mathfrak{p}/\mathfrak{p}^{n+1} C_\mathfrak{p}$ is non-zero. In particular, the image of $f$ in $C$ is non-zero. □

**Proposition (B.2).** *Let $S$ be a scheme and let $X \to S$ be a smooth morphism. Then the family of all subschemes $Z_\lambda \hookrightarrow X$ which are étale over $Y$, is schematically dominant.*

*Proof.* It is enough to show that the family is schematically dominant when $X$ and $S$ are affine. Let $f, g \in \Gamma(X, \mathcal{O}_X)$ such that $f$ is non-zero in $\Gamma(X_g, \mathcal{O}_X)$. The above lemma gives a locally closed subscheme $X_{f,g} \hookrightarrow X_g$ such that $X_{f,g} \to S$ is étale and such that the pull-back of $f$ to $X_{f,g}$ is non-zero. It follows that the family $(X_{f,g} \hookrightarrow X)$ is schematically dominant. □

**Corollary (B.3).** *Let $S$ be a scheme, $S' \to S$ a smooth morphism and $B' \to S'$ a flat morphism, locally of finite presentation. Then there is a family of locally closed subschemes $S'_\lambda \hookrightarrow S'$ such that $S'_\lambda \to S$ is étale and such that $(S'_\lambda \times'_S B' \hookrightarrow B')$ is schematically dominant.*

*Proof.* Take a schematically dominant family $(S'_\lambda \hookrightarrow S')$ as in the proposition. Then the pull-back family $(S'_\lambda \times'_S B' \hookrightarrow B')$ is schematically dominant as well [EGA$_{\mathrm{IV}}$, Thm. 11.10.5]. □

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