REPRESENTABILITY OF HILBERT SCHEMES AND HILBERT STACKS OF POINTS

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Abstract. We show that the Hilbert functor of points on an arbitrary *separated* algebraic stack is an algebraic space. We also show the algebraicity of the Hilbert stack of points on an algebraic stack and the algebraicity of the Weil restriction of an algebraic stack along a finite flat morphism. For the latter two results, no separation assumptions are necessary.

Introduction

The purpose of this note is to give a short and elementary proof of the algebraicity of the Hilbert functor of points $\mathcal{H}ilb^d_X$ for an arbitrary separated algebraic space or stack X, cf. Theorem (4.1). The key fact is that given an étale (resp. smooth) presentation $f\colon U \to X$ there is an open subspace $\mathcal{H}ilb^d_{U\to X}$ of $\mathcal{H}ilb^d_U$ and a representable étale (resp. smooth) surjective map $f_*\colon \mathcal{H}ilb^d_{U\to X} \to \mathcal{H}ilb^d_X$. To show that $\mathcal{H}ilb^d_X$ is algebraic, it is thus enough to show that $\mathcal{H}ilb^d_U$ is algebraic. When U is affine, this is well-known [Nor78, GLS07a].

When X is not separated, the Hilbert functor is not representable [LS08]. A replacement for the Hilbert functor is then the Hilbert stack which was briefly introduced by M. Artin [Art74, App.]. Applying the same method as for the Hilbert functor, we deduce the algebraicity of the Hilbert stack of points from the affine case, cf. Theorems (4.3) and (4.4). Along the way we also prove the algebraicity of the Weil restriction of an algebraic stack along a finite flat morphism and the algebraicity of Hom-stacks when the source is finite flat, cf. Theorems (3.7) and (3.12). In Section 5, we show that the open substack parameterizing étale families coincides with the stack quotient of the d^{th} fiber product X^d by the symmetric group. This is the stack of zero-dimensional branchvarieties [AK10].

Date: 2010-03-31.

2000 *Mathematics Subject Classification.* Primary 14C05; Secondary 14A20.

Key words and phrases. Hilbert scheme, Hilbert stack, Weil restriction, Hom stack, non-separated.

The final version of this paper is to appear in Comm. Alg.

Let us indicate the relationship between our methods and more standard representation techniques. Thus consider the following diagram of subcategories of algebraic spaces over some fixed base scheme:

$$\begin{tikzcd}
 & \mathbf{QProj}_{\mathrm{fp}} \arrow[r, hook] \arrow[d, hook] & \mathbf{AlgSp}_{\mathrm{fp,sep}} \arrow[r, hook] \arrow[d, hook] & \mathbf{AlgSp}_{\mathrm{fp}} \arrow[d, hook] \\
\mathbf{Aff} \arrow[r, hook] & \mathbf{AF\text{-}Sch} \arrow[r, hook] & \mathbf{AlgSp}_{\mathrm{sep}} \arrow[r, hook] & \mathbf{AlgSp}.
\end{tikzcd}$$

Here **Aff** (resp. **QProj**, resp. **AF-Sch**, resp. **AlgSp**) denotes the category of affine schemes (resp. quasi-projective schemes, resp. AF-schemes, resp. algebraic spaces). A scheme is AF if every finite set of points lies in an affine open subscheme. The subscripts "fp" and "sep" stand for "locally of finite presentation" and "separated".

Let X be an algebraic space in one of these categories and consider a functor or stack of objects on X such as the Hilbert functor $\mathcal{H}ilb_X$. The classical approach is to work in the category $\mathbf{QProj}_{\mathrm{fp}}$ and show that the functor can be embedded into a projective scheme [FGA, No. 221]. Alternatively, we can work in the bigger category $\mathbf{AlgSp}_{\mathrm{fp,sep}}$ where Artin's algebraization theorems [Art69, Art74] apply. In this category, the algebraicity of the Hilbert functor, Hilbert stack and Weil restriction is well-known, cf. [Art69, Cor. 6.2], [Sta06, Rmk. 4.5] and [Lie06, §2.1].

In this paper we will primarily be interested in the second line of the diagram and show that in the *zero-dimensional case* we can use more elementary and constructive methods to deduce algebraicity. As a bonus, we need neither finiteness nor separatedness assumptions. Similar methods were applied in [GLS07a, GLS07b] for Hilbert and Quot schemes of affine and AF-schemes.

In a subsequent paper, we will turn the attention to the category $\mathbf{AlgSp}_{\mathrm{fp}}$ and use Artin's algebraization theorem to show the algebraicity of the Hilbert stack parameterizing *higher-dimensional* families on a given, possibly *non-separated*, scheme or algebraic space. This relies on variants of Chow's lemma and Grothendieck's existence theorem for non-separated spaces.

As it is easier to first establish the algebraicity of the Hilbert stack and then *a posteriori* verify that it is quasi-separated we work with general algebraic spaces and algebraic stacks without any separation assumptions, cf. Appendix A.

Acknowledgments. I would like to thank D. Laksov, R. Skjelnes and the referee for useful comments and discussions.

1. The Hilbert functor and the Hilbert stack

For simplicity, we work over a fixed base *scheme* S. If X is a scheme or an algebraic stack over S, then a property of X/S always refers to a property of the structure morphism $X \to S$.

Definition (1.1). We say that a finite morphism $f\colon X \to Y$ is *flat of rank* d if $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_Y$-module of constant rank d.

Definition (1.2). Let X/S be a *separated* scheme (resp. separated algebraic space, resp. separated algebraic stack). The *Hilbert functor of points* $\mathcal{H}ilb^d_{X/S}$ is the functor which to an S-scheme T assigns the set of closed subschemes (resp. subspaces, resp. substacks) $Z \hookrightarrow X \times_S T$ such that the second projection $p\colon Z \to T$ is finite and flat of rank d.

Remark **(1.3).** It is easily seen, using [$\mathrm{EGA_{IV}}$, Thm. 12.2.1 (i), (ii)], that $\mathcal{H}ilb^d_{X/S}$ is an open and closed subfunctor of the full Hilbert functor $\mathcal{H}ilb_{X/S}$ which parameterizes closed subspaces (or substacks) which are flat, proper and of finite presentation. Note that an object of $\mathcal{H}ilb^d_{X/S}(T)$ is a closed substack $Z \hookrightarrow X \times_S T$ such that $p\colon Z \to T$ is finite and hence Z is always a *scheme* even if X is an algebraic stack.

Remark **(1.4).** Using Artin's criteria for algebraicity it can be shown that the Hilbert functor $\mathcal{H}ilb_{\mathscr{X}/S}$ is a separated algebraic space locally of finite presentation when $\mathscr{X}/S$ is a separated algebraic stack locally of finite presentation. In this generality, the result is due to M. Olsson [Ols05, Thm. 1.5] but also see [OS03] for boundedness results when $\mathscr{X}$ is a Deligne–Mumford stack.

We will now define the Hilbert *stack* of points [Art74, App.]. The difference between the Hilbert stack and the Hilbert functor is that in the stack we consider flat families $Z \to T$ with morphisms $Z \to X$ without the condition that $Z \to X \times_S T$ is a closed immersion.

Definition (1.5). Given an algebraic stack $\mathscr{X}/S$, let $\mathscr{H}^d_{\mathscr{X}}$ be the category with objects pairs of morphisms $(p\colon Z \to T, q\colon Z \to \mathscr{X})$ where T is an S-scheme and p is finite and flat of rank d. The morphisms are triples (φ, ψ, τ) fitting into a 2-commutative diagram

$$\begin{tikzcd}
Z_1 \arrow[r, "\varphi"] \arrow[rr, bend left, "q_1"] \arrow[d, "p_1"] \arrow[dr, phantom, "\square"] & Z_2 \arrow[r, "q_2"] \arrow[d, "p_2"] \arrow[r, phantom, shift left=4, "\Downarrow\tau"] & \mathscr{X} \\
T_1 \arrow[r, "\psi"] & T_2 &
\end{tikzcd}$$

such that the square is cartesian. The category $\mathscr{H}^d_{\mathscr{X}}$ is fibered in groupoids over $\mathbf{Sch}_{/S}$ and by étale descent of affine schemes [$\mathrm{SGA_1}$, Exp. VIII, Thm. 2.1] it follows that $\mathscr{H}^d_{\mathscr{X}}$ is a stack. We call $\mathscr{H}^d_{\mathscr{X}}$ the *Hilbert stack of d points* on $\mathscr{X}$.

When $\mathscr{X}/S$ is a *separated* algebraic stack, then the Hilbert functor $\mathcal{H}ilb^d_{\mathscr{X}/S}$ is an open substack of the Hilbert stack $\mathscr{H}^d_{\mathscr{X}}$, cf. Proposition (1.9) below.

Example (1.6). The Hilbert stack $\mathscr{H}^1_{\mathscr{X}}$ is equivalent to $\mathscr{X}$. If $\mathscr{X}$ is separated, then the Hilbert functor $\mathcal{H}ilb^1_{\mathscr{X}}$ equals the automorphism-free locus of $\mathscr{X}$.

In the remainder of this section we will review more general Hilbert stacks present in the literature. These are not used in the subsequent sections.

Definition (1.7). Let $\mathscr{X}$ be an algebraic stack. The Hilbert stack $\mathscr{H}_{\mathscr{X}}$ is the stack parameterizing flat and proper algebraic stacks $p\colon \mathscr{Z} \to T$ of finite presentation together with a representable morphism $q\colon \mathscr{Z} \to \mathscr{X}$. The substack of objects such that $(q,p)\colon \mathscr{Z} \to \mathscr{X} \times_S T$ is locally quasi-finite (resp. unramified) is denoted $\mathscr{H}_{\mathscr{X}}^{\mathrm{qfin}}$ (resp. $\mathscr{H}_{\mathscr{X}}^{\mathrm{unram}}$). The substack of objects such that p is representable, i.e., $\mathscr{Z}$ is an algebraic space, is denoted $\mathscr{H}_{\mathscr{X}}^{\mathrm{repr}}$.

Lemma (1.8). *Let $f\colon X \to Y$ be a morphism of algebraic spaces, locally of finite type.*

(i) *The locus of points in X where f is quasi-finite is open.*
(ii) *The locus of points in X where f is unramified is open.*
(iii) *If f is proper, then the subfunctor $Y' \subseteq Y$ consisting of morphisms $T \to Y$ such that $X \times_Y T \to T$ is a closed immersion, is an open subspace.*

Proof. (i) is [$\mathrm{EGA_{IV}}$, Cor. 13.1.4] as the question is étale-local on X and Y. The locus where f is unramified is the complement of the support of Ω^1_f and thus open. To show (iii) we can assume that f is quasi-finite and hence finite [LMB00, Cor. A.2.1] since the locus of Y where f is quasi-finite is open by (i). That $Y' \subseteq Y$ is an open subspace now follows by Nakayama's lemma. □

Proposition (1.9). *Let $\mathscr{X}/S$ be an algebraic stack. Then*

(i) *$\mathscr{H}_{\mathscr{X}}^{\mathrm{repr}} \subset \mathscr{H}_{\mathscr{X}}$ is an open substack.*
(ii) *$\mathscr{H}_{\mathscr{X}}^{\mathrm{d}} \subset \mathscr{H}_{\mathscr{X}}^{\mathrm{repr}}$ is an open and closed substack.*
(iii) *$\mathscr{H}_{\mathscr{X}}^{\mathrm{unram}} \subset \mathscr{H}_{\mathscr{X}}^{\mathrm{qfin}} \subset \mathscr{H}_{\mathscr{X}}$ are open substacks.*
(iv) *If $\mathscr{X}/S$ is a* separated *algebraic stack, then the Hilbert functor $\mathcal{H}ilb_{\mathscr{X}}$ is an open subfunctor of the Hilbert stack $\mathscr{H}_{\mathscr{X}}$.*

Proof. Let T be a scheme and let $(p\colon \mathscr{Z} \to T, q\colon \mathscr{Z} \to \mathscr{X})$ be an object of $\mathscr{H}_{\mathscr{X}}(T)$. We let $T^{\mathrm{repr}} = T \times_{\mathscr{H}_{\mathscr{X}}} \mathscr{H}_{\mathscr{X}}^{\mathrm{repr}} \subseteq T$ and similarly for T^{unram} and T^{qfin}.

Note that since p is separated, the inertia stack $I_{\mathscr{Z}} \to \mathscr{Z}$ is proper. By Lemma (1.8) (iii) applied to $I_{\mathscr{Z}} \to \mathscr{Z}$ the automorphism-free locus $\mathscr{Z}' \subseteq |\mathscr{Z}|$ is open. Thus $T^{\mathrm{repr}} = T \setminus p(\mathscr{Z} \setminus \mathscr{Z}')$ is an open subscheme of T. This shows (i). The second statement follows from [$\mathrm{EGA_{IV}}$, Thm. 12.1.1 (i), (ii)].

To show (iii), let $\mathscr{Z}' \subseteq \mathscr{Z}$ be the locus where $(q,p)\colon \mathscr{Z} \to \mathscr{X} \times_S T$ is locally quasi-finite (resp. unramified). This locus is open by Lemma (1.8) and thus T^{qfin} (resp. T^{unram}) is the open subscheme $T \setminus p(\mathscr{Z} \setminus \mathscr{Z}')$.

(iv) If $\mathscr{X}/S$ is separated, then $(q,p)\colon \mathscr{Z} \to \mathscr{X} \times_S T$ is proper and by Lemma (1.8) (iii), the locus of $\mathscr{X} \times_S T$ over which (q,p) is a closed immersion is open. We let $\mathscr{Z}' \subseteq \mathscr{Z}$ be the inverse image of this locus. Then $T \times_{\mathscr{H}_{\mathscr{X}}} \mathcal{H}ilb_{\mathscr{X}} = T \setminus p(\mathscr{Z} \setminus \mathscr{Z}')$ is an open subscheme of T. □

The stack $\mathscr{H}_{\mathscr{X}}$ is not algebraic in general as Grothendieck's existence theorem does not hold without any projectivity assumptions on either p or (q,p). The algebraicity of $\mathscr{H}_{\mathscr{X}}^{\mathrm{qfin,repr}}$ for a *separated* algebraic stack $\mathscr{X}$ with finite diagonal, locally of finite presentation over S, is proved by J. Starr in [Sta06, Rmk. 4.5]. A. Vistoli studies the substack $\mathscr{H}_{\mathscr{X}}^{\mathrm{unram,repr}}$ when $\mathscr{X}$ is a separated Deligne–Mumford stack, locally of finite presentation over S although he does not prove algebraicity [Vis91].

A sketch of the proof of the algebraicity of $\mathscr{H}_{\mathscr{X}}^{\mathrm{qfin}}$ for a *separated* algebraic stack, locally of finite presentation over S, is given by M. Lieblich in [Lie06, Thm. 2.1, Cor. 2.5]. More generally, it is expected that $\mathscr{H}_{\mathscr{X}}^{\mathrm{qfin}}$ is algebraic when $\mathscr{X}/S$ is an algebraic stack locally of finite presentation with separated and quasi-finite diagonal. Note that in this case $(q,p)\colon \mathscr{Z} \to \mathscr{X} \times_S T$ is quasi-finite and separated, hence quasi-affine so that $\mathcal{O}_{\mathscr{Z}}$ is (q,p)-ample.

The *stack of branchvarieties*, introduced by V. Alexeev and A. Knutson [AK10], is the open substack of $\mathscr{H}_{\mathscr{X}}^{\mathrm{qfin}}$ parameterizing families $\mathscr{Z} \to T$ with geometrically reduced fibers.

The *space of husks* of a separated algebraic space X is the open subspace of $\mathscr{H}_{X}^{\mathrm{qfin}}$ parameterizing families $Z \to T$ with (S_1)-fibers such that $(q,p)\colon Z \to X \times_S T$ is birational onto its image. This is the space of algebra husks of quotients of $\mathcal{O}_X$ introduced by J. Kollár [Kol09, Rmk. 10]. When limited to 1-dimensional husks this is M. Hønsen's space of Cohen–Macaulay curves [Høn04].

2. Representability of the Hilbert scheme

In this section we show that the Hilbert functor $\mathcal{H}ilb_X^d$ is separated and that it is represented by a scheme if X is an AF-scheme. As we do not know *a priori* that $\mathcal{H}ilb_X^d$ is quasi-separated, it is not enough to check the valuative criterion for separatedness.

Lemma (2.1). *Let S be a scheme and let $\mathscr{X} \to S$ be a separated algebraic stack. Then the diagonal of $\mathcal{H}ilb^d_{\mathscr{X}/S} \to S$ is a closed immersion.*

Proof. Let T be an S-scheme and let $T \to \mathcal{H}ilb^d_{\mathscr{X}/S} \times_S \mathcal{H}ilb^d_{\mathscr{X}/S}$ be a morphism corresponding to closed subschemes $Z_1 \hookrightarrow \mathscr{X} \times_S T$ and $Z_2 \hookrightarrow \mathscr{X} \times_S T$. Let $T_{12}\colon \mathbf{Sch}_{/T} \to \mathbf{Set}$ be the functor defined as follows: for a T-scheme T', we let $T_{12}(T')$ be the one-point set if $Z_1 \times_T T' = Z_2 \times_T T'$ and the empty set otherwise. Then $T_{12} \hookrightarrow T$ is represented by a closed subscheme [EGA$_{\mathrm{I}}$, Lem. 9.7.9.1]. Since the natural diagram

$$\begin{tikzcd}
\mathcal{H}ilb^d_{\mathscr{X}/S} \arrow[r, hook, "\Delta"] & \mathcal{H}ilb^d_{\mathscr{X}/S} \times_S \mathcal{H}ilb^d_{\mathscr{X}/S} \\
T_{12} \arrow[u] \arrow[r, hook] & T \arrow[u, "{[Z_1],[Z_2]}"']
\end{tikzcd}$$

is cartesian, it follows that $\Delta_{\mathcal{H}ilb^d_{\mathscr{X}/S}}$ is a closed immersion. □

Definition (2.2). Let X be a scheme. We say that X is an AF-scheme if every finite set of points $Z \subseteq X$ is contained in an affine open subset of X.

Remark **(2.3).** If S is an affine scheme and $X \to S$ is a locally quasi-finite and separated morphism of algebraic spaces, then it follows from Zariski's main theorem [LMB00, Thm. A.2] that every finite subset $Z \subseteq X$ is contained in a quasi-affine open subscheme of X. It then follows from [$\mathrm{EGA}_{\mathrm{II}}$, Cor. 4.5.4] that X is an AF-scheme.

Theorem (2.4). *Let S be an affine scheme and let X/S be an* AF*-scheme. Then $\mathcal{H}ilb^d_{X/S}$ is represented by a separated scheme* $\mathrm{Hilb}^d(X/S)$*.*

Proof. When X is an affine scheme then it is known that $\mathcal{H}ilb^d_{X/S}$ is represented by a scheme [Nor78, GLS07a] and this also follows independently from Theorem (4.3). If X is an AF-scheme, then let $X = \bigcup_\alpha U_\alpha$ be an open cover of X by affines such that every subset of d points of X lies in some U_α. It is then easily seen that $\coprod_\alpha \mathcal{H}ilb^d_{U_\alpha} \to \mathcal{H}ilb^d_X$ is a Zariski covering and thus $\mathcal{H}ilb^d_X$ is represented by a scheme. That $\mathrm{Hilb}^d(X/S)$ is separated is Lemma (2.1). □

3. Weil restriction and push-forward of Hilbert stacks

In this section we show that the *Weil restriction* of an algebraic space or algebraic stack along a finite flat morphism is algebraic. As a consequence the *push-forward* of Hilbert stacks is algebraic. We also provide a list of properties for the Weil restriction and the push-forward of Hilbert stacks. This section generalizes the results of Bosch, Lütkebohmert and Raynaud [BLR90, §7.6].

Definition (3.1). Let $X \to S'$ and $g\colon S' \to S$ be morphisms of algebraic spaces. The *Weil restriction* $\mathbf{R}_{S'/S}(X)$ is the functor from S-schemes to sets that takes an S-scheme T to the set of sections of $X_T \to S'_T$, i.e.,

$$\mathbf{R}_{S'/S}(X)(T) = \mathrm{Hom}_{S'}(S' \times_S T, X).$$

Similarly, if $\mathscr{X} \to S'$ is an algebraic stack, then $\mathbf{R}_{S'/S}(\mathscr{X})$ is the stack with T-points the groupoid $\mathbf{Hom}_{S'}(S' \times_S T, \mathscr{X})$ and the natural notion of pull-back. If $f\colon \mathscr{X} \to \mathscr{Y}$ is a morphism of algebraic stacks, then there is a natural morphism of stacks

$$\mathbf{R}_{S'/S}(f)\colon \mathbf{R}_{S'/S}(\mathscr{X}) \to \mathbf{R}_{S'/S}(\mathscr{Y})$$

taking a morphism $s\colon S' \times_S T \to \mathscr{X}$ to the morphism $f \circ s\colon S' \times_S T \to \mathscr{Y}$.

The Weil restriction is also sometimes denoted $\Pi_{S'/S}X$ or g_*X and is also known as *restriction of scalars*, cf. [FGA, No. 195, §C 2] and [Ols06]. The Weil restriction can also be defined on 2-morphisms so that $\mathbf{R}_{S'/S}$ becomes a strict 2-functor from the 2-category of stacks over S' to the 2-category of stacks over S. This functor is *left exact*, i.e., takes 2-fiber products to 2-fiber products and the terminal object S' to the terminal object S. Indeed, $\mathbf{R}_{S'/S}$ is a right 2-adjoint to the pull-back functor g^{-1}.

Definition (3.2). Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks. There is a natural morphism $f_*\colon \mathscr{H}^d_{\mathscr{X}} \to \mathscr{H}^d_{\mathscr{Y}}$ taking an object $(p\colon Z \to T, q\colon Z \to \mathscr{X})$ to $(p, f \circ q)$.

Definition (3.3). Let P be a property of morphisms of stacks. We say that P is *stable under base change* if for every morphism $f\colon \mathscr{X} \to \mathscr{Y}$ with P and every morphism $\mathscr{Y}' \to \mathscr{Y}$ the base change $f'\colon \mathscr{X} \times_{\mathscr{Y}} \mathscr{Y}' \to \mathscr{Y}'$ has P. We say that P *can be checked on affines*, if a morphism $f\colon \mathscr{X} \to \mathscr{Y}$ has P if and only if $f\colon \mathscr{X} \times_{\mathscr{Y}} T \to T$ has P for every affine scheme T and morphism $T \to \mathscr{Y}$.

A stable property which is fppf-local on the base can be checked on affines. An example of a non-fppf local property that can be checked on affines is the property "strongly representable", i.e., represented by schemes.

Lemma (3.4). *Let P and Q be two properties of morphisms of stacks. Assume that these properties are stable under base change and that Q is a property that can be checked on affines. Let d be a positive integer. The following are equivalent:*

(i) *If $S' \to S$ is a finite flat morphism of rank d between algebraic spaces and if $f\colon \mathscr{X} \to \mathscr{Y}$ is a morphism of algebraic stacks over S' with property P then $\mathbf{R}_{S'/S}(f)\colon \mathbf{R}_{S'/S}(\mathscr{X}) \to \mathbf{R}_{S'/S}(\mathscr{Y})$ has property Q.*

(ii) *If S is an affine scheme, if $S' \to S$ is finite flat of rank d and if $\mathscr{X} \to S'$ is a morphism of stacks with property P, then $\mathbf{R}_{S'/S}(\mathscr{X}) \to S$ has property Q.*

(iii) *If $f\colon \mathscr{X} \to \mathscr{Y}$ is a morphism of algebraic stacks over S with property P then $f_*\colon \mathscr{H}^d_{\mathscr{X}/S} \to \mathscr{H}^d_{\mathscr{Y}/S}$ has property Q.*

Proof. We have that (i) $\Longrightarrow$ (ii) since $S = \mathbf{R}_{S'/S}(S')$.

(ii) $\Longrightarrow$ (iii): Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic S-stacks, let T be an affine S-scheme and let $h\colon T \to \mathscr{H}^d_{\mathscr{Y}/S}$ be a morphism corresponding to a family $(Z \to T, Z \to \mathscr{Y})$. Then the diagram

(3.4.1)
$$\begin{tikzcd}
\mathbf{R}_{Z/T}(\mathscr{X} \times_{\mathscr{Y}} Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "h"] \\
\mathscr{H}^d_{\mathscr{X}/S} \arrow[r, "f_*"] & \mathscr{H}^d_{\mathscr{Y}/S}
\end{tikzcd}$$

is cartesian and thus (ii) $\Longrightarrow$ (iii).

(iii) $\Longrightarrow$ (i): Let $g\colon S' \to S$ be finite flat of rank d and let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic S'-stacks. Let $h\colon S \to \mathscr{H}^d_{S'/S}$ be the morphism corresponding to the family $(g, \mathrm{id}_{S'})$. Then from the cartesian diagram (3.4.1) we obtain the cartesian diagram

$$\begin{tikzcd}
\mathbf{R}_{S'/S}(\mathscr{X}) \arrow[r, "\mathbf{R}_{S'/S}(f)"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbf{R}_{S'/S}(\mathscr{Y}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \cong \mathcal{H}ilb^d_{S'/S} \arrow[d, "h"] \\
\mathscr{H}^d_{\mathscr{X}/S} \arrow[r, "f_*"] & \mathscr{H}^d_{\mathscr{Y}/S} \arrow[r] & \mathscr{H}^d_{S'/S}
\end{tikzcd}$$

and the lemma follows. $\square$

Proposition (3.5). *Let S be a scheme and let $S' \to S$ be a finite flat morphism of finite presentation. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of stacks over S'. If f has one of the properties:*

(i) *locally of finite presentation,*

(ii) *formally étale,*

(iii) *formally unramified,*

(iv) *formally smooth,*
(v) *surjective and smooth,*
(vi) *a closed immersion,*
(vii) *an open immersion,*
(viii) *an isomorphism,*
(ix) *affine;*

then so has $\mathbf{R}_{S'/S}(f)$ *(also see Proposition* (3.8) *for further properties).*

Proof. Let P be one of the properties and assume that f has P. Since P is Zariski-local, we can assume that $S' \to S$ has constant rank d. By Lemma (3.4) we can further assume that S is affine, and that $\mathscr{Y} = S'$.

Properties (i)–(v) of $\mathbf{R}_{S'/S}(\mathscr{X})$ are verified using the functorial characterization of morphisms which are locally of finite presentation [$\mathrm{EGA}_{\mathrm{IV}}$, Prop. 8.14.2], [LMB00, Prop. 4.15] and the infinitesimal criteria for formally étale, unramified and smooth maps. Property (vi) follows from [$\mathrm{EGA}_{\mathrm{I}}$, Lem. 9.7.9.1] and properties (vii) and (viii) are obvious. For (ix) it is by (vi) enough to show that $\mathbf{R}_{S'/S}(X)$ is represented by a scheme affine over S when X is the spectrum of a polynomial ring over $\mathcal{O}_{S'}$. This is straight-forward. We refer to [BLR90, §7.6, Prop. 2, pf. of Thm. 4, Prop. 5] for details. □

Theorem (3.6) ([BLR90, §7.6, Thm. 4])**.** *Let S be an affine scheme and let $g\colon S' \to S$ be a finite flat morphism of finite presentation. Let $X \to S'$ be a morphism of schemes. If X is an* AF*-scheme then $\mathbf{R}_{S'/S}(X)$ is an* AF*-scheme.*

Proof. Let $X = \bigcup_\alpha U_\alpha$ be an open cover of X by affines such that every finite subset of points of X lies in some U_α. By Proposition (3.5), we have that $\mathbf{R}_{S'/S}(U_\alpha)$ is affine and that $\mathbf{R}_{S'/S}(U_\alpha) \to \mathbf{R}_{S'/S}(X)$ is an open immersion. It is then easily seen that $\coprod_\alpha \mathbf{R}_{S'/S}(U_\alpha) \to \mathbf{R}_{S'/S}(X)$ is a Zariski covering so that $\mathbf{R}_{S'/S}(X)$ is a scheme. Moreover, $\mathbf{R}_{S'/S}(X)$ is an AF-scheme since a finite number of points in $\mathbf{R}_{S'/S}(X)$ corresponds to a morphism $S' \times_S \coprod_{i=1}^n \mathrm{Spec}(k_i) \to X$ and this factors through one of the U_α's. □

We say that a morphism of stacks $f\colon \mathscr{X} \to \mathscr{Y}$ is *algebraic* if for every affine scheme T and morphism $T \to \mathscr{Y}$ the stack $\mathscr{X} \times_{\mathscr{Y}} T$ is an algebraic stack.

Theorem (3.7). *Let $S' \to S$ be a finite flat morphism of finite presentation between algebraic spaces. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of S'-stacks.*

(i) *If f is representable, locally quasi-finite and separated then $\mathbf{R}_{S'/S}(f)$ is strongly representable.*
(ii) *If f is representable then so is $\mathbf{R}_{S'/S}(f)$.*
(iii) *If f is algebraic then so is $\mathbf{R}_{S'/S}(f)$.*

In particular, if $\mathscr{X}$ is an algebraic space (resp. an algebraic stack) then so is $\mathbf{R}_{S'/S}(\mathscr{X})$.

Proof. By Lemma (3.4) we can assume that S is affine and $\mathscr{Y} = S'$. If f is as in (i), then $\mathscr{X}$ is an AF-scheme by Remark (2.3) so that $\mathbf{R}_{S'/S}(\mathscr{X})$ is represented by a scheme according to Theorem (3.6).

(ii) If f is representable, then $\mathscr{X}$ is an algebraic space. Choose an étale presentation $U \to \mathscr{X}$ with U an AF-scheme (e.g., a disjoint union of affine schemes). By Theorem (3.6) we have that $\mathbf{R}_{S'/S}(U)$ is a scheme. Furthermore, by Proposition (3.5) and (i) we have that $\mathbf{R}_{S'/S}(U) \to \mathbf{R}_{S'/S}(\mathscr{X})$ is étale, surjective and strongly representable. Thus, by definition $\mathbf{R}_{S'/S}(\mathscr{X})$ is an algebraic space.

(iii) If f is algebraic, then $\mathscr{X}$ is an algebraic stack. Choose a smooth presentation $U \to \mathscr{X}$ with U an algebraic space. Then $\mathbf{R}_{S'/S}(U)$ is an algebraic space by (ii) and $\mathbf{R}_{S'/S}(U) \to \mathbf{R}_{S'/S}(\mathscr{X})$ is smooth, surjective and representable by Proposition (3.5) and (ii). Thus $\mathbf{R}_{S'/S}(\mathscr{X})$ is an algebraic stack. $\square$

We now complement Proposition (3.5) with some additional properties.

Proposition (3.8). *Let S be a scheme and let $S' \to S$ be a finite flat morphism of finite presentation. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks over S' so that $\mathbf{R}_{S'/S}(f)$ is a morphism of algebraic stacks over S. If f has one of the properties:*

- (x) *representable,*
- (xi) *locally of finite type,*
- (xii) *quasi-compact,*
- (xiii) *quasi-affine,*
- (xiv) *of finite type,*
- (xv) *of finite presentation,*
- (xvi) *monomorphism,*
- (xvii) *representable and separated,*
- (xviii) *quasi-separated,*
- (xix) *separated diagonal,*
- (xx) *affine diagonal,*
- (xxi) *quasi-affine diagonal,*
- (xxii) *unramified diagonal (i.e., relatively Deligne–Mumford);*

then so has $\mathbf{R}_{S'/S}(f)$.

Proof. As before we can assume that S is an affine scheme and that $\mathscr{Y} = S'$. Property (x) is Theorem (3.7) (ii).

(xi) Take a smooth surjective morphism $U \to \mathscr{X}$ such that U is a disjoint union of affine schemes. If f is locally of finite type, then $U \to S'$ factors through a closed immersion $U \hookrightarrow W$ and a morphism $W \to S'$ which is locally of finite presentation. Thus by (vi), (i) and (v), it follows that $\mathbf{R}_{S'/S}(\mathscr{X})$ is locally of finite type.

Similarly, for property (xii) take a smooth surjective morphism $U \to \mathscr{X}$ with U affine and the quasi-compactness of $\mathbf{R}_{S'/S}(\mathscr{X})$ follows from (ix). Property (xiii) is the conjunction of properties (vii), (ix) and (xii).

As $\mathbf{R}_{S'/S}$ preserves fiber products we have that $\mathbf{R}_{S'/S}(\Delta_f) = \Delta_{\mathbf{R}_{S'/S}(f)}$. Property (xvi) [resp. (xvii)] is equivalent to the diagonal being an isomorphism [resp. a closed immersion]. Property (xviii) is equivalent to the quasi-compactness of the diagonal and its diagonal. Properties (xvi)–(xxii) thus follows from applying the Proposition to the diagonal Δ_f and its diagonal Δ_{Δ_f} with the properties (iii), (vi), (viii), (ix), (xii), (xiii).

Finally, properties (xiv) and (xv) follow from properties (i), (xi), (xii) and (xviii). $\square$

Remark **(3.9).** If $S' \to S$ is finite and étale then Proposition (3.5) also holds for the properties "proper", "flat" and "separated" [BLR90, §7.6, Prop. 5].

Example (3.10). Proposition (3.5) does not hold for the property "proper" nor for the property "finite and étale". In fact, let S be arbitrary and let $S' \to S$ be a finite flat ramified cover of degree d. Then $\mathbf{R}_{S'/S}(S' \amalg S') \to S$ is étale and has generic rank 2^d but has lower rank over the branch locus of $S' \to S$. Thus $\mathbf{R}_{S'/S}(S' \amalg S') \to S'$ is not proper.

Similarly if $\mathscr{X}$ is a separated algebraic stack, i.e., has proper diagonal, then $\mathbf{R}_{S'/S}(\mathscr{X})$ need not be separated unless S'/S is étale. For example, let S'/S be a finite flat ramified covering of degree d as before and let $\mathscr{X} = \mathrm{B}G(S')$ where G is a finite constant group. Then $\mathbf{R}_{S'/S}(\mathscr{X})$ is an étale gerbe over S with generic geometric automorphism group G^d but with automorphism group of lower rank over the points of S where S'/S is ramified.

Theorem (3.11). *Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of algebraic stacks.*

(i) *The morphism $f_*\colon \mathscr{H}^d_X \to \mathscr{H}^d_Y$ is algebraic.*
(ii) *If f is representable, locally quasi-finite and separated, then f_* is strongly representable.*
(iii) *If f has one of the properties* (i)–(xxii) *of Propositions* (3.5) *and* (3.8)*, then so has f_*.*

Proof. This is the conjunction of Lemma (3.4), Theorem (3.7) and Propositions (3.5) and (3.8). □

Theorem (3.12). *Let $X \to S$ be a finite flat morphism of finite presentation and let $\mathscr{Y} \to S$ be an arbitrary algebraic stack. Then the stack $\mathscr{H}om_S(X, \mathscr{Y})$ is algebraic. If $\mathscr{Y} \to S$ has one of the properties in Propositions* (3.5) *and* (3.8)*, then so has $\mathscr{H}om_S(X, \mathscr{Y}) \to S$.*

Proof. As $\mathscr{H}om_S(X, \mathscr{Y}) = \mathbf{R}_{X/S}(X \times_S \mathscr{Y})$ this follows immediately from Theorem (3.7). □

Remark **(3.13).** The argument in [Ols06, §3.3] shows that Theorem (3.12) remains true if X is an *algebraic stack* and $X \to S$ is proper, quasi-finite and flat of finite presentation. Indeed, any such stack admits, étale-locally on S, a finite flat presentation.

4. Algebraicity of the Hilbert functor and the Hilbert stack

Let $f\colon \mathscr{U} \to \mathscr{X}$ be a morphism of separated algebraic stacks. Let $\mathcal{H}ilb^d_{\mathscr{U}\to\mathscr{X}} \subseteq \mathcal{H}ilb^d_{\mathscr{U}}$ be the subfunctor parameterizing families $Z \hookrightarrow \mathscr{U} \times_S T$ such that the composition $Z \hookrightarrow \mathscr{U} \times_S T \to \mathscr{X} \times_S T$ is a closed immersion. Then

$$\begin{tikzcd}
\mathcal{H}ilb^d_{\mathscr{U}\to\mathscr{X}} \arrow[r, "f_*"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{H}ilb^d_{\mathscr{X}/S} \arrow[d] \\
\mathscr{H}^d_{\mathscr{U}/S} \arrow[r, "f_*"] & \mathscr{H}^d_{\mathscr{X}/S}
\end{tikzcd}$$

is cartesian and the two vertical morphisms are open immersions.

Theorem (4.1). *Let $\mathscr{X}/S$ be a* separated *algebraic stack. Then $\mathcal{H}ilb^d_{\mathscr{X}/S}$ is a separated algebraic space.*

Proof. We can assume that S is affine. Let $f\colon U = \coprod_\alpha U_\alpha \to \mathscr{X}$ be a smooth presentation such that the U_α's are affine. Then U is an AF-scheme and $\mathcal{H}ilb^d_{U/S}$ is represented by a scheme according to Theorem (2.4). As f is representable, smooth and surjective, so is $f_*\colon \mathcal{H}ilb^d_{U\to\mathscr{X}} \to \mathcal{H}ilb^d_{\mathscr{X}/S}$ by Theorem (3.11). Thus $\mathcal{H}ilb^d_{\mathscr{X}/S}$ is an algebraic space. That $\mathcal{H}ilb^d_{\mathscr{X}/S}$ is separated is Lemma (2.1). □

As for the Hilbert functor, the algebraicity of the Hilbert stack will be an immediate consequence of Theorem (3.11) after we have verified that the Hilbert stack of an affine scheme is algebraic.

In the finitely presented case, the following results follow from the more general results of [Lie06, §2.1]. In the affine case treated below, the proofs are a matter of elementary algebra.

Lemma (4.2). *Let B be an A-algebra and let M be a locally free A-module of finite rank. Then there is an A-algebra Q which represents B-algebra structures on M. That is, for every A-algebra A', there is a functorial one-to-one correspondence between $B' = B \otimes_A A'$-algebra structures on $M' = M \otimes_A A'$ and homomorphisms $Q \to A'$. If B is an A-algebra of finite type (resp. of finite presentation), then so is Q.*

Proof. A B'-algebra structure on M' is given by multiplication maps $\mu\colon M'\otimes_{A'} M' \to M'$, $m\colon B' \otimes_{A'} M' \to M'$, and a unit $\eta\colon A' \to M'$. Such triples of maps correspond to A-module homomorphisms

$$(M \otimes_A M \otimes_A M^\vee) \oplus (B \otimes_A M \otimes_A M^\vee) \oplus M^\vee \to A'$$

and are thus represented by the symmetric algebra

$$P := \operatorname{Sym}\big((M \otimes_A M \otimes_A M^\vee) \oplus (B \otimes_A M \otimes_A M^\vee) \oplus M^\vee\big).$$

That the multiplication μ is commutative, associative and compatible with m and η can be expressed as the vanishing of the A'-homomorphisms

$$\begin{aligned}
\mu - \mu\circ\tau \ &: M' \otimes_{A'} M' \to M' \\
\mu\circ(\mu\otimes \mathrm{id}_{M'}) - \mu\circ(\mathrm{id}_{M'}\otimes\mu) \ &: M' \otimes_{A'} M' \otimes_{A'} M' \to M' \\
\mu\circ(m\otimes \mathrm{id}_{M'}) - m\circ(\mathrm{id}_{B'}\otimes\mu) \ &: B' \otimes_{A'} M' \otimes_{A'} M' \to M' \\
\mu\circ(\eta\otimes \mathrm{id}_{M'}) - \mathrm{id}_{M'} \ &: M' \to M'
\end{aligned}$$

where $\tau\colon M' \otimes_{A'} M' \to M' \otimes_{A'} M'$ swaps the two factors. This vanishing is represented by a quotient Q of P according to [$\mathrm{EGA_I}$, Lem. 9.7.9.1].

If B is of finite type, then clearly so is P and hence Q. If B is of finite presentation we use a limit argument to reduce to the noetherian case and it follows that Q is of finite presentation. □

Theorem (4.3). *Let X and S be affine schemes. Then $\mathscr{H}^d_{X/S}$ is a quasi-compact algebraic stack with affine diagonal. If X/S is of finite type (resp. of finite presentation) then so is $\mathscr{H}^d_{X/S}$.*

Proof. There is a natural morphism $\mathscr{H}^d_{X/S} \to \mathrm{BGL}_d(S)$ which maps (Z, p, q) to the locally free $\mathcal{O}_T$-module $p_*\mathcal{O}_Z$. The stack $\mathrm{BGL}_d(S)/S$ is a finitely presented algebraic stack with affine diagonal. Lemma (4.2) shows that $\mathscr{H}^d_{X/S} \to \mathrm{BGL}_d(S)$ is represented by affine morphisms and the theorem follows. $\square$

Theorem (4.4). *Let $\mathscr{X}/S$ be an algebraic stack. Then $\mathscr{H}^d_{\mathscr{X}/S}$ is algebraic. If $\mathscr{X}/S$ has one of the properties: quasi-compact, quasi-separated, locally of finite presentation, locally of finite type, separated diagonal, affine diagonal, quasi-affine diagonal; then so has $\mathscr{H}^d_{\mathscr{X}/S} \to S$.*

Proof. By Theorem (4.3), we have that $\mathscr{H}^d_{S/S}$ is an algebraic stack over S of finite presentation and with affine diagonal. The algebraicity of $\mathscr{H}^d_{\mathscr{X}/S}$ thus follows from Theorem (3.11) (i). The properties of $\mathscr{H}^d_{\mathscr{X}/S} \to S$ follows from Theorem (3.11) (iii). $\square$

Note that $\mathscr{H}^d_{\mathscr{X}/S}$ is not Deligne–Mumford nor has quasi-finite diagonal. On the other hand, it can be seen that the open substack $\mathscr{H}^{d,\mathrm{unram}}_{\mathscr{X}/S}$ is Deligne–Mumford.

5. Étale families

Let S be a scheme and let X/S be an algebraic space. The *stack of branchvarieties* [AK10] on X is the open substack of the Hilbert stack $\mathscr{H}_{X/S}$ parameterizing families $(p\colon Z \to T, q\colon Z \to X)$ such that the geometric fibers of p are reduced. For zero-dimensional families of rank d, this is the open substack $\mathscr{E}t^d_{X/S}$ of $\mathscr{H}^d_{X/S}$ parameterizing étale families of rank d. If X/S is separated, it is natural to also study the subspace $\mathrm{ET}^d_{X/S}$ parameterizing étale families $Z \to T$ of rank d such that $Z \to X \times_S T$ is a closed immersion. Following the notation of [LMB00, Cons. 6.6] we let $\mathrm{SEC}^d_{X/S}$ be the open subset of $(X/S)^d = X \times_S \cdots \times_S X$ which is the complement of the diagonals. The symmetric group $\mathfrak{S}_d$ acts by permutations on $(X/S)^d$ and this action is free over $\mathrm{SEC}^d_{X/S}$. We have the following descriptions of these stacks:

Theorem (5.1). *Let X/S be an arbitrary algebraic space. There is a natural isomorphism $\mathscr{E}t^d_{X/S} \to [(X/S)^d/\mathfrak{S}_d]$. If X/S is separated, then the open substack $\mathrm{ET}^d_{X/S}$ is identified with the algebraic space $\mathrm{SEC}^d_{X/S}/\mathfrak{S}_d$.*

Proof. We will construct canonical morphisms in both directions. Let $(p\colon Z \to T, q\colon Z \to X)$ be a T-point of $\mathscr{E}t^d_{X/S}$. The scheme $(Z/T)^d$ is étale of rank d^d over T. The diagonals of this scheme are open and closed, and their complement $\mathrm{SEC}^d_{Z/T}$ is étale of rank $d!$. This can be verified over algebraically closed points where it is trivial. The scheme $\mathrm{SEC}^d_{Z/T}$ is an $\mathfrak{S}_d$-torsor and comes with an $\mathfrak{S}_d$-equivariant morphism $\mathrm{SEC}^d_{Z/T} \to (X/S)^d$. This defines a T-point of $[(X/S)^d/\mathfrak{S}_d]$.

Conversely, let $W \to T$ be a T-point of $[(X/S)^d/\mathfrak{S}_d]$, i.e., let W/T be an $\mathfrak{S}_d$-torsor together with an $\mathfrak{S}_d$-equivariant morphism $W \to (X/S)^d$. Let $\mathfrak{S}_{d-1}$ be the subgroup of $\mathfrak{S}_d$ acting by permuting the first $d-1$ factors of $(X/S)^d$. This group acts freely on W and the quotient $Z = W/\mathfrak{S}_{d-1}$ is an algebraic space, étale of rank d over T. Moreover, the composition of $W \to (X/S)^d$ with the last projection is $\mathfrak{S}_{d-1}$-invariant and induces a morphism $Z \to X$. We have thus constructed a T-point of $\mathscr{E}t^d_{X/S}$.

It is clear that these constructions are functorial and thus defines morphisms $F\colon \mathscr{E}t^d_{X/S} \to [(X/S)^d/\mathfrak{S}_d]$ and $G\colon [(X/S)^d/\mathfrak{S}_d] \to \mathscr{E}t^d_{X/S}$. It is not difficult to show that these are (quasi-)inverses. In fact, if (Z,p,q) is a T-point of $\mathscr{E}t^d_{X/S}$, then we have a canonical morphism $\mathrm{SEC}^d_{Z/T}/\mathfrak{S}_{d-1} \to Z$ and that this is an isomorphism can be checked over algebraically closed points. Conversely, if W/T is an $\mathfrak{S}_d$-torsor, then we obtain a morphism $W \to ((W/\mathfrak{S}_{d-1})/T)^d$ where the i^{th} factor is the composition of $\tau_{in}\colon W \to W$ and the quotient $W \to W/\mathfrak{S}_{d-1}$. Again, it is easily verified that $W \to ((W/\mathfrak{S}_{d-1})/T)^d$ induces an isomorphism of W onto $\mathrm{SEC}^d_{(W/\mathfrak{S}_{d-1})/T}$. □

Remark **(5.2).** The universal $\mathfrak{S}_d$-torsor of $[(X/S)^d/\mathfrak{S}_d]$ is $(X/S)^d$. The above isomorphism shows that $[(X/S)^d/\mathfrak{S}_{d-1}] = [(X/S)^{d-1}/\mathfrak{S}_{d-1}] \times_S X$ is the universal étale rank d family on $\mathscr{E}t^d_{X/S}$.

Appendix A. Algebraic spaces and stacks

A sheaf of sets F on the category of schemes **Sch** with the étale topology is an *algebraic space* if there exists a scheme X and a morphism $X \to F$ which is represented by surjective étale morphisms of schemes [RG71, Déf. 5.7.1], i.e., for any scheme T and morphism $T \to F$, the fiber product $X \times_F T$ is a scheme and $X \times_F T \to T$ is surjective and étale. A *stack* $\mathscr{X}$ is a category fibered in groupoids over **Sch** with the étale topology satisfying the usual sheaf condition [LMB00, Déf. 3.1].

A morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of stacks is *representable* if for any scheme T and morphism $T \to \mathscr{Y}$, the 2-fiber product $\mathscr{X} \times_{\mathscr{Y}} T$ is an algebraic space. A stack $\mathscr{X}$ is *algebraic* if there exists a smooth presentation, i.e., a smooth, surjective and representable morphism $U \to \mathscr{X}$ where U is a scheme (or algebraic space). A morphism $f\colon \mathscr{X} \to \mathscr{Y}$ of stacks is *quasi-separated* if the diagonal $\Delta_{\mathscr{X}/\mathscr{Y}}$ is quasi-compact and quasi-separated, i.e., if both $\Delta_{\mathscr{X}/\mathscr{Y}}$ and its diagonal are quasi-compact.

We do not require that algebraic spaces and stacks are quasi-separated nor that the diagonal of an algebraic stack is separated. The diagonal of a (not necessarily quasi-separated) algebraic space is represented by schemes. This follows by effective fppf-descent of monomorphisms which are locally of finite type. Indeed, more generally the class of locally quasi-finite and separated morphisms is an effective class in the fppf-topology (cf. [Mur66, App.], [SGA$_3$, Exp. X, Lem. 5.4] or [RG71, pf. of 5.7.2]). The diagonal of an algebraic stack $\mathscr{X}$ is representable. This follows by [LMB00, pf. of Prop. 4.3.1] as [LMB00, Cor. 1.6.3] generalizes to arbitrary algebraic spaces.

In particular, if X is an algebraic space (resp. an algebraic stack) and S and T are schemes (resp. algebraic spaces) with morphisms $S \to X$ and $T \to X$, then $S \times_X T$ is a scheme (resp. an algebraic space).

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