FUNCTIONAL FLATIFICATION OF PROPER MORPHISMS

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Abstract. For proper morphisms, we give a functorial flatification algorithm by blow-ups in the spirit of Hironaka's flatification algorithm. In characteristic zero, this gives functorial flatification by blow-ups in smooth centers. We also give a functorial étalification algorithm by Kummer blow-ups in characteristic zero.

Introduction

The celebrated flatification theorem of Raynaud–Gruson [RG71, Thm. 5.2.2] states that any morphism $f\colon X \to S$ of finite type between schemes can be flatified by a sequence of blow-ups. This sequence depends on many choices and is highly non-canonical. Extending this result to the case where S is an algebraic space or an algebraic stack is therefore non-trivial. This was nevertheless accomplished for algebraic spaces by Raynaud–Gruson [RG71, Thm. 5.7.9], for stacks with finite stabilizers in [Ryd11b] and for arbitrary stacks in [Ryd16b].

Shortly after Raynaud–Gruson's result, Hironaka gave a different proof of the theorem when f is a *proper* morphism of analytic spaces [Hir75]. Hironaka's proof is long and complicated but the basic idea is simple. Inspired by his ideas, we give the following *functorial* flatification result:

Theorem A. *Let $f\colon X \to S$ be a proper morphism of noetherian schemes. Let $U \subseteq S$ be the largest open substack such that $f|_U$ is flat. Then there exists a sequence of blow-ups $\widetilde{S} \to S$ with centers disjoint from U such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{S}$ is flat. Moreover, this sequence is* functorial *with respect to flat morphisms $S' \to S$ between noetherian schemes.*

Using functorial embedded resolution of singularities [BM08], we immediately obtain the following smooth variant:

Theorem B. *Let k be a field of characteristic zero and let S be a smooth k-scheme of finite type. Let $f\colon X \to S$ be a proper morphism. Let $U \subseteq S$ be the largest open substack such that $f|_U$ is flat. Then there exists a sequence of blow-ups $\widetilde{S} \to S$ with* smooth *centers disjoint from U such that the strict transform $\widetilde{f}\colon \widetilde{X} \to \widetilde{S}$ is flat. Moreover, this sequence is* functorial *with respect to smooth morphisms $S' \to S$ of finite type.*

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By functoriality, these results immediately extend to algebraic stacks. They also extend to proper non-representable morphisms and even to non-separated universally closed morphisms (e.g., good moduli space morphisms). There is also functorial flatification of coherent sheaves of $\mathcal{O}_X$-modules. See Theorem 1.1 for the precise statement.

A very important use case is flatification of *modifications*, i.e., representable proper birational maps. The key point is that a flat modification is an isomorphism. For *stack-theoretic modifications*, i.e., non-representable proper birational maps, this is no longer the case. Prominent examples are root stacks. But an étale stack-theoretic modification is an isomorphism.

To make a flat ramified map étale, we need something more than blow-ups. In this paper, we only treat étalification in characteristic zero and then *Kummer blow-ups* are enough. A Kummer blow-up is a blow-up followed by a *root stack* along the exceptional divisor. In addition, we only treat the smooth case:

Theorem C (Functorial étalification of proper morphisms in characteristic zero)**.** *Let S be a noetherian algebraic stack, smooth over a field of characteristic zero. Let $f\colon X \to S$ be a proper morphism with finite diagonal. Let $U \subseteq S$ be the largest open substack such that $f|_U$ is étale. Then there exists a commutative diagram*

$$\begin{tikzcd} \widetilde{X} \arrow[r, "q"] \arrow[d, "\widetilde{f}"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "f"] \\ \widetilde{S} \arrow[r, "p"] & S \end{tikzcd}$$

where $\widetilde{f}$ is étale, p is a sequence of Kummer blow-ups with smooth centers disjoint from U and q is a sequence of Kummer blow-ups with centers disjoint from $f^{-1}(U)$. Moreover, these sequences are functorial *with respect to smooth morphisms $S' \to S$ of finite type.*

Applications. Our first application is that any (stack-theoretic) modification becomes a (Kummer) blow-up after replacing the source with a (Kummer) blow-up and this procedure is functorial (Theorems 3.1 and 3.2).

Our second application is that the indeterminacy locus of a birational map $f\colon X \dashrightarrow Y$ has a functorial resolution by (Kummer) blow-ups when the target is proper. Resolving the indeterminacy locus reduces the problem of weak factorization of f to the situation where f is a sequence of blow-ups. The latter situation can be solved functorially, resulting in a proof of weak factorization of birational morphisms of Deligne–Mumford stacks in characteristic zero that is completely functorial [Ryd15]. It is also likely that the algorithm in [AKMW02] becomes functorial if Hironaka's flatification theorem is replaced with ours.

Our third application is a general Chow lemma (Theorem 3.5).

On the proofs. For *projective* morphisms there is an easy functorial flatification algorithm. There is a canonical stratification of S in locally closed substacks S_P indexed by Hilbert polynomials. If one blows up $\overline{S_P}$ then the non-flatness over S_P improves (Lemma 1.18). We obtain a functorial algorithm by blowing up the $\overline{S_P}$ starting with the largest P and continuing in decreasing order.

In general, there is also a minimal modification that flatifies $X \to S$. This is obtained by taking the closure of U in the Hilbert scheme $\mathrm{Hilb}(X/S)$ (or the Quot scheme $\mathrm{Quot}(\mathcal{F}/X/S)$ for coherent sheaves). This, however, only gives a flattening modification, not a sequence of blow-ups.

Hironaka's algorithm uses the following three key facts:

(i) There is a canonical *filtration* of S in open subschemes such that f is flat over the reduced strata. This follows from generic flatness since S is noetherian (Remark 1.4). In the complex-analytic setting this is a theorem of Frisch.

(ii) Étale-locally around any point $s \in S$, there is a maximal closed subscheme Z of S passing through s with the following universal property: $f|_Z$ is flat and any morphism $g\colon S' \to S$ from a connected scheme S' such that $f_{S'}$ is flat and $s \in g(S')$, factors through Z.

(iii) Blowing up Z as in (ii) improves the non-flatness at every point of the inverse image of Z.

Raynaud–Gruson proves (ii) for schemes using their method of dévissage [RG71, Thms. 4.1.2]. Hironaka proves (ii) and (iii) for analytic spaces using pseudo-free presentations which is a complex-analytic analogue of the dévissage method [Hir75, Thm. 2.4]. In our treatment, we deduce (ii) from the existence of the universal flattening (Proposition 1.5) without dévissage. To establish (iii), however, dévissage is used (Lemma 1.18).

The ultimate goal of both Hironaka's algorithm and our algorithm is to make Z globally defined so that it can be blown-up. We do this by "resolving the flattening monomorphism" (Proposition 1.12) and this uses the canonical flattening filtration. Hironaka accomplishes this in [Hir75, §4], also using the canonical flattening filtration but with a more complicated algorithm. His blow-ups are in *permissible* centers D. In particular, $f|_D$ is flat whereas this is essentially never the case in our algorithm (Example 1.13). Hironaka also assumes that S is reduced.

Functorial étalification is proven in Section 2 and follows from functorial flatification and the generalized Abhyankar lemma.

Remarks.

Dévissage and flatness. The dévissage of Raynaud–Gruson is a somewhat complicated machinery. We use dévissage to give a very simple characterization of flatness (Theorem 1.14) that could be of independent interest.

Functoriality of Hironaka's algorithm. In [AKMW02, 1.2.4], the authors remark that Hironaka's algorithm does not depend on any choices and is invariant under isomorphisms but is not functorial with respect to smooth morphisms, nor under localizations. A crucial point where it is seems that the algorithm is not functorial with respect to localizations is when picking $\beta \geq \alpha$ to form the center D_β [Hir75, p. 542 and Lem. 3.2].

Hironaka's example of a non-projective threefold (Example 1.6) gives rise to a non-projective but locally projective birational morphism $f\colon X \to Y$.

The naïve flatification algorithm for projective morphisms, using flattening stratifications, does not glue to a flatification algorithm for f. However, even in the projective case, Hironaka's algorithm does not equal the naïve flatification algorithm (Example 1.13). Thus, this example does not prove that Hironaka's algorithm is not functorial (as [AKMW02, 1.2.4] seems to suggest).

Functorial étalification for singular stacks. In [Ryd11b, Ryd16b], étalification is proved in characteristic zero and also in positive characteristic under tameness assumptions. There is no smoothness assumption but the algorithm is highly non-functorial. It is natural to ask whether there is a functorial étalification algorithm for proper maps without the smoothness assumption. Ideally, one would have something that guides the algorithm analogously to the universal flattening.

Functorial étalification in positive characteristic. The author has proved étalification in positive characteristic using Artin–Schreier stacks [Ryd12]. Again, this is highly non-functorial and one could ask whether there is a functorial algorithm. This seems very difficult.

Noetherian assumption. This article is throughout written in the context of noetherian schemes and stacks. Although the majority of the paper is true in the non-noetherian setting under suitable finiteness assumptions, there is one crucial place where the noetherian assumption is necessary. In the noetherian setting we have a *canonical* flattening filtration (Remark 1.4). In the non-noetherian setting, we have non-canonical flattening quasi-compact filtrations, see [Ryd16a, Thm. 8.3]. These can be obtained by pulling-back the canonical flattening filtration of a noetherian approximation but are not unique. This is essentially due to the fact that the map $S \to S_0$ from a non-noetherian scheme to a noetherian approximation S_0 is not flat. Similarly, we cannot deduce a non-noetherian version of the main theorem by noetherian approximation since the result is only functorial with respect to flat morphisms. In fact, we give a non-noetherian example for which there cannot be a functorial flattening algorithm by blow-ups (Example 1.3).

Comments. This paper was mostly written in 2015 after reading Hironaka's paper [Hir75] with the purpose of obtaining *functorial* weak factorization of Deligne–Mumford stacks [Ryd15]. The paper was then finished almost ten years later.

There is a recent preprint by Michael McQuillan [McQ24], also inspired by Hironaka's paper, which gives a smooth-functorial flatification similar to Theorem A, also including formal stacks. The proofs are similar but McQuillan uses a "minimal compactification" of the flattening monomorphism instead of the étale envelope and étale dévissage employed in the proof of Proposition 1.12.

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1. Functorial flatification

Let $f\colon X\to S$ be a morphism of finite type between noetherian algebraic stacks and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let U be the maximal open substack of S such that $\mathcal{F}|_{f^{-1}(U)}$ is U-flat. If S is reduced, then U is dense. If S is non-reduced, it may happen that $U=\emptyset$. The *f-torsion* of $\mathcal{F}$ is the kernel of the adjunction morphism $\mathcal{F}\to j_*j^*\mathcal{F}$ where $j\colon f^{-1}(U)\to X$ is the inclusion. We will frequently let $\widetilde{\mathcal{F}}$ denote $\mathcal{F}$ modulo its f-torsion. When S is a smooth curve, then $\widetilde{\mathcal{F}}$ is S-flat. If $S'\to S$ is a morphism, then we let $\mathcal{F}\times_S S'$ denote the pull-back of $\mathcal{F}$ along $X\times_S S'\to X$. The strict transform of $\mathcal{F}$ is $\widetilde{\mathcal{F}\times_S S'}$, that is, the sheaf $\mathcal{F}\times_S S'$ modulo its f'-torsion where $f'\colon X'\to S'$ is the pull-back of f.

Theorem (1.1). *Let S be a noetherian algebraic stack, let $f\colon X\to S$ be a universally closed morphism of finite type (e.g., proper) and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $U\subseteq S$ be the largest open substack such that $\mathcal{F}|_{f^{-1}(U)}$ is U-flat. Then there exist a sequence of blow-ups $S'\to S$ with centers disjoint from U such that the strict transform of $\mathcal{F}$ is S'-flat. This sequence is functorial with respect to flat morphisms $S'\to S$ between noetherian algebraic stacks.*

If in addition S is smooth over a field of characteristic zero, then the centers can be taken to be smooth and this sequence is functorial with respect to smooth morphisms.

Remark **(1.2).** A natural generalization of the latter part of the theorem would be to allow S to be a regular and quasi-excellent $\mathbb{Q}$-stack and obtain a sequence of blow-ups in regular centers, functorial with respect to regular morphisms. Temkin has proven functorial desingularization of quasi-excellent $\mathbb{Q}$-stacks [Tem12, Tem18] but unfortunately he has not proven strong principalization [Tem18, §1.1.10] which would be required to deduce this generalization.

Example (1.3) (Non-noetherian counter-example)**.** Let S be the scheme $\operatorname{Spec} k[x,y_1,y_2,\dots]/(xy_1,xy_2,\dots)$ which has two irreducible components, $V(x)$ and $V(y_1,y_2,\dots)$, meeting in a point P. Let $X=V(x)\hookrightarrow S$ be the inclusion of the first component which is of finite presentation. Then $\mathcal{O}_X$ is flat over $U=S\smallsetminus P$. Any functorial algorithm is necessarily U-admissible. But there are no finitely generated ideals I such that $V(I)=P$, hence no U-admissible blow-ups.

1.1. Universal flattening and flattening stratifications.

A crucial ingredient for the proof is the *universal flattening* (also called the universal flatificator) of $\mathcal{F}$, a monomorphism $\operatorname{Flat}_{\mathcal{F}/S}\to S$ that universally "makes $\mathcal{F}$ flat". This has appeared many times, e.g., in [Mur66, Thm. 2], [RG71, Thms. 4.1.2 & 4.3.1], [OS03, §3, Eqn. 13], [Kre13] and [SP, 05MW, 05UH].

For f and $\mathcal{F}$ as in Theorem 1.1, consider the functor $\operatorname{Flat}_{\mathcal{F}/S}\colon \mathbf{Sch}_{/S}^{\mathrm{op}}\to \mathbf{Set}$ such that $\operatorname{Flat}_{\mathcal{F}/S}(T)$ is the one-point set if $\mathcal{F}\times_S T$ is flat over T and the empty set otherwise. This means that a morphism $T\to S$ factors through $\operatorname{Flat}_{\mathcal{F}/S}$ if and only if $\mathcal{F}\times_S T$ is flat over T and then the factorization is unique. The functor $\operatorname{Flat}_{\mathcal{F}/S}$ is a *bijective monomorphism* $\operatorname{Flat}_{\mathcal{F}/S}\hookrightarrow S$, that is,

$$\mathrm{Flat}_{\mathcal{F}/S}(T) \to \mathrm{Mor}_S(T,S) = \{*\}$$

is injective for every S-scheme T, and bijective if T is the spectrum of a field.

Remark **(1.4).** There is always a canonical *flattening filtration*. This is a sequence of open subsets $U_0 = \emptyset \subset U_1 \subset U_2 \subset \cdots \subset U_n = |S|$ such that $\mathcal{F}$ is flat over $(U_i \smallsetminus U_{i-1})_{\mathrm{red}}$. Given U_i, we define U_{i+1} as follows. Let $Z_i = (S \smallsetminus U_i)_{\mathrm{red}}$ and let $V \subseteq Z_i$ be the largest open subset such that $\mathcal{F} \times_S V$ is flat over V. Then let $U_{i+1} = V \cup U_i$. Since S is noetherian, it has the ascending chain condition on open subsets so the sequence is finite.

The flattening filtration gives rise to a canonical reduced *flattening stratification* $T = \coprod_{i=1}^{n} (U_i \smallsetminus U_{i-1})_{\mathrm{red}}$ of S.

Proposition (1.5). *$\mathrm{Flat}_{\mathcal{F}/S} \to S$ is representable and of finite type.*

Proof. We give two different proofs of representability.

The first proof requires that f is separated, that is, **proper**. We have a natural map $\mathrm{Flat}_{\mathcal{F}/S} \to \mathrm{Quot}(\mathcal{F}/X/S)$ taking T to the quotient $\mathcal{F} \times_S T \xrightarrow{\cong} \mathcal{F} \times_S T$ with trivial kernel. This is an open subfunctor: if T is any S-scheme and $\varphi\colon \mathcal{F} \times_S T \twoheadrightarrow \mathcal{G}$ is a surjection onto a finitely presented sheaf $\mathcal{G}$ that is flat over T then the subfunctor of S where φ becomes an isomorphism is open. The Quot-functor is represented by an algebraic space that is separated and locally of finite presentation over S, for any proper morphism $X \to S$ [Ols05, Thm. 1.5]. This proves that $\mathrm{Flat}_{\mathcal{F}/S} \to S$ is representable and locally of finite presentation.

For **non-separated** f, the Quot functor is not representable so a different proof is required. We will use Murre's representability theorem. The question is local on S so we can assume that S is a scheme. Let $p\colon X' \to X$ be a smooth surjective morphism from a scheme. Then $\mathrm{Flat}_{\mathcal{F}/S} = \mathrm{Flat}_{p^*\mathcal{F}/S}$ so [Mur66, Thm. 2] applies and states that $\mathrm{Flat}_{\mathcal{F}/S}$ is representable if and only if $\mathrm{Flat}_{\mathcal{F}/S}(\operatorname{Spec} A) \to \varprojlim_n \mathrm{Flat}_{\mathcal{F}/S}(\operatorname{Spec} A/\mathfrak{m}^n)$ is bijective for every complete local noetherian ring A with maximal ideal $\mathfrak{m}$. Suppose that the right hand side is non-empty. Then, by the local criterion of flatness, $\mathcal{F} \times_S \operatorname{Spec} A$ is A-flat at every point of the special fiber, hence in an open neighborhood of the special fiber. Since f is universally closed, every open neighborhood of the special fiber is $X \times_S \operatorname{Spec} A$ so $\mathrm{Flat}_{\mathcal{F}/S}(\operatorname{Spec} A)$ is non-empty.

To see that $\mathrm{Flat}_{\mathcal{F}/S} \to S$ is quasi-compact, we consider the flattening stratification $T \to S$ of Remark 1.4. This gives rise to a surjection $T \to \mathrm{Flat}_{\mathcal{F}/S}$ so $\mathrm{Flat}_{\mathcal{F}/S}$ is quasi-compact since T is a finite union of locally closed subschemes. □

When f is *projective* with a specified ample line bundle, taking the Hilbert polynomials of the fibers $\mathcal{F}|_s$ is an upper-semicontinuous function on S. If $T \to S$ is a morphism with T reduced, then $\mathcal{F} \times_S T$ is flat over T if and only if the Hilbert polynomial is locally constant. Thus, we have a filtration of open substacks $S_{\leq P} \subseteq S$ such that f is flat over the induced reduced strata $(S_P)_{\mathrm{red}} = S_{\leq P} \smallsetminus S_{<P}$. Moreover, it follows that the universal flattening is a disjoint union of locally closed substacks S_P indexed by Hilbert polynomials P. Note that the flattening filtration $S_{\leq P}$ need not be compatible with the canonical flattening filtration, see Example 1.13.

The following examples show that, in general, the universal flattening $\mathrm{Flat}_{\mathcal{F}/S} \to S$ is not a disjoint union of locally closed subschemes, contrary to the expectation of [OS03, Rmk (1) after Thm. 3.2].

Example (1.6) (Hironaka [Hir75, Ex. 2])**.** Let Y be a complex manifold of dimension 3 and let $C \hookrightarrow Y$ be a curve with a single node P. Locally around P, the curve C is a union of two smooth irreducible curves C_1 and C_2 meeting at P. Let $f\colon X \to Y$ be the modification where we, locally around P, first blow up C_1 and then blow up the strict transform of C_2. Outside P, this is simply the blow-up of C.

Then $\mathrm{Flat}_{\mathcal{O}_X/Y} = (Y \smallsetminus C) \amalg (\widetilde{C} \smallsetminus P_2)$ where P_2 is the point of the normalization $\widetilde{C}$ corresponding to the local branch C_2. Indeed, locally around P, the restriction to the first branch $f|_{C_1}$ is flat, whereas the restriction to the second branch $f|_{C_2}$ is not flat. (The first restriction is a flat family of $\mathbb{P}^1$s degenerating to two intersecting $\mathbb{P}^1$s whereas the second restriction is a $\mathbb{P}^1$-bundle with an extra irreducible component over P.)

The canonical reduced flattening stratification is $(Y \smallsetminus C) \amalg (C \smallsetminus P) \amalg P$ but it is not the universal flattening.

Example (1.7) (Kresch [Kre10])**.** Let $g\colon Y \to S$ be an étale covering of degree 2 between projective smooth threefolds. Choose a curve D in S with a single node such that the preimage $g^{-1}(D)$ is the union of two smooth curves C_1 and C_2 meeting transversally at two points P and Q — the preimages of the node.

Let $p\colon X \to Y$ be Hironaka's construction of a non-projective 3-fold [Har77, B.3.4.1]). That is, over P we first blow up C_1 and then the strict transform of C_2 and over Q we first blow up C_2 and then the strict transform of C_1. This gives a smooth proper 3-dimensional scheme X which is not projective. As in the previous example, $p|_{C_1}$ is flat except at Q and $p|_{C_2}$ is flat except at P. We have that $\mathrm{Flat}_{\mathcal{O}_X/Y} = \big(Y \smallsetminus (C_1 \cup C_2)\big) \amalg (C_1 \smallsetminus Q) \amalg (C_2 \smallsetminus P)$.

Now, consider the composition $f\colon X \to Y \to S$. Then $\mathrm{Flat}_{\mathcal{O}_X/S} = (S \smallsetminus D) \amalg (\widetilde{D} \smallsetminus R)$ where R is the point of the normalization $\widetilde{D}$ that is the image of $Q \in C_1$ and $P \in C_2$. In particular, the universal flattening is not a disjoint union of locally closed subschemes.

Example (1.8). In [Kre13, §4], Kresch uses partial stabilization of families of prestable curves to construct proper morphisms of schemes whose universal flattening is not a stratification.

Lemma (1.9). *The canonical flattening filtration of Remark 1.4 commutes with flat base change $S' \to S$.*

Proof. For smooth base change, this follows directly since taking reduced subschemes commutes with smooth base change. For general flat base change we argue as follows.

Consider the universal flattening $F := \mathrm{Flat}_{\mathcal{F}/S}$. Since $F \to S$ is a bijective monomorphism of finite type, the following properties are equivalent: proper, finite, closed immersion, nil-immersion. Let V be the largest open subset of S such that $F|_V \to V$ is finite. Then V is also the largest open subset such that $F|_{V_{\mathrm{red}}} \to V_{\mathrm{red}}$ is an isomorphism, or equivalently, the largest open subset such that $\mathcal{F}|_{V_{\mathrm{red}}}$ is flat over V_{red}. Since $F \to S$ is of finite presentation, $s \in |V|$ if and only if $F \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \to \operatorname{Spec}(\mathcal{O}_{S,s})$ is finite.

If $s' \in |S'|$ such that $F \times_S \operatorname{Spec}(\mathcal{O}_{S',s'}) \to \operatorname{Spec}(\mathcal{O}_{S',s'})$ is finite, then by fpqc descent, $F \times_S \operatorname{Spec}(\mathcal{O}_{S,s}) \to \operatorname{Spec}(\mathcal{O}_{S,s})$ is also finite. Thus the formation of V commutes with flat base change. It now follows by construction that the canonical flattening filtration commutes with flat base change. $\square$

1.2. **Flatifying modules.** In this section, we prove the main theorem when $X = S$. We thus assume that $\mathcal{F}$ is a coherent sheaf on S. We will later only use the case when $\mathcal{F}$ is an ideal sheaf.

For $s\colon \operatorname{Spec} k \to S$, let $\operatorname{rk}_{\mathcal{F}}(s) := \dim_k(\mathcal{F} \otimes_{\mathcal{O}_S} k)$. This only depends on $s \in |S|$. The rank function $\operatorname{rk}_{\mathcal{F}}$ is upper semi-continuous and the nth Fitting ideal $F_n(\mathcal{F})$ cuts out the locus where $\operatorname{rk}_{\mathcal{F}} > n$, cf. [RG71, §5.4.1].

Proposition (1.10). *Let S be a noetherian algebraic stack and $\mathcal{F}$ a coherent $\mathcal{O}_S$-module. Let $\Delta \subseteq \mathbb{N}$ be a subset and let $U \subseteq S$ be the largest open substack such that $\mathcal{F}|_U$ is locally free with ranks in Δ. Then there exists a single U-admissible blow-up $S' = \operatorname{Bl}_Z S \to S$ such that the strict transform of $\mathcal{F}$ is locally free with ranks in Δ. Moreover, $|Z| = S \setminus U$ and Z is functorial with respect to flat morphisms $S' \to S$ between noetherian algebraic stacks.*

Proof. Let $U_\delta \subseteq S$ be the largest open substack where $\mathcal{F}|_U$ is locally free of rank δ. Then $U = \coprod_{\delta\in\Delta} U_\delta$ and U_δ is empty for all but finitely many δ. Let $\overline{U_\delta}$ be the schematic closure of U_δ. There are two obstructions to the flatness of $\mathcal{F}$: (i) the open substack U need not be schematically dense, and (ii) the rank could exceed δ on $\overline{U_\delta}$. In particular, the different $\overline{U_\delta}$ could intersect.

Let J_δ be the ideal defining $\overline{U_\delta}$. We take the center Z as the closed substack defined by the ideal

$$I = F_0\Big(\textstyle\bigcap_{\delta\in\Delta} J_\delta\Big) \cdot \displaystyle\prod_{\delta\in\Delta} \Big(F_\delta(\mathcal{F}) + J_\delta\Big).$$

Then $I|_U = \mathcal{O}_U$ and I commutes with flat base change since schematic closures, finite intersections and Fitting ideals commute with flat base change.

Let $p\colon S' = \operatorname{Bl}_Z S \to S$. The closed subset $V(F_0(\bigcap_{\delta\in\Delta} J_\delta))$ equals the support of $\bigcap_{\delta\in\Delta} J_\delta$ which is exactly the locus where U is not schematically dense. It follows that $U' := p^{-1}(U)$ is schematically dense in S'. Let $\overline{U'_\delta}$ be the schematic closure of $U'_\delta := p^{-1}(U_\delta)$ in S'.

Let $\delta, \delta' \in \Delta$ and $\delta' > \delta$. On $\overline{U_\delta} \cup \overline{U_{\delta'}}$, we have that $V(J_\delta) = \overline{U_\delta}$ and $V(F_\delta(\mathcal{F})) \supseteq \overline{U_{\delta'}}$. It follows that $\overline{U'_\delta}$ and $\overline{U'_{\delta'}}$ are disjoint, cf. [RG71, Lem. 5.1.5].

Let $\mathcal{F}' = \mathcal{F} \times_S S'$. On $\overline{U'_\delta}$, we have that $F_\delta(\mathcal{F})\mathcal{O}_{S'} = F_\delta(\mathcal{F}')$ is invertible and that $\mathcal{F}'$ is locally free of rank δ on the schematically dense open subset U'_δ. It follows that the strict transform is locally free of rank δ [RG71, Lem. 5.4.3]. $\square$

Remark **(1.11).** Example 1.3 shows that Proposition 1.10 is false for S non-noetherian even if $\mathcal{F}$ is of finite presentation. In the example, $\mathcal{F} = \mathcal{O}_S/(x)$, $J_0 = (y_1, y_2, \dots)$, $J_1 = (x)$, $F_0(\mathcal{F}) = (x)$, $F_1(\mathcal{F}) = (1)$ and the problem is that $F_0(\mathcal{F}) + J_0$ is not of finite type.

Similarly, if instead $S = \operatorname{Spec} k[x, y_1, y_2, \dots]/(x^2, xy_1, xy_2, \dots)$ and $\mathcal{F} = \mathcal{O}_S/(x)$. Then $J_0 = (1)$, $J_1 = (x)$, $F_0(\mathcal{F}) = (x)$, $F_1(\mathcal{F}) = (1)$ but $F_0(J_0 \cap J_1) = (x, y_1, y_2, \dots)$ is not of finite type.

1.3. **Resolving monomorphisms.** Recall that a monomorphism $F \to S$, locally of finite type, is unramified, that is, $\Omega_{F/S} = 0$. An unramified morphism $h\colon F \to S$ is étale-locally on F and S a closed immersion. In fact, there is even a canonical factorization $h = e \circ i\colon F \hookrightarrow E \to S$ where $i\colon F \hookrightarrow E$ is a closed immersion and $e\colon E \to S$ is étale [Ryd11a].

The closed immersion i is a regular embedding of codimension δ at $x \in F$ if and only if $F \to S$ is a local complete intersection at x and $\mathcal{H}^{-1}(L_{F/S})$ is locally free of rank δ at x. In this situation, we say that $F \to S$ is a local regular embedding of codimension δ. If $F \to S$ is a local regular embedding at every point, then $\mathcal{H}^{-1}(L_{F/S})$ is locally free and hence F is a disjoint union $\coprod_\delta F_\delta$ where $F_\delta \to S$ is a local regular embedding of codimension δ. A local regular embedding of codimension 1 is a "local Cartier divisor". The following are equivalent for an unramified morphism $F \to S$ and a point $x \in F$:

(i) $F \to S$ is a local regular embedding of codimension 0 at x.
(ii) $F \to S$ is étale at x, and
(iii) $F \to S$ is flat (and locally of finite presentation) at x,

If $h\colon F \to S$ is a monomorphism, then (i)–(iii) at x are equivalent to

(iv) $F \to S$ is an isomorphism in an open neighborhood of $h(x)$.

If $F = \operatorname{Flat}_{\mathcal{F}/S}$, then the largest open substack $U \subseteq S$ such that $\mathcal{F}$ is flat over U coincides with the largest open substack $U \subseteq S$ such that $h|_U$ is an isomorphism.

We will now resolve the monomorphism $h\colon F := \operatorname{Flat}_{\mathcal{F}/S} \to S$ by which we mean a modification $S' \to S$ such that $F \times_S S' \to S'$ is a local regular embedding of codimension ≤ 1.

Proposition (1.12). *Let S be a noetherian algebraic stack and let $h\colon F \hookrightarrow S$ be a monomorphism of finite type. Let $U \subseteq S$ be the largest open substack such that $h|_U$ is a local regular embedding of codimension ≤ 1. Then there exists a U-admissible sequence of blow-ups $S' \to S$, functorial with respect to flat morphisms, such that $F' := F \times_S S' \to S'$ is a local regular embedding of codimension ≤ 1.*

Note that U contains the largest open substack over which h is an isomorphism and that $F' = F'_0 \amalg F'_1$ where $F'_0 \to S'$ is an open immersion and $F'_1 \to S'$ is a local regular embedding of codimension 1.

Proof of Proposition 1.12. First assume that h is a **closed immersion** and let $I \subseteq \mathcal{O}_S$ be the ideal defining h. Then $U = U_0 \amalg U_1$ where I is locally free of rank δ over U_δ. By Proposition 1.10 (with $\Delta = \{0,1\}$) there exists a canonical U-admissible blow-up $S' = \operatorname{Bl}_Z S \to S$ that flatifies I, that is, such that the strict transform of I is locally free. The strict transform of I is nothing but the inverse image $I\mathcal{O}_{S'}$ so $F' \to S'$ is a local regular embedding.

In **general**, we consider the *étale envelope* of h, that is, the canonical factorization of h as a closed immersion $i\colon F \hookrightarrow E$ followed by an étale non-separated map $e\colon E \to S$ which has a canonical section $j\colon S \to E$ whose image is the complement of $i(F)$ [Ryd11a]. By the special case, there is a $j(S) \cup e^{-1}(U)$-admissible blow-up $\mathrm{Bl}_Z E \to E$ that makes the pull-back of i regular. This blow-up can be dominated by a sequence of U-admissible blow-ups on S by [Ryd11b, Prop. 4.14(b)]. To avoid dependence on [Ryd11b] and to see that the sequence on S is functorial, we will repeat the proof which also is simpler in our case.

Let $\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = S$ be the canonical flattening filtration of $F \to S$ (Remark 1.4). In particular, $h|_{(U_i \smallsetminus U_{i-1})_{\mathrm{red}}}$ is an isomorphism for $i = 1, 2, \dots, n$ so that $e^{-1}(U_i \smallsetminus U_{i-1})_{\mathrm{red}}$ is the disjoint union of two copies of $(U_i \smallsetminus U_{i-1})_{\mathrm{red}}$.

Let $K \subseteq \mathcal{O}_E$ be the ideal of the center Z. We will show that there is a functorial sequence of U-admissible blow-ups $S' \to S$ such that $K\mathcal{O}_{E \times_S S'}$ becomes invertible. Note that $K = \mathcal{O}_E$, hence invertible, over $j(S) \cup e^{-1}(U)$.

By induction, we may assume that $K|_{e^{-1}(U_{i-1})}$ is invertible. By Proposition 1.10, applied to $K|_{e^{-1}(U_i)}$ and $\Delta = \{1\}$, there is a canonical $j(U_i) \cup e^{-1}(U_{i-1})$-admissible blow-up $\mathrm{Bl}_W e^{-1}(U_i) \to e^{-1}(U_i)$ such that the inverse image of $K|_{e^{-1}(U_i)}$ becomes invertible. Since $|W| \subseteq |i(F) \cap e^{-1}(U_i)| \smallsetminus e^{-1}(U_{i-1})$, it follows that $e|_W \colon W \to U_i$ is a closed immersion with image disjoint from U_{i-1}. We blow up the scheme-theoretic closure $Q := \overline{e(W)}$.

Since $e^{-1}(Q \cap U_i) = W \amalg j(e(W))$, the blow-up of Q will transform W into a Cartier divisor so the inverse image of K becomes invertible over $e^{-1}(U_i)$. We conclude by induction. $\square$

Example (1.13). Let S be a smooth threefold with two smooth curves C_1 and C_2 meeting transversally at P. Let $f \colon X \to S$ be the blow-up of C_1 followed by the blow-up of the strict transform of C_2. Then the universal flattening is $F := \mathrm{Flat}_{\mathcal{O}_X/S} = \big(S \setminus (C_1 \cup C_2)\big) \amalg C_1 \amalg (C_2 \setminus P)$ whereas the canonical flattening filtration is $U_1 = S \setminus (C_1 \cup C_2)$, $U_2 = S \setminus P$, $U_3 = S$. The algorithm of Proposition 1.12 first blows up $C_1 \cup C_2$ and then blows up a point above P. The simple algorithm for projective morphisms, that blows up $\overline{S_P}$ in decreasing order, blows up C_1 followed by the strict transform of C_2. Hironaka's algorithm first blows up P and then the strict transform of $C_1 \cup C_2$. The latter two algorithms have centers over which f is flat in contrast to the first algorithm: f is not flat over $C_1 \cup C_2$.

1.4. **Dévissage.** Let (S, s) be a henselian noetherian local scheme and let $f \colon X \to S$ be a morphism of schemes, locally of finite type, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let $x \in f^{-1}(s)$. Then by [RG71, Prop. 1.2.3] there exists an étale neighborhood $g \colon (U, u) \to (X, x)$, such that $g^{-1}(\mathcal{F})$ admits a *total dévissage at* u:

$$D_i = \big(X_i \xrightarrow{p_i} T_i, \mathcal{L}_i \xrightarrow{\alpha_i} \mathcal{N}_i \to \mathcal{P}_i, t_i\big)$$

where $(T_0, t_0) := (U, u)$, $\mathcal{P}_0 := g^*\mathcal{F}$, and for $i = 1, 2, .., r$:

(i) $X_i := V(\mathrm{Ann}(\mathcal{P}_{i-1})) \hookrightarrow T_{i-1}$, and $t_{i-1} \in X_i$,
(ii) $T_i \to S$ is smooth and affine with geometrically connected fibers,
(iii) $p_i \colon X_i \to T_i$ is finite, $t_i = p_i(t_{i-1})$ and $p_i^{-1}(t_i) = \{t_{i-1}\}$ as sets,
(iv) $\mathcal{L}_i$ is a free sheaf of finite rank on T_i,
(v) $\mathcal{N}_i := (p_i)_*\mathcal{P}_{i-1}$,

(vi) $\alpha_i \colon \mathcal{L}_i \to \mathcal{N}_i$ is a homomorphism such that $\alpha_i \otimes k(\tau)$ is bijective where τ is the unique generic point of $(T_i)_s$, and $\mathcal{P}_i = \operatorname{coker}(\alpha_i)$.

Finally, we also have $\mathcal{P}_r = 0$.

Then $\mathcal{F}_x$ is S-flat if and only if α_i is injective for every $i = 1, 2, \dots, r$ [RG71, Cor. 2.3], cf. [SP, 05I2]. Likewise, if (S_1, s_1) is a local scheme, $(S', s') \to (S, s)$ is a local morphism, $X' = X \times_S S'$ and x' is a point above x, then the pull-back of the dévissage $(D_i)_{i=1,\dots,r}$ is a total dévissage $(D'_i)_{i=1,\dots,r}$ at x' and $\mathcal{F}'_{x'}$ is S'-flat if and only if α'_i is injective for every $i = 1, 2, \dots, r$.

The morphism $T_i \to S$ is *pure* [RG71, Def. 3.3.3, Ex. 3.3.4]. Equivalently, $L_i := \Gamma(T_i, \mathcal{L}_i)$ is a *projective* A-module [RG71, Thm. 3.3.5]. Equivalently, since A is local, L_i is a *free* A-module [Kap58].

Let $P_i = \Gamma(T_i, P_i)$. Then we have exact sequences

$$L_i \xrightarrow{\alpha_i} P_{i-1} \to P_i \to 0$$

for $i = 1, 2, \dots, r$. Since L_i are free, we can lift the α_i to maps $\beta_i \colon L_i \to P_0 = \Gamma(U, g^*\mathcal{F})$. If we let $\beta \colon L_1 \oplus \dots \oplus L_r \to P_0$ be the sum, then β is surjective. As mentioned above, $\mathcal{F}_x$ is S-flat if and only if all the α_i are injective, that is, if and only if β is an isomorphism. We can now summarize the situation as follows.

Theorem (1.14) (Free presentation and flatness). *Let $(S, s) = \operatorname{Spec} A$ be a henselian noetherian local scheme, let X be an algebraic stack, locally of finite type over S, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let $x \in |X|$ be a point over s. Then there exists a smooth morphism $p \colon \operatorname{Spec} B \to X$, a free A-module L, usually of infinite rank, and a surjection $\beta \colon L \to M = \Gamma(\operatorname{Spec} B, p^*\mathcal{F})$ such that β is an isomorphism if and only if $\mathcal{F}$ is S-flat at x.*

Moreover, if $(S', s') \to (S, s)$ is a local morphism, then the pull-back $\beta' \colon L' \to M'$ of β to S' is an isomorphism if and only if the pull-back $\mathcal{F}'$ of $\mathcal{F}$ to $X \times_S S'$ is S'-flat at some (or equivalently every) point x' over x. In particular, $\beta|_s$ is an isomorphism.

Proof. Let $p \colon U \to X$ be a smooth presentation and pick a point u above x. After replacing U with an étale neighborhood, there is a total dévissage. This gives $\beta \colon L \to M$ where $L = L_1 \oplus \dots \oplus L_r$ and $M = \Gamma(U, p^*\mathcal{F})$ as described above. □

Remark **(1.15).** The theorem also implies that $\mathcal{F}$ is flat at x if and only if $\mathcal{F}$ is flat over the image of p. In the setting of complex spaces, Hironaka replaces the free L_i with *pseudo-free* modules. The dévissage is replaced with a pseudo-free presentation [Hir75, Thm. 2.1]. If $X = S = \operatorname{Spec} A$ is local, then Theorem 1.14 is elementary: we can choose a basis $\kappa(s)^n \cong \mathcal{F}|_s$ and any lift $\beta \colon A^n \twoheadrightarrow \Gamma(S, \mathcal{F})$ is surjective by Nakayama's lemma and gives a presentation with the requested properties.

1.5. **Flattening in the local case.** Let (S, s) be a henselian noetherian local scheme, let X be an algebraic stack, locally of finite type over S, let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module and let x be a point above s. Let $\operatorname{Flat}_{\mathcal{F}/S,x}$ be the functor from local schemes (S', s') above (S, s) to sets that "makes $\mathcal{F}$ flat at x". That is, if $x' \in X' = X \times_S S'$ is a point above x, then $\operatorname{Flat}_{\mathcal{F}/S,x}(S', s')$ is the one-point set if $\mathcal{F}$ is S'-flat at x' and the empty set otherwise.

Proposition (1.16). *The functor* $\mathrm{Flat}_{\mathcal{F}/S,x}$ *is represented by a closed subscheme* $\mathrm{Spec}(A/I)$ *of* $S = \mathrm{Spec}\, A$*. Let* $\widetilde{\mathcal{F}}$ *be the quotient of* $\mathcal{F}$ *by the* I*-torsion. If* I *is invertible, then* $\mathcal{F} \otimes \kappa(s) \twoheadrightarrow \widetilde{\mathcal{F}} \otimes \kappa(s)$ *is not an isomorphism.*

Proof. Pick $\beta \colon L \to M$ as in Theorem 1.14. Then $\mathrm{Flat}_{\mathcal{F}/s,x}$ is equivalent to the functor that "makes β an isomorphism". Let $(e_\alpha)_\alpha$ be a basis of L. For $g \in L$, let $g = \sum_\alpha g_\alpha e_\alpha$. Let $I \subseteq A$ be the ideal generated by the g_α for every $g \in \ker \beta$. Then $\mathrm{Flat}_{\mathcal{F}/s,x} = V(I)$. Indeed, I is the smallest ideal such that $\ker \beta \subseteq IL$.

If $I = (a)$ and a is not a zero-divisor, then for every $g \in \ker \beta$, there is a unique $g' \in L$ such that $g = ag'$. Since $(g'_\alpha)_{g,\alpha} = (1)$ and A is local, there exists some g and α such that g'_α is a unit. For such a g, we have that $g' \notin \mathfrak{m}L$ where $\mathfrak{m}$ denotes the maximal ideal of A. In particular, $\overline{g'} \in L/\mathfrak{m}L = M/\mathfrak{m}M$ is non-zero whereas ag' is zero in M. It follows that $\mathcal{F} \otimes \kappa(s) \twoheadrightarrow \widetilde{\mathcal{F}} \otimes \kappa(s)$ has non-trivial kernel. □

Remark **(1.17).** The first part of Proposition 1.16 is [RG71, Thm. 4.1.2] whereas the second part does not seem to appear there. In the setting of complex spaces, the analogue of both parts of Proposition 1.16 is [Hir75, Thm. 2.4].

1.6. **Proof of the main theorem.**

Lemma (1.18). *Let* S*,* $f \colon X \to S$ *and* $\mathcal{F}$ *be as in Theorem 1.1. Suppose that* $\mathrm{Flat}_{\mathcal{F}/S} = F_0 \amalg F_1$ *where* $F_0 \to S$ *is an open immersion and* $F_1 \to S$ *is a local regular embedding of codimension* 1*. Let* $\widetilde{\mathcal{F}}$ *be the quotient of* $\mathcal{F}$ *by its torsion, relative to some* $U \subseteq F_0$*. Then* $\mathcal{F} \otimes \kappa(s) \twoheadrightarrow \widetilde{\mathcal{F}} \otimes \kappa(s)$ *is not an isomorphism for every* $s \in |F_1|$*.*

Proof. The statement is smooth-local on S so we can assume that $S = \mathrm{Spec}\, A$ is an affine scheme and we may furthermore replace S with the henselization $\mathrm{Spec}\, \mathcal{O}^{\mathrm{h}}_{S,s}$ at some $s \in |F_1|$ and assume that A is local henselian. Then there is a non-zero divisor $a \in A$ such that $F_1 = V(a) \amalg F'_1$ where $F'_1 \to S$ factors through $D(a)$ [EGA$_{\mathrm{IV}}$, Thm. 18.5.11c].

For every $x \in f^{-1}(s)$ we have $\mathrm{Flat}_{\mathcal{F}/S,x} = V(I_x)$ (Proposition 1.16) which makes $\mathcal{F}$ flat in a neighborhood of x. Since f is universally closed and S is local, we have that $\mathcal{F}$ is flat if and only if $\mathcal{F}$ is flat at every point x above s. Thus, $V(a) = \bigcap_x V(I_x)$. Since a is a non-zero divisor and A is local, there exists an x such that $I_x = (a)$. The result now follows from Proposition 1.16 since killing torsion with respect to U also kills a-torsion since $U \subseteq D(a)$. □

Remark **(1.19).** The results of Sections 1.4 and 1.5 are only used for the proof of Lemma 1.18 and there is perhaps a more elementary proof. It is not difficult to see that $\mathcal{F} \otimes A/(a) \to \widetilde{\mathcal{F}} \otimes A/(a)$ is not an isomorphism and this only requires that $\mathrm{Spec}\, A/(a) \hookrightarrow F$.

Proof of Theorem 1.1. We will apply the following algorithm.

(i) Let $U \subseteq S$ be the maximal open substack such that $\mathcal{F}$ is flat over U. Let $S_0 = S$ and let $n = 0$.
(ii) Let $\mathcal{F}_n = \mathcal{F} \mathbin{\widetilde{\times_S}} S_n$ be the strict transform and let $F_n = \mathrm{Flat}_{\mathcal{F}_n/S_n}$.

(iii) Apply Proposition 1.12 to produce a blow-up sequence $S_{n+1} \to S_n$ such that $F_n \times_{S_n} S_{n+1} \to S_{n+1}$ is a local regular embedding of codimension ≤ 1.
(iv) If $S_{n+1} \neq S_n$, then increase n by 1 and repeat from (ii).

Let $Z_n \subseteq |S_n|$ be the non-flat locus of $\mathcal{F}_n$. Then the Z_n are closed and $Z_{n+1} \subseteq Z_n \times_{S_n} S_{n+1}$. Moreover, for every point $s_{n+1} \in Z_{n+1}$, the surjection $\mathcal{F}_n \times_{S_n} \kappa(s_{n+1}) \twoheadrightarrow \mathcal{F}_{n+1} \times_{S_{n+1}} \kappa(s_{n+1})$ is not an isomorphism by Lemma 1.18.

Suppose that the algorithm does not terminate. Consider the topological space $\widetilde{S} = \varprojlim_n |S_n|$. This is a quasi-compact space since the space $\varprojlim_n |S_n|_{\text{cons}}$ has the same underlying set but a finer topology which is compact and Hausdorff. Here $|S_n|_{\text{cons}}$ denotes the constructible topology, where open sets are arbitrary unions of constructible sets, which is compact and Hausdorff [EGA$_{\text{IV}}$, Prop. 1.9.15].

Since $\widetilde{S}$ is quasi-compact and $Z_n \times_{S_n} \widetilde{S}$ is a filtered system of closed subsets, its intersection is non-empty. Thus, there exists an infinite sequence of closed points $s_n \in |Z_n|$, $n = 0, 1, 2, \dots$ such that s_{n+1} lies above s_n. Let k be an algebraic closure of $\kappa(s_0)$ and choose embeddings $\kappa(s_n) \hookrightarrow k$. Then we have a sequence of non-trivial surjections

$$\mathcal{F}_0 \times_{S_0} k \twoheadrightarrow \mathcal{F}_1 \times_{S_1} k \twoheadrightarrow \dots$$

But this is a sequence of coherent sheaves on the noetherian stack $X \times_S k$ so cannot be infinite. This finishes the proof for general S.

If S is smooth, every ideal on S can be principalized by a sequence of blow-ups in smooth centers that is functorial with respect to smooth morphisms, see [Kol07, Thm. 3.26] or [BM08, Thm. 1.3]. If the flatification algorithm blows-up I_1, I_2, $\dots$, I_n, then we apply principalization to I_1, then to the inverse image of I_2 etc. This gives the result. □

Remark **(1.20).** We have only used that f is universally closed to prove that $\mathrm{Flat}_{\mathcal{F}/S}$ is representable (Proposition 1.5) and to compare it with $\mathrm{Flat}_{\mathcal{F}/S,x}$ (proof of Lemma 1.18). In both situations it is enough that f is of finite type and $\mathcal{F}$ is *pure* [RG71, Def. 3.3.3]. Nevertheless, the algorithm does not work in this setting. The problem is that the strict transform of a pure sheaf need not be pure as the following example shows.

Example (1.21). Let $S = \mathbb{A}^2 = \operatorname{Spec} k[s,t]$ and $\overline{X} = \mathbb{P}^2_S = \operatorname{Proj} k[s,t,x,y,z]$. Let $X = \overline{X} \smallsetminus \{s = t = x = 0\}$. Let $Z = V(s) \cup V(xy - sz^2) \hookrightarrow X$. Then it is readily verified that $Z \to S$ is pure but killing s-torsion gives $\widetilde{Z} = V(xy - sz^2)$ which is not pure because above $s = 0$, we have the irreducible component $x = 0$ whose fiber over the origin is empty.

2. Functorial étalification

In this section we prove Theorem C. As the other main theorems, the proof does not require that f is separated, only that f is universally closed, of finite type, and has quasi-finite diagonal. In particular, f is relatively Deligne–Mumford. Since the algorithm we construct will be functorial with respect to smooth morphisms, we can assume that S is a scheme.

We begin with blowing up $X \smallsetminus f^{-1}(U)$ with the reduced structure. After this blow-up, $f^{-1}(U)$ is schematically dense and will remain so throughout the algorithm. If U is not dense, we blow up the components of S that U does not intersect. After this U is dense in S.

Recall that if $S' = \mathrm{Bl}_W S \to S$ is a blow-up, then the strict transform X' is simply the blow-up of X in $f^{-1}(W)$. Since $f^{-1}(U)$ is schematically dense, the strict transform of a U-admissible blow-up is also the closure of $f^{-1}(U)$ in $X \times_S S'$.

We now apply Theorem B to flatify $X \to S$ by a sequence of smooth U-admissible blow-ups. We can thus assume that $X \to S$ is flat. In particular $X \to S$ is now quasi-finite.

Let $\widetilde{X}$ be the normalization of X. Note that $\widetilde{X} \to X$ is an isomorphism over U. Let $V \subseteq S$ be the locus where $\widetilde{X} \to S$ is étale.

Let $Z = S \smallsetminus V$ with the reduced scheme structure. Resolve Z, that is, perform a sequence of V-admissible blow-ups $S' \to S$ with smooth centers such that $(Z \times_S S')_{\mathrm{red}}$ becomes a divisor $D_1 \cup D_2 \cup \dots \cup D_n$ where the D_i are smooth and meet with simple normal crossings. Replace S, X and $\widetilde{X}$ with S', $X \times_S S'$ and the normalization of $X \times_S S'$.

Let $s \in |S|$ be a point of codimension 1. Then we can define the ramification index at s as follows. Any $x \in |\widetilde{X}|$ above s is normal of codimension 1, so the strict local ring $\mathcal{O}^{\mathrm{sh}}_{\widetilde{X},x}$ is a discrete valuation ring. Therefore $\mathfrak{m}_s\mathcal{O}^{\mathrm{sh}}_{\widetilde{X},x} = (\mathfrak{m}_x)^{e(x)}$ for some integer $e(x) \geq 1$. We let $e(s)$ be the least common multiple of the $e(x)$ for all x above s. Note that $e(s) = 1$ if $s \in V$.

Now start with $i = 1$. Then start with $r = 2$. Take the rth root stack of all connected components of D_i with ramification index r. Increase r by 1 and repeat until all components of D_i with $e(s) > 1$ has been rooted. Then increase i by 1 and repeat until $i = n$. When we take a root stack $S' := S\bigl(\sqrt[r]{D}\bigr) \to S$ we also take the root stack $X \times_S S' = X\bigl(\sqrt[r]{f^{-1}(D)}\bigr) \to X$.

If $S' \to S$ is the composition of all these root stacks, replace S with S' and X with $X \times_S S'$ and $\widetilde{X}$ with the normalization of $X \times_S S'$. A local analysis, using that in characteristic zero, an extension of strictly henselian discrete valuation rings $A \to B$ with ramification index r is an rth Kummer extension, that is, $B = A[x]/(x^r - t)$ where $t \in A$ is a uniformizer, shows that $\widetilde{X}$ is now étale in codimension 1.

Let $Y \to \widetilde{X}$ be an étale presentation. By the Zariski–Nagata purity theorem [SGA$_1$, Exp. X, Thm. 3.1], [SGA$_2$, Exp. X, Thm. 3.4], $Y \to S$ is étale, hence so is $\widetilde{X} \to S$.

To finish the proof, we need to replace the normalization $n\colon \widetilde{X} \to X$ with a sequence of blow-ups. Firstly, use Theorem A to find a functorial sequence $X' \to X$ of $f^{-1}(U)$-admissible blow-ups that flatify n, that is, makes n into an isomorphism. We obtain the commutative diagram

$$\begin{tikzcd} \widetilde{X}' \arrow[r] \arrow[d, "\cong"'] & \widetilde{X} \arrow[d, "n"] \\ X' \arrow[r] & X \end{tikzcd}$$

where the bottom row is a sequence of blow-ups in centers W_i, $i = 1, 2, \ldots, r$ and the top row is a sequence of blow-ups in centers $n^{-1}(W_i)$.

Secondly, use Theorem B to find a functorial sequence $S' \to S$ of U-admissible blow-ups in smooth centers that flatify $X' \to X \to S$. Let $X'' \to X'$ be the strict transform, which also is a sequence of blow-ups. This gives us the commutative diagram

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X' \arrow[d] \\ \widetilde{X} \times_S S' \arrow[d] & X \arrow[d] \\ S' \arrow[r] & S. \end{tikzcd}$$

Since $X'' \to S'$ is flat and $\widetilde{X} \times_S S' \to S'$ is étale, it follows that the modification $X'' \to \widetilde{X} \times_S S'$ is flat, hence an isomorphism. We have thus obtained a sequence of blow-ups $X'' \to X' \to X$ and a sequence of blow-ups with smooth centers $S' \to S$ such that $X'' \to S'$ is étale.

3. Applications of functorial flatification

3.1. **Cofinality of blow-ups among modifications.** Our first applications are immediate consequences of the three main theorems since flat modifications and étale stack-theoretic modifications are isomorphisms.

Theorem (3.1) (Blow-ups are functorially cofinal)**.** *Let S be a noetherian algebraic stack and let $f\colon X \to S$ be a modification. That is, f is proper, representable, and $f|_U$ is an isomorphism for some open substack $U \subseteq S$. Then there exists a sequence of U-admissible blow-ups $X' \to X$ such that the composition $X' \to S$ is a sequence of U-admissible blow-ups. Moreover,*

(i) *The sequences are functorial with respect to flat base change $S' \to S$.*
(ii) *If S is smooth over a field of characteristic zero, there are sequences of blow-ups where $X' \to S$ has smooth centers, and they are functorial with respect to smooth base change $S' \to S$.*

Theorem (3.2) (Kummer blow-ups are functorially cofinal)**.** *Let S be a smooth algebraic stack over a field of characteristic zero. Let $f\colon X \to S$ be a stack-theoretic modification, that is, f is proper, not necessarily representable, with finite diagonal, and $f|_U$ is an isomorphism for some open substack $U \subseteq S$. Then there exists a sequence of U-admissible Kummer blow-ups $X' \to X$ such that the composition $X' \to S$ is a sequence of U-admissible Kummer blow-ups with smooth centers. The sequences are functorial with respect to smooth base change $S' \to S$.*

3.2. **Resolution of the indeterminacy locus.** As a consequence we obtain resolution of the indeterminacy locus by a functorial sequence of blow-ups.

Theorem (3.3). *Let S be a noetherian algebraic stack, let X be a noetherian S-stack, let $Y \to S$ be a proper morphism and let $f\colon X \dashrightarrow Y$ be a rational map over S, defined on an open substack $U \subseteq X$.*

(i) *If $Y \to S$ is representable, then there exists a sequence of U-admissible blow-ups $p\colon X' \to X$ such that the map $f \circ p\colon X' \dashrightarrow Y$ is defined everywhere. The sequence is functorial with respect to flat base change $X' \to X$.*
(ii) *If in addition, X is smooth over a field of characteristic zero, then the map p can be chosen as a sequence of blow-ups in smooth centers which is functorial with respect to smooth base change $X' \to X$.*
(iii) *If $Y \to S$ is not representable but relatively Deligne–Mumford, and X is smooth over a field of characteristic zero, then there exists a sequence of U-admissible Kummer blow-ups $p\colon X' \to X$ such that the map $f \circ p\colon X' \dashrightarrow Y$ is defined everywhere. The sequence is functorial with respect to smooth base change $X' \to X$.*

Proof. Consider the proper morphism $X \times_S Y \to X$ which has a section over U induced by f. In case (i) and (ii), the section is a closed immersion and we let W be the closure of its image. This gives us a modification $W \to X$ and Theorem 3.1 gives us a functorial sequence of blow-ups $p\colon X' \to X$ such that $f \circ p$ factors as $X' \to W \to Y$ and thus is defined everywhere.

In case (iii), the section over U is a finite morphism $U \to U \times_S Y$ and we let W be the normalization of $X \times_S Y$ in U. Then $W \to X \times_S Y$ is finite because $X \times_S Y$ is of finite type over a field and U is smooth. The composition $W \to X$ is a stack-theoretic modification with finite diagonal. We now conclude as before using Theorem 3.2. □

Remark **(3.4).** *Birational case* — If in addition $X \to S$ is proper and f is birational, then $X' \to Y$ becomes a (stack-theoretic) modification and we can apply Theorem 3.1 or 3.2 to obtain a sequence of (Kummer) blow-ups $X'' \to X'$ (not necessarily in smooth centers) such that $X'' \to Y$ becomes a sequence of (Kummer) blow-ups.

In the smooth case, we can then continue by taking a sequence of (Kummer) blow-ups $X''' \to X''$ such that the composition $X''' \to X'$ is a sequence of (Kummer) blow-ups in smooth centers. But we cannot simultaneously arrange so that the blow-up sequence $X''' \to Y$ has smooth centers: this would amount to the strong factorization conjecture.

3.3. **A general Chow lemma.** Our last application is a Chow lemma. In contrast to the other results, this is *not* functorial, since the starting point is a quasi-projective open substack and there is neither a unique maximal quasi-projective open substack, nor a functorial projective compactification of this open substack. To which extent there could be a functorial Chow lemma is not clear to the author.

Theorem (3.5). *Let S be a noetherian algebraic stack and let $f\colon X \to S$ be a representable proper morphism. If there exist an open substack $U \subseteq X$ such that $f|_U$ is quasi-projective, then there exists a sequence of U-admissible blow-ups $X' \to X$ such that $X' \to S$ is projective.*

Proof. Let $\mathcal{L}$ be an S-ample line bundle on U. There is a, non-canonical, coherent $\mathcal{O}_S$-module $\mathcal{F}$ and an immersion $j\colon U \hookrightarrow \mathbb{P}(\mathcal{F})$ such that $\mathcal{L} = j^*\mathcal{O}(1)$, cf. [Ryd16a, Thm. 8.6(i)]. The theorem follows from Theorem 3.3(i) and Remark 3.4 applied to the birational map $X \dashrightarrow \overline{U} \hookrightarrow \mathbb{P}(\mathcal{F})$. □

Proposition (3.6). *Let S be a noetherian algebraic stack and let $f\colon X \to S$ be a representable and separated morphism of finite type. Assume that*

(i) *S has quasi-finite and separated diagonal, or*
(ii) *S has affine stabilizers and X is reduced, or*
(iii) *S has affine stabilizers and the generic points of X has linearly reductive stabilizers.*

Then there exists an open dense $U \subseteq X$ such that $f|_U$ is quasi-projective.

Proof. (i) If S has quasi-finite and separated diagonal, then so has X so there is a quasi-finite flat presentation $p\colon X' \to X$ where X' is an affine scheme [Ryd11c, Thm. 7.1]. There is a dense open substack $U \subseteq X$ such that $p|_U$ is finite. The morphism $p^{-1}(U) \to S$ is quasi-affine since $p^{-1}(U)$ is quasi-affine and S has quasi-affine diagonal. It follows that $U \to S$ is quasi-affine [Ryd11c, Lem. C.1].

(ii) By standard approximation techniques, it is enough to show that for every generic point $x \in |X|$, the residual gerbe $\mathcal{G}_x$ admits an S-ample line bundle. We have a factorization $\mathcal{G}_x \to \mathcal{G}_s \hookrightarrow S$ where $s = f(x)$. Since $\mathcal{G}_s \hookrightarrow S$ is quasi-affine [Ryd11c, Thm. B.2], it is enough to show that $\mathcal{G}_x$ admits a $\mathcal{G}_s$-ample line bundle.

We have a factorization $\mathcal{G}_x \to \mathcal{G}_s \times_{\kappa(s)} \kappa(x) \to \mathcal{G}_s$. Since the second map is affine, we can replace $\mathcal{G}_s$ with $\mathcal{G}_s \times_{\kappa(s)} \kappa(x)$ and assume that $\kappa(x) = \kappa(s)$. Let $k/\kappa(s)$ be a finite field extension that neutralizes both gerbes. Then it is enough to prove that $\mathcal{G}_x \times_{\kappa(s)} k \to \mathcal{G}_s \times_{\kappa(s)} k$ is quasi-projective because the norm of a $\mathcal{G}_s \times_{\kappa(s)} k$-ample line bundle on $\mathcal{G}_x \times_{\kappa(s)} k$ gives a $\mathcal{G}_s$-ample line bundle on $\mathcal{G}_x$ [$\mathrm{EGA_{II}}$, Prop. 6.6.1].

It is thus enough to show that if G is an algebraic group scheme over k and H is a closed subgroup, then $BH \to BG$ is quasi-projective. By a theorem of Chevalley, there exists a finite-dimensional G-representation V and a 1-dimensional subspace $L \subseteq V$ whose stabilizer is H (as a group scheme) [Mil17, Thm. 4.27]. This gives a monomorphism $BH \to [\mathbb{P}(V)/G]$ which is a locally closed immersion [Mil17, Prop. 1.65(b)]. It follows that $BH \to BG$ is quasi-projective.

(iii) Let $x \in |X|$ be a generic point. The intersection of all open neighborhoods of x is a 1-point stack X_x and $(X_x)_{\mathrm{red}} = \mathcal{G}_x$. By (ii), $\mathcal{G}_x$ carries an S-ample line bundle $\mathcal{L}$. Since $\mathcal{G}_x$ has linearly reductive stabilizer, it has cohomological dimension zero. In particular, if I is the ideal sheaf of $\mathcal{G}_x \hookrightarrow X_x$, then $\mathrm{Ext}^2(\mathcal{G}_x, \mathcal{L}^\vee \otimes \mathcal{L} \otimes I^n/I^{n+1}) = 0$ for all n. Thus, the obstruction for extending the line bundle $\mathcal{L}$ to X_x vanishes. An extension to X_x is also S-ample [$\mathrm{EGA_{II}}$, Prop. 4.6.15] and it spreads out to an S-ample line bundle on some open neighborhood of x. □

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