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Integral Equations and Operator Theory

An Operator-valued Berezin Transform and the Class of n-Hypercontractions

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Abstract. We study an operator-valued Berezin transform corresponding to certain standard weighted Bergman spaces of square integrable analytic functions in the unit disc. The study of this operator-valued Berezin transform relates in a natural way to the study of the class of n-hypercontractions on Hilbert space introduced by Agler. To an n-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ we associate a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_{n,T}$ supported on the closed unit disc $\bar{\mathbb{D}}$ in a way that generalizes the above notion of operator-valued Berezin transform. This construction of positive operator measures $d\omega_{n,T}$ gives a natural functional calculus for the class of n-hypercontractions. We revisit also the operator model theory for the class of n-hypercontractions. The new results here concern certain canonical features of the theory. The operator model theory for the class of n-hypercontractions gives information about the structure of the positive operator measures $d\omega_{n,T}$.

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Contents

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0. Introduction

Let $\mathcal{H}$ be a general not necessarily separable complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of all bounded linear operators on $\mathcal{H}$. Let $n \geq 1$ be an integer. An operator $T \in \mathcal{L}(\mathcal{H})$ is called an n-hypercontraction if the operator inequality

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^k \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H})$$

holds true for every $1 \leq m \leq n$. In this terminology a 1-hypercontraction is a contraction, but for $n \geq 2$ the class of n-hypercontractions is a more restricted class of operators. The class of n-hypercontractions was first introduced by Agler [1, 2] whereas the study of contractions on Hilbert space is a classical topic of which the book [27] by Sz.-Nagy and Foias is a standard reference.

Let $n \geq 1$ be an integer. We shall need the Hilbert space $A_n(\mathbb{D})$ of analytic functions in the unit disc $\mathbb{D}$ with reproducing kernel

$$K_n(z, \zeta) = \frac{1}{(1 - \bar{\zeta} z)^n}, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D}.$$

The space $A_1(\mathbb{D})$ is just the standard Hardy space $H^2(\mathbb{D})$, and for $n \geq 2$ the space $A_n(\mathbb{D})$ is the standard weighted Bergman space of square integrable analytic functions in $\mathbb{D}$ corresponding to the weighted area measure

$$d\mu_n(z) = (n-1)(1 - |z|^2)^{n-2} dA(z), \quad z \in \mathbb{D};$$

here $dA(z) = dxdy/\pi$, $z = x + iy$, is the usual planar Lebesgue area measure normalized so that the unit disc $\mathbb{D}$ is of unit area. For notational reasons we let also $d\mu_1$ denote the normalized Lebesgue arc length measure on the unit circle $\mathbb{T} = \partial\mathbb{D}$. For $n \geq 1$, an analytic function f in $\mathbb{D}$ belongs to the space $A_n(\mathbb{D})$ if and only if the norm

$$\|f\|_{A_n}^2 = \lim_{r \to 1} \int_{\bar{\mathbb{D}}} |f(rz)|^2 d\mu_n(z)$$

is finite. Notice that this norm can also be written

$$\|f\|_{A_n}^2 = \sum_{k \geq 0} |a_k|^2 \mu_{n;k},$$

where a_k is the k-th Taylor coefficient $f \in A_n(\mathbb{D})$ (see (0.5) below) and $\{\mu_{n;k}\}_{k \geq 0}$ is the sequence of moments of the measure $d\mu_n$ defined by

$$\mu_{n;k} = \int_{\bar{\mathbb{D}}} |z|^{2k} d\mu_n(z) = 1 \Big/ \binom{k+n-1}{k}, \quad k \geq 0. \tag{0.1}$$

A standard reference for Bergman spaces on the unit disc is the recent book [19] by Hedenmalm, Korenblum and Zhu.

The function B_n defined by

$$B_n(z,\zeta) = \frac{|K_n(z,\zeta)|^2}{K_n(z,z)} = \frac{(1-|z|^2)^n}{|1-\bar{\zeta}z|^{2n}}, \quad (z,\zeta) \in \mathbb{D} \times \bar{\mathbb{D}},$$

is called the Berezin kernel associated to the kernel function K_n. The corresponding transform defined by

$$B_n[f](z) = \int_{\bar{\mathbb{D}}} B_n(z,\zeta) f(\zeta)\, d\mu_n(\zeta), \quad z \in \mathbb{D},$$

for, say, $f \in L^1(\mu_n)$ is called the Berezin transform. Notice that for $n = 1$ the so-called Poisson transform is obtained. It is well-known that the Berezin transform reproduces harmonic functions. In the literature the Berezin transform has attracted some attention because of its use in the study of Toeplitz operators; see for instance [3, 7].

In this paper we shall consider certain related operator-valued Berezin transforms that we now proceed to define. Let $T \in \mathcal{L}(\mathcal{H})$ be an operator with spectral radius $r(T) = \max_{z\in\sigma(T)} |z|$ strictly less than 1. The operator-valued Berezin kernel $B_n(T,\cdot)$ is the function defined by

$$B_n(T,\zeta) = (I - \zeta T^*)^{-n} \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k \Big) (I - \bar{\zeta} T)^{-n}, \quad \zeta \in \bar{\mathbb{D}}. \tag{0.2}$$

Notice that $B_n(T,\zeta) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if T is an n-hypercontraction. We have an associated operator-valued Berezin transform defined by

$$B_n[f](T) = \int_{\bar{\mathbb{D}}} B_n(T,\zeta) f(\zeta)\, d\mu_n(\zeta)$$

for, say, $f \in C(\bar{\mathbb{D}})$. Throughout the paper we denote by $C(\bar{\mathbb{D}})$ the space of continuous functions on the closed unit disc $\bar{\mathbb{D}}$.

We shall associate to an n-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_{n,T}$ supported on the closed unit disc $\bar{\mathbb{D}}$. The Berezin transform of a monomial $\bar{\zeta}^j \zeta^k$, $j,k \geq 0$, has the power series expansion

$$B_n[\bar{\zeta}^j \zeta^k](z) = \bar{z}^{j-\min(j,k)} \Big(\sum_{m\geq 0} W_{n;m;j,k} |z|^{2m} \Big) z^{k-\min(j,k)}, \quad z \in \mathbb{D}.$$

This equality clearly determines the numbers $W_{n;m;j,k}$ uniquely. The operator measure $d\omega_{n,T}$ is defined by its action on monomials by the requirement that

$$\int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k\, d\omega_{n,T}(\zeta) = T^{*(j-\min(j,k))} \Big(\sum_{m\geq 0} W_{n;m;j,k} T^{*m} T^m \Big) T^{k-\min(j,k)} \tag{0.3}$$

for $j,k \geq 0$ (see Theorem 3.1). We remark that there is a decay estimate

$$W_{n;m;j,k} = O(m^{-(n+1)}) \quad \text{as} \quad m \to \infty$$

so that formula (0.3) makes sense (see Lemma 3.1). Notice also that since the space $\mathbb{C}[z,\bar{z}]$ of polynomials in z and $\bar{z}$ is dense in $C(\bar{\mathbb{D}})$ (Stone-Weierstrass) the operator measure $d\omega_{n,T}$ is uniquely determined by its action on monomials.

The operator measure $d\omega_{n,T}$ extends the above notion of operator-valued Berezin transform in the sense that the equality

$$\int_{\bar{\mathbb{D}}} f(\zeta)d\omega_{n,T}(\zeta) = B_n[f](T), \quad f \in C(\bar{\mathbb{D}}),$$

holds when $r(T) < 1$ (see Corollary 3.1). Recall that $C(\bar{\mathbb{D}})$ denotes the space of continuous functions on the closed unit disc $\bar{\mathbb{D}}$.

We remark that for $n = 1$ the operator measure $d\omega_{1,T}$ obtained in this way is supported by the unit circle $\mathbb{T}$ and coincides with a certain operator measure on $\mathbb{T}$ denoted by $d\omega_T$ (see Proposition 4.1). We mention that the operator measure $d\omega_T$ is closely related to the unitary dilation of the contraction T and was called the harmonic spectral measure by Foias [15, 16] in the 1950's (see Section 4).

The operator measures $d\omega_{n,T}$ have the continuity property that the map

$$C(\bar{\mathbb{D}}) \times \mathcal{C}_n \ni (f,T) \mapsto \int_{\bar{\mathbb{D}}} f(\zeta)d\omega_{n,T}(\zeta) \in \mathcal{L}(\mathcal{H})$$

is continuous; here $\mathcal{C}_n$ denotes the set of all n-hypercontractions in $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H})$ is equipped the uniform operator topology (see Theorem 3.2). The operator measures $d\omega_{n,T}$ are also shown to have a property of conformal invariance with respect to conformal automorphisms of the unit disc (see Corollary 3.2). This property of conformal invariance is inherited from corresponding conformal invariance properties of the class of n-hypercontractions (see Section 1).

The above considerations yield also a natural functional calculus for the class of n-hypercontractions. Let the function u in $\mathbb{D}$ be the Berezin transform of $f \in C(\bar{\mathbb{D}})$: $u = B_n[f]$. The function u is real-analytic in $\mathbb{D}$ and has a power series expansion

$$u(z) = \sum_{j,k\geq 0} c_{jk}\bar{z}^j z^k, \quad z \in \mathbb{D}.$$

The function u now operates on the class of n-hypercontractions $T \in \mathcal{L}(\mathcal{H})$ in the sense that

$$u(T) := \lim_{r\to 1} \sum_{j,k\geq 0} r^{j+k} c_{jk} T^{*j} T^k = \int_{\bar{\mathbb{D}}} f(\zeta)d\omega_{n,T}(\zeta) \quad \text{in } \mathcal{L}(\mathcal{H}) \tag{0.4}$$

(see Theorem 3.3). We emphasize that the limit in (0.4) is computed in the uniform operator topology, that is, in operator norm. A basic property coming from the positivity of the operator measure $d\omega_{n,T}$ and $\omega_{n,T}(\bar{\mathbb{D}}) = I$ is the norm inequality

$$\Big\| \int_{\bar{\mathbb{D}}} f(\zeta)d\omega_{n,T}(\zeta)\Big\| \leq \|f\|_\infty, \quad f \in C(\bar{\mathbb{D}}),$$

which by (0.4) extends the classical von Neumann inequality [25].

For reasons of modeling a general n-hypercontraction we shall need to consider also Hilbert space valued versions of the spaces $A_n(\mathbb{D})$. Let $\mathcal{E}$ be a Hilbert space and denote by $A_n(\mathcal{E}) = A_n(\mathbb{D}, \mathcal{E})$ the space of all $\mathcal{E}$-valued analytic functions

$$f(z) = \sum_{k \geq 0} a_k z^k, \quad z \in \mathbb{D}; \tag{0.5}$$

here $a_k \in \mathcal{E}$ for $k \geq 0$, with finite norm

$$\|f\|_{A_n}^2 = \sum_{k \geq 0} \|a_k\|^2 \mu_{n;k},$$

where $\{\mu_{n;k}\}_{k \geq 0}$ is the sequence of moments of $d\mu_n$ given by (0.1). Notice that this is consistent with the previous description of the space $A_n(\mathbb{D})$. On the space $A_n(\mathcal{E})$ we have a natural shift operator $S = S_n$ defined by

$$(S_n f)(z) = z f(z) = \sum_{k \geq 1} a_{k-1} z^k, \quad z \in \mathbb{D},$$

for $f \in A_n(\mathcal{E})$ given by (0.5). It turns out that the shift operator S_n acts boundedly on the space $A_n(\mathcal{E})$ in such a way that the adjoint operator S_n^* is an n-hypercontraction with the property that $\lim_{k \to \infty} S_n^{*k} = 0$ in the strong operator topology (see Proposition 5.1).

In Sections 6 and 7 we shall revisit some operator model theory relating to the class of n-hypercontractions. Recall that an operator $A \in \mathcal{L}(\mathcal{H})$ is part of an operator $B \in \mathcal{L}(\mathcal{K})$ if $\mathcal{H}$ is a B-invariant subspace of $\mathcal{K}$ and $A = B|_{\mathcal{H}}$; the operator B is then called an extension of A. As pointed out in the previous paragraph the adjoint shift operator S_n^* is an n-hypercontraction with the property that $\lim_{k \to \infty} S_n^{*k} = 0$ in the strong operator topology. It is also clear that every isometry is an n-hypercontraction. The principal modeling result of n-hypercontractions due to Agler [1, 2] asserts that an operator $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction if and only if it is part of an operator of the form $S_n^* \oplus U$, where U is an isometry. As a byproduct of this result one has that an operator $T \in \mathcal{L}(\mathcal{H})$ is part of an adjoint shift operator S_n^* if and only T is an n-hypercontraction such that $\lim_{k \to \infty} T^k = 0$ in the strong operator topology. This result by Agler was first proved using C^*-algebra methods. The purpose of our presentation in Sections 6 and 7 is to show that there is a certain uniqueness property and associated canonical construction behind this modeling result of n-hypercontractions.

To describe our results we need some more notation. For an operator $T \in \mathcal{L}(\mathcal{H})$ such that the limit $\lim_{k \to \infty} \|T^k x\|^2$ exists for every $x \in \mathcal{H}$ we consider the operator

$$Q = \Big(\lim_{k \to \infty} T^{*k} T^k \Big)^{1/2} \quad \text{in } \mathcal{L}(\mathcal{H}),$$

where the positive square root is used and the limit is computed in the weak operator topology. We denote by $\mathcal{Q}$ the closure in $\mathcal{H}$ of the range of Q, that is, $\mathcal{Q} = \overline{Q(\mathcal{H})}$. On the space $\mathcal{Q}$ we have a natural isometry U defined by $U : Qx \mapsto QTx$ for $x \in \mathcal{H}$ and continuity.

In Section 6 we consider the more general problem of modeling an operator $T \in \mathcal{L}(\mathcal{H})$ as part of an operator of the form $T_1^* \oplus T_2$, where $T_j \in \mathcal{L}(\mathcal{H}_j)$ $(j = 1, 2)$ are operators such that $\lim_{k\to\infty} T_1^{*k} = 0$ in the strong operator topology and T_2 is an isometry. This more general modeling problem amounts to that of finding an isometry

$$V = (V_1, V_2) : \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2 \tag{0.6}$$

of $\mathcal{H}$ into $\mathcal{H}_1 \oplus \mathcal{H}_2$ satisfying the intertwining relation

$$VT = (T_1^* \oplus T_2)V. \tag{0.7}$$

It turns out that there is a canonical choice of V_2 and $(T_2, \mathcal{H}_2)$ given by $V_2 = Q$, $T_2 = U$ and $\mathcal{H}_2 = \mathcal{Q}$, and that the general modeling problem (0.6) and (0.7) reduces to that of finding a bounded linear operator $V_1 : \mathcal{H} \to \mathcal{H}_1$ satisfying the norm equality

$$\|x\|^2 - \|Tx\|^2 = \|V_1x\|^2 - \|T_1^*V_1x\|^2, \quad x \in \mathcal{H},$$

and the intertwining relation $V_1T = T_1^*V_1$ (see Theorems 6.1 and 6.2).

For an n-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ we consider the defect operators

$$D_{m,T} = \Big(\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*k}T^k\Big)^{1/2} \quad \text{in } \mathcal{L}(\mathcal{H})$$

for $1 \le m \le n$, where the positive square root is used. We have an associated defect space $\mathcal{D}_{n,T}$ defined as the closure in $\mathcal{H}$ of the range of $D_{n,T}$, that is, $\mathcal{D}_{n,T} = \overline{D_{n,T}(\mathcal{H})}$.

In Section 7 we specialize the modeling problem (0.6) and (0.7) further to the case when $\mathcal{H}_1 = A_n(\mathcal{E})$ and $T_1 = S_n$ is the shift operator acting on this space. It turns out that there is a canonical choice of coefficient space $\mathcal{E}$ and operator $V_1 : \mathcal{H} \to A_n(\mathcal{E})$ given by $\mathcal{E} = \mathcal{D}_{n,T}$ and $V_1x = V_{1,n}x$ for $x \in \mathcal{H}$, where for $x \in \mathcal{H}$ the $\mathcal{D}_{n,T}$-valued analytic function $V_{1,n}x$ is defined by the formula

$$(V_{1,n}x)(z) = D_{n,T}(I - zT)^{-n}x = \sum_{k\ge 0}\binom{k+n-1}{k}(D_{n,T}T^kx)z^k, \quad z \in \mathbb{D} \tag{0.8}$$

(see Theorem 7.1). To some extent formula (0.8) is also motivated by the explicit form of the operator-valued Berezin kernel (0.2).

In the case of an n-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ we show that the map $V_1 = V_{1,n} : x \mapsto V_{1,n}x$ given by (0.8) is admissible for the above modeling problem (0.6) and (0.7) in the sense that the map

$$V = (V_{1,n}, Q) : \mathcal{H} \to A_n(\mathcal{D}_{n,T}) \oplus \mathcal{Q}$$

defined by $Vx = (V_{1,n}x, Qx)$ for $x \in \mathcal{H}$ is an isometry of $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T}) \oplus \mathcal{Q}$ satisfying the interwining relation

$$VT = (S_n^* \oplus U)V$$

(see Theorem 7.2).

In Section 8 we use the operator model theory for the class of n-hypercontractions to give some more detailed results describing the structure of the operator measures $d\omega_{n,T}$. Let us denote by $\mathfrak{S}$ the σ-algebra of planar Borel sets. The operator measure $d\omega_{n,T}$ naturally decomposes as

$$\omega_{n,T}(S) = V_{1,n}^* \omega_{n,S_n^*}(S) V_{1,n} + Q\omega_U(S)Q, \quad S \in \mathfrak{S},$$

and for $n \geq 2$ we further have that

$$V_{1,n}^* \omega_{n,S_n^*}(S) V_{1,n} = \int_{\mathbb{D} \bigcap S} B_n(T,\zeta) d\mu_n(\zeta), \quad S \in \mathfrak{S}$$

(see Theorems 8.1 and 8.2). Notice that this gives that $\omega_{n,T}(S) = 0$ in $\mathcal{L}(\mathcal{H})$ if S is a Borel subset of $\mathbb{D}$ of planar Lebesgue area measure zero. In particular, we have that

$$d\omega_{n,T}(\zeta) = B_n(T,\zeta) d\mu_n(\zeta), \quad \zeta \in \bar{\mathbb{D}},$$

if $n \geq 2$ and $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction such that $\lim_{k\to\infty} T^k = 0$ in the strong operator topology (see Corollary 8.1). Invoking a classical theorem of Sz.-Nagy and Foias we deduce that the operator measure $d\omega_U$ is absolutely continuous with respect to Lebesgue arc length measure on $\mathbb{T}$ if $T \in \mathcal{L}(\mathcal{H})$ is a completely non-unitary contraction.

The method of construction of operator models used here goes back at least to work of de Branges and Rovnyak [10, Theorem 1] in the 1960's; see also [27, Section I.10.1]. In this context we also want to mention more recent related work by Müller [22], Vasilescu [28, 29], Müller and Vasilescu [23], and Curto and Vasilescu [13, 14] concerned with modeling of operators in terms of weighted shifts, and also the papers Ambrozie, Engliš and Müller [4] and Arazy and Engliš [5]. Also, operator models of this type form an integral part in recent work on constrained von Neumann inequalities by Badea and Cassier [8].

It was shown by Agler [2, Theorem 3.1] that an operator $T \in \mathcal{L}(\mathcal{H})$ is a subnormal contraction if and only if it is an n-hypercontraction for every $n \geq 1$. In Section 9 we derive this characterization of subnormal contractions as a limit case of our study of operator-valued Berezin transforms (see Theorem 9.1). As an application of this result by Agler we consider two operator-valued moment problems of Hausdorff type (see Theorem 9.2 and Proposition 9.3).

At several places in this paper we encounter operators T such that $\lim_{k\to\infty} T^k = 0$ in the strong operator topology. We mention that an operator $T \in \mathcal{L}(\mathcal{H})$ is said to belong to the class $C_{0\cdot}$ if $\lim_{k\to\infty} T^k = 0$ in the strong operator topology (see [27, Section II.4]).

In a recent paper [26] we have studied a related positive operator measure $d\omega_T$ on the unit n-torus $\mathbb{T}^n$ associated to an n-tuple $T = (T_1, \ldots T_n)$ of commuting contractions in $\mathcal{L}(\mathcal{H})$ having a so-called regular unitary dilation. The more involved construction of the operator measure $d\omega_{n,T}$ in this paper using the numbers $W_{n;m;j,k}$ and the decay of these numbers as $m \to \infty$ (see Lemma 3.1) is due to a more complicated regularity behavior of Berezin transforms compared to the case of the Poisson transform. Eventhough $f \in \mathbb{C}[z,\bar{z}]$ is a polynomial, the Berezin transform $B_2[f]$ is in general not C^2-smooth up to the boundary $\mathbb{T} = \partial\mathbb{D}$ (see Remark 3.2 for an example). A similar regularity behavior is known to present itself in the study of the Dirichlet problem for the so-called invariant Laplacian (the Laplace-Beltrami operator) for the unit ball in $\mathbb{C}^n$ (see [21, Chapter 6]).

Preliminaries. Let us recall the notions of weak, strong and uniform operator topology. The uniform operator topology on $\mathcal{L}(\mathcal{H})$ is the usual topology on $\mathcal{L}(\mathcal{H})$ defined by the operator norm. The strong operator topology (SOT) on $\mathcal{L}(\mathcal{H})$ is the topology on $\mathcal{L}(\mathcal{H})$ defined by the semi-norms

$$\mathcal{L}(\mathcal{H}) \ni T \mapsto \|Tx\| \in [0, \infty), \quad x \in \mathcal{H}.$$

Notice that $T_k \to T$ (SOT) means that $T_k x \to Tx$ in $\mathcal{H}$ for every $x \in \mathcal{H}$. The weak operator topology (WOT) on $\mathcal{L}(\mathcal{H})$ is the topology on $\mathcal{L}(\mathcal{H})$ defined by the semi-norms $T \mapsto |\langle Tx, y\rangle|$ for $x, y \in \mathcal{H}$.

In the paper we shall need some facts from the theory of integration in Hilbert space. Let $\mathfrak{S}$ be the σ-algebra of planar Borel sets. A finitely additive set function $\mu : \mathfrak{S} \to \mathcal{L}(\mathcal{H})$ is called a positive operator measure if $\mu(S) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for every $S \in \mathfrak{S}$ and the set functions $\mu_{x,y}$, $x, y \in \mathcal{H}$, defined by $\mu_{x,y}(S) = \langle \mu(S)x, y\rangle$ for $S \in \mathfrak{S}$, are all complex regular Borel measures. A positive operator measure μ is of finite semi-variation

$$|\mu|(S) := \sup_{\|x\|,\|y\|\leq 1} |\mu_{x,y}|(S) = \|\mu(S)\|, \quad S \in \mathfrak{S},$$

where $|\mu_{x,y}|$ is the total variation of the complex measure $\mu_{x,y}$. The integral $\int_S f d\mu$ is defined as an operator in $\mathcal{L}(\mathcal{H})$ by the duality requirement that $\langle \int_S f d\mu x, y\rangle = \int_S f d\mu_{x,y}$ for all $x, y \in \mathcal{H}$. An important property of the integral is the norm inequality

$$\Big\| \int_S f(s) d\mu(s) \Big\| \leq |\mu|(S) \|f\|_\infty,$$

where $\| \cdot \|_\infty$ denotes the norm of essential supremum on S. We refer to [26] for some more details.

We shall use the following operator version of the F. Riesz representation theorem: If Λ is a linear map from $C(\bar{\mathbb{D}})$ into $\mathcal{L}(\mathcal{H})$ which is positive in the sense that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $f \geq 0$ in $\bar{\mathbb{D}}$, then there exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\lambda$ on $\bar{\mathbb{D}}$ which represents Λ in the sense that

$$\Lambda(f) = \int_{\bar{\mathbb{D}}} f(z) d\lambda(z), \quad f \in C(\bar{\mathbb{D}})$$

(see [26]).

1. Invariance properties of n-hypercontractions

The purpose of this section is to discuss some invariance properties of the class of n-hypercontractions and the related operator-valued Berezin kernel. Let $n \geq 1$ be an integer, and recall that an operator $T \in \mathcal{L}(\mathcal{H})$ is called an n-hypercontraction if

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^k \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H}) \tag{1.1}$$

for $1 \leq m \leq n$. Notice that the defining property (1.1) of an n-hypercontraction is equivalently formulated that

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^k x\|^2 \geq 0, \quad x \in \mathcal{H},$$

for $1 \leq m \leq n$.

Let us consider the backward shift operator λ acting on sequences $a = \{a_k\}_{k=0}^{\infty}$ by $(\lambda a)_k = a_{k+1}$ for $k \geq 0$. We notice that an operator $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction if and only if $(I - \lambda)^m a \geq 0$ for $1 \leq m \leq n$ and every sequence $a = \{a_k\}_{k=0}^{\infty}$ of the form $a_k = \|T^k x\|^2$ for $k \geq 0$, where $x \in \mathcal{H}$.

It is known that if $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction, then so is rT for every $0 \leq r < 1$ (see [2, Lemma 1.9]). For the sake of completeness we include a proof of this fact.

Proposition 1.1. *If $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction, then so is rT for every $0 \leq r < 1$.*

Proof. We consider the backward shift operator λ acting on sequences $a = \{a_k\}_{k=0}^{\infty}$ by $(\lambda a)_k = a_{k+1}$ for $k \geq 0$. By the binomial theorem we have that

$$(I - r^2\lambda)^m = \sum_{k=0}^{m} \binom{m}{k} (1 - r^2)^{m-k} r^{2k} (I - \lambda)^k. \tag{1.2}$$

Consider now a sequence $a = \{a_k\}_{k=0}^{\infty}$ of the form $a_k = \|T^k x\|^2$ for $k \geq 0$, where $x \in \mathcal{H}$. Since the operator T is an n-hypercontraction we have that $(I - \lambda)^m a \geq 0$ for $1 \leq m \leq n$. By the binomial identity (1.2) we conclude that $(I - r^2\lambda)^m a \geq 0$ for $1 \leq m \leq n$. This yields the conclusion that the operator rT is an n-hypercontraction. □

The following lemma gives a kind of stability property of n-hypercontractions.

Lemma 1.1. *Let $n \geq 2$. Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H}),$$

and $\|T^k x\|^2 = o(k)$ as $k \to \infty$ for every $x \in \mathcal{H}$. Then the operator T is an n-hypercontraction, that is, inequality (1.1) holds for $1 \leq m \leq n$.

Proof. Let us first recall a simple fact about convex sequences: If $a = \{a_k\}_{k=0}^{\infty}$ is a convex sequence and $\limsup_{k\to\infty} a_k/k \leq 0$, then $\{a_k\}_{k=0}^{\infty}$ is decreasing. In terms of the backward shift λ this fact gives that $(I - \lambda)^2 a \geq 0$ and $a_k = o(k)$ implies $(I - \lambda)a \geq 0$.

We now consider a sequence $a = \{a_k\}_{k=0}^{\infty}$ of the form $a_k = \|T^k x\|^2$ for $k \geq 0$, where $x \in \mathcal{H}$. By assumption we know that $(I-\lambda)^n a \geq 0$ and $a_k = o(k)$. By repeated applications of the observation in the previous paragraph, we conclude that $(I-\lambda)^m(a) \geq 0$ for $1 \leq m \leq n$. This yields the conclusion of the lemma. □

We mention in passing that Lemma 1.1 relaxes the growth assumption of T power bounded used in [22, Corollary 3.6].

Lemma 1.1 gives the following converse to Proposition 1.1.

Corollary 1.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that*

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} r^{2j} T^{*j} T^j \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H})$$

for $r = r_k \to 1$, $0 \leq r_k < 1$, and $\limsup_{k\to\infty} \|T^k x\|^{2/k} \leq 1$ for every $x \in \mathcal{H}$. Then the operator T is an n-hypercontraction.

Proof. By an application of Lemma 1.1 we conclude that the operator $r_k T$ is an n-hypercontraction. Now letting $r_k \to 1$ the conclusion of the corollary follows. □

We shall now consider some properties of invariance with respect to conformal automorphisms of the unit disc. First we need a lemma.

Lemma 1.2. *Let $n \geq 1$ be an integer, and let $T \in \mathcal{L}(\mathcal{H})$ an operator such that $r(T) \leq 1$. Then the equality*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \varphi_\alpha(T)^{*k} \varphi_\alpha(T)^k$$
$$= (1-|\alpha|^2)^n (I-\alpha T^*)^{-n} \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k \Big) (I-\bar{\alpha} T)^{-n}$$

holds for every conformal automorphism φ_α of the unit disc of the form $\varphi_\alpha(z) = (z-\alpha)/(1-\bar{\alpha}z)$ for $z \in \mathbb{D}$, where $\alpha \in \mathbb{D}$.

Proof. To simplify some formulas in the proof we shall write

$$S_n(T) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k.$$

With this notation the assertion of the lemma reads as

$$S_n(\varphi_\alpha(T)) = (1-|\alpha|^2)^n (I-\alpha T^*)^{-n} S_n(T) (I-\bar{\alpha} T)^{-n}. \tag{1.3}$$

We shall prove formula (1.3) by induction on $n \geq 0$. Notice that by the standard formula $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for binomial coefficients we have

$$S_n(T) = S_{n-1}(T) - T^* S_{n-1}(T) T. \tag{1.4}$$

Let us now turn to the proof of (1.3). For $n = 0$ there is nothing to prove. Assume $n \geq 1$ and that formula (1.3) holds true with n replaced by $n-1$. Using (1.4) and the induction hypothesis we compute that

$$\begin{aligned}
S_n(\varphi_\alpha(T)) &= S_{n-1}(\varphi_\alpha(T)) - \varphi_\alpha(T)^* S_{n-1}(\varphi_\alpha(T))\varphi_\alpha(T) \\
&= (1-|\alpha|^2)^{n-1}(I-\alpha T^*)^{-n+1} S_{n-1}(T)(I-\bar{\alpha}T)^{-n+1} \\
&\quad - (1-|\alpha|^2)^{n-1}\varphi_\alpha(T)^*(I-\alpha T^*)^{-n+1} S_{n-1}(T)(I-\bar{\alpha}T)^{-n+1}\varphi_\alpha(T) \\
&= (1-|\alpha|^2)^{n-1}(I-\alpha T^*)^{-n}\Big\{(I-\alpha T^*)S_{n-1}(T)(I-\bar{\alpha}T) \\
&\quad - (T^*-\bar{\alpha}I)S_{n-1}(T)(T-\alpha I)\Big\}(I-\bar{\alpha}T)^{-n} \\
&= (1-|\alpha|^2)^{n}(I-\alpha T^*)^{-n}\big\{S_{n-1}(T) - T^* S_{n-1}(T)T\big\}(I-\bar{\alpha}T)^{-n} \\
&= (1-|\alpha|^2)^{n}(I-\alpha T^*)^{-n} S_n(T)(I-\bar{\alpha}T)^{-n}.
\end{aligned}$$

By the principle of induction this completes the proof of formula (1.3). □

Let us denote by $\mathrm{Aut}(\mathbb{D})$ the set of all conformal automorphisms φ of the unit disc $\mathbb{D}$. It is well-known that every $\varphi \in \mathrm{Aut}(\mathbb{D})$ can be written in the form

$$\varphi(z) = e^{i\theta}\varphi_\alpha(z), \quad z \in \mathbb{D},$$

where $e^{i\theta} \in \mathbb{T}$, $\alpha \in \mathbb{D}$ and $\varphi_\alpha(z) = (z-\alpha)/(1-\bar{\alpha}z)$.

We can now conclude that the conformal automorphisms operate on the class of n-hypercontractions.

Corollary 1.2. *If $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction, then so is $\varphi(T)$ for every $\varphi \in \mathrm{Aut}(\mathbb{D})$.*

Proof. This is clear by Lemma 1.2. □

We mention here that Corollary 1.2 is contained in the statement of [12, Theorem 2.1].

Recall that the operator-valued Berezin kernel is the function defined by the formula

$$B_n(T,\zeta) = (I-\zeta T^*)^{-n}\Big(\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k\Big)(I-\bar{\zeta}T)^{-n}, \quad \zeta \in \bar{\mathbb{D}};$$

here $T \in \mathcal{L}(\mathcal{H})$ is an operator such that $r(T) < 1$. We shall now prove a property of invariance of this operator-valued Berezin kernel.

Proposition 1.2. *Let $n \geq 1$, and let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$. Then the operator-valued Berezin kernel has the invariance property that*

$$B_n(\varphi(T),\varphi(\zeta))(1-|\varphi(\zeta)|^2)^n = B_n(T,\zeta)(1-|\zeta|^2)^n, \quad \zeta \in \bar{\mathbb{D}},$$

for every $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Proof. It is easy to see that it suffices to consider φ of the form $\varphi = \varphi_\alpha$. We first compute that

$$(I - \overline{\varphi_\alpha(\zeta)}\varphi_\alpha(T))^{-n} = \Big(\frac{1-\alpha\bar{\zeta}}{1-|\alpha|^2}\Big)^n (I - \bar{\alpha}T)^n (I - \bar{\zeta}T)^{-n}.$$

Using Lemma 1.2 we now compute that

$$\begin{aligned} B_n(\varphi_\alpha(T), \varphi_\alpha(\zeta)) &= \Big(\frac{1-\bar{\alpha}\zeta}{1-|\alpha|^2}\Big)^n (I - \alpha T^*)^n (I - \zeta T^*)^{-n} \\ &\quad \times (1-|\alpha|^2)^n (I - \alpha T^*)^{-n} \Big(\sum_{k=0}^n (-1)^k \binom{n}{k} T^{*k} T^k\Big)(I - \bar{\alpha}T)^{-n} \\ &\quad \times \Big(\frac{1-\alpha\bar{\zeta}}{1-|\alpha|^2}\Big)^n (I - \bar{\alpha}T)^n (I - \bar{\zeta}T)^{-n} \\ &= \frac{|1-\bar{\alpha}\zeta|^{2n}}{(1-|\alpha|^2)^n} (I - \zeta T^*)^{-n} \Big(\sum_{k=0}^n (-1)^k \binom{n}{k} T^{*k} T^k\Big)(I - \bar{\zeta}T)^{-n} \\ &= \frac{|1-\bar{\alpha}\zeta|^{2n}}{(1-|\alpha|^2)^n} B_n(T, \zeta). \end{aligned}$$

By the well-known formula

$$1 - |\varphi_\alpha(\zeta)|^2 = \frac{(1-|\alpha|^2)(1-|\zeta|^2)}{|1-\bar{\alpha}\zeta|^2}$$

the conclusion of the proposition now follows. □

We remark that in Proposition 1.2 we have $r(\varphi(T)) < 1$ by the spectral mapping theorem.

Associated to the Berezin kernel $B_n(T, \cdot)$ we have the operator-valued Berezin transform defined by

$$B_n[f](T) = \int_{\mathbb{D}} B_n(T, \zeta) f(\zeta) d\mu_n(\zeta), \quad f \in C(\bar{\mathbb{D}}). \tag{1.5}$$

We shall now prove that this operator-valued Berezin transform (1.5) commutes with the action of conformal automorphisms.

Theorem 1.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$. Then the operator-valued Berezin transform has the invariance property that*

$$B_n[f \circ \varphi](T) = B_n[f](\varphi(T)), \quad f \in C(\bar{\mathbb{D}}),$$

for every $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Proof. We assume that $n \geq 2$. The case $n = 1$ is handled similarly. By a change of variables we see that

$$B_n[f \circ \varphi](T) = (n-1) \int_{\mathbb{D}} B_n(T, \varphi^{-1}(\zeta)) f(\zeta) |(\varphi^{-1})'(\zeta)|^2 (1 - |\varphi^{-1}(\zeta)|^2)^{n-2} dA(\zeta).$$

Notice that $T = \varphi^{-1}(\varphi(T))$. By an application of Proposition 1.2 we now conclude that

$$B_n[f \circ \varphi](T) = (n-1) \int_{\mathbb{D}} B_n(\varphi(T), \zeta) f(\zeta) \frac{|(\varphi^{-1})'(\zeta)|^2}{(1 - |\varphi^{-1}(\zeta)|^2)^2} (1 - |\zeta|^2)^n dA(\zeta).$$

By an invariance property of the Bergman kernel function we know that

$$\frac{|(\varphi^{-1})'(\zeta)|^2}{(1 - |\varphi^{-1}(\zeta)|^2)^2} = \frac{1}{(1 - |\zeta|^2)^2}, \quad \zeta \in \mathbb{D}.$$

This completes the proof of the theorem. □

We remark that in Theorem 1.1 we have $r(\varphi(T)) < 1$ by the spectral mapping theorem.

We shall consider also the variant of the operator-valued Berezin kernel defined by

$$B_n(T, \zeta) = (I - \zeta T^*)^{-n} \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k \Big) (I - \bar{\zeta} T)^{-n}, \quad \zeta \in \mathbb{D}, \tag{1.6}$$

where $T \in \mathcal{L}(\mathcal{H})$ is an operator such that $r(T) \leq 1$. Notice that this modified operator-valued Berezin kernel given by (1.6) has the corresponding invariance property that

$$B_n(\varphi(T), \varphi(\zeta))(1 - |\varphi(\zeta)|^2)^n = B_n(T, \zeta)(1 - |\zeta|^2)^n, \quad \zeta \in \mathbb{D},$$

for $\varphi \in \mathrm{Aut}(\mathbb{D})$ (see Proposition 1.2).

Notice also that the Berezin kernel $B_n(T, \zeta)$ given by (1.6) is positive in $\mathcal{L}(\mathcal{H})$ if T is an n-hypercontraction. We shall need the following lemma.

Lemma 1.3. *Let $n \geq 2$, and let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then the function $B_n(T, \cdot)$ defined by* (1.6) *is integrable with respect to the measure $d\mu_n$. Furthermore, we have that*

$$\int_{\mathbb{D}} |\langle B_n(T, \zeta) x, y \rangle| d\mu_n(\zeta) \leq \|x\| \|y\|, \quad x, y \in \mathcal{H}.$$

Proof. Let $0 \leq r < 1$. By Proposition 1.1 the Berezin kernel $B_n(rT, \zeta)$ is positive in $\mathcal{L}(\mathcal{H})$. By Corollary 2.1 in the next section with $f = 1$ we have that

$$\int_{\mathbb{D}} \langle B_n(rT, \zeta) x, x \rangle d\mu_n(\zeta) = \|x\|^2, \quad x \in \mathcal{H}.$$

Letting $r \to 1$ an application of Fatou's lemma gives that

$$\int_{\mathbb{D}} \langle B_n(T, \zeta) x, x \rangle d\mu_n(\zeta) \leq \|x\|^2.$$

By the Cauchy-Schwarz inequality we now have that

$$\begin{aligned} \int_{\mathbb{D}} |\langle B_n(T, \zeta) x, y \rangle| d\mu_n(\zeta) &\leq \int_{\mathbb{D}} \langle B_n(T, \zeta) x, x \rangle^{1/2} \langle B_n(T, \zeta) y, y \rangle^{1/2} d\mu_n(\zeta) \\ &\leq \|x\| \|y\|, \quad x, y \in \mathcal{H}. \end{aligned}$$

This completes the proof of the lemma. □

For $T \in \mathcal{L}(\mathcal{H})$ an n-hypercontraction and $n \geq 2$, the Berezin transform $B_n[f](T)$ defined by (1.5) and (1.6) is well-defined by Lemma 1.3. The invariance property of Theorem 1.1 remains true in this context.

Proposition 1.3. *Let $n \geq 2$, and let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then the operator-valued Berezin transform defined by* (1.5) *and* (1.6) *has the invariance property that*

$$B_n[f \circ \varphi](T) = B_n[f](\varphi(T)), \quad f \in C(\bar{\mathbb{D}}),$$

for every $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Proof. See the proof of Theorem 1.1. We omit the details. □

We remark that in Proposition 1.3 the operator $\varphi(T)$ is an n-hypercontraction by Corollary 1.2.

We wish to point out that similar conformal invariance properties of operators have been studied in the context of the unit ball in $\mathbb{C}^n$ by Curto and Vasilescu [12]. The principal object of study in [12] is the operator-valued $\mathcal{M}$-harmonic Poisson kernel introduced in [28].

2. The Berezin transform for a general radial measure

In this section we shall derive some formulas for the Berezin transform in the context of a general radial measure μ on the closed unit disc $\bar{\mathbb{D}}$. We assume throughout the section that the associated kernel function K_μ is non-vanishing in $\mathbb{D} \times \mathbb{D}$.

Let μ be a finite positive radial measure on the closed unit disc $\bar{\mathbb{D}}$ such that $\mu(\bar{\mathbb{D}} \setminus r\mathbb{D}) > 0$ for every $0 \leq r < 1$. The Bergman space $A_\mu(\mathbb{D})$ is the space of all analytic functions

$$f(z) = \sum_{k \geq 0} a_k z^k, \quad z \in \mathbb{D},$$

with finite norm

$$\|f\|_{A_\mu}^2 = \lim_{r \to 1} \int_{\bar{\mathbb{D}}} |f(rz)|^2 d\mu(z) = \sum_{k \geq 0} |a_k|^2 \mu_k,$$

where $\{\mu_k\}_{k \geq 0}$ is the sequence of moments of μ defined by

$$\mu_k = \int_{\bar{\mathbb{D}}} |z|^{2k} d\mu(z), \quad k \geq 0.$$

Notice that $\lim_{k \to \infty} \mu_k^{1/k} = 1$.

The function K_μ defined by

$$K_\mu(z, \zeta) = \sum_{k \geq 0} \frac{1}{\mu_k} (\bar{\zeta} z)^k, \quad (z, \zeta) \in \mathbb{D} \times \mathbb{D},$$

is known as the kernel function for the Bergman space $A_\mu(\mathbb{D})$. The corresponding function B_μ defined by

$$B_\mu(z,\zeta) = \frac{|K_\mu(z,\zeta)|^2}{K_\mu(z,z)}, \quad (z,\zeta) \in \mathbb{D} \times \bar{\mathbb{D}},$$

is called the Berezin kernel.

In what follows we assume that the kernel function K_μ is non-vanishing in $\mathbb{D} \times \mathbb{D}$, that is, $K_\mu(z,\zeta) \neq 0$ for $(z,\zeta) \in \mathbb{D}^2$. We write

$$\frac{1}{K_\mu(z,\zeta)} = \sum_{k\geq 0} c_k(\bar{\zeta}z)^k, \quad (z,\zeta) \in \mathbb{D} \times \mathbb{D}.$$

The assumption of non-vanishing of the kernel function K_μ is a non-trivial assumption which means that $\limsup_{k\to\infty} |c_k|^{1/k} \leq 1$, so that the above series is convergent. Notice that

$$c_0/\mu_0 = 1 \quad \text{and} \quad \sum_{k=0}^{n} c_{n-k}/\mu_k = 0 \tag{2.1}$$

for $n \geq 1$.

We now proceed to define the Berezin transform for operator-valued arguments. Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$. We set

$$B_\mu(T,\zeta) = \Big(\sum_{k\geq 0} \frac{1}{\mu_k}\zeta^k T^{*k}\Big)\Big(\sum_{k\geq 0} c_k T^{*k}T^k\Big)\Big(\sum_{k\geq 0} \frac{1}{\mu_k}\bar{\zeta}^k T^k\Big), \quad \zeta \in \bar{\mathbb{D}}. \tag{2.2}$$

Notice that by the spectral radius formula the sums in (2.2) are absolutely convergent in $\mathcal{L}(\mathcal{H})$. We shall be interested in operator-valued Berezin transforms of the type

$$B_\mu[f](T) = \int_{\bar{\mathbb{D}}} B_\mu(T,\zeta) f(\zeta)\, d\mu(\zeta),$$

where, say, the function f is in $C(\bar{\mathbb{D}})$.

Lemma 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$. Then*

$$B_\mu(T,\zeta) = \sum_{r,s\geq 0} p_{rs}(\zeta)\, T^{*r}T^s, \quad \zeta \in \bar{\mathbb{D}}, \tag{2.3}$$

where the polynomials

$$p_{rs}(\zeta) = \sum_{l=0}^{\min(r,s)} \frac{1}{\mu_{r-l}\mu_{s-l}} c_l \zeta^{r-l}\bar{\zeta}^{s-l}$$

satisfy the growth bound $\limsup_{r,s\to\infty} \|p_{rs}\|_{C(\bar{\mathbb{D}})}^{1/(r+s)} \leq 1$. In particular, we have that

$$\int_{\bar{\mathbb{D}}} B_\mu(T,\zeta) f(\zeta)\, d\mu(\zeta) = \sum_{r,s\geq 0} \int_{\bar{\mathbb{D}}} p_{rs}(\zeta) f(\zeta)\, d\mu(\zeta)\, T^{*r}T^s \tag{2.4}$$

for, say, $f \in C(\bar{\mathbb{D}})$.

Proof. Expanding formula (2.2) for $B_\mu(T,\zeta)$ we have that

$$B_\mu(T,\zeta) = \sum_{j,k,n\geq 0} \frac{1}{\mu_j\mu_k} c_n \zeta^j \bar{\zeta}^k T^{*(j+n)} T^{k+n}.$$

Notice that by the spectral radius formula this sum is absolutely convergent uniformly in $\zeta \in \bar{\mathbb{D}}$. By a change of order of summation we obtain the series expansion (2.3). Indeed, if we set $r = j+n$, $s = k+n$ and $l = n$, then $r, s \geq 0$ and $0 \leq l \leq \min(r,s)$, which gives (2.3). The growth bound for the polynomials p_{rs} follows by $\lim_{k\to\infty} \mu_k^{1/k} = 1$ and $\limsup_{k\to\infty} |c_k|^{1/k} \leq 1$. The last formula (2.4) follows by termwise integration of (2.3). $\square$

We shall now compute the Berezin transform of a monomial.

Proposition 2.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$ and fix $j, k \geq 0$. Then*

$$\int_{\bar{\mathbb{D}}} B_\mu(T,\zeta)\bar{\zeta}^j\zeta^k d\mu(\zeta) = T^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{m;j,k} T^{*m} T^m\Big) T^{k-\min(j,k)},$$

where

$$W_{m;j,k} = \sum_{l=0}^{m} \frac{\mu_{j+k-\min(j,k)+l}}{\mu_{j-\min(j,k)+l}\,\mu_{k-\min(j,k)+l}} c_{m-l}.$$

Proof. Recall formula (2.4) in Lemma 2.1. Notice that the polynomial p_{rs} has the homogeneity property that $p_{rs}(e^{i\theta}\zeta) = e^{i(r-s)\theta} p_{rs}(\zeta)$ for $e^{i\theta} \in \mathbb{T}$. Since the measure μ is radial we have that

$$\int_{\bar{\mathbb{D}}} p_{rs}(\zeta)\bar{\zeta}^j\zeta^k d\mu(\zeta) = 0$$

whenever $r + k \neq s + j$. Assume now that $r = j - \min(j,k) + m$ and $s = k - \min(j,k) + m$, where $m \geq 0$. For such r, s we have that

$$\begin{aligned}
&\int_{\bar{\mathbb{D}}} p_{rs}(\zeta)\bar{\zeta}^j\zeta^k d\mu(\zeta) \\
&\quad = \sum_{l=0}^{m} \frac{c_l}{\mu_{j-\min(j,k)+m-l}\,\mu_{k-\min(j,k)+m-l}} \int_{\bar{\mathbb{D}}} |\zeta|^{2(j+k-\min(j,k)+m-l)} d\mu(\zeta) = W_{m;j,k},
\end{aligned}$$

where the last equality follows by a change of order of summation. Going back to formula (2.4) we have that

$$\int_{\bar{\mathbb{D}}} B_\mu(T,\zeta)\bar{\zeta}^j\zeta^k d\mu(\zeta) = T^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{m;j,k} T^{*m} T^m\Big) T^{k-\min(j,k)}.$$

This completes the proof of the lemma. $\square$

We remark that Proposition 2.1 gives the power series expansion for the Berezin transform of a monomial

$$\int_{\bar{\mathbb{D}}} B_\mu(z,\zeta)\bar{\zeta}^j\zeta^k d\mu(\zeta) = \bar{z}^{j-\min(j,k)}\Big(\sum_{m\geq 0} W_{m;j,k}|z|^{2m}\Big)z^{k-\min(j,k)}, \quad z\in\mathbb{D},$$

where $j,k\geq 0$.

Lemma 2.2. *Assume that $j=0$ or $k=0$. Then*

$$W_{m;j,k} = \sum_{l=0}^{m} \frac{\mu_{j+k-\min(j,k)+l}}{\mu_{j-\min(j,k)+l}\mu_{k-\min(j,k)+l}} c_{m-l} = \delta_{m,0},$$

where $\delta_{0,0}=1$ and $\delta_{m,0}=0$ for $m\geq 1$ is the Kronecker's delta.

Proof. We have that

$$W_{m;j,k} = \sum_{l=0}^{m} \frac{\mu_{j+k-\min(j,k)+l}}{\mu_{j-\min(j,k)+l}\mu_{k-\min(j,k)+l}} c_{m-l} = \sum_{l=0}^{m} c_{m-l}/\mu_l,$$

and the conclusion follows by (2.1). □

We now conclude that the Berezin transform reproduces harmonic polynomials.

Corollary 2.1. *Let $T\in\mathcal{L}(\mathcal{H})$ be an operator such that $r(T)<1$. Then for every harmonic function $f=\sum c_k r^{|k|}e^{ik\theta}$ ($z=re^{i\theta}$) in $C(\bar{\mathbb{D}})$ we have that*

$$\int_{\bar{\mathbb{D}}} B_\mu(T,\zeta)f(\zeta)d\mu(\zeta) = \sum_{k=-\infty}^{\infty} c_k T(k),$$

where $T(k)=T^k$ for $k\geq 0$ and $T(k)=T^{*|k|}$ for $k<0$.

Proof. By Proposition 2.1 and Lemma 2.2 we have that

$$\begin{aligned}\int_{\bar{\mathbb{D}}} B_\mu(T,\zeta)f(\zeta)d\mu(\zeta) &= \lim_{r\to 1}\int_{\bar{\mathbb{D}}} B_\mu(T,\zeta)f(r\zeta)d\mu(\zeta)\\ &= \lim_{r\to 1}\sum_{k=-\infty}^{\infty} c_k r^{|k|}T(k) = \sum_{k=-\infty}^{\infty} c_k T(k) \quad \text{in } \mathcal{L}(\mathcal{H}),\end{aligned}$$

where the limits are computed in the uniform operator topology. □

We wish to point out that the assumption of non-vanishing of the kernel function K_μ in $\mathbb{D}\times\mathbb{D}$ is of a non-trivial nature even for simple measures μ. Let

$$d\mu = c\,d\delta_0 + d\mu_n,$$

where $c\geq 0$ is a positive parameter and $d\delta_0$ is the unit Dirac mass at 0. A straightforward computation shows that

$$K_\mu(z,\zeta) = \Big(\frac{1}{c+1}-1\Big) + \frac{1}{(1-\bar{\zeta}z)^n}, \quad (z,\zeta)\in\mathbb{D}\times\mathbb{D}.$$

It is a straightforward matter to verify that for c large the function K_μ has zeroes in $\mathbb{D} \times \mathbb{D}$. This example has been communicated to the author by Carl Sundberg (private discussion).

On the other hand, Hedenmalm and Perdomo [20] have shown that if the weight function $w : \mathbb{D} \to (0, \infty)$ is such that the function

$$\mathbb{D} \ni z \mapsto \log\big(w(z)/(1-|z|^2)\big)$$

is subharmonic, then the corresponding kernel function K_w is non-vanishing in $\mathbb{D} \times \mathbb{D}$.

3. Construction of the operator measure $d\omega_{n,T}$

The purpose of this section is to construct the operator measure $d\omega_{n,T}$ and discuss some of its properties. Let $n \geq 1$ and let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$. The operator-valued Berezin kernel is defined by the formula

$$B_n(T,\zeta) = (I - \zeta T^*)^{-n}\Big(\sum_{k=0}^{n}(-1)^k\binom{n}{k}T^{*k}T^k\Big)(I-\bar{\zeta}T)^{-n}, \quad \zeta \in \bar{\mathbb{D}}.$$

Recall from Section 2 that

$$\int_{\bar{\mathbb{D}}} B_n(T,\zeta)\bar{\zeta}^j\zeta^k d\mu_n(\zeta) = T^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{n;m;j,k}T^{*m}T^m\Big)T^{k-\min(j,k)}$$

for $j,k \geq 0$, where

$$W_{n;m;j,k} = \sum_{l=0}^{\min(m,n)} \frac{\mu_{n;j+k-\min(j,k)+m-l}}{\mu_{n;j-\min(j,k)+m-l}\mu_{n;k-\min(j,k)+m-l}}(-1)^l\binom{n}{l}$$

(see Proposition 2.1); here $\mu_{n;k} = 1/\binom{k+n-1}{k}$ for $k \geq 0$ are the moments of $d\mu_n$.

We shall need the following lemma.

Lemma 3.1. *Let $n \geq 1$ and $j,k \geq 0$ be integers. Then*

$$W_{n;m;j,k} = O\big(m^{-(n+1)}\big) \quad \text{as } m \to \infty.$$

Proof. We set

$$a_l = \frac{\mu_{n;j+k-\min(j,k)+l}}{\mu_{n;j-\min(j,k)+l}\mu_{n;k-\min(j,k)+l}}.$$

Since

$$\frac{1}{\mu_{n;k}} = \binom{k+n-1}{k} = \frac{1}{(n-1)!}\prod_{s=1}^{n-1}(k+s),$$

we have that

$$a_l = \frac{1}{(n-1)!}\frac{\prod_{s=1}^{n-1}(j-\min(j,k)+l+s)\prod_{s=1}^{n-1}(k-\min(j,k)+l+s)}{\prod_{s=1}^{n-1}(j+k-\min(j,k)+l+s)}.$$

In particular, the coefficient a_l is a rational function in l. By the division algorithm, we see that a_l has the form

$$a_l = p(l) + \frac{q(l)}{r(l)}, \tag{3.1}$$

where p, q, r are polynomials with $\deg(p) \le n-1$ and $\deg(q) < \deg(r) \le n-1$.

We now consider the shift operator σ acting on sequences $a = \{a_k\}_{k=0}^{\infty}$ by $(\sigma a)_k = a_{k-1}$ for $k \ge 1$ and $(\sigma a)_0 = 0$. In terms of this shift operator we have that

$$W_{n;m;j,k} = \sum_{l=0}^{\min(m,n)} (-1)^l \binom{n}{l} a_{m-l} = \big((I-\sigma)^n a\big)_m, \quad m \ge 0.$$

Now using the above algebraic form (3.1) of the coefficients a_l we see that

$$\big((I-\sigma)^n a\big)_m = O\big(m^{-(n+1)}\big) \quad \text{as } m \to \infty.$$

The conclusion of the lemma follows. □

Remark 3.1. For $n = 1$ the numbers $W_{n;m;j,k}$ are easily computable. We have that $W_{1;0;j,k} = 1$ and $W_{1;m;j,k} = 0$ for $m \ge 1$.

Remark 3.2. Let us also consider the case when $n = 2$ and $j = k = 1$. A computation gives that $W_{2;0;1,1} = 1/2$, $W_{2;1;1,1} = 1/3$ and

$$W_{2;m;1,1} = \frac{1}{m+2} - \frac{2}{m+1} + \frac{1}{m} = \frac{2}{(m+2)(m+1)m}$$

for $m \ge 2$. In particular, we see that decay estimate in Lemma 3.1 gives the right order of magnitude in this case. A further computation using Proposition 2.1 gives that

$$B_2[|\zeta|^2](z) = \sum_{m\ge 0} W_{2;m;1,1}|z|^{2m} = 2 - \frac{1}{|z|^2} + \Big(1 - \frac{1}{|z|^2}\Big)^2 \log\Big(\frac{1}{1-|z|^2}\Big), \quad z \in \mathbb{D},$$

which is not C^2-smooth up to the boundary $\mathbb{T} = \partial\mathbb{D}$.

We are now ready to construct the operator measure $d\omega_{n,T}$.

Theorem 3.1. *Let $n \ge 1$ be an integer, and let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then there exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_{n,T}$ on the closed unit disc $\bar{\mathbb{D}}$ such that*

$$\int_{\bar{\mathbb{D}}} \bar\zeta^j \zeta^k d\omega_{n,T}(\zeta) = T^{*(j-\min(j,k))}\Big(\sum_{m\ge 0} W_{n;m;j,k} T^{*m}T^m\Big) T^{k-\min(j,k)},$$

where

$$W_{n;m;j,k} = \sum_{l=0}^{\min(m,n)} \frac{\mu_{n;j+k-\min(j,k)+m-l}}{\mu_{n;j-\min(j,k)+m-l}\,\mu_{n;k-\min(j,k)+m-l}} (-1)^l \binom{n}{l}$$

for $j, k \ge 0$; here $\mu_{n;k} = 1/\binom{k+n-1}{k}$. Furthermore, the operator measure $d\omega_{n,T}$ is uniquely determined by this action on monomials.

Proof. We set

$$\Lambda(\bar{z}^j z^k) = T^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{n;m;j,k} T^{*m} T^m\Big) T^{k-\min(j,k)}$$

for monomials $\bar{z}^j z^k$, $j,k \geq 0$, and extend this map Λ linearly to a linear map Λ from the space $\mathbb{C}[z,\bar{z}]$ of polynomials in z and $\bar{z}$ into $\mathcal{L}(\mathcal{H})$. We shall show below that this map Λ extends uniquely to a bounded linear map from $C(\bar{\mathbb{D}})$ into $\mathcal{L}(\mathcal{H})$ of norm less than or equal to 1 with the property that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for $0 \leq f \in C(\bar{\mathbb{D}})$.

By an operator version of the F. Riesz representation theorem (see the preliminaries in the introduction) it then follows that there exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_{n,T}$ on $\bar{\mathbb{D}}$ such that

$$\Lambda(f) = \int_{\bar{\mathbb{D}}} f(z) d\mu_{n,T}(z), \quad f \in C(\bar{\mathbb{D}}).$$

Clearly, this operator measure $d\omega_{n,T}$ has the action on monomials described in the theorem. Since the polynomials in $\mathbb{C}[z,\bar{z}]$ is dense in $C(\bar{\mathbb{D}})$ (Stone-Weierstrass) it is clear that the operator measure $d\omega_{n,T}$ is uniquely determined by its action on monomials.

We now proceed to prove the estimates needed. Let $f(z) = \sum_{j,k\geq 0} c_{jk}\bar{z}^j z^k$ be a polynomial in $\mathbb{C}[z,\bar{z}]$ and $0 \leq r < 1$. By Proposition 2.1 we have that

$$\begin{aligned} &\int_{\bar{\mathbb{D}}} B_n(rT,\zeta) f(\zeta) d\mu_n(\zeta) \\ &\quad = \sum_{j,k\geq 0} c_{jk} (rT)^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{n;m;j,k}(rT)^{*m}(rT)^m\Big)(rT)^{k-\min(j,k)}. \end{aligned} \tag{3.2}$$

By Proposition 1.1 the Berezin kernel $B_n(rT,\zeta)$ is positive in $\mathcal{L}(\mathcal{H})$. We have now that the left-hand side in (3.2) is of norm less than or equal to $\|f\|_{C(\bar{\mathbb{D}})}$ (see the preliminaries in the introduction) and is positive in $\mathcal{L}(\mathcal{H})$ if $f \geq 0$ in $\mathbb{D}$. Also, the right-hand side in (3.2) tends to $\Lambda(f)$ in $\mathcal{L}(\mathcal{H})$ as $r \to 1$. Passing to the limit as $r \to 1$ we conclude that $\|\Lambda(f)\| \leq \|f\|_{C(\bar{\mathbb{D}})}$ for $f \in \mathbb{C}[z,\bar{z}]$ and that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $f \in \mathbb{C}[z,\bar{z}]$ is such that $f \geq 0$ in $\mathbb{D}$. Since the polynomials in $\mathbb{C}[z,\bar{z}]$ is dense in $C(\bar{\mathbb{D}})$ (Stone-Weierstrass), the map Λ extends uniquely to a continuous linear map $\Lambda : C(\bar{\mathbb{D}}) \to \mathcal{L}(\mathcal{H})$ of norm less than or equal to 1. Let us verify the positivity property that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $0 \leq f \in C(\bar{\mathbb{D}})$.

Since $0 \leq f \in C(\bar{\mathbb{D}})$, also the function $\sqrt{f}$ is in $C(\bar{\mathbb{D}})$ and we can find a sequence $\{p_j\}$ of polynomials in $\mathbb{C}[z,\bar{z}]$ such that $p_j \to \sqrt{f}$ in $C(\bar{\mathbb{D}})$. Now the polynomial $f_j = |p_j|^2$ is positive and we have that $f_j \to f$ in $C(\bar{\mathbb{D}})$. Now $\Lambda(f) = \lim_{j\to\infty} \Lambda(f_j) \geq 0$ in $\mathcal{L}(\mathcal{H})$. This completes the proof of the theorem. □

The operator measure $d\omega_{n,T}$ is positive and $\omega_{n,T}(\bar{\mathbb{D}}) = \int d\omega_{n,T} = I$. By these properties we have the inequality

$$\Big\| \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) \Big\| \leq \|f\|_\infty, \quad f \in C(\bar{\mathbb{D}}) \tag{3.3}$$

(see the preliminaries in the introduction). We also have that $\int f d\omega_{n,T} \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $f \geq 0$ in $\bar{\mathbb{D}}$.

We next observe that the operator measure $d\omega_{n,T}$ generalizes the notion of operator-valued Berezin transform.

Corollary 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction such that $r(T) < 1$. Then*

$$d\omega_{n,T}(\zeta) = B_n(T,\zeta) d\mu_n(\zeta), \quad \zeta \in \bar{\mathbb{D}}.$$

Proof. By the formulas stated in the first paragraph in this section the operator measures $B_n(T,\cdot)d\mu_n$ and $d\omega_{n,T}$ have the same action on monomials. The corollary follows by the uniqueness assertion of Theorem 3.1. □

We remark that in terms of action on test functions the assertion of Corollary 3.1 means that

$$\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) = \int_{\bar{\mathbb{D}}} B_n(T,\zeta) f(\zeta) d\mu_n(\zeta), \quad f \in C(\bar{\mathbb{D}}).$$

The operator measures $d\omega_{n,T}$ enjoy the following continuity property.

Theorem 3.2. *Denote by $\mathcal{C}_n$ the set all n-hypercontractions in $\mathcal{L}(\mathcal{H})$, and let $\mathcal{C}_n$ and $\mathcal{L}(\mathcal{H})$ be equipped with the uniform operator topology. Then the map*

$$C(\bar{\mathbb{D}}) \times \mathcal{C}_n \ni (f,T) \mapsto \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) \in \mathcal{L}(\mathcal{H})$$

is continuous.

Proof. Let $\{T_m\}$ be a sequence of n-hypercontractions such that $T_m \to T_0$ in $\mathcal{L}(\mathcal{H})$. Using Lemma 3.1 it is straightforward to check that

$$\int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{n,T_m}(\zeta) \to \int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{n,T_0}(\zeta) \quad \text{in } \mathcal{L}(\mathcal{H})$$

as $m \to \infty$. By linearization we see that $\int P d\omega_{n,T_m} \to \int P d\omega_{n,T_0}$ in $\mathcal{L}(\mathcal{H})$ for every polynomial P in $\mathbb{C}[z,\bar{z}]$.

The proof is now completed by a standard approximation argument. Let also $f_m \to f_0$ in $C(\bar{\mathbb{D}})$, and fix $\varepsilon > 0$. By approximation (Stone-Weierstrass) we can find a polynomial P in $\mathbb{C}[z,\bar{z}]$ such that $\|f_0 - P\|_{C(\bar{\mathbb{D}})} < \varepsilon/4$. Recall the inequality (3.3). We now have that

$$\begin{aligned}
\Big\| \int_{\bar{\mathbb{D}}} f_m(\zeta) d\omega_{n,T_m}(\zeta) - \int_{\bar{\mathbb{D}}} f_0(\zeta) d\omega_{n,T_0}(\zeta) \Big\| &\leq \Big\| \int_{\bar{\mathbb{D}}} \big(f_m(\zeta) - f_0(\zeta)\big) d\omega_{n,T_m}(\zeta) \Big\| \\
&+ \Big\| \int_{\bar{\mathbb{D}}} \big(f_0(\zeta) - P(\zeta)\big) d\omega_{n,T_m}(\zeta) \Big\| + \Big\| \int_{\bar{\mathbb{D}}} P(\zeta) d\omega_{n,T_m}(\zeta) - \int_{\bar{\mathbb{D}}} P(\zeta) d\omega_{n,T_0}(\zeta) \Big\| \\
&+ \Big\| \int_{\bar{\mathbb{D}}} \big(P(\zeta) - f_0(\zeta)\big) d\omega_{n,T_0}(\zeta) \Big\| < \varepsilon
\end{aligned}$$

for m large. □

The previous results yield in particular a uniform functional calculus for the class of n-hypercontractions. Let $u = B_n[f]$ be the Berezin transform of $f \in C(\bar{\mathbb{D}})$. The function u is real-analytic in $\mathbb{D}$ and has a power series expansion

$$u(z) = \sum_{j,k \geq 0} c_{jk} \bar{z}^j z^k, \quad z \in \mathbb{D}.$$

For an operator $T \in \mathcal{L}(\mathcal{H})$ such that $r(T) < 1$ we set

$$u(T) := \sum_{j,k \geq 0} c_{jk} T^{*j} T^k \quad \text{in } \mathcal{L}(\mathcal{H}). \tag{3.4}$$

Notice that by the spectral radius formula the series in (3.4) is absolutely convergent in $\mathcal{L}(\mathcal{H})$. This functional calculus extends naturally to the class of n-hypercontractions.

Theorem 3.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction and let $u = B_n[f]$ be the Berezin transform of $f \in C(\bar{\mathbb{D}})$. Then*

$$\lim_{r \to 1} u(rT) = \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) \quad \textit{in } \mathcal{L}(\mathcal{H}).$$

Proof. We consider first the case when $T \in \mathcal{L}(\mathcal{H})$ is an operator such that $r(T) < 1$. By Lemma 2.1 we have that

$$\int_{\bar{\mathbb{D}}} B_n(T, \zeta) f(\zeta) d\mu_n(\zeta) = \sum_{r,s \geq 0} \int_{\bar{\mathbb{D}}} p_{rs}(\zeta) f(\zeta) d\mu_n(\zeta)\, T^{*r} T^s, \tag{3.5}$$

where

$$p_{rs}(\zeta) = \sum_{l=0}^{\min(n,r,s)} \frac{1}{\mu_{n;r-l} \mu_{n;s-l}} (-1)^l \binom{n}{l} \zeta^{r-l} \bar{\zeta}^{s-l}.$$

If we substitute zI, $z \in \mathbb{D}$, for T in (3.5) we obtain the power series expansion of u, that is,

$$u(z) = \sum_{r,s \geq 0} c_{rs} \bar{z}^r z^s, \quad z \in \mathbb{D},$$

where $c_{rs} = \int p_{rs} f d\mu_n$. We now conclude that

$$u(T) = \int_{\bar{\mathbb{D}}} B_n(T, \zeta) f(\zeta) d\mu_n(\zeta),$$

where $u(T)$ is defined by (3.4).

Let us now consider the case when $T \in \mathcal{L}(\mathcal{H})$ is an n-hypercontraction. By the result of the previous paragraph we have that

$$u(rT) = \int_{\bar{\mathbb{D}}} B_n(rT, \zeta) f(\zeta) d\mu_n(\zeta) = \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,rT}(\zeta),$$

where the last equality follows by Corollary 3.1. Notice that the operator rT is an n-hypercontraction by Proposition 1.1, so that $d\omega_{n,rT}$ exists by Theorem 3.1. The conclusion of the theorem now follows by Theorem 3.2. □

Remark 3.3. We remark that in the first paragraph in the proof of Theorem 3.3 we showed that

$$u(T) = \int_{\bar{\mathbb{D}}} B_n(T,\zeta) f(\zeta) d\mu_n(\zeta)$$

when $u(T)$ is defined by (3.4) and $T \in \mathcal{L}(\mathcal{H})$ is an arbitrary operator such that $r(T) < 1$.

The operator measures $d\omega_{n,T}$ have a property of invariance with respect to conformal automorphisms of the unit disc.

Corollary 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then we have the invariance property that*

$$\int_{\bar{\mathbb{D}}} (f \circ \varphi)(\zeta) d\omega_{n,T}(\zeta) = \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,\varphi(T)}(\zeta), \quad f \in C(\bar{\mathbb{D}}),$$

for every $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Proof. Let $0 \leq r < 1$. By Theorem 1.1 we have that

$$\int_{\bar{\mathbb{D}}} B_n(rT,\zeta)(f \circ \varphi)(\zeta) d\mu_n(\zeta) = \int_{\bar{\mathbb{D}}} B_n(\varphi(rT),\zeta) f(\zeta) d\mu_n(\zeta),$$

which we can restate as

$$\int_{\bar{\mathbb{D}}} (f \circ \varphi)(\zeta) d\omega_{n,rT}(\zeta) = \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,\varphi(rT)}(\zeta).$$

Notice that the operators rT and $\varphi(rT)$ are n-hypercontractions by Proposition 1.1 and Corollary 1.2. Letting $r \to 1$ the conclusion of the corollary now follows by Theorem 3.2. □

Notice that $\varphi(T)$ is an n-hypercontraction by Corollary 1.2 so that the operator measure $d\omega_{n,\varphi(T)}$ exists by Theorem 3.1.

4. Relations with the operator measure $d\omega_T$

It is well-known that every contraction $T \in \mathcal{L}(\mathcal{H})$ has a unitary dilation, that is, there exists a unitary operator $U \in \mathcal{L}(\mathcal{K})$ on some larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace such that

$$T^k = PU^k|_{\mathcal{H}}, \quad k \geq 0,$$

where P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$. This unitary dilation $U \in \mathcal{L}(\mathcal{K})$ can be chosen minimal in the sense that $\mathcal{K} = \bigvee_{k=-\infty}^{\infty} U^k(\mathcal{H})$ and is then uniquely determined up to isomorphism. We refer to [27, Chapter I] for details of the construction.

Let now $U \in \mathcal{L}(\mathcal{K})$ be a unitary dilation of a contraction $T \in \mathcal{L}(\mathcal{H})$. By the spectral theorem the unitary operator $U \in \mathcal{L}(\mathcal{K})$ has an $\mathcal{L}(\mathcal{K})$-valued spectral measure dE supported by the unit circle $\mathbb{T}$. Compressing the spectral measure dE

down to $\mathcal{H}$ we obtain a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\omega_T$ on $\mathbb{T}$ by the requirement that

$$\omega_T(S) = PE(S)|_{\mathcal{H}}, \quad S \in \mathfrak{S};$$

here as above P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$ and $\mathfrak{S}$ is the σ-algebra of planar Borel sets. This operator measure $d\omega_T$ does not depend on the particular choice of unitary dilation $U \in \mathcal{L}(\mathcal{K})$ of $T \in \mathcal{L}(\mathcal{H})$ (see the formula for the Fourier coefficients $\hat{\omega}_T(k)$ below).

In terms of Fourier coefficients the operator measure $d\omega_T$ is characterized by the requirement that

$$\hat{\omega}_T(k) = \int_{\mathbb{T}} e^{-ik\theta} d\omega_T(e^{i\theta}) = \begin{cases} T^{*k} & \text{for } k \geq 0, \\ T^{|k|} & \text{for } k < 0. \end{cases}$$

In the case of a contraction $T \in \mathcal{L}(\mathcal{H})$ such that $r(T) < 1$ the operator measure $d\omega_T$ has the explicit form

$$d\omega_T(e^{i\theta}) = P(T, e^{i\theta})\, d\theta/2\pi, \quad e^{i\theta} \in \mathbb{T},$$

where $P(T, \cdot)$ is the operator-valued Poisson kernel given by the formula

$$P(T, e^{i\theta}) = (I - e^{i\theta}T^*)^{-1}(I - T^*T)(I - e^{-i\theta}T)^{-1}, \quad e^{i\theta} \in \mathbb{T}.$$

We refer to the paper [26] for the similar construction in the context of the unit polydisc $\mathbb{D}^n$ in $\mathbb{C}^n$. In the terminology of Foias [15, 16] the operator measure $d\omega_T$ is called the harmonic spectral measure for the contraction T with respect to the spectral set $\bar{\mathbb{D}}$.

We next observe that $d\omega_{1,T} = d\omega_T$.

Proposition 4.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction. Then*

$$d\omega_{1,T}(\zeta) = d\omega_T(\zeta), \quad \zeta \in \bar{\mathbb{D}}.$$

In particular, the operator measure $d\omega_{1,T}$ is supported by the unit circle $\mathbb{T}$.

Proof. By Remark 3.1 we know that $W_{1;0;j,k} = 1$ and $W_{1;m;j,k} = 0$ for $m \geq 1$. By Theorem 3.1 we now have that

$$\int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{1,T}(\zeta) = T^{*(j-\min(j,k))}T^{k-\min(j,k)} = \hat{\omega}_T(j-k) = \int_{\mathbb{T}} \bar{\zeta}^j \zeta^k d\omega_T(\zeta)$$

for all $j, k \geq 0$. By linearization we see that $\int P d\omega_{1,T} = \int P d\omega_T$ for every polynomial $P \in \mathbb{C}[z, \bar{z}]$ and an approximation argument (Stone-Weierstrass) gives that $\int f d\omega_{1,T} = \int f d\omega_T$ for every $f \in C(\bar{\mathbb{D}})$. This completes the proof of the proposition. □

We remark that in terms of action on test functions the assertion of Proposition 4.1 means that

$$\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{1,T}(\zeta) = \int_{\mathbb{T}} f(e^{i\theta}) d\omega_T(e^{i\theta}), \quad f \in C(\bar{\mathbb{D}})$$

(compare Proposition 4.2 below).

We shall need the following lemma.

Lemma 4.1. *Let the numbers* $W_{n;m;j,k}$ *be as in Section* 3. *Then*

$$\sum_{m\geq 0} W_{n;m;j,k} = 1.$$

Proof. We shall use a property of the Berezin transform of a continuous function. Namely, if $f \in C(\bar{\mathbb{D}})$, then $B_n[f] \in C(\bar{\mathbb{D}})$ and $B_n[f] = f$ on $\mathbb{T}$. For a proof of this fact we refer to [19, Proposition 2.3].

Let us now turn to the proof of the lemma. Notice first that the sum in the lemma is absolutely convergent by Lemma 3.1. By Proposition 2.1 we have the power series expansion

$$\int_{\bar{\mathbb{D}}} B_n(z,\zeta)\bar{\zeta}^j\zeta^k d\mu_n(\zeta) = \bar{z}^{j-\min(j,k)}\Big(\sum_{m\geq 0} W_{n;m;j,k}|z|^{2m}\Big)z^{k-\min(j,k)}, \quad z\in\mathbb{D}.$$

Now letting $z \to 1$ using the property of the Berezin transform quoted in the previous paragraph the conclusion of the lemma follows. □

We shall now consider the case of an isometry.

Proposition 4.2. *Let* $T \in \mathcal{L}(\mathcal{H})$ *be an isometry. Then*

$$d\omega_{n,T}(\zeta) = d\omega_T(\zeta), \quad \zeta \in \bar{\mathbb{D}},$$

for $n \geq 1$. *In particular, the operator measure* $d\omega_{n,T}$ *does not depend on* $n \geq 1$ *and is supported by the unit circle* $\mathbb{T}$.

Proof. By the construction of the operator measure $d\omega_{n,T}$ in Theorem 3.1 we know that

$$\int_{\bar{\mathbb{D}}} \bar{\zeta}^j\zeta^k d\omega_{n,T}(\zeta) = T^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{n;m;j,k}T^{*m}T^m\Big)T^{k-\min(j,k)}. \tag{4.1}$$

Since T is an isometry, meaning that $T^*T = I$, we have by (4.1) and Lemma 4.1 that

$$\int_{\bar{\mathbb{D}}} \bar{\zeta}^j\zeta^k d\omega_{n,T}(\zeta) = T^{*(j-\min(j,k))}T^{k-\min(j,k)}$$

for all $j,k \geq 0$. The conclusion of the proposition now follows by a linearization and approximation argument (see the proof of Proposition 4.1). □

5. The space $A_n(\mathcal{E})$ and its shift operator S_n

In this section we shall discuss some properties of the shift operator S_n and its adjoint S_n^* acting on the space $A_n(\mathcal{E})$. Let $\mathcal{E}$ be a Hilbert space. We denote by $A_n(\mathcal{E})$ the Hilbert space of all $\mathcal{E}$-valued analytic functions

$$f(z) = \sum_{k\geq 0} a_k z^k, \quad z \in \mathbb{D}; \tag{5.1}$$

here $a_k \in \mathcal{E}$ for $k \geq 0$, with finite norm

$$\|f\|_{A_n}^2 = \sum_{k\geq 0} \|a_k\|^2 \mu_{n;k},$$

where $\mu_{n;k} = 1/\binom{k+n-1}{k}$ for $k \geq 0$. The norm of $A_n(\mathcal{E})$ can also be written

$$\|f\|_{A_n}^2 = \lim_{r\to 1} \int_{\bar{\mathbb{D}}} \|f(rz)\|^2 d\mu_n(z).$$

The measure $d\mu_1$ is the normalized (mass 1) Lebesgue arc length measure on the unit circle $\mathbb{T}$, and for $n \geq 2$ the measure $d\mu_n$ is the weighted area measure given by

$$d\mu_n(z) = (n-1)(1-|z|^2)^{n-2} dA(z), \quad z \in \mathbb{D},$$

where $dA(z) = dxdy/\pi$, $z = x + iy$, is the normalized Lebesgue area measure.

On the space $A_n(\mathcal{E})$ we have a natural shift operator $S = S_n$ defined by

$$(S_n f)(z) = zf(z) = \sum_{k\geq 1} a_{k-1} z^k, \quad z \in \mathbb{D},$$

for $f \in A_n(\mathcal{E})$ given by (5.1). In fact, by the formula

$$\frac{1}{\mu_{n;k}} = \binom{k+n-1}{k} = \frac{1}{(n-1)!} \prod_{j=1}^{n-1} (k+j)$$

we see that the weight sequence $\{\mu_{n;k}\}_{k\geq 0}$ is decreasing in $k \geq 0$ and that the ratio $\mu_{n;k+1}/\mu_{n;k}$ tends to 1 as $k \to \infty$. Therefore the operator S_n is bounded on $A_n(\mathcal{E})$ of norm equal to 1. The adjoint operator S_n^* of S_n has the form

$$(S_n^* f)(z) = \sum_{k\geq 0} \frac{\mu_{n;k+1}}{\mu_{n;k}} a_{k+1} z^k, \quad z \in \mathbb{D}, \tag{5.2}$$

where $f \in A_n(\mathcal{E})$ is given by (5.1) above.

For later use it will be convenient to have available the following lemma.

Lemma 5.1. *Let S_n be as above and let $f \in A_n(\mathcal{E})$ be given by* (5.1). *Then*

$$\sum_{j=0}^{m} (-1)^j \binom{m}{j} \|S_n^{*j} f\|_{A_n}^2 = \sum_{k\geq 0} \binom{k+n-m-1}{k} \mu_{n;k}^2 \|a_k\|^2$$

for $1 \leq m < n$, and

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} \|S_n^{*j} f\|_{A_n}^2 = \|a_0\|^2.$$

Proof. By formula (5.2) we have that

$$\|S_n^{*j} f\|_{A_n}^2 = \sum_{k\geq 0} \frac{\mu_{n;k+j}^2}{\mu_{n;k}} \|a_{k+j}\|^2.$$

Let $1 \le m \le n$. A computation shows that

$$\sum_{j=0}^{m}(-1)^j\binom{m}{j}\|S_n^{*j}f\|_{A_n}^2 = \sum_{j=0}^{m}(-1)^j\binom{m}{j}\sum_{k\ge 0}\frac{\mu_{n;k+j}^2}{\mu_{n;k}}\|a_{k+j}\|^2 \tag{5.3}$$

$$= \sum_{k\ge 0}\Big(\sum_{j=0}^{\min(m,k)}(-1)^j\binom{m}{j}\frac{1}{\mu_{n;k-j}}\Big)\mu_{n;k}^2\|a_k\|^2,$$

where the last equality follows by a change of order of summation. We now notice that the sum

$$\sum_{j=0}^{\min(m,k)}(-1)^j\binom{m}{j}\frac{1}{\mu_{n;k-j}}$$

equals the k-th coefficient in the power series expansion of the function

$$(1-z)^m\frac{1}{(1-z)^n} = \frac{1}{(1-z)^{n-m}}.$$

We conclude that

$$\sum_{j=0}^{\min(m,k)}(-1)^j\binom{m}{j}\frac{1}{\mu_{n;k-j}} = \binom{k+n-m-1}{k}$$

for $1 \le m < n$, and that

$$\sum_{j=0}^{\min(n,k)}(-1)^j\binom{n}{j}\frac{1}{\mu_{n;k-j}} = \delta_{k,0},$$

where $\delta_{0,0} = 1$ and $\delta_{k,0} = 0$ for $k \ge 1$ is the Kronecker's delta. Substituting the values of these last two sums into (5.3) we arrive at the formulas in the lemma. □

The following proposition establishes two basic properties of the adjoint shift operator S_n^*.

Proposition 5.1. *The operator $S_n^* : A_n(\mathcal{E}) \to A_n(\mathcal{E})$ is an n-hypercontraction such that $\lim_{k\to\infty} S_n^{*k} = 0$ in the strong operator topology.*

Proof. Let us first verify that $S_n^{*k} \to 0$ (SOT). If $f \in A_n(\mathcal{E})$ is a polynomial, then clearly $S_n^{*k}f = 0$ for k large. Since $\|S_n^{*k}\| \le 1$ we conclude by an approximation argument that $S_n^{*k}f \to 0$ in $A_n(\mathcal{E})$ for every $f \in A_n(\mathcal{E})$. The assertion that the operator S_n^* is an n-hypercontraction is evident by Lemma 5.1. □

We shall now compute the operator measure $d\omega_{n,S_n^*}$.

Proposition 5.2. *For $n \ge 2$, we have that*

$$d\omega_{n,S_n^*}(\zeta) = B_n(S_n^*,\zeta)d\mu_n(\zeta), \quad \zeta \in \bar{\mathbb{D}}.$$

Proof. By Lemma 1.3 we know that the function $B_n(S_n^*, \cdot)$ is integrable with respect to $d\mu_n$. To prove the proposition it suffices to show that

$$\int_{\bar{\mathbb{D}}} \bar\zeta^j \zeta^k d\omega_{n,S_n^*}(\zeta) = \int_{\mathbb{D}} B_n(S_n^*, \zeta)\bar\zeta^j\zeta^k d\mu_n(\zeta) \tag{5.4}$$

for $j, k \geq 0$. The conclusion of the proposition then follows by a linearization and approximation argument (see the proof of Proposition 4.1).

Let now $f, g \in A_n(\mathcal{E})$ be polynomials and $0 \leq r < 1$. Since $S_n^{*k} f = 0$ for k large for such an f, the resolvent sum

$$(I - r\bar\zeta S_n^*)^{-n} f = \sum_{k\geq 0} \frac{1}{\mu_{n;k}} r^k \bar\zeta^k S_n^{*k} f$$

is finite. Therefore, the function

$$\langle B_n(rS_n^*, \zeta)f, g\rangle = \langle\Big(\sum_{k=0}^{n} (-1)^k r^{2k} \binom{n}{k} S_n^k S_n^{*k}\Big)(I - r\bar\zeta S_n^*)^{-n} f, (I - r\bar\zeta S_n^*)^{-n} g\rangle$$

is a polynomial in r, ζ and $\bar\zeta$. By Corollary 3.1 we now have that

$$\langle \int_{\bar{\mathbb{D}}} \bar\zeta^j\zeta^k d\omega_{n,rS_n^*}(\zeta) f, g\rangle = \langle \int_{\mathbb{D}} B_n(rS_n^*, \zeta)\bar\zeta^j\zeta^k d\mu_n(\zeta) f, g\rangle$$
$$= \int_{\mathbb{D}} \langle B_n(rS_n^*, \zeta) f, g\rangle \bar\zeta^j\zeta^k d\mu_n(\zeta) \to \int_{\mathbb{D}} \langle B_n(S_n^*, \zeta) f, g\rangle \bar\zeta^j\zeta^k d\mu_n(\zeta)$$

as $r \to 1$. Since also

$$\int_{\bar{\mathbb{D}}} \bar\zeta^j\zeta^k d\omega_{n,rS_n^*}(\zeta) \to \int_{\bar{\mathbb{D}}} \bar\zeta^j\zeta^k d\omega_{n,S_n^*}(\zeta) \quad \text{in } \mathcal{L}(\mathcal{H})$$

as $r \to 1$ by Theorem 3.2, we conclude that

$$\langle \int_{\mathbb{D}} B_n(S_n^*, \zeta)\bar\zeta^j\zeta^k d\mu_n(\zeta) f, g\rangle = \langle \int_{\bar{\mathbb{D}}} \bar\zeta^j\zeta^k d\omega_{n,S_n^*}(\zeta) f, g\rangle$$

for f, g polynomials in $A_n(\mathcal{E})$. By approximation (5.4) follows. □

We remark that in terms of action on test functions the assertion of Proposition 5.2 means that

$$\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,S_n^*}(\zeta) = \int_{\mathbb{D}} B_n(S_n^*, \zeta) f(\zeta) d\mu_n(\zeta), \quad f \in C(\bar{\mathbb{D}}).$$

6. Operator model theory. General considerations

In this section we consider the problem of modeling an operator $T \in \mathcal{L}(\mathcal{H})$ as part of an operator of the form $T_1^* \oplus T_2$, where $T_j \in \mathcal{L}(\mathcal{H}_j)$ $(j = 1, 2)$ are operators such that $T_1^{*k} \to 0$ in the strong operator topology and T_2 is an isometry. The principal results in this section are Theorems 6.1 and 6.2 below. Let us begin with some general considerations.

Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that the limit

$$\lim_{k\to\infty} \|T^k x\|^2$$

exists for every $x \in \mathcal{H}$. Since $\|T^k x\|^2 = \langle T^{*k}T^k x, x\rangle$ for $x \in \mathcal{H}$, we have by polarization that the operator limit $\lim_{k\to\infty} T^{*k}T^k$ exists in the weak operator topology. We can now introduce the operator

$$Q = \Big(\lim_{k\to\infty} T^{*k}T^k\Big)^{1/2} \quad \text{in } \mathcal{L}(\mathcal{H}),$$

where the positive square root is used. Notice that

$$\|Qx\|^2 = \lim_{k\to\infty} \|T^k x\|^2, \quad x \in \mathcal{H}. \tag{6.1}$$

Associated to the operator Q we have the range space $\mathcal{Q}$ defined as the closure in $\mathcal{H}$ of the range of Q, that is, $\mathcal{Q} = \overline{Q(\mathcal{H})}$. By (6.1) we see that the formula

$$U : Qx \mapsto QTx$$

gives a well-defined map which by continuity extends uniquely to an isometry U on $\mathcal{Q}$ satisfying the intertwining relation $QT = UQ$.

We have the following theorem.

Theorem 6.1. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $T_j \in \mathcal{L}(\mathcal{H}_j)$ $(j = 1, 2)$ be operators such that $T_1^{*k} \to 0$ in the strong operator topology and T_2 is an isometry on $\mathcal{H}_2$. Assume that we have an isometry*

$$V = (V_1, V_2) : \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

of $\mathcal{H}$ into $\mathcal{H}_1 \oplus \mathcal{H}_2$ satisfying the intertwining relation

$$VT = (T_1^* \oplus T_2)V. \tag{6.2}$$

Then

- *the limit $\lim_{k\to\infty} \|T^k x\|^2$ exists for every $x \in \mathcal{H}$,*
- *the map $V_2 : \mathcal{H} \to \mathcal{H}_2$ factorizes as $V_2 = \hat{V}_2 Q$, where $\hat{V}_2 : \mathcal{Q} \to \mathcal{H}_2$ is an isometry, in such a way that the intertwining relation $\hat{V}_2 U = T_2 \hat{V}_2$ holds,*
- *the operator $V_1 : \mathcal{H} \to \mathcal{H}_1$ satisfies the norm equality*

$$\|x\|^2 - \|Tx\|^2 = \|V_1 x\|^2 - \|T_1^* V_1 x\|^2, \quad x \in \mathcal{H}.$$

Furthermore, the operator limit

$$Q^2 = V_2^* V_2 = \lim_{k\to\infty} T^{*k}T^k$$

exists in the strong operator topology.

Proof. Since the map V is an isometry, we have by the intertwining relation (6.2) and the isometry property of T_2 that

$$\|T^k x\|^2 = \|T_1^{*k} V_1 x\|^2 + \|V_2 x\|^2, \quad x \in \mathcal{H},$$

for $k \geq 0$. By this equality and the assumption that $T_1^{*k} \to 0$ (SOT), we have that the limit $\lim_{k\to\infty} \|T^k x\|^2$ exists and equals $\|V_2 x\|^2$ for every $x \in \mathcal{H}$. We thus have that

$$\|Qx\|^2 = \lim_{k\to\infty} \|T^k x\|^2 = \|V_2 x\|^2, \quad x \in \mathcal{H}.$$

By this last equality we see that the formula $\hat{V}_2 : Qx \mapsto V_2 x$ gives a well-defined map which by continuity extends uniquely to an isometry $\hat{V}_2 : \mathcal{Q} \to \mathcal{H}_2$ such that $\hat{V}_2 Q = V_2$. Also

$$\hat{V}_2 UQx = \hat{V}_2 QTx = V_2 Tx = T_2 V_2 x = T_2 \hat{V}_2 Qx, \quad x \in \mathcal{H},$$

which gives that $\hat{V}_2 U = T_2 \hat{V}_2$.

Let us now verify the norm equality for the operator V_1. Since the operators $V = (V_1, V_2)$ and $T_2 \in \mathcal{L}(\mathcal{H}_2)$ are isometries we have using the intertwining relations $T_2 V_2 = V_2 T$ and $V_1 T = T_1^* V_1$ that

$$\begin{aligned} \|x\|^2 - \|V_1 x\|^2 &= \|V_2 x\|^2 = \|T_2 V_2 x\|^2 = \|V_2 T x\|^2 \\ &= \|Tx\|^2 - \|V_1 T x\|^2 = \|Tx\|^2 - \|T_1^* V_1 x\|^2, \quad x \in \mathcal{H}. \end{aligned}$$

This gives the norm equality for V_1.

Let us now turn to the last limit assertion of the theorem. By the intertwining relation (6.2) we have

$$VT^k = (T_1^{*k} \oplus T_2^k)V,$$

and passing to the adjoint operator we see that

$$T^{*k} V^* = V^*(T_1^k \oplus T_2^{*k}).$$

We now have that

$$T^{*k} T^k = T^{*k} V^* V T^k = V^*(T_1^k T_1^{*k} \oplus T_2^{*k} T_2^k)V = V^*(T_1^k T_1^{*k} \oplus I_{\mathcal{H}_2})V,$$

where in the last step we used that $T_2^* T_2 = I$. Since $T_1^{*k} \to 0$ (SOT) we have that also $T_1^k T_1^{*k} \to 0$ (SOT). Passing to the limit we now conclude that

$$\lim_{k\to\infty} T^{*k} T^k = V^*(0 \oplus I_{\mathcal{H}_2})V = V_2^* V_2$$

in the strong operator topology. □

We remark that in the statement of Theorem 6.1 and also in Theorem 6.2 below the existence of the limit $\lim_{k\to\infty} \|T^k x\|^2$ for every $x \in \mathcal{H}$ is included merely to ensure the existence of the operator Q. As pointed out in the paragraph preceding Theorem 6.1 the limit assertion (6.1) can be rephrased saying that

$$Q^2 = \lim_{k\to\infty} T^{*k} T^k \tag{6.3}$$

in the weak operator topology. The last conclusion of Theorem 6.1 says that the limit (6.3) holds also in the stronger sense of convergence in the strong operator topology.

By Theorem 6.1 we see that there is a natural choice of space $\mathcal{H}_2$ and operator T_2 given by $\mathcal{H}_2 = \mathcal{Q}$ and $T_2 = U$. We shall now show that this choice $(T_2, \mathcal{H}_2) = (U, \mathcal{Q})$ does the job.

Theorem 6.2. *Let $T \in \mathcal{L}(\mathcal{H})$ and let $T_1 \in \mathcal{L}(\mathcal{H}_1)$ be an operator such that $T_1^{*k} \to 0$ in the strong operator topology. Assume that there exists a bounded linear operator $V_1 : \mathcal{H} \to \mathcal{H}_1$ satisfying the norm equality*

$$\|x\|^2 - \|Tx\|^2 = \|V_1 x\|^2 - \|T_1^* V_1 x\|^2, \quad x \in \mathcal{H}, \tag{6.4}$$

as well as the intertwining relation

$$V_1 T = T_1^* V_1.$$

Then the limit $\lim_{k\to\infty} \|T^k x\|^2$ exists for every $x \in \mathcal{H}$ and the map

$$V = (V_1, Q) : \mathcal{H} \to \mathcal{H}_1 \oplus \mathcal{Q}$$

defined by $V : x \mapsto (V_1 x, Qx)$ for $x \in \mathcal{H}$ is an isometry of $\mathcal{H}$ into $\mathcal{H}_1 \oplus \mathcal{Q}$ satisfying the intertwining relation

$$VT = (T_1^* \oplus U)V;$$

here Q, $\mathcal{Q}$ and U are as in the discussion preceding Theorem 6.1.

Proof. Substituting $T^j x$ for x in (6.4) we obtain using the intertwining relation $V_1 T = T_1^* V_1$ that

$$\|T^j x\|^2 - \|T^{j+1} x\|^2 = \|T_1^{*j} V_1 x\|^2 - \|T_1^{*(j+1)} V_1 x\|^2$$

for $j \geq 0$. Summing these equalities for $j = 0, \ldots, k-1$ we see that

$$\|x\|^2 - \|T^k x\|^2 = \|V_1 x\|^2 - \|T_1^{*k} V_1 x\|^2, \quad x \in \mathcal{H}.$$

Now since $T_1^{*k} \to 0$ (SOT), we see that the limit $\lim_{k\to\infty} \|T^k x\|^2$ exists for every $x \in \mathcal{H}$. Furthermore, by a passage to the limit we conclude that

$$\|x\|^2 = \|V_1 x\|^2 + \lim_{k\to\infty} \|T^k x\|^2 = \|V_1 x\|^2 + \|Qx\|^2, \quad x \in \mathcal{H}.$$

This last equality shows that the map $V = (V_1, Q)$ is an isometry of $\mathcal{H}$ into $\mathcal{H}_1 \oplus \mathcal{Q}$. The intertwining relation $VT = (T_1^* \oplus U)V$ follows by $V_1 T = T_1^* V_1$ and $QT = UQ$. This completes the proof of the theorem. □

7. Operator model theory. n-hypercontractions

In this section we continue the study of operator model theory from Section 6. Of particular concern here is the modeling of a general n-hypercontraction $T \in \mathcal{L}(\mathcal{H})$ as part of an operator of the form $S_n^* \oplus U$; here $U \in \mathcal{L}(\mathcal{Q})$ is the canonical isometry associated to T described in Section 6 and S_n is the shift operator on a space $A_n(\mathcal{E})$. In the notation of Section 6 we have that $\mathcal{H}_1 = A_n(\mathcal{E})$ and $T_1 = S_n$.

Recall that the adjoint shift operator S_n^* on $A_n(\mathcal{E})$ is an n-hypercontraction such that $\lim_{k\to\infty} S_n^{*k} = 0$ in the strong operator topology (see Proposition 5.1).

Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction, and consider the defect operators

$$D_{m,T} = \Big(\sum_{k=0}^{m} (-1)^k \binom{m}{k} T^{*k} T^k \Big)^{1/2} \quad \text{in } \mathcal{L}(\mathcal{H})$$

for $1 \leq m \leq n$, where the positive square root is used. We write $\mathcal{D}_{n,T}$ for the defect space defined as the closure in $\mathcal{H}$ of the range of $D_{n,T}$, that is, $\mathcal{D}_{n,T} = \overline{D_{n,T}(\mathcal{H})}$.

Theorem 7.1. *Let $T \in \mathcal{L}(\mathcal{H})$, and let $V_1 : \mathcal{H} \to A_n(\mathcal{E})$ be a bounded linear operator satisfying the norm equality*

$$\|x\|^2 - \|Tx\|^2 = \|V_1 x\|^2_{A_n} - \|S_n^* V_1 x\|^2_{A_n}, \quad x \in \mathcal{H}, \tag{7.1}$$

as well as the intertwining relation $V_1 T = S_n^* V_1$. Then the operator T is an n-hypercontraction and there exists an isometry $\hat{V}_1 : \mathcal{D}_{n,T} \to \mathcal{E}$ such that the operator V_1 admits the representation

$$(V_1 x)(z) = \hat{V}_1 D_{n,T} (I - zT)^{-n} x, \quad z \in \mathbb{D},$$

for $x \in \mathcal{H}$.

Proof. By Theorem 6.2 with $T_1 = S_n$ and $\mathcal{H}_1 = A_n(\mathcal{E})$ the operator T is part of the operator $S_n^* \oplus U$, where U is an isometry. Therefore the operator T is an n-hypercontraction.

The operator $\hat{V}_1$ in the theorem is defined by the formula

$$\hat{V}_1 : D_{n,T} x \mapsto (V_1 x)(0)$$

for $x \in \mathcal{H}$. We shall show that this formula gives a well-defined map which by continuity extends uniquely to an isometry $\hat{V}_1 : \mathcal{D}_{n,T} \to \mathcal{E}$. To accomplish this it suffices to prove the norm equality $\|D_{n,T} x\|^2 = \|(V_1 x)(0)\|^2$ for $x \in \mathcal{H}$.

Notice first that the standard identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for binomial coefficients gives us the formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \|T^k x\|^2 = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \big(\|T^k x\|^2 - \|T^{k+1} x\|^2 \big), \quad x \in \mathcal{H}, \tag{7.2}$$

which is valid for an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$.

Let us now turn to the proof of the norm equality $\|D_{n,T} x\|^2 = \|(V_1 x)(0)\|^2$ for $x \in \mathcal{H}$. Let $x \in \mathcal{H}$ and write $f = V_1 x$, where $f \in A_n(\mathcal{E})$ is given by (5.1). Substituting $T^k x$ for x in the norm equality (7.1) we obtain using the intertwining relation $V_1 T = S_n^* V_1$ that

$$\|T^k x\|^2 - \|T^{k+1} x\|^2 = \|S_n^{*k} f\|^2_{A_n} - \|S_n^{*(k+1)} f\|^2_{A_n}.$$

By formula (7.2) we have that

$$\begin{aligned} \|D_{n,T} x\|^2 &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \|T^k x\|^2 = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \big(\|T^k x\|^2 - \|T^{k+1} x\|^2 \big) \\ &= \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \big(\|S_n^{*k} f\|^2_{A_n} - \|S_n^{*(k+1)} f\|^2_{A_n} \big) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \|S_n^{*k} f\|^2_{A_n}. \end{aligned}$$

By Lemma 5.1 we have that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\|S_n^{*k}f\|_{A_n}^2 = \|a_0\|^2.$$

We now conclude that $\|D_{n,T}x\|^2 = \|a_0\|^2 = \|(V_1x)(0)\|^2$. This gives the asserted norm equality.

We now turn our attention to the representation formula for the operator V_1. By the intertwining relation $V_1T = S_n^*V_1$ we have that

$$\hat{V}_1 D_{n,T}T^kx = (V_1T^kx)(0) = (S_n^{*k}V_1x)(0) = \mu_{n;k}a_k,$$

where in the last step we have used (5.2); here $V_1x = f \in A_n(\mathcal{E})$ is given by (5.1). A computation now gives that

$$f(z) = \sum_{k\geq 0} a_kz^k = \sum_{k\geq 0}\frac{1}{\mu_{n;k}}(\hat{V}_1D_{n,T}T^kx)z^k = \hat{V}_1D_{n,T}(I - zT)^{-n}, \quad z \in \mathbb{D}.$$

This completes the proof of the theorem. □

Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. For an element $x \in \mathcal{H}$ we consider the $\mathcal{D}_{n,T}$-valued analytic function $V_{1,n}x$ defined by the formula

$$(V_{1,n}x)(z) = D_{n,T}(I - zT)^{-n}x = \sum_{k\geq 0}\binom{k+n-1}{k}(D_{n,T}T^kx)z^k, \quad z \in \mathbb{D}. \tag{7.3}$$

The explicit form of this function $V_{1,n}x$ given by (7.3) is of course strongly suggested by Theorem 7.1. The formula (7.3) is also to some extent motivated by the explicit form of the operator-valued Berezin kernel (0.2) studied earlier.

Our next task is to model a general n-hypercontraction using the map $V_{1,n} : x \mapsto V_{1,n}x$. The following proposition gives a norm bound for this operator $V_{1,n}$.

Proposition 7.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then the above map $V_{1,n} : x \mapsto V_{1,n}x$ defined by (7.3) maps $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T})$ in such a way that*

$$\|V_{1,n}x\|_{A_n}^2 \leq \|x\|^2, \quad x \in \mathcal{H},$$

and the intertwining relation $V_{1,n}T = S_n^*V_{1,n}$ holds.

Proof. Let us first verify the intertwining relation $V_{1,n}T = S_n^*V_{1,n}$. By (5.2) and (7.3) we have that

$$\begin{aligned}(S_n^*V_{1,n}x)(z) &= \sum_{k\geq 0}\frac{\mu_{n;k+1}}{\mu_{n;k}}\frac{1}{\mu_{n;k+1}}(D_{n,T}T^{k+1}x)z^k \\ &= \sum_{k\geq 0}\frac{1}{\mu_{n;k}}(D_{n,T}T^{k+1}x)z^k = (V_{1,n}Tx)(z), \quad z \in \mathbb{D}.\end{aligned}$$

This gives the conclusion that $S_n^*V_{1,n} = V_{1,n}T$.

Let us now turn our attention to the norm bound of the operator $V_{1,n}$. Let $0 \leq r < 1$ and fix $x \in \mathcal{H}$. By Corollary 2.1 with $f = 1$ we have that

$$\int_{\bar{\mathbb{D}}} \langle B_n(rT, \zeta)x, x\rangle d\mu_n(\zeta) = \|x\|^2.$$

Recall that also the operator rT is an n-hypercontraction (see Proposition 1.1). In particular, the defect operator $D_{n,rT}$ is defined, and by the defining formula for the operator-valued Berezin kernel (0.2) we have that

$$\langle B_n(rT, \zeta)x, x\rangle = \|D_{n,rT}(I - \bar{\zeta}rT)^{-n}x\|^2.$$

A change of variables and the Parseval formula now shows that

$$\sum_{k\geq 0} \frac{1}{\mu_{n;k}} \|D_{n,rT}(rT)^k x\|^2 = \int_{\bar{\mathbb{D}}} \|D_{n,rT}(I - \zeta rT)^{-n}x\|^2 d\mu_n(\zeta) = \|x\|^2.$$

Now letting $r \to 1$ an application of Fatou's lemma gives that

$$\sum_{k\geq 0} \frac{1}{\mu_{n;k}} \|D_{n,T}T^k x\|^2 \leq \|x\|^2, \quad x \in \mathcal{H}.$$

This completes the proof of the proposition. □

We shall need the following lemma.

Lemma 7.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then we have the norm equality*

$$\|x\|^2 - \|Tx\|^2 = \sum_{k\geq 0} \frac{1}{\mu_{n-1;k}} \|D_{n,T}T^k x\|^2, \quad x \in \mathcal{H}.$$

Proof. Let us first make a few preparatory remarks. Recall that

$$\|D_{m,T}x\|^2 = \sum_{j=0}^{m} (-1)^j \binom{m}{j} \|T^j x\|^2, \quad x \in \mathcal{H},$$

for $1 \leq m \leq n$. Using a standard formula for binomial coefficients it is a straightforward matter to verify that

$$\|D_{m+1,T}x\|^2 = \|D_{m,T}x\|^2 - \|D_{m,T}Tx\|^2, \quad x \in \mathcal{H}, \tag{7.4}$$

for $1 \leq m < n$. We also notice that since the limit $\lim_{k\to\infty} \|T^k x\|^2$ exists (the operator T is a contraction) we have that $\lim_{k\to\infty} \|D_{m,T}T^k x\|^2 = 0$ for $1 \leq m \leq n$.

Let us now turn to the proof of the lemma. Substituting $T^j x$ for x in formula (7.4) we see that

$$\|D_{m+1,T}T^j x\|^2 = \|D_{m,T}T^j x\|^2 - \|D_{m,T}T^{j+1}x\|^2, \quad x \in \mathcal{H},$$

for $j \geq 0$. Summing these equalities for $j = 0, \ldots, k-1$ we obtain that

$$\begin{aligned}\sum_{j=0}^{k-1} \|D_{m+1,T}T^j x\|^2 &= \sum_{j=0}^{k-1} \left(\|D_{m,T}T^j x\|^2 - \|D_{m,T}T^{j+1}x\|^2\right) \\ &= \|D_{m,T}x\|^2 - \|D_{m,T}T^k x\|^2.\end{aligned}$$

Now letting $k \to \infty$, using that $\|D_m T^k x\|^2 \to 0$ (see the previous paragraph), we conclude that

$$\|D_{m,T}x\|^2 = \sum_{k\geq 0} \|D_{m+1,T}T^k x\|^2, \quad x \in \mathcal{H},$$

for $1 \leq m < n$. Iterating this last equality, we arrive at

$$\|D_{1,T}x\|^2 = \sum_{k_1,\dots,k_{n-1}\geq 0} \|D_{n,T}T^{k_1+\dots+k_{n-1}}x\|^2 = \sum_{k\geq 0} \frac{1}{\mu_{n-1;k}} \|D_{n,T}T^k x\|^2,$$

where in the last step we have used a standard property of binomial coefficients. This completes the proof of the lemma. □

We can now model a general n-hypercontraction.

Theorem 7.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction, and consider the map $V_{1,n} : x \mapsto V_{1,n}x$ given by formula* (7.3)*. Then the map*

$$V = (V_{1,n}, Q) : \mathcal{H} \to A_n(\mathcal{D}_{n,T}) \oplus \mathcal{Q}$$

defined by $Vx = (V_{1,n}x, Qx)$ for $x \in \mathcal{H}$ is an isometry of $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T}) \oplus \mathcal{Q}$ satisfying the intertwining relation

$$VT = (S_n^* \oplus U)V;$$

here Q, $\mathcal{Q}$ and U are as in the discussion preceding Theorem 6.1*.*

Proof. We shall apply Theorem 6.2 with $V_1 = V_{1,n}$ and $T_1 = S_n$ acting on the space $\mathcal{H}_1 = A_n(\mathcal{D}_{n,T})$. Recall that $S_n^{*k} \to 0$ (SOT) (see Proposition 5.1). By Proposition 7.1 the map $V_{1,n} : \mathcal{H} \to A_n(\mathcal{D}_{n,T})$ is bounded of norm less than or equal to 1 and the intertwining relation $V_{1,n}T = S_n^* V_{1,n}$ holds.

It remains to verify the norm equality (6.4) in our case. Let $x \in \mathcal{H}$. We have that

$$\begin{aligned}\|V_{1,n}x\|_{A_n}^2 - \|S_n^* V_{1,n}x\|_{A_n}^2 &= \sum_{k\geq 0} \frac{1}{\mu_{n;k}} \|D_{n,T}T^k x\|^2 - \sum_{k\geq 0} \frac{1}{\mu_{n;k}} \|D_{n,T}T^{k+1}x\|^2 \\ &= \|D_{n,T}x\|^2 + \sum_{k\geq 1} \Big(\frac{1}{\mu_{n;k}} - \frac{1}{\mu_{n;k-1}}\Big) \|D_{n,T}T^k x\|^2.\end{aligned}$$

By the standard identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ for binomial coefficients we further conclude that

$$\|V_{1,n}x\|_{A_n}^2 - \|S_n^* V_{1,n}x\|_{A_n}^2 = \sum_{k\geq 0} \frac{1}{\mu_{n-1;k}} \|D_{n,T}T^k x\|^2.$$

By Lemma 7.1 this last sum on the right hand-side equals $\|x\|^2 - \|Tx\|^2$. This completes the proof of (6.4) in our case. □

In the proof of Theorem 7.2 above we needed the boundedness of $V_{1,n}$ as an operator from $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T})$. The proof of this boundedness property we gave in Proposition 7.1 used properties of the operator-valued Berezin kernel studied earlier. Adapting the argument from the proof of Theorem 6.2 instead, we can prove this boundedness of $V_{1,n}$ directly without reference to operator-valued Berezin kernels. In fact, we have the following proposition.

Proposition 7.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction. Then we have the norm equality*

$$\|x\|^2 = \sum_{k\geq 0} \frac{1}{\mu_{n;k}} \|D_{n,T} T^k x\|^2 + \lim_{k\to\infty} \|T^k x\|^2, \quad x \in \mathcal{H}.$$

Proof. Notice first that since T is a contraction the limit $\lim_{k\to\infty} \|T^k x\|^2$ exists. Substituting $T^j x$ for x in Lemma 7.1 we obtain that

$$\|T^j x\|^2 - \|T^{j+1} x\|^2 = \sum_{k\geq 0} \frac{1}{\mu_{n-1;k}} \|D_{n,T} T^{k+j} x\|^2, \quad x \in \mathcal{H},$$

for $j \geq 0$. Summing these equalities for $j = 0, \ldots, l-1$ we conclude that

$$\|x\|^2 - \|T^l x\|^2 = \sum_{j=0}^{l-1} \sum_{k=0}^{\infty} \frac{1}{\mu_{n-1;k}} \|D_{n,T} T^{k+j} x\|^2, \quad x \in \mathcal{H}.$$

Now letting $l \to \infty$, noticing that

$$\sum_{j,k\geq 0} \frac{1}{\mu_{n-1;k}} \|D_{n,T} T^{k+j} x\|^2 = \sum_{k\geq 0} \frac{1}{\mu_{n;k}} \|D_{n,T} T^k x\|^2,$$

the conclusion of the proposition follows. □

Notice that the conclusion of Proposition 7.2 can be rephrased saying that the map $V = (V_{1,n}, Q)$ in Theorem 7.2 is an isometry of $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T}) \oplus \mathcal{Q}$.

Theorem 7.2 has the following corollary when the operator T is also in the class $C_{0\cdot}$.

Corollary 7.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction such that $\lim_{k\to\infty} T^k = 0$ in the strong operator topology. Then the map $V_{1,n} : x \mapsto V_{1,n}x$ defined by (7.3) is an isometry of $\mathcal{H}$ into $A_n(\mathcal{D}_{n,T})$ satisfying the intertwining relation $V_{1,n}T = S_n^* V_{1,n}$.*

Proof. The operator Q in Theorem 7.2 vanish. □

8. Structure properties of the operator measure $d\omega_{n,T}$

The purpose of this section is to discuss some results describing the structure of the operator measure $d\omega_{n,T}$ in some more detail. We denote by $\mathfrak{S}$ the σ-algebra of planar Borel sets.

Theorem 8.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction, and let*

$$V = (V_{1,n}, Q) : \mathcal{H} \to A_n(\mathcal{D}_n) \oplus \mathcal{Q}$$

be the isometry in Theorem 7.2. Then

$$\omega_{n,T}(S) = V_{1,n}^* \omega_{n,S_n^*}(S) V_{1,n} + Q\omega_U(S)Q, \quad S \in \mathfrak{S};$$

here the operator U is as in the discussion preceding Theorem 6.1.

Proof. A computation using the intertwining relation $VT = (S_n^* \oplus U)V$ shows that

$$T^{*r}T^s = V_{1,n}^* S_n^r S_n^{*s} V_{1,n} + QU^{*r}U^sQ \tag{8.1}$$

for $r, s \geq 0$. By Theorem 3.1 we have that

$$\begin{aligned}\int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{n,T}(\zeta) &= T^{*(j-\min(j,k))}\Big(\sum_{m\geq 0} W_{n;m;j,k} T^{*m}T^m\Big)T^{k-\min(j,k)} \\ &= V_{1,n}^*\Big(\int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{n,S_n^*}(\zeta)\Big)V_{1,n} + Q\Big(\int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{n,U}(\zeta)\Big)Q\end{aligned}$$

for $j, k \geq 0$. An approximation argument gives that

$$\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) = V_{1,n}^*\Big(\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,S_n^*}(\zeta)\Big)V_{1,n} + Q\Big(\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,U}(\zeta)\Big)Q$$

for $f \in C(\bar{\mathbb{D}})$ (see the proof of Proposition 4.1). By Proposition 4.2 we know that $d\omega_{n,U} = d\omega_U$. This completes the proof of the theorem. □

We remark that in terms of action on test functions the assertion of Theorem 8.1 means that

$$\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) = V_{1,n}^*\Big(\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,S_n^*}(\zeta)\Big)V_{1,n} + Q\Big(\int_{\mathbb{T}} f(e^{i\theta}) d\omega_U(e^{i\theta})\Big)Q$$

for $f \in C(\bar{\mathbb{D}})$.

Theorem 8.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction for some $n \geq 2$, and let $V = (V_{1,n}, Q)$ be as in Theorem 8.1. Then*

$$\omega_{n,T}(S) = \int_{\mathbb{D}\bigcap S} B_n(T,\zeta) d\mu_n(\zeta) + Q\omega_U(S)Q, \quad S \in \mathfrak{S};$$

here the operator U is as in the discussion preceding Theorem 6.1.

Proof. By Proposition 5.2 we have that

$$d\omega_{n,S_n^*}(\zeta) = B_n(S_n^*,\zeta) d\mu_n(\zeta), \quad \zeta \in \bar{\mathbb{D}},$$

which by Theorem 8.1 allows us to conclude that

$$\int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) = V_{1,n}^*\Big(\int_{\mathbb{D}} B_n(S_n^*,\zeta) f(\zeta) d\mu_n(\zeta)\Big)V_{1,n} + Q\Big(\int_{\mathbb{T}} f(e^{i\theta}) d\omega_U(e^{i\theta})\Big)Q \tag{8.2}$$

for $f \in C(\bar{\mathbb{D}})$.

We shall now consider the Berezin kernel

$$B_n(T,\zeta) = (I - \zeta T^*)^{-n}\Big(\sum_{k=0}^n (-1)^k \binom{n}{k} T^{*k}T^k\Big)(I - \bar{\zeta}T)^{-n}, \quad \zeta \in \mathbb{D},$$

in some more detail. By formula (8.1) and Lemma 2.1 we have that

$$\begin{aligned} B_n(T,\zeta) &= \sum_{r,s\geq 0} p_{rs}(\zeta)T^{*r}T^s \\ &= V_{1,n}^*\Big(\sum_{r,s\geq 0} p_{rs}(\zeta)S_n^r S_n^{*s}\Big)V_{1,n} + Q\Big(\sum_{r,s\geq 0} p_{rs}(\zeta)U^{*r}U^s\Big)Q \\ &= V_{1,n}^* B_n(S_n^*,\zeta)V_{1,n} + QB_n(U,\zeta)Q, \quad \zeta\in\mathbb{D}. \end{aligned}$$

Now since U is an isometry, that is, $U^*U = I$, we have that $B_n(U,\zeta) = 0$ for all $\zeta \in \mathbb{D}$. We conclude that

$$B_n(T,\zeta) = V_{1,n}^* B_n(S_n^*,\zeta)V_{1,n}, \quad \zeta\in\mathbb{D}. \tag{8.3}$$

By formulas (8.2) and (8.3) the conclusion of the theorem follows. □

Notice that by Theorem 8.2 we have that $\omega_{n,T}(S) = 0$ for every Borel subset S of $\mathbb{D}$ of planar Lebesgue measure zero.

We remark that the operator measure $B_n(T,\zeta)d\mu_n(\zeta)$ appearing in Theorem 8.2 has an invariance property with respect to conformal automorphisms of the unit disc (see Proposition 1.3).

We have the following corollary when the operator T is in the class $C_{0\cdot}$.

Corollary 8.1. *Let $n \geq 2$, and let $T \in \mathcal{L}(\mathcal{H})$ be an n-hypercontraction such that $\lim_{k\to\infty} T^k = 0$ in the strong operator topology. Then*

$$d\omega_{n,T}(\zeta) = B_n(T,\zeta)d\mu_n(\zeta), \quad \zeta\in\bar{\mathbb{D}}.$$

Proof. In this case the operator U is not present (see Corollary 7.1). The corollary follows by Theorem 8.2. □

We remark that in terms of action on test functions the assertion of Corollary 8.1 means that

$$\int_{\bar{\mathbb{D}}} f(\zeta)d\omega_{n,T}(\zeta) = \int_{\mathbb{D}} B_n(T,\zeta)f(\zeta)d\mu_n(\zeta), \quad f\in C(\bar{\mathbb{D}}).$$

We recall that a contraction $T \in \mathcal{L}(\mathcal{H})$ is said to be completely non-unitary (c.n.u.) if for every element $x \in \mathcal{H}$ the equalities

$$\|T^{*k}x\|^2 = \|x\|^2 = \|T^k x\|^2, \quad k\geq 0,$$

imply that $x = 0$ (see [27, Section I.3]). A classical result of Sz.-Nagy and Foias asserts that the spectral measure for the minimal unitary dilation of a c.n.u. contraction is absolutely continuous with respect to Lebesgue arc length measure on the unit circle (see [27, Theorem II.6.4]).

Let $T \in \mathcal{L}(\mathcal{H})$ be a completely non-unitary contraction, and let U be as in the discussion preceding Theorem 6.1. Using the result of Sz.-Nagy and Foias qouted in the previous paragraph it is straightforward to see that the operator measure $d\omega_U$ is absolutely continuous with respect to Lebesgue arc length measure on $\mathbb{T}$, that is, $\omega_U(S) = 0$ in $\mathcal{L}(\mathcal{H})$ for every planar Borel set S such that $\mu_1(S) = 0$. We omit the details.

9. Subnormal contractions and the Hausdorff moment problem

A theorem of Agler [2, Theorem 3.1] asserts that an operator $T \in \mathcal{L}(\mathcal{H})$ is a subnormal contraction if and only if it is an n-hypercontraction for every $n \geq 1$. In this section we shall reconsider this characterization of subnormal contractions and derive it as a limit case of our study of operator-valued Berezin transforms (see Theorem 9.1 below). As an application of this result we shall also consider two operator-valued moment problems of Hausdorff type (see Theorem 9.2 and Proposition 9.3).

First we need a lemma.

Lemma 9.1. *Let $n \geq 2$, and let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that $r(T) < 1$. Then*

$$\int_{\mathbb{D}} B_n(T,\zeta)\bar{\zeta}^j\zeta^k d\mu_n(\zeta) = \int_{\mathbb{D}} (T^* - \bar{\zeta}I)^j(I-\zeta T^*)^{-k}(T-\zeta I)^k(I-\bar{\zeta}T)^{-j} d\mu_n(\zeta)$$

for all integers $j,k \geq 0$.

Proof. We have a series expansion

$$(T-\zeta I)^k(I-\bar{\zeta}T)^{-j} = \sum_{l\geq 0} p_{j,k;l}(\zeta)\, T^l,$$

where the $p_{j,k;l}(\zeta)$'s are polynomials in $\mathbb{C}[\zeta,\bar{\zeta}]$. A more detailed analysis shows that the maximum $\max_{|\zeta|\leq 1} |p_{j,k;l}(\zeta)|$ grows at most like a polynomial in l. We now have the series expansion

$$(T^* - \bar{\zeta}I)^j(I-\zeta T^*)^{-k}(T-\zeta I)^k(I-\bar{\zeta}T)^{-j} = \sum_{l_1,l_2\geq 0} p_{k,j;l_1}(\bar{\zeta})p_{j,k;l_2}(\zeta)\, T^{*l_1}T^{l_2},$$

which we integrate term by term to obtain that

$$\begin{aligned}\int_{\mathbb{D}} (T^* - \bar{\zeta}I)^j(I-\zeta T^*)^{-k}&(T-\zeta I)^k(I-\bar{\zeta}T)^{-j} d\mu_n(\zeta) \qquad (9.1)\\ &= \sum_{l_1,l_2\geq 0} \int_{\mathbb{D}} p_{k,j;l_1}(\bar{\zeta})p_{j,k;l_2}(\zeta) d\mu_n(\zeta)\, T^{*l_1}T^{l_2}.\end{aligned}$$

We shall now consider in some more detail the integrals appearing on the right-hand side in (9.1). Substituting zI, $z \in \mathbb{D}$, for T in (9.1) we obtain that

$$\int_{\mathbb{D}} \overline{\varphi_z(\zeta)}^j \varphi_z(\zeta)^k d\mu_n(\zeta) = \sum_{l_1,l_2\geq 0} \int_{\mathbb{D}} p_{k,j;l_1}(\bar{\zeta})p_{j,k;l_2}(\zeta) d\mu_n(\zeta)\, \bar{z}^{l_1} z^{l_2}, \quad z \in \mathbb{D},$$

where $\varphi_z(\zeta) = (z-\zeta)/(1-\bar{z}\zeta)$ is a conformal automorphism of the unit disc. The change of variables $w = \varphi_z(\zeta)$ gives that

$$\int_{\mathbb{D}} \overline{\varphi_z(\zeta)}^j \varphi_z(\zeta)^k d\mu_n(\zeta) = \int_{\mathbb{D}} B_n(z,w)\bar{w}^j w^k d\mu_n(w)$$

(see [19, Section 2.1]). By Proposition 2.1 we have the power series expansion

$$\int_{\mathbb{D}} B_n(z,\zeta)\bar{\zeta}^j\zeta^k d\mu_n(\zeta) = \bar{z}^{j-\min(j,k)}\Big(\sum_{m\geq 0} W_{n;m;j,k}|z|^{2m}\Big) z^{k-\min(j,k)}, \quad z \in \mathbb{D}.$$

Comparing coefficients we conclude that

$$\int_{\mathbb{D}} p_{k,j;l_1}(\bar{\zeta}) p_{j,k;l_2}(\zeta) d\mu_n(\zeta) = W_{n;m;j,k}$$

for $l_1 = j - \min(j,k) + m$, $l_2 = k - \min(j,k) + m$, $m \geq 0$, and that

$$\int_{\mathbb{D}} p_{k,j;l_1}(\bar{\zeta}) p_{j,k;l_2}(\zeta) d\mu_n(\zeta) = 0$$

for all other values of $l_1, l_2 \geq 0$. Going back to (9.1) we conclude that

$$\int_{\mathbb{D}} (T^* - \bar{\zeta} I)^j (I - \zeta T^*)^{-k} (T - \zeta I)^k (I - \bar{\zeta} T)^{-j} d\mu_n(\zeta)$$
$$= T^{*(j - \min(j,k))} \Big(\sum_{m \geq 0} W_{n;m;j,k} T^{*m} T^m \Big) T^{k - \min(j,k)} = \int_{\mathbb{D}} B_n(T, \zeta) \bar{\zeta}^j \zeta^k d\mu_n(\zeta),$$

where the last equality holds by Proposition 2.1. This completes the proof of the lemma. □

Let us recall the notion of a subnormal operator. An operator $T \in \mathcal{L}(\mathcal{H})$ is called subnormal if there exists a normal operator $N \in \mathcal{L}(\mathcal{K})$ on some larger Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a closed subspace such that $T = N|_{\mathcal{H}}$; the operator N is then called a normal extension of T. A normal extension N of T can be chosen minimal in the sense that $\mathcal{K} = \bigvee_{j,k \geq 0} N^{*j} N^k(\mathcal{H})$ and is then uniquely determined up to unitary equivalence. By the spectral theorem the normal operator N has a spectral measure dE supported on the spectrum $\sigma(N)$ of N. If we set

$$\mu(S) = PE(S)|_{\mathcal{H}}, \quad S \in \mathfrak{S}; \tag{9.2}$$

here $\mathfrak{S}$ is the σ-algebra of planar Borel sets and P is the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$, then $d\mu$ is a positive $\mathcal{L}(\mathcal{H})$-valued operator measure on $K = \sigma(N)$ which represents the operator T in the sense that

$$T^{*j} T^k = \int_K \bar{z}^j z^k d\mu(z), \quad j, k \geq 0. \tag{9.3}$$

Notice that (9.3) determines the operator measure $d\mu$ uniquely (Stone-Weierstrass). Conversely, assume that (9.3) holds for some (compactly supported) positive operator measure $d\mu$. By a theorem of Naimark [24] there exists an $\mathcal{L}(\mathcal{K})$-valued spectral measure dE also supported by K such that (9.2) holds. This spectral measure dE is then the spectral measure for the normal extension $N = \int_K z dE(z)$ of T with $\sigma(N) \subset K$. A standard reference for subnormal operators is the book Conway [11]; see also Bram [9].

Theorem 9.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be an operator such that the inequality*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k \geq 0 \quad \textit{in } \mathcal{L}(\mathcal{H}) \tag{9.4}$$

holds for every $n \geq 1$. *Then there exists a positive* $\mathcal{L}(\mathcal{H})$*-valued operator measure* $d\mu$ *on* $\bar{\mathbb{D}}$ *such that*

$$T^{*j}T^k = \int_{\bar{\mathbb{D}}} \bar{z}^j z^k d\mu(z), \quad j,k \geq 0.$$

In particular, the operator T *is a subnormal contraction.*

Proof. We consider the map Λ with values in $\mathcal{L}(\mathcal{H})$ defined for polynomials $f(z) = \sum_{j,k\geq 0} c_{jk}\bar{z}^j z^k$ in $\mathbb{C}[z,\bar{z}]$ by

$$\Lambda(f) = \sum_{j,k\geq 0} c_{jk} T^{*j}T^k.$$

We shall show below that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $f(z) \geq 0$ for all $z \in \mathbb{D}$. By approximation the map Λ then extends uniquely to a continuous linear map Λ from $C(\bar{\mathbb{D}})$ into $\mathcal{L}(\mathcal{H})$ such that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $0 \leq f \in C(\bar{\mathbb{D}})$ (see the proof of Theorem 3.1). By an operator version of the F. Riesz representation theorem it then follows that that there exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\mu$ on $\bar{\mathbb{D}}$ such that

$$\Lambda(f) = \int_{\bar{\mathbb{D}}} f(z) d\mu(z), \quad f \in C(\bar{\mathbb{D}})$$

(see the preliminaries in the introduction). This gives then the conclusion of the theorem.

Let us prove the estimate needed. We consider first the case of an operator $T \in \mathcal{L}(\mathcal{H})$ such that $r(T) < 1$ satisfying (9.4) for $n \geq 1$. Notice that the sequence $\{\mu_n\}$ of probability measures converges weak* to the Dirac measure δ_0 in $M(\bar{\mathbb{D}}) = C(\bar{\mathbb{D}})^*$, that is,

$$\lim_{n\to\infty} \int_{\mathbb{D}} f(z) d\mu_n(z) = f(0), \quad f \in C(\bar{\mathbb{D}}).$$

By Lemma 9.1 we have that

$$\begin{aligned}\lim_{n\to\infty} \int_{\bar{\mathbb{D}}} \bar{\zeta}^j \zeta^k d\omega_{n,T}(\zeta) &= \lim_{n\to\infty} \int_{\mathbb{D}} (T^* - \bar{\zeta}I)^j (I - \zeta T^*)^{-k} (T - \zeta I)^k (I - \bar{\zeta}T)^{-j} d\mu_n(\zeta) \\ &= T^{*j}T^k \quad \text{in } \mathcal{L}(\mathcal{H})\end{aligned}$$

for $j,k \geq 0$. Taking linear combinations we see that

$$\lim_{n\to\infty} \int_{\bar{\mathbb{D}}} f(\zeta) d\omega_{n,T}(\zeta) = \Lambda(f) \quad \text{in } \mathcal{L}(\mathcal{H})$$

for every $f \in \mathbb{C}[\zeta,\bar{\zeta}]$. Since the $d\omega_{n,T}$'s are positive operator measures, we conclude that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ if $f \in \mathbb{C}[z,\bar{z}]$ is positive in $\mathbb{D}$.

Let us now consider the case of an arbitrary operator $T \in \mathcal{L}(\mathcal{H})$ satisfying (9.4) for $n \geq 1$. Let $0 \leq r < 1$. By Proposition 1.1 the operator rT satisfies (9.4) for $n \geq 1$, and by the result of the previous paragraph we have that

$$\sum_{j,k\geq 0} c_{jk} r^{j+k} T^{*j}T^k \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H})$$

for every $f(z) = \sum_{j,k\geq 0} c_{jk}\bar{z}^j z^k$ in $\mathbb{C}[z,\bar{z}]$ such that $f \geq 0$ on $\mathbb{D}$. Letting $r \to 1$ we conclude that $\Lambda(f) \geq 0$ in $\mathcal{L}(\mathcal{H})$ for every polynomial f in $\mathbb{C}[z,\bar{z}]$ which is positive in $\mathbb{D}$. This completes the proof of the theorem. □

It is known that a subnormal contraction is an n-hypercontraction for every $n \geq 1$. For the sake of completeness we include some details of proof.

Proposition 9.1. *If $T \in \mathcal{L}(\mathcal{H})$ is a subnormal operator such that $\|T\| \leq 1$, then T is an n-hypercontraction for every $n \geq 1$.*

Proof. Let $d\mu$ be as in (9.3). For $c > 1$ we have that

$$I \geq T^{*k}T^k = \int_K |z|^{2k} d\mu(z) \geq c^{2k}\mu(\{z \in K : \ |z| > c\}) \quad \text{in } \mathcal{L}(\mathcal{H})$$

for $k \geq 1$. Letting $k \to \infty$ we see that $\mu(\{z \in K : \ |z| > c\}) = 0$. This shows that the operator measure $d\mu$ is supported by $\bar{\mathbb{D}}$. We now have that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k}T^k = \int_{\bar{\mathbb{D}}} (1-|z|^2)^k d\mu(z) \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H}),$$

which shows that the operator T is an n-hypercontraction for every $n \geq 1$. □

We want to mention here that the relationship between the spectrum of a subnormal operator and the spectrum of its minimal normal extension has been studied by Halmos [17] and Bram [9] in the 1950's; see also [11, Theorem II.2.11]. Notice that the spectrum of the minimal normal extension N of a subnormal operator $T \in \mathcal{L}(\mathcal{H})$ equals the support of the operator measure $d\mu$ given by (9.3) (see the discussion preceding Theorem 9.1).

We shall now turn to a discussion of some related moment problems. We say that an infinite matrix $\{L_{jk}\}_{j,k\geq 0}$ with entries L_{jk} in $\mathcal{L}(\mathcal{H})$ is positive definite if

$$\sum_{j,k=0}^{N} \langle L_{jk}x_j, x_k\rangle \geq 0$$

for every choice of $x_0, \ldots, x_N \in \mathcal{H}$.

As an application of Theorem 9.1 we have the following variation of a moment problem considered by Atzmon [6]; see also [30, Theorem 3.7].

Theorem 9.2. *Let $\{L_{jk}\}_{j,k\geq 0}$ be an infinite matrix with entries L_{jk} in $\mathcal{L}(\mathcal{H})$ such that the matrices*

$$\Big\{ \sum_{m=0}^{n} (-1)^m \binom{n}{m} L_{j+m,k+m} \Big\}_{j,k\geq 0}, \quad n \geq 0, \tag{9.5}$$

are all positive definite. Then there exists a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\lambda$ on $\bar{\mathbb{D}}$ such that

$$L_{jk} = \int_{\bar{\mathbb{D}}} z^j \bar{z}^k d\lambda(z), \quad j,k \geq 0. \tag{9.6}$$

Proof. By (9.5) with $n = 0$ we have that the matrix $\{L_{jk}\}_{j,k\geq 0}$ is positive definite. On the space of $\mathcal{H}$-valued analytic polynomials

$$f(z) = \sum_{k\geq 0} a_k z^k; \tag{9.7}$$

here $a_k \in \mathcal{H}$ for $k \geq 0$, we consider the semi-norm defined by

$$\|f\|^2 = \sum_{j,k\geq 0} \langle L_{jk} a_j, a_k\rangle.$$

This semi-norm induces in a natural way a Hilbert space A_L in which the equivalence classes of $\mathcal{H}$-valued polynomials form a dense subset.

We now consider the shift operator S defined by

$$(Sf)(z) = zf(z) = \sum_{k\geq 1} a_{k-1} z^k \tag{9.8}$$

for f a polynomial given by (9.7). A computation shows that

$$\|S^m f\|^2 = \sum_{j,k\geq m} \langle L_{jk} a_{j-m}, a_{k-m}\rangle = \sum_{j,k\geq 0} \langle L_{j+m,k+m} a_j, a_k\rangle,$$

and that

$$\sum_{m=0}^{n} (-1)^m \binom{n}{m} \|S^m f\|^2 = \sum_{j,k\geq 0} \langle \Big(\sum_{m=0}^{n} (-1)^m \binom{n}{m} L_{j+m,k+m} \Big) a_j, a_k\rangle;$$

here f is given by (9.7) and $n \geq 1$. By (9.5) with $n = 1$ we have that the shift operator S induces a well-defined contraction on the space A_L which we also denote by S.

Invoking the full strength of (9.5) for $n \geq 1$ we have that the induced operator S on A_L is an n-hypercontraction for every $n \geq 1$. By Theorem 9.1 we conclude that there exists a positive $\mathcal{L}(A_L)$-valued operator measure $d\mu$ on $\bar{\mathbb{D}}$ such that

$$S^{*j} S^k = \int_{\bar{\mathbb{D}}} \bar{z}^j z^k d\mu(z), \quad j,k \geq 0.$$

We have a natural map A_0 mapping the element $x \in \mathcal{H}$ to the corresponding constant element x in A_L, that is, $A_0 x = f$, where f is given by (9.7) with $a_0 = x$ and $a_k = 0$ for $k \geq 1$. We now set $\lambda(F) = A_0^* \mu(F) A_0$ for $F \in \mathfrak{S}$ (Borel sets). This gives us a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\lambda$ on $\bar{\mathbb{D}}$. We proceed to show that (9.6) holds with this choice of $d\lambda$. We have that

$$\int_{\bar{\mathbb{D}}} z^j \bar{z}^k d\lambda(z) = A_0^* \Big(\int_{\bar{\mathbb{D}}} z^j \bar{z}^k d\mu(z) \Big) A_0 = A_0^* S^{*k} S^j A_0,$$

and $\langle A_0^* S^{*k} S^j A_0 x, y\rangle = \langle S^j A_0 x, S^k A_0 y\rangle = \langle L_{jk} x, y\rangle$ for $x, y \in \mathcal{H}$, which gives that $A_0^* S^{*k} S^j A_0 = L_{jk}$. This completes the proof of the theorem. □

We remark that the method of proof of Theorem 9.2 is adapted from Atzmon [6]. We also remark that an operator measure $d\lambda$ is uniquely determined by (9.6) (Stone-Weierstrass); the same uniqueness remark applies in the context of Proposition 9.3 below.

Theorem 9.2 has the following converse.

Proposition 9.2. *Let the infinite matrix $\{L_{jk}\}_{j,k\geq 0}$ with entries L_{jk} in $\mathcal{L}(\mathcal{H})$ be a Hausdorff moment sequence in the sense that* (9.6) *holds for some positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\lambda$ on $\bar{\mathbb{D}}$. Then the infinite matrices* (9.5) *are all positive definite.*

Proof. By a theorem of Naimark [24] there exists an $\mathcal{L}(\mathcal{K})$-valued spectral measure dE on $\bar{\mathbb{D}}$ and a bounded linear operator $A : \mathcal{H} \to \mathcal{K}$ such that

$$\lambda(S) = A^* E(S) A, \quad S \in \mathfrak{S};$$

here $\mathfrak{S}$ is the σ-algebra of planar Borel sets.

A computation shows that

$$\begin{aligned} L_{j,k;n} &= \sum_{m=0}^{n} (-1)^m \binom{n}{m} L_{j+m,k+m} = \int_{\bar{\mathbb{D}}} \Big(\sum_{m=0}^{n} (-1)^m \binom{n}{m} z^{j+m} \bar{z}^{k+m} \Big) d\lambda(z) \\ &= A^* \Big(\int_{\bar{\mathbb{D}}} z^j \bar{z}^k (1-|z|^2)^m dE(z) \Big) A. \end{aligned}$$

Let $x_0, \ldots, x_N \in \mathcal{H}$. We now have that

$$\begin{aligned} \sum_{j,k\geq 0} \langle L_{j,k;n} x_j, x_k \rangle &= \sum_{j,k\geq 0} \langle A^* \Big(\int_{\bar{\mathbb{D}}} z^j \bar{z}^k (1-|z|^2)^m dE(z) \Big) A x_j, x_k \rangle \\ &= \Big\| \sum_{j\geq 0} \Big(\int_{\bar{\mathbb{D}}} z\, dE(z) \Big)^j \Big(\int_{\bar{\mathbb{D}}} (1-|z|^2)^{m/2} dE(z) \Big) A x_j \Big\|^2 \geq 0. \end{aligned}$$

This completes the proof of the proposition. □

We can adapt the proof of Theorem 9.2 to yield also the following version of the operator-valued Hausdorff moment problem.

Proposition 9.3. *Let $\{L_k\}_{k\geq 0}$ be a sequence of operators in $\mathcal{L}(\mathcal{H})$ such that*

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{j+k} \geq 0 \quad \textit{in } \mathcal{L}(\mathcal{H}) \tag{9.9}$$

for all integers $n, j \geq 0$. Then there exists a radial $\mathcal{L}(\mathcal{H})$-valued positive operator measure $d\lambda$ on $\bar{\mathbb{D}}$ such that

$$L_k = \int_{\bar{\mathbb{D}}} |z|^{2k} d\lambda(z), \quad k \geq 0. \tag{9.10}$$

Furthermore, by a change of variables, we have a positive $\mathcal{L}(\mathcal{H})$-valued operator measure $d\nu$ on the closed unit interval $[0,1]$ such that

$$L_k = \int_{[0,1]} x^k d\nu(x), \quad k \geq 0. \tag{9.11}$$

Proof. By (9.9) with $n = 0$, the operators L_k are all positive. On the space of $\mathcal{H}$-valued analytic polynomials f of the form (9.7) we consider the semi-norm defined by

$$\|f\|^2 = \sum_{k\geq 0} \langle L_k a_k, a_k \rangle.$$

This semi-norm induces in a natural way a Hilbert space A_L in which the equivalence classes of $\mathcal{H}$-valued polynomials form a dense subset.

We consider the shift operator S defined by (9.8). By (9.9) with $n = 1$, the shift operator S induces a contraction on A_L which we also denote by S. A computation shows that

$$\|S^k f\|^2 = \sum_{j\geq 0} \langle L_{k+j} a_j, a_j \rangle$$

and therefore we have that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \|S^k f\|^2 = \sum_{j\geq 0} \langle \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{k+j} \Big) a_j, a_j \rangle \geq 0$$

for f given by (9.7) and $n \geq 1$. We thus have that the induced operator S on A_L is an n-hypercontraction for every $n \geq 1$, and by Theorem 9.1 we conclude that there exists a positive $\mathcal{L}(A_L)$-valued operator measure $d\mu$ on $\bar{\mathbb{D}}$ such that

$$S^{*j} S^k = \int_{\bar{\mathbb{D}}} \bar{z}^j z^k d\mu(z), \quad j, k \geq 0.$$

Consider the natural map A_0 mapping the element $x \in \mathcal{H}$ to the corresponding constant element x in A_L, that is, $A_0 x = f$, where f is given by (9.7) with $a_0 = x$ and $a_k = 0$ for $k \geq 1$. We now set $\lambda(F) = A_0^* \mu(F) A_0$ for $F \in \mathfrak{S}$ (Borel sets). A computation shows that

$$\int_{\bar{\mathbb{D}}} z^j \bar{z}^k d\lambda(z) = A_0^* \Big(\int_{\bar{\mathbb{D}}} z^j \bar{z}^k d\mu(z) \Big) A_0 = A_0^* S^{*k} S^j A_0 = \begin{cases} 0 & \text{for } j \neq k, \\ L_k & \text{for } j = k \end{cases}$$

(see the proof of Theorem 9.2). We conclude that $d\lambda$ is a radial positive operator measure satisfying (9.10). The last conclusion (9.11) of the proposition is evident by a change of variables. □

We mention that a sequence $\{L_k\}_{k\geq 0}$ satisfying (9.9) for $n, j \geq 0$ is sometimes called totally monotone (see [18, Section 11.6]). Notice that if $\{L_k\}_{k\geq 0}$ is a Hausdorff moment sequence in the sense of (9.10) or (9.11), then the sequence $\{L_k\}_{k\geq 0}$ is totally monotone. Indeed, if (9.10) holds we have that

$$\begin{aligned} \sum_{k=0}^{n} (-1)^k \binom{n}{k} L_{j+k} &= \int_{\bar{\mathbb{D}}} \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} |z|^{2(j+k)} \Big) d\lambda(z) \\ &= \int_{\bar{\mathbb{D}}} (1 - |z|^2)^n |z|^{2j} d\lambda(z) \geq 0 \quad \text{in } \mathcal{L}(\mathcal{H}) \end{aligned}$$

for $n, j \geq 0$, and similarly in the case of (9.11).

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