Quantum

Lecture 5

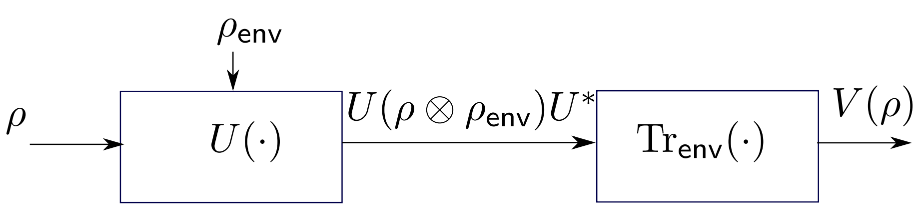
- Noisy systems
- Quantum operations

Noisy Systems

So far: Isolated/closed/noiseless quantum systems

Noiseless dynamics: If the state is $|\psi_1\rangle$ at time t_1, then the state at time t_2 is $|\psi_2\rangle = U|\psi_1\rangle$ where U is unitary

Open/noisy system: Noise = part of system we cannot control = *environment* $\Rightarrow$ noisy dynamics, $\rho \to V(\rho)$



Quantum channel from ρ to $V(\rho)$

More on the partial trace

Consider two Hilbert spaces $\mathcal{A}$ and $\mathcal{B}$ and an operator T on $\mathcal{A} \otimes \mathcal{B}$

Let $\{|e_k\rangle\}$ be a basis for $\mathcal{B}$ and I an identity operator on $\mathcal{A}$. Then for $|x\rangle \in \mathcal{A}$, we have

$$\mathrm{Tr}_{\mathcal{B}}(T)(|x\rangle) = \sum_k (I \otimes \langle e_k|)\, T\, (|x\rangle \otimes |e_k\rangle)$$

That is, with some abuse of notation we can write

$$\mathrm{Tr}_{\mathcal{B}}(T) = \sum_k \langle e_k|T|e_k\rangle$$

where it is understood that $\langle e_k|$ and $|e_k\rangle$ affect only space $\mathcal{B}$, that is $\langle e_k| = I \otimes \langle e_k|$ and $|e_k\rangle$ "$=$" $I \otimes |e_k\rangle$

In finite dimensions (I a matrix, $|e\rangle$ a vector) the "$=$" makes perfect sense. However, we will use this approach as short-hand also in the general case.

For the quantum channel $V(\rho) = \mathrm{Tr}_{\text{env}} U(\rho \otimes \rho_{\text{env}})U^*$, let $\{|e_k\rangle\}$ be a basis for the environment and assume $\rho_{\text{env}} = |e_0\rangle\langle e_0|$. We get

$$\begin{aligned}
&\mathrm{Tr}_{\text{env}}(U(\rho \otimes |e_0\rangle\langle e_0|)U^*) \\
&\quad = \sum_n \langle e_n|U(\rho \otimes |e_0\rangle\langle e_0|)U^*|e_n\rangle \\
&\quad = \sum_n (I \otimes \langle e_n|UI \otimes |e_0\rangle)(\rho \otimes I)(I \otimes \langle e_n|UI \otimes |e_0\rangle)^* \\
&\quad = \sum_n E_n(\rho \otimes I)E_n^*
\end{aligned}$$

with $E_n = (I \otimes \langle e_n|)U(I \otimes |e_0\rangle) = \langle e_n|U|e_0\rangle$

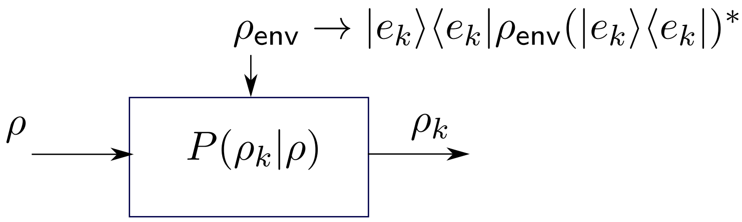
System in state ρ. Measure the environment in the basis $\{|e_n\rangle\}$, assume the outcome is $k \Rightarrow$ the principal system is now in state

$$\rho_k = \frac{E_k \rho E_k^*}{\mathrm{Tr}(E_k \rho E_k^*)}$$

with probability $\mathrm{Tr}(E_k \rho E_k^*)$. That is

$$P(\rho_k|\rho) = \mathrm{Tr}(E_k \rho E_k^*)$$

C.f. noisy transmission over a channel $P(\cdot|\cdot)$



Measurement on the noisy system: Principal system in $\mathcal{Q}$ and environment in $\mathcal{R}$. Initial states $\rho \in \mathcal{Q}$ and $\sigma \in \mathcal{R} \Rightarrow$ joint state $\rho \otimes \sigma \in \mathcal{Q} \otimes \mathcal{R}$. Combined noisy dynamics $U(\rho \otimes \sigma)U^*$

Measurement on the combined system $\{M_k\}$, outcome $m \Rightarrow$ state in $\mathcal{Q}$ after dynamic evolution and measurement,

$$\frac{\mathrm{Tr}_{\mathcal{R}}(M_m U(\rho \otimes \sigma)U^* M_m^*)}{\mathrm{Tr}(M_m U(\rho \otimes \sigma)U^* M_m^*)}$$

Letting $\mathcal{E}_m(\rho) = \mathrm{Tr}_{\mathcal{R}}(M_m U(\rho \otimes \sigma)U^* M_m^*)$, the combined noisy dynamics + measurement maps ρ to $\mathcal{E}_m(\rho)/\mathrm{Tr}\mathcal{E}_m(\rho)$ with probability $\mathrm{Tr}\mathcal{E}_m(\rho)$

If $\sigma = \sum_i q_i |s_i\rangle\langle s_i|$ and $\{|e_\ell\rangle\}$ is a basis for $\mathcal{Q}$, then

$$\mathcal{E}_m(\rho) = \sum_{k\ell} E_{k\ell} \rho E_{k\ell}^*$$

with $E_{k\ell} = \sqrt{q_k}\langle e_\ell|M_m U|s_k\rangle$

General definition: (Noisy) quantum operation

$\mathcal{E}(\rho)$ is a mapping from density operators ρ on $\mathcal{A}$ to (unnormalized) density operators $\mathcal{E}(\rho)$ on $\mathcal{B}$, representing a random transformation

$\mathcal{E}(\cdot)$ is completely positive:

- $\mathcal{E}(\rho)$ on $\mathcal{B}$ is a positive operator for any ρ
- for the identity map I on a third system $\mathcal{Q}$,

  $(I \otimes \mathcal{E})(O)$ is positive for any positive O on $\mathcal{Q} \otimes \mathcal{A}$

For $\sum_i p_i = 1$ and a set $\{\rho_i\}$ of densities, $\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$

Given ρ, the event represented by $\mathcal{E}$ occurs with probability $\mathrm{Tr}\mathcal{E}(\rho)$, and $\mathcal{E}(\rho)/\mathrm{Tr}\mathcal{E}(\rho)$ is the resulting density on $\mathcal{B}$

The case $\mathrm{Tr}\,\mathcal{E} < 1$ is allowed, with interpretation that $\mathcal{E}$ does not provide a complete description, there are *other possible outcomes*

$\iff$ "measurements"

In general, we will however assume that quantum operations are trace preserving, i.e. $\mathrm{Tr}\mathcal{E}(\rho) = 1$, if not stated otherwise

$\mathrm{Tr}\,\mathcal{E} = 1 \iff$ deterministic

$\mathrm{Tr}\,\mathcal{E} < 1 =$ trace preserving + generalized measurement

Trace preserving mappings are deterministic only in the quantum sense, we are dealing with density operators so there is still classical uncertainty/probabilities $(\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|)$

A general $\mathcal{E}(\rho)$ always has the operator–sum representation

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$$

where $\sum_i E_i^* E_i \le I$ (i.e., $\langle x|(I - \sum_i E_i^* E_i)|x\rangle \ge 0$ for all $|x\rangle$)
$= I$ when trace preserving (as assumed in general)

Here $\{E_i\}$ are the operation elements, or Kraus operators, for $\mathcal{E}$

If the input space $\mathcal{A}$ is of finite dimension d, then the sum contains at most d^2 elements

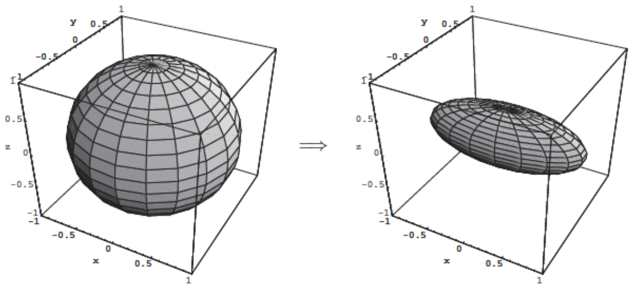
Standard Operations on Qubits

Assume both the input and output spaces are qubits

$\psi = \alpha|0\rangle + \beta|1\rangle, \quad \rho = |\psi\rangle\langle\psi|$

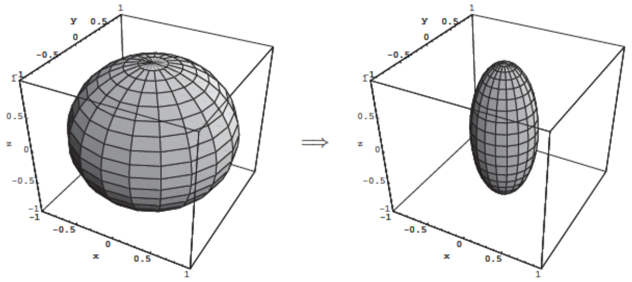
bit flip:

$$E_0 = \sqrt{p}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad E_1 = \sqrt{1-p}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$



phase flip:

$$E_0 = \sqrt{p}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad E_1 = \sqrt{1-p}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

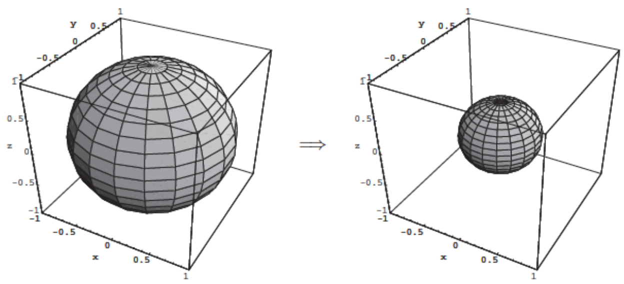


bit-phase flip:

$$E_0 = \sqrt{p}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad E_1 = \sqrt{1-p}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

depolarizing:

$$\mathcal{E}(\rho) = \frac{p}{2}I + (1-p)\rho$$



All operations on qubits can written in terms of the Pauli matrices

$$\sigma_0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \ \sigma_x = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \ \sigma_y = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \ \sigma_z = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$

For depolarization, we can use

$$\frac{I}{2} = \frac{1}{4}(\sigma_0\rho\sigma_0 + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z)$$

Quantum Communication