Quantum

Lecture 4

- The density operator
- Quantum teleportation
- EPR and Bell

The Density Operator

Consider an ensemble $\{p_i, |\psi_i\rangle\}$. The system is in state $|\psi_i\rangle \in \mathcal{H}$ with probability p_i

Models the situation that the state $|\psi\rangle$ is not known precisely (has not been precisely prepared), but it is known that $|\psi\rangle \in \{|\psi_i\rangle\}$

system state known (has been prepared) $\Rightarrow$ pure state $|\psi\rangle$

known only that $|\psi\rangle \in \{|\psi_i\rangle\}$ $\Rightarrow$ mixed state (a *classical* mix)

The density operator characterizing a mixed (or pure) state is

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

A more general definition (for an uncountable mix): a linear operator ρ is a density operator if it is positive and $\mathrm{Tr}\rho = 1$

Postulates in terms of ρ

Associated to any isolated quantum system is a Hilbert space $\mathcal{H}$ over $\mathbb{C}$, the state space. The system state is completely characterized by a density operator ρ

The evolution (in time) of a closed system is fully described by a unitary linear operator U, such that if the state is ρ at time t_1 then at time t_2 $(> t_1)$ it has evolved to $U\rho U^*$

Measurements are described by linear operators $\{M_m\}$. If the system is measured in state ρ then the probability of the mth outcome is $\pi(m) = \mathrm{Tr}(M_m^* M_m \rho)$. When observing outcome m the state ρ collapses to $\pi(m)^{-1} M_m \rho M_m^*$

The composition of two systems associated with $(\mathcal{H}_1, \rho_1)$ and $(\mathcal{H}_2, \rho_2)$, respectively, is described by $(\mathcal{H}_1 \otimes \mathcal{H}_2, \rho_1 \otimes \rho_2)$

Projective measurements: For a pure state $|\psi\rangle$, $\rho = |\psi\rangle\langle\psi|$, and a projective measurement

$$M = \sum_m \lambda_m P_m$$

the probability for outcome λ_m is $\langle\psi|P_m|\psi\rangle = \mathrm{Tr}(P_m \rho)$, and the expected outcome is

$$\langle M \rangle = \langle\psi|M|\psi\rangle = \mathrm{Tr}(M\rho) = \sum_i \lambda_m \mathrm{Tr}(P_m \rho)$$

This generalizes to mixed states: The probability of measurement m is $\mathrm{Tr}(P_m \rho)$ and $\langle M \rangle = \mathrm{Tr}(M\rho)$

We know how to form a composite system from two subsystems

We can also go from composite to sub as follows: Assume $\mathcal{H}_1 \otimes \mathcal{H}_2$ is described by ρ, then the reduced density operator for $\mathcal{H}_1$ is

$$\rho_1 = \mathrm{Tr}_{\mathcal{H}_2} \rho$$

If M is a measurement on $\mathcal{H}_1$, and $\tilde{M}$ is a measurement on $\mathcal{H}_1 \otimes \mathcal{H}_2$ that measures the same physical quantity ("belonging to" $\mathcal{H}_1$), then $\tilde{M} = M \otimes I_2$ (where I_2 is the unity operator on $\mathcal{H}_2$). The operator $\rho_1 = \mathrm{Tr}_{\mathcal{H}_2} \rho$ on $\mathcal{H}_1$ is the unique operator such that

$$\mathrm{Tr}(M\rho_1) = \mathrm{Tr}(\tilde{M}\rho) = \mathrm{Tr}((M \otimes I_2)\rho)$$

That is, the expected outcome is the same

Purification: For a *mixed* state ρ in $\mathcal{H}$, there is always a space $\mathcal{R}$ and a state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{R}$ such that

$$\rho = \mathrm{Tr}_{\mathcal{R}} |\psi\rangle\langle\psi|$$

If $\rho = \sum_i \lambda_i |x_i\rangle\langle x_i|$ is a spectral decomposition for ρ, and $\{|y_j\rangle\}$ is a basis for $\mathcal{R}$ (dim of $\mathcal{R} \geq$ dim of $\mathcal{H}$), then we can choose

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |x_i\rangle|y_i\rangle$$

Note that ρ mixed $\Rightarrow |\psi\rangle$ *entangled*

Separation: A state of the form

$$\rho = \sum_i p_i \rho_i \otimes \sigma_i$$

is called separable $=$ a classical mix of non-entangled states

No cloning: For any Hilbert space $\mathcal{H}$ there is no unitary transformation U such that for $|\psi\rangle, |\psi\rangle' \in \mathcal{H}$,

$$U(|\psi\rangle \otimes |\psi\rangle') = |\psi\rangle \otimes |\psi\rangle$$

Quantum Teleportation

Two qubit systems $\mathcal{H}_1$ and $\mathcal{H}_2$ with bases $(|0\rangle_i, |1\rangle_i)$. Prepare the state

$$|\phi\rangle = \frac{|0\rangle_1|0\rangle_2 + |1\rangle_1|1\rangle_2}{\sqrt{2}} \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

Alice and Bob share and then split $|\phi\rangle$ in the sense that Alice has access to the $|\cdot\rangle_1$ qubit and Bob to the $|\cdot\rangle_2$

Later, when Alice and Bob are no longer co-located, Alice is in possession of a state $|\psi\rangle = \alpha|0\rangle_1 + \beta|1\rangle_1 \in \mathcal{H}_1$

Consider the state

$$|\psi\rangle|\phi\rangle = \frac{1}{\sqrt{2}}\alpha|0\rangle_1(|0\rangle_1|0\rangle_2+|1\rangle_1|1\rangle_2)+\frac{1}{\sqrt{2}}\beta|1\rangle_1(|0\rangle_1|0\rangle_2+|1\rangle_1|1\rangle_2)$$

where Alice can influence the $|\cdot\rangle_1$ qubits and Bob the $|\cdot\rangle_2$ qubit

By linear operations, Alice can map $|\psi\rangle|\beta\rangle$ into

$$\frac{1}{2}|0\rangle_1|0\rangle_1(\alpha|0\rangle_2 + \beta|1\rangle_2) + \frac{1}{2}|0\rangle_1|1\rangle_1(\alpha|1\rangle_2 + \beta|0\rangle_2) + \frac{1}{2}|1\rangle_1|0\rangle_1(\alpha|0\rangle_2 - \beta|1\rangle_2) + \frac{1}{2}|1\rangle_1|1\rangle_1(\alpha|1\rangle_2 - \beta|0\rangle_2)$$

only touching her own qubits

By measuring in the $|\cdot\rangle_1|\cdot\rangle_1$ basis, Alice will make Bob's qubit collapse into one of the states $\alpha|0\rangle_2 \pm \beta|1\rangle_2$ and $\beta|0\rangle_2 \pm \alpha|1\rangle_2$

$\Rightarrow$ Alice has transformed Bob's qubit into one of four states, from all of which he can conclude the pair (α, β)

This transformation happens instantaneously as Alice performs her measurement, irrespective of where Alice and Bob are physically located at that point in time

However, before Bob has got separate classical information about the outcome of Alice's measurement, the appropriate description of the system $\mathcal{H}_1 \otimes \mathcal{H}_1 \otimes \mathcal{H}_2$ from Bob's point of view is as a (classical) mix over four states described by ρ

Bob's own qubit is thus in the state

$$\rho' = \mathrm{Tr}_{\mathcal{H}_1\otimes\mathcal{H}_1}(\rho) = \frac{|0\rangle_2\langle 0|_2 + |1\rangle_2\langle 1|_2}{2}$$

$\Rightarrow$ he cannot yet conclude anything about ψ

Still, after Bob receives two classical bits informing him about the outcome of Alice's measurement, he can transform his qubit into $|\psi\rangle$, thus conveying "infinite bandwidth *quantum* information"

Note that Alice's copy of ψ has however collapsed, so the state has not been cloned

Faster-than-light communication is (always) impossible; infinite bandwidth *quantum* information transfer at finite resolution is seemingly possible; cloning is (always) impossible

EPR and Bell

Einstein, Rosen and Podolsky (EPR) published the paper "Can quantum-mechanical description of physical reality be considered complete," in 1935. They argued that any valid description of physical reality must obey two basic principles

Realism: Physical entities have numerical values which exist independent of observation

Locality: Two well-separated physical systems cannot influence each-other instantaneously

Then they used examples similar to quantum teleportation to argue that quantum mechanics is not complete (e.g. since in quantum teleportation Alice knows instantaneously the state of Bob's qubit)

The Bell inequality

Consider four classical binary random variables Q, R, S, T, with values in $\{\pm 1\}$ and with joint pmf $p(q, r, s, t)$

Assume this models an experiment where Alice and Bob share two pieces of a physical entity, and then separate

Alice decides at random to measure either property Q or R on her piece. Similarly Bob independently decides between S and T

They perform their experiments simultaneously, each observing a value in $\{\pm 1\}$

The physical properties measured are objective in the sense that they exist with certain values also when not observed

Simple calculations result in the Bell inequality

$$E(QS) + E(RS) + E(RT) - E(QT) \le 2$$

Note the assumptions made

- Joint classical pmf
- Locality
- Realism

A corresponding quantum inequality

Consider the Pauli (spin) matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

as linear operators on qubits in $\mathcal{H}$. Define the Bell–CHSH operator

$$B = \sigma_x \otimes (\sigma_x + \sigma_y) + \sigma_y \otimes (\sigma_y - \sigma_x)$$

Assume a pure state $|\psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ corresponding to a density operator $\rho = |\psi\rangle\langle\psi|$, then $|\langle B \rangle| = |\mathrm{Tr}\rho B|$ and

$$\mathrm{Tr}\rho(B - \langle B \rangle I)^2 \ge 0 \Rightarrow (\mathrm{Tr}\rho B)^2 \le \mathrm{Tr}(\rho B^2) \le 8$$

That is, $|\langle B \rangle| \le 2\sqrt{2}$

Notice that both σ_x and σ_y have eigenvalues ± 1, so when individually measured these are the possible outcomes

In $B = a \otimes (a' + b') + b \otimes (b' - a')$, $\{a, b\}$ belong to one qubit space ("Alice") and $\{a', b'\}$ to the other ("Bob")

In an attempt to describe the system as a classical system, we can assign $a \to R$, $b \to Q$, $a' \to T$, $b' \to S$, and

$$\tilde{B} = R(T + S) + Q(S - T)$$

to arrive at the Bell inequality $E[\tilde{B}] \leq 2$

However, in 1982 an experiment was carried out that fits the above quantum mechanical setup, and resulting in $2 < |\langle B \rangle| < 2\sqrt{2}$

$\Rightarrow$ the quantum system *cannot be described in terms of a classical probability distribution*

Entanglement as a non-classical resource

The quantum inequality for the Bell–CHSH operator: $|\langle B \rangle| \leq 2\sqrt{2}$

Since $\langle B \rangle = \langle \psi | B | \psi \rangle$ the value of $|\langle B \rangle|$ depends on the prepared state $|\psi\rangle$. Bell's inequality ($\langle B \rangle \leq 2$) can only be *violated when $|\psi\rangle$ is an entangled state*

As an example, if $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is a basis for $\mathcal{H} \otimes \mathcal{H}$, then

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \Rightarrow |\langle B \rangle| = 2\sqrt{2}$$

With this as the initial state shared by Alice and Bob, at least one of *the assumptions that led to the Bell inequality must be invalid* when trying to model the corresponding quantum system as a classical system