# Quantum

## Lecture 3

- Tensors
- Composite systems

# Tensor Products of Linear Spaces

For linear spaces $X$, $Y$ and $Z$ over the same field $F$ ($\mathbb{R}$ or $\mathbb{C}$), a mapping $L(x, y)$ from $X \times Y$ to $Z$ is bilinear if

$$L(a_1x_1 + a_2x_2, y) = a_1L(x_1, y) + a_2L(x_2, y)$$
$$L(x, b_1y_1 + b_2x_2) = b_1L(x, y_1) + b_2L(x, y_2)$$

for all $x_1, x_2, x \in X$, $y_1, y_2, y \in Y$ and scalars $a_1, a_2, b_1, b_2$

When $Z = F$ the mapping is a bilinear form

Let $\mathcal{B}(X \times Y)$ denote the set of bilinear forms over $X \times Y$

For $x \in X$ and $y \in Y$ let $x \otimes y$ denote the functional $f : \mathcal{B}(X \times Y) \to F$ defined as

$$f(L) = L(x, y)$$

for each bilinear form $L$ on $X \times Y$

For fixed $x$ and $y$, $x \otimes y$ is called a simple tensor

The tensor product $X \otimes Y$ of the spaces $X$ and $Y$ is the linear space spanned by the the simple tensors

The members of $X \otimes Y$ are called tensors

Any tensor $u \in X \otimes Y$ can thus be written as

$$u = \sum_i a_i x_i \otimes y_i$$

for some $a_i \in F$, $x_i \in X$ and $y_i \in Y$

Interpreting $x \otimes y$ as a 'product' on $X \times Y$ with values in $X \otimes Y$, we can conclude

$$(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$$
$$x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$$
$$a(x \otimes y) = (ax) \otimes y = x \otimes (ay)$$

Since $a_i(x_i \otimes y_i) = (a_i x_i) \otimes (y_i) = x_i \otimes (a_i y_i)$, each tensor has an equivalent representation

$$u = \sum_i x_i \otimes y_i$$

for some (other) $x_i \in X$ and $y_i \in Y$

If $\{a_i\}$ and $\{b_j\}$ are bases for $X$ and $Y$, then $\{a_i \otimes b_j\}$ is a basis for $X \otimes Y$

If $X$ and $Y$ are finite-dimensional of dimensions $d_1$ and $d_2$ then $X \otimes Y$ has dimension $d_1 d_2$. Also, for $u \in X \otimes Y$ there is then a smallest $n < \infty$, the rank of $u$, such that $u = \sum_{i=1}^{n} x_i \otimes y_i$

For each bilinear form $L : X \times Y \to F$ there is a unique linear functional $\tilde{L} : X \otimes Y \to F$ such that $L(x, y) = \tilde{L}(x \otimes y)$

Hence, $x \otimes y$ can be identified as the nonlinear combination of $x$ and $y$ that linearizes the mapping $(x, y) \to L(x, y)$

## Finite (real) dimensions

$X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, with elements interpreted as column vectors, inner product $\langle x_1, x_2 \rangle = x_1^T x_2$, etc., and $Z = \mathbb{R}$

Bilinear forms have the representation $L(x, y) = x^T A y$ for some matrix $A$

If we identify a bilinear $L$ with its matrix representation $A(L)$, then $x \otimes y(L)$ assigns the value $x^T A(L) y$

For each bilinear $L$, there is a unique linear $\tilde{L}$ such that $L(x, y) = x^T A y = \tilde{L}(x \otimes y)$

Since $x^T A y = \mathrm{Tr}(A y x^T)$, we can identify $x \otimes y$ with the outer vector product $y x^T$

We can also identify $x \otimes y$ with the Kronecker (or tensor) matrix product

$$x \otimes y = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \otimes y = \begin{bmatrix} x_1 y \\ x_2 y \\ \vdots \\ x_n y \end{bmatrix} \in \mathbb{R}^{nm}$$

between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$

Then the linear functional $\tilde{L}$ corresponding to $L(x, y)$ has the representation

$$\tilde{L}(x \otimes y) = a^T x \otimes y$$

for some $a \in \mathbb{R}^{nm}$

The outer and Kronecker product representations are in fact equivalent, since $\mathrm{Tr}(Ayx^T) = a^T x \otimes y \iff a = \mathrm{vec} A^T$ (column-wise vectorization). Also note that $yx^T = y \otimes x^T$

# Tensor Product of Hilbert Spaces

Consider two Hilbert spaces $(\mathcal{H}_1, g_1)$ and $(\mathcal{H}_2, g_2)$, over $\mathbb{C}$

Hilbert spaces are linear spaces, so we have a corresponding tensor product space (as a linear space)

On this space we can define an inner product as

$$\langle x_1 \otimes y_1, x_2 \otimes y_2 \rangle = g_1(x_1, x_2) g_2(y_1, y_2)$$

for simple tensors $x_1 \otimes y_1$ and $x_2 \otimes y_2$, with obvious extension to linear combinations

The (completion of) this inner product space is a Hilbert space, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$: the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$

For linear operators $A$ and $B$ on $\mathcal{H}_1$ and $\mathcal{H}_2$, we also define $A \otimes B$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ via

$$(A \otimes B)(x \otimes y) = A(x) \otimes B(y)$$

with obvious extension to linear combinations

## Partial trace

For two Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$, and a fixed $g \in \mathcal{G}$, define $T_g : \mathcal{H} \to \mathcal{H} \otimes \mathcal{G}$ as

$$T_g(h) = h \otimes g$$

The corresponding adjoint $T_g^* : \mathcal{H} \otimes \mathcal{G} \to \mathcal{H}$ is obtained as

$$T_g^*(u \otimes v) = \langle g, v \rangle u$$

(with extension to linear combinations)

If $T$ is a trace-class linear operator on $\mathcal{H} \otimes \mathcal{G}$ then for all $g \in \mathcal{G}$ the operator $S : \mathcal{H} \to \mathcal{H}$ defined by

$$S_g(x) = T_g^*(T(T_g(x)))$$

is trace-class on $\mathcal{H}$

For a trace-class operator $T$ on $\mathcal{H} \otimes \mathcal{G}$, and any orthonormal basis $\{g_i\}$ in $\mathcal{G}$ define

$$\mathrm{Tr}_{\mathcal{G}}(T) = \sum_i S_{g_i}$$

(as an operator from $\mathcal{H}$ to $\mathcal{H}$). The definition does not depend on the choice of basis

$\mathrm{Tr}_{\mathcal{G}}(T)$ is the partial trace of $T$ w.r.t. $\mathcal{G}$

the space $\mathcal{G}$ has been "traced out"

$\mathrm{Tr}_{\mathcal{G}}(T)$ is the unique trace-class operator on $\mathcal{H}$ such that

$$\mathrm{Tr}(\mathrm{Tr}_{\mathcal{G}}(T)B) = \mathrm{Tr}(T(B \otimes I))$$

for all bounded $B : \mathcal{H} \to \mathcal{H}$ (and $I$ the identity)

For the special case $T = T_1 \otimes T_2$ we get

$$\mathrm{Tr}_{\mathcal{G}}(T)(x) = \sum_i \langle g_i, T_2(g_i)\rangle T_1(x) = \mathrm{Tr}(T_2) T_1(x)$$

In general, linear operators on $\mathcal{H}_1 \otimes \mathcal{H}_2$ are of the form

$$T = \sum_{ij} a_{ij} T_i^{(1)} \otimes T_j^{(2)}$$

for $T_k^{(i)} : \mathcal{H}_i \to \mathcal{H}_i$. The partial trace then has the representation

$$\mathrm{Tr}_{\mathcal{H}_1}(T) = \sum_{ij} a_{ij} \mathrm{Tr}(T_i^{(1)}) T_j^{(2)}$$

In finite dimensions: For $\mathcal{H}$ of dimension $m$ and $\mathcal{G}$ of dimension $n$, let $\{h_i\}$ be a basis for $\mathcal{H}$ and $\{g_j\}$ one for $\mathcal{G}$. Let $T$ be a linear mapping from $\mathcal{H} \otimes \mathcal{G}$ to a linear mapping on $\mathcal{H}$, and let $t_{k\ell,ij}$, $1 \leq k, i \leq m$, $1 \leq \ell, j \leq n$, be a matrix representation for $T$ relative to the basis $h_k \otimes g_\ell$ in $\mathcal{H} \otimes \mathcal{G}$. Define the matrix

$$s_{k,i} = \sum_{j=1}^{n} a_{kj,ij}$$

for $1 \leq k, i \leq m \Rightarrow$ the corresponding linear mapping defines the partial trace $\mathrm{Tr}_{\mathcal{G}} T$ of $T$ over $\mathcal{G}$

In general, with matrices $\{H_i\}$ $(m \times m)$ and $\{G_j\}$ $(n \times n)$, and for $T = \sum_{ij} c_{ij} H_i \otimes G_j$ we get

$$\mathrm{Tr}_{\mathcal{G}} T = \sum_{ij} c_{ij} \mathrm{Tr}(G_j) H_i \quad (m \times m)$$

# Composite Quantum Systems

Postulate 4: Assume that two different isolated quantum systems have states in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Then the composite system representing the simultaneous characterization of both systems has states in $\mathcal{H}_1 \otimes \mathcal{H}_2$

# Two Qubit Systems

Consider the simultaneous description of two qubit systems, with individual states in $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. The composite system is characterized by the 4-dimensional space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$

If $\{|0\rangle_i, |1\rangle_i\}$ is a basis for $\mathcal{H}_i$ then

$$|00\rangle = |0\rangle_1|0\rangle_2, \;\; |01\rangle = |0\rangle_1|1\rangle_2, \;\; |10\rangle = |1\rangle_1|0\rangle_2, \;\; |11\rangle = |1\rangle_1|1\rangle_2$$

is a basis for $\mathcal{H}$

# Entanglement

In $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, consider the state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

This can be verified to be a rank-2 tensor in $\mathcal{H}$, that is, $|\psi\rangle$ cannot be written as $|a\rangle|b\rangle$ for any $|a\rangle \in \mathcal{H}_1$ and $|b\rangle \in \mathcal{H}_2$

The rank-2 states $|\psi\rangle$ in $\mathcal{H}$ (rank-$r$, $r > 1$, in general) are called entangled states