Quantum Information Theory
Spring semester, 2017

Assignment 4
Assigned: Friday, April 7, 2017
Due: Friday, April 21, 2017 M. Skoglund

Problem 4.1: Explain and interpret the concepts of *density operator* and *reduced* density operator.

Problem 4.2: Discuss how *entanglement* is a non-classical concept: In what sense is entanglement stronger than classical correlation? How does entanglement relate to the EPR paradox, and/or the Bell inequality?

Problem 4.3: Prove that the state

$$|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

is entangled.

Problem 4.4: Prove that any mixed state ρ in $\mathcal{H}$ can be *purified*, $\rho = \mathrm{Tr}_{\mathcal{R}}|\psi\rangle\langle\psi|$, by the state

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |x_i\rangle|y_i\rangle$$

in $\mathcal{H} \otimes \mathcal{R}$, given a spectral decomposition $\rho = \sum_i \lambda_i |x_i\rangle\langle x_i|$ for ρ and where $\{|y_i\rangle\}$ is a basis for $\mathcal{R}$.

Problem 4.5: [2.71 in NC] For any density operator (state) ρ, show that $\mathrm{Tr}(\rho^2) \leq 1$, with equality iff ρ is a pure state.

Problem 4.6: [2.73 in NC] The *rank* of a compact self-adjoint operator $O : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is equal to the dimension of $\{|y\rangle : |y\rangle = O|x\rangle,\ |x\rangle \in \mathcal{H}\}$. For a given density operator ρ of rank $r < \infty$, a representation

$$\rho = \sum_{i=1}^{s} p_i |\psi_i\rangle\langle\psi_i|$$

is *minimal* if $s = r$. Let $\{|x_i\rangle\}$ be all eigenvectors corresponding to non-zero eigenvalues of ρ, and let $|\psi\rangle$ be an arbitrary vector in $\mathrm{span}\{|x_i\rangle\}$. Show that there exists a minimal representation $\{p_i, |\psi_i\rangle\}$ where one of the $|\psi_i\rangle$'s can be chosen as $|\psi\rangle$. Also express the probabilities p_i in terms of $|\psi_i\rangle$ and ρ.

Problem 4.7: Prove the *no-cloning theorem*: For any Hilbert space $\mathcal{H}$ there is no unitary transformation U such that for $|\psi\rangle, |\psi\rangle' \in \mathcal{H}$,

$$U(|\psi\rangle \otimes |\psi\rangle') = |\psi\rangle \otimes |\psi\rangle$$