Probability and Random Processes

Lecture 4

- General integration theory

Measurable Extended Real-valued Functions

- $\mathbb{R}^*$ = the extended real numbers; a subset $O \subset \mathbb{R}^*$ is open if it can be expressed as a countable union of intervals of the form (a, b), $[-\infty, b)$, $(a, \infty]$
- A measurable space $(\Omega, \mathcal{A})$; an extended real-valued function $f : \Omega \to \mathbb{R}^*$ is measurable if $f^{-1}(O) \subset \mathcal{A}$ for all open $O \subset \mathbb{R}^*$
- A sequence $\{f_n\}$ of measurable extended real-valued functions: for any x, $\limsup f_n(x)$ and $\liminf f_n(x)$ are measurable $\Rightarrow$ if $f_n \to g$ pointwise, then g is measurable
  - Hence, with the definition above, e.g.

$$f_n(x) = \frac{n}{\sqrt{2\pi}} \exp\left(-\frac{(nx)^2}{2}\right)$$

    converges to a measurable function on $(\mathbb{R}, \mathcal{B})$ or $(\mathbb{R}, \mathcal{L})$

Measurable Simple Function

- An $\mathcal{A}$-measurable function s is a simple function if its range is a finite set $\{a_1, \dots, a_n\}$. With $A_k = \{x : s(x) = a_k\}$, we get

$$s(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)$$

(since s is measurable, $A_k \in \mathcal{A}$)

Integral of a Nonnegative Simple Function

- A measure space $(\Omega, \mathcal{A}, \mu)$ and $s : \Omega \to \mathbb{R}$ a nonnegative simple function which is $\mathcal{A}$-measurable, represented as

$$s(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x)$$

The integral of s over Ω with respect to μ is defined as

$$\int s(x) d\mu(x) = \sum_{k=1}^{n} a_k \mu(A_k)$$

Approximation by a Simple Function

- For any nonnegative extended real-valued and $\mathcal{A}$-measurable function f, there is a nondecreasing sequence of nonnegative $\mathcal{A}$-measurable simple functions that converges pointwise to f,

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x)$$
$$f(x) = \lim_{n \to \infty} s_n(x)$$

- If f is the pointwise limit of an increasing sequence of nonnegative $\mathcal{A}$-measurable simple functions, then f is an extended real-valued $\mathcal{A}$-measurable function

$\Longleftrightarrow$ The nonnegative extended real-valued $\mathcal{A}$-measurable functions are exactly the ones that can be approximated using sequences of $\mathcal{A}$-measurable simple functions

Integral of a Nonnegative Function

- A measure space $(\Omega, \mathcal{A}, \mu)$ and $f : \Omega \to \mathbb{R}^*$ a nonnegative extended real-valued function which is $\mathcal{A}$-measurable. The integral of f over Ω is defined as

$$\int_\Omega f d\mu = \sup_s \int_\Omega s d\mu$$

where the supremum is over all nonnegative $\mathcal{A}$-measurable simple functions dominated by f.

- Integral over an arbitrary set $E \in \mathcal{A}$,

$$\int_E f d\mu = \int_\Omega f \chi_E d\mu$$

Convergence Results

- MCT: if $\{f_n\}$ is a monotone nondecreasing sequence of nonnegative extended real-valued $\mathcal{A}$-measurable functions, then
$$\int_E \lim f_n d\mu = \lim \int_E f_n d\mu$$
for any $E \in \mathcal{A}$
- Fatou: if $\{f_n\}$ is a sequence of nonnegative extended real-valued $\mathcal{A}$-measurable functions, then
$$\int_E \liminf f_n d\mu \leq \liminf \int_E f_n d\mu$$
for any $E \in \mathcal{A}$

Integral of a General Function

- Let f be an extended real-valued $\mathcal{A}$-measurable function, and let $f^+ = \max\{f, 0\}, f^- = -\min\{f, 0\}$, then the integral of f over E is defined as
$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$
for any $E \in \mathcal{A}$
- f is integrable over E if
$$\int_E |f| d\mu = \int_E f^+ d\mu + \int_E f^- d\mu < \infty$$

Integral of a Function Defined A.E.

- A measure space $(\Omega, \mathcal{A}, \mu)$, and a function f defined μ-a.e. on Ω (if D is the domain of f then $\mu(D^c) = 0$). If there is an extended real-valued $\mathcal{A}$-measurable function g such that $g = f$ μ-a.e., then define the integral of f as

$$\int_E f d\mu = \int_E g d\mu$$

for any $E \in \mathcal{A}$.

DCT, General Version

- A measure space $(\Omega, \mathcal{A}, \mu)$, and a sequence $\{f_n\}$ of extended real-valued $\mathcal{A}$-measurable functions that converges pointwise μ-a.e. Assume that there is a nonnegative integrable function g such that $|f_n| \leq g$ μ-a.e. for each n. Then

$$\int_E \lim f_n d\mu = \lim \int_E f_n d\mu$$

for any $E \in \mathcal{A}$

DCT: Proof

- Let $f(x) = \lim f_n(x)$ if $\lim f_n(x)$ exists, and $f(x) = 0$ o.w., then f is measurable and $f_n \to f$ μ-a.e. Hence

$$\int_E \lim f_n d\mu = \int_E f d\mu$$

- Fatou $\Rightarrow$

$$\int (g - f) d\mu \leq \liminf_{n\to\infty} \int (g - f_n) d\mu = \int g d\mu - \limsup_{n\to\infty} \int f_n d\mu$$

$\Rightarrow \limsup \int f_n d\mu \leq \int f d\mu$

- Fatou $\Rightarrow$

$$\int (g + f) d\mu \leq \liminf_{n\to\infty} \int (g + f_n) d\mu \Rightarrow \int f d\mu \leq \liminf_{n\to\infty} \int f_n d\mu$$

DCT for Convergence in Measure

- A measure space $(\Omega, \mathcal{A}, \mu)$, and a sequence $\{f_n\}$ of extended real-valued $\mathcal{A}$-measurable functions that converges in measure to the $\mathcal{A}$-measurable function f. Assume that there is a nonnegative integrable function g such that $|f_n| \leq g$ μ-a.e. for each n. Then

$$\int_E f d\mu = \lim \int_E f_n d\mu$$

for any $E \in \mathcal{A}$

Distribution Functions

- Let μ be a finite measure on $(\mathbb{R}, \mathcal{B})$, then the distribution function of μ is defined as

$$F_\mu(x) = \mu((-\infty, x])$$

- A (general) real-valued function F on $\mathbb{R}$ is called a distribution function if the following holds
  1. F is monotone nondecreasing
  2. F is right continuous
  3. F is bounded
  4. $\lim_{x \to -\infty} F(x) = 0$
- Each distribution function is the distribution function corresponding to a unique finite measure on $(\mathbb{R}, \mathcal{B})$
- The finite measure μ corresponding to F is called the Lebesgue–Stieltjes measure corresponding to F

The Lebesgue–Stieltjes Integral

- Let F be a distribution function with corresponding Lebesgue–Stieltjes measure μ. Let f be a Borel measurable function, then the Lebesgue–Stieltjes integral of f w.r.t. F is defined as

$$\int f(x) dF(x) = \int f(x) d\mu(x)$$

The Lebesgue–Stieltjes Integral: Example

- Take the Dirac measure

$$\delta_b(E) = \begin{cases} 1, & b \in E \\ 0, & \text{o.w.} \end{cases}$$

  and restrict it to $\mathcal{B}$, then the corresponding distribution function is

$$F(x) = \begin{cases} 1, & x \geq b \\ 0, & \text{o.w.} \end{cases}$$

- Let f be finite and Borel measurable, then

$$\int f(x) dF(x) = f(b)$$

- A way of handling discrete (random) variables and expectation, without having to resort to 'Dirac δ-functions'