Probability and Random Processes

Lecture 2

- The Lebesgue integral on the real line

Simple Functions

- A Lebesgue measurable function s that takes on only a finite number of values $\{s_i\}$ is called a simple function
- Let $S_i = \{x : s(x) = s_i\}$ and

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & \text{o.w.} \end{cases}$$

then

$$s(x) = \sum_i s_i \chi_{S_i}(x)$$

- For any Lebesgue measurable and nonnegative function f, there is a nondecreasing sequence of simple nonnegative functions that converges pointwise to f,

$$0 \leq s_1(x) \leq s_2(x) \leq \cdots \leq f(x)$$
$$f(x) = \lim_{n\to\infty} s_n(x)$$

- If f is the pointwise limit of an increasing sequence of simple nonnegative functions, then f is Lebesgue measurable

$\iff$ The nonnegative Lebesgue measurable functions are exactly the ones that can be approximated using sequences of simple functions

The Lebesgue Integral

- The integral of a simple nonnegative function

$$\int s(x)d\lambda(x) = \sum_i s_i \lambda(S_i)$$

- The integral of a Lebesgue measurable nonnegative function

$$\int f(x)d\lambda(x) = \lim_{n\to\infty} \int s_n(x)d\lambda(x)$$

where $\{s_n(x)\}$ are simple functions that approximate $f(x)$, or

$$\int f(x)d\lambda(x) = \sup \int s(x)d\lambda(x)$$

over nonnegative simple functions $s(x) \leq f(x)$

- A general (positive and negative) Lebesgue measurable function f, define
  - $f^+ = \max\{0, f\}$; $f^- = -\min\{0, f\}$
- Obviously

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-$$

  and, furthermore, f^+ and f^- are Lebesgue measurable if f is, and vice versa
- The general Lebesgue integral

$$\int f d\lambda = \int f^+ d\lambda - \int f^- d\lambda$$

  (where the integrals on the r.h.s. are defined as before)
- Integral over a set $E \in \mathcal{L}$

$$\int_E f d\lambda = \int \chi_E f d\lambda$$

- f Lebesgue measurable and

$$\int |f| d\lambda = \int f^+ d\lambda + \int f^- d\lambda < \infty$$

  $\Rightarrow f$ Lebesgue integrable
- For any function $f(x) < \infty$ of bounded support, iff

$$\lambda(\{x : f \text{ is discontinous at } x\}) = 0$$

  that is, f is continuous Lebesgue almost everywhere (λ-a.e.), then $f(x)$ is both Riemann and Lebesgue integrable, and the integrals are equal
- $f = g$ λ-a.e. and f measurable $\Rightarrow g$ measurable, and if f is integrable then g is too and the integrals are equal

Convergence Theorems

- One of the most useful properties of Lebesgue integration theory: powerful convergence theorems for a sequence of functions $\{f_i\}$ with pointwise limit f
- A statement like

$$\int \lim_{i\to\infty} f_i(x)dx = \lim_{i\to\infty} \int f_i(x)dx$$

  may not make sense if the integral is a Riemann integral, since f is in general not Riemann integrable even if all the f_i's are continuous
- If the integral is instead a Lebesgue integral, then under "mild" conditions, the statement is usually true

Monotone Convergence Theorem (MCT)

- Assume $\{f_i(x)\}$ is a monotone nondecreasing sequence of nonnegative Lebesgue measurable functions and that $\lim_{i\to\infty} f_i(x) = f(x) < \infty$ pointwise. Then

$$\int \lim_{i\to\infty} f_i d\lambda = \int f d\lambda = \lim_{i\to\infty} \int f_i d\lambda$$

Monotone Convergence Theorem (MCT): Proof

- $\{f_i\}$ measurable $\Rightarrow f$ measurable $\Rightarrow K = \int f d\lambda$ exists but can be ∞
- $\{f_i\}$ nondecreasing $\Rightarrow \{\int f_i d\lambda\}$ nondecreasing $\Rightarrow L = \lim \int f_i d\lambda$ exists but can be ∞
- Since $f_i \leq f, L \leq K$
- For $\alpha \in (0,1)$, let $0 \leq s \leq f$ be a simple function and $E_n = \{x : f_n(x) \geq \alpha s(x)\}$

- $\{f_i\}$ nondecreasing $\Rightarrow E_1 \subset E_2 \subset \cdots$. Also, $\cup_n E_n = \mathbb{R}$. Hence

$$\alpha \int s d\lambda = \lim_{n\to\infty} \int_{E_n} \alpha s d\lambda \leq \limsup_{n\to\infty} \int_{E_n} f_n d\lambda \leq L$$

- Consequently $\int s d\lambda \leq \alpha^{-1} L$ and thus

$$\int f d\lambda = \sup_{0 \leq s \leq f} \int s d\lambda \leq \alpha^{-1} L$$

for each $\alpha \in (0,1)$; letting $\alpha \to 1 \Rightarrow K \leq L$

Fatou's Lemma

- Let $\{f_i\}$ be a sequence of nonnegative Lebesgue measurable functions, with $\lim f_i = f$ pointwise. Then

$$\int \lim_{i\to\infty} f_i d\lambda \le \liminf_{i\to\infty} \int f_i d\lambda$$

- Proof:
  - Let $g_n = \inf_{k\ge n} f_k \Rightarrow \{g_n\}$ is a nondecreasing sequence of nonnegative measurable functions and $\lim g_n = f$ pointwise
  - Thus by the MCT

$$\int f d\lambda = \lim_{i\to\infty} \int g_i d\lambda$$

  - However, since $g_i \le f_i$ it also holds that

$$\lim_{i\to\infty} \int g_i d\lambda \le \liminf_{i\to\infty} \int f_i d\lambda$$

Dominated Convergence Theorem (DCT)

- Assume $\{f_i\}$ are Lebesgue measurable functions, that $f = \lim_{i\to\infty} f_i$ exists pointwise, and that there is a Lebesgue measurable and integrable function $g \ge 0$ such that $|f_i| \le g$, then f is Lebesgue measurable and integrable and

$$\int \lim_{i\to\infty} f_i d\lambda = \int f d\lambda = \lim_{i\to\infty} \int f_i d\lambda$$

Dominated Convergence Theorem: Proof

- $|f_n| \leq g$ and g integrable $\Rightarrow f_n$ and f integrable
- $g - f_n \to g - f$ pointwise; Fatou's lemma gives

$$\int (g-f)d\lambda \leq \liminf_{n\to\infty} \int (g-f_n)d\lambda = \int g d\lambda - \limsup_{n\to\infty} \int f_n d\lambda$$

$\Rightarrow \limsup \int f_n d\lambda \leq \int f d\lambda$

- $g + f_n \geq 0$ and $g + f_n \to g + f$ pointwise; Fatou's lemma gives

$$\int (g+f)d\lambda \leq \liminf_{n\to\infty} \int (g+f_n)d\lambda \Rightarrow \int f d\lambda \leq \liminf_{n\to\infty} \int f_n d\lambda$$