Probability and Random Processes
Spring semester, 2024

Assignment 9
Assigned: Thursday, March 21, 2024
Due: Thursday, April 11, 2024 M. Skoglund

Problem 9.1: Given a set Ω, a semialgebra $\mathcal{C}$ of subsets and a pre-measure m on $\mathcal{C}$; explain the steps that need to be taken to extend m to a measure μ on a σ-algebra that contains $\mathcal{C}$.

Problem 9.2: Given the measure constructed in problem 1, state a condition (involving $\mathcal{C}$ and m) that ensures that there exists a unique extension of m from $\mathcal{C}$ to $\sigma(\mathcal{C})$.

Problem 9.3: Introduce and explain the concept product measure space.

Problem 9.4: Let $\mathcal{I}$ be the collection of all intervals $\subset \mathbb{R}$ and let $\ell(I) =$ length of $I \in \mathcal{I}$. Show that λ (Lebesgue measure) is the unique extension of ℓ from $\mathcal{I}$ to $\mathcal{L}$ (the Lebesgue sets). Note that $\mathcal{B} = \sigma(I) \neq \mathcal{L}$ (where $\mathcal{B} =$ the Borel sets).

Problem 9.5: Again, let $\mathcal{I}$ be the collection of all intervals $\subset \mathbb{R}$. Given a nonnegative function $g : \mathbb{R} \to \mathbb{R}^+$ that satisfies $\int_{(-\infty,n)} g d\lambda < \infty$ for $n = 0, 1, 2, \ldots$, let

$$m(I) = \int_I g d\lambda$$

for any $I \in \mathcal{I}$. Show that there is a unique extension μ of m from $\mathcal{I}$ to $\mathcal{B}$ (the Borel sets) that satisfies $\mu(B) = \int_B g d\lambda$ for $B \in \mathcal{B}$.

Problem 9.6: Let $\mathcal{B}_2 =$ smallest σ-algebra of subsets of $\mathbb{R}^2$ that contains all the open sets in $\mathbb{R}^2$. Show that $\mathcal{B}_2 = \mathcal{B} \times \mathcal{B}$. Assume that μ and ν are two finite measures on $\mathcal{B}_2$, and that

$$\mu(A \times B) = \nu(A \times B)$$

for all $A, B \in \mathcal{B}$, prove that $\mu = \nu$.