Probability and Random Processes
Spring semester, 2024

Assignment 8
Assigned: Thursday, March 14, 2024
Due: Thursday, March 21, 2024 M. Skoglund

Problem 8.1: Given a set Ω and a class $\mathcal{T}$ of subsets. Describe the conditions for $\mathcal{T}$ to be a topology. Given two topological spaces $(\Omega, \mathcal{T})$ and $(\Gamma, \mathcal{S})$, and a function $f : \Omega \to \Gamma$, define what it means for f to be continuous. Finally also describe how a topological space can be defined based on a given metric space, and what it means for a topological space to be metrizable.

Problem 8.2: Given a topological space $(\Omega, \mathcal{T})$, define the corresponding Borel space. Given two topological spaces $(\Omega, \mathcal{T})$ and $(\Gamma, \mathcal{S})$, and a function $f : \Omega \to \Gamma$. Define what it means for f to be Borel measurable. Relate the class of Borel measurable functions to the continuous functions.

Problem 8.3: Define what it means for a sequence to converge in a general topological space. Also specialize to a metric space. Define Cauchy sequence in a metric space, and the concept of a complete space.

Problem 8.4: Given a topological space $(\Omega, \mathcal{T})$ and a subset $E \subset \Omega$, define what it means for E to be dense in Ω. Also define the concept separable topological space. Finally also define the concept Polish space.

Problem 8.5: Define the concept standard measurable space and relate it to the concept Polish space.

Problem 8.6: Given two topological spaces $(\Omega, \mathcal{T})$ and $(\Gamma, \mathcal{S})$, the two spaces are said to be homeomorphic if there is a function $f : \Omega \to \Gamma$ such that

- f is 1-to-1; f is continuous; f^{-1} is continuous

Prove that $(0, 1)$ and $\mathbb{R}$ are homeomorphic.

Problem 8.7: Given two topological spaces $(\Omega, \mathcal{T})$ and $(\Gamma, \mathcal{S})$, and a function $f : \Omega \to \Gamma$, let $\mathcal{X}$ be the set of all sequences that converge in $(\Omega, \mathcal{T})$. Prove that

- f is continuous $\Rightarrow \lim_n f(x_n) \to f(\lim_n x_n)$ for all $\{x_n\} \in \mathcal{X}$
- if $(\Omega, \mathcal{T})$ is metrizable, then f is continuous $\iff$ $\lim_n f(x_n) \to f(\lim_n x_n)$ for all $\{x_n\} \in \mathcal{X}$