Probability and Random Processes
Spring semester, 2024

Assignment 10

Assigned: Thursday, April 11, 2024
Due: Thursday, April 18 20, 2024 M. Skoglund

Problem 10.1: Describe the two different (but equivalent) definitions of a random process used in the lecture.

Problem 10.2: State and explain Kolmogorov's extension theorem.

Problem 10.3: Given $(\Omega, \mathcal{A}, P)$, a sequence $\mathcal{A}_0, \mathcal{A}_1, \ldots$, such that

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}$$

is called a *filtration*. A random process/sequence $\{X_n\}$ ($X_n : \Omega \to \mathbb{R}$) is *adapted* to a given filtration if X_n is $\mathcal{A}_n$-measurable. A process adapted to a filtration is called a *martingale* if $E(|X_n|) < \infty$ and

$$E[X_n | \mathcal{A}_{n-1}] = X_{n-1}$$

with probability one, for $n \geq 1$.

Let $\{X_n\}_{n=1}^{\infty}$ be zero-mean and independent variables, with $E(|X_n|) < \infty$. Let $S_n = \sum_{i=1}^{n} X_i$ and $\mathcal{A}_n = \sigma(X_1, \ldots, X_n)$. Show that $\{S_n\}$ is a martingale (given $\{\mathcal{A}_n\}$).

Problem 10.4: Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of variables $X_n : (\Omega, \mathcal{A}, P) \to \{0, 1\}$, such that $\{X_{n_1}, X_{n_2}, \ldots, X_{n_N}\}$ are mutually independent for any $1 \leq n_1 < n_2 < \cdots < n_N < \infty$ and $2 \leq N < \infty$, and $P(X_n = 1) = P(X_n = 0) = 1/2$ for all n. Let

$$Y = \sum_{n=1}^{\infty} 2^{-n} X_n$$

Show that the unique distribution of Y on $([0, 1], \mathcal{B}([0, 1])$ is Lebesgue measure.

Problem 10.5: Consider a continuous-time random process $\{X_t\}_{t \in T}$ with $T = [0, \infty)$. A continuously indexed family of σ-algebras $\{\mathcal{A}_t\}_{t \in T}$ is a filtration if $\mathcal{A}_u \subset \mathcal{A}_v$ for $u < v$. Assume that X_t is $\mathcal{A}_t$ measurable, and that $E(|X_t|) < \infty$, then $\{X_t\}$ is a martingale if $E[X_t | \mathcal{A}_s] = X_s$ (with probability one) for $t > s$.

The process $\{X_t\}$ is a *Brownian motion* (or Wiener process) if $\{X_t\}$ is zero-mean Gaussian and $E[X_u X_v] = \min(u, v)$. Show that if $\{X_t\}$ is a Brownian motion with $E(|X_t|) < \infty$ then it is also a martingale w.r.t. $\mathcal{A}_t = \sigma(\{X_s, s \leq t\})$.