Overview

Lecture 1-4: Channel capacity
  - Gaussian channels
  - Fading Gaussian channels
  - Multiuser Gaussian channels
  - Multiuser diversity

Lecture 5: Antenna diversity and MIMO capacity
Diversity

Multiuser diversity (lecture 4)
- Transmissions over independent fading channels.
- Sum capacity increases with the number of users.
  → High probability that at least one user will have a strong channel.

Fading channels (point-to-point links)
- Use diversity to mitigate the effect of (deep) fading.
- Diversity: let symbols pass through multiple paths.
- Time diversity: interleaving and coding, repetition coding.
- Frequency diversity: for example OFDM.
- Antenna Diversity.

Antenna/Spatial Diversity

Motivation: For narrowband channels with large coherence time or delay constraints, time diversity and frequency diversity cannot be exploited!

Antenna diversity
- Multiple transmit/receive antennas with sufficiently large spacing:
  - Mobiles: rich scattering → 1/2...1 carrier wavelength.
  - Base stations on high towers: tens of carrier wavelength.
- Receive diversity: multiple receive antennas,
  → single-input/multiple-output (SIMO) systems.
- Transmit diversity: multiple transmit antennas,
  → multiple-input/single-output (MISO) systems.
- Multiple transmit and receive antennas,
  → multiple-input/multiple-output (MIMO) systems.
Antenna/Spatial Diversity

- **Receive Diversity (SIMO)**

  - Channel model: flat fading channel, 1 transmit antenna, L receive antennas:
    \[
    y[m] = h[m] \cdot x[m] + w[m], \\
    y[l][m] = h[l][m] \cdot x[m] + w[l][m], \quad l = 1, \ldots, L
    \]
    with
    - additive noise \(w[l][m] \sim CN(0, N_0)\), independent across antennas,
    - Rayleigh fading coefficients \(h[l][m]\).
  - Optimal diversity combining: maximum-ratio combining (MRC)
    \[
    r[m] = h[m]^\ast \cdot y[m] = ||h[m]||^2 \cdot x[m] + h[l][m]w[l][m]
    \]
  - Error probability for BPSK (conditioned on \(h[m]\))
    \[
    Pr(x[m] \neq \text{sign}(r[m])) = Q(\sqrt{2||h||^2 \text{SNR}})
    \]
    with the (instantaneous) SNR
    \[
    \gamma = ||h||^2 \text{SNR} = ||h||^2 \mathbb{E}(|x|^2)/N_0 = L \text{SNR} \cdot \frac{1}{L} ||h||^2
    \]
    \(
    \rightarrow \text{Diversity gain due to } \frac{1}{L} ||h||^2 \text{ and power/array gain } L \text{SNR.}
    \)
    \(
    \rightarrow 3 \text{ dB gain by doubling the number of antennas.}
    \)

Antenna/Spatial Diversity

- **Transmit Diversity (MISO), Space-Time Coding**

  Channel model
  
  Flat fading channel, L transmit antennas, 1 receive antenna:
  \[
  y[m] = h^T[m] \cdot x[m] + w[m], \quad \text{with}
  \]
  - additive noise \(w[m] \sim CN(0, N_0)\),
  - vector \(h[m]\) of Rayleigh fading coefficients \(h_i[m]\).
  
  Alamouti scheme
  
  - Rate-1 space-time block code (STBC) for transmitting two data symbols \(u_1, u_2\) over two symbol times with \(L = 2\) transmit antennas.
  - Transmitted symbols: \(x[1] = [u_1, u_2]^T\) and \(x[2] = [-u_2^*, u_1^*]^T\).
  - Channel observations at the receiver (with channel coefficients \(h_1, h_2\)):
    \[
    [y[1], y[2]] = [h_1, h_2] \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + [w[1], w[2]].
    \]
Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

- Alternative formulation
  \[
  \begin{bmatrix}
  y[1] \\
  y[2]
  \end{bmatrix} = \begin{bmatrix}
  h_1 & h_2 \\
  h_2^* & -h_1^*
  \end{bmatrix} \begin{bmatrix}
  u_1 \\
  u_2
  \end{bmatrix} + \begin{bmatrix}
  w[1] \\
  w[2]
  \end{bmatrix}
  \]

  \[
  \begin{bmatrix}
  u_1 + (h_2 - h_1^*) u_2 + [w[1]] \\
  \end{bmatrix} = \begin{bmatrix}
  v_1 \\
  v_2
  \end{bmatrix}
  \]

  \[
  y = y[1] + y[2]*
  \]

  \[
  y = \begin{bmatrix}
  y[1] \quad \ldots \quad y[N]
  \end{bmatrix}
  \]

  \[
  h = \begin{bmatrix}
  h_1 \quad \ldots \quad h_L
  \end{bmatrix}
  \]

  \[
  w = \begin{bmatrix}
  w_1 \quad \ldots \quad w_L
  \end{bmatrix}
  \]

  → \( v_1 \) and \( v_2 \) are orthogonal; i.e., the AS spreads the information onto two dimensions of the received signal space.

- Matched-filter receiver\(^2\): correlate with \( v_1 \) and \( v_2 \)
  \[
  r_i = v_i^H y = \|h\|^2 u_i + \tilde{w}_i, \quad \text{for } i = 1, 2,
  \]
  with independent \( \tilde{w}_i \sim \text{CN}(0, \|h\|^2 N_0) \).

- SNR (under power constraint \( E\{\|x\|^2\} = P_0 \)):
  \[
  \text{SNR} = \frac{\|h\|^2}{P_0} \frac{2}{N_0} \rightarrow \text{diversity gain of } 2!
  \]

\(^2\)The textbook uses a projection on the orthonormal basis \( v_i/\|v_i\|, v_j/\|v_j\| \).

Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

Determinant criterion for space-time code design

- Model: codewords of a space-time code with \( L \) transmit antennas and \( N \) time slots: \( X_i, (L \times N) \) matrix.

\[
Y^T = h^*X_i + w^T
\]

with \( \begin{bmatrix}
Y^T \\
h^* \\
w^T
\end{bmatrix} = \begin{bmatrix}
y[1], \ldots, y[N] \\
h_1, \ldots, h_L \\
w_1, \ldots, w_L
\end{bmatrix} \]

Example: Alamouti scheme: Repetition coding:

\[
X_i = \begin{bmatrix}
u_1 & -u_2^* \\
u_2 & u_1^*
\end{bmatrix}
\]

\[
X_i = \begin{bmatrix}
u & 0 \\
0 & u
\end{bmatrix}
\]

- Pairwise error probability of confusing \( X_A \) with \( X_B \) given \( h \)

\[
Pr(X_A \rightarrow X_B|h) = Q\left(\sqrt{\frac{\|h^*(X_A - X_B)\|^2}{2N_0}}\right)
\]

\[
= Q\left(\sqrt{\frac{\text{SNR} h^*(X_A - X_B)(X_A - X_B)^* h}{2}}\right)
\]

(Normalization: unit energy per symbol → SNR = 1/N_0)
• Average pairwise error probability
\[ \Pr(X_A \rightarrow X_B) = E\{\Pr(X_A \rightarrow X_B|\mathbf{h})\} \]

• Some useful facts...
  - \((X_A - X_B)(X_A - X_B)^*\) is Hermitian (i.e., \(Z^* = Z\)).
  - \((X_A - X_B)(X_A - X_B)^*\) can be diagonalized by an unitary transform,
    \((X_A - X_B)(X_A - X_B)^* = U \Lambda U^*,\)
    where \(U\) is unitary (i.e., \(U^* U = U U^* = I\)) and \(\Lambda = \text{diag}\{\lambda_1^2, \ldots, \lambda_L^2\}\),
    with the singular values \(\lambda_l\) of \(X_A - X_B\).

• And we get (with \(\tilde{\mathbf{h}} = U^* \mathbf{h}\))
\[ \Pr(X_A \rightarrow X_B) = E \left\{ Q \left( \sqrt{\frac{\text{SNR} \sum_{l=1}^{L} |h_l|^2 \lambda_l^2}{2}} \right) \right\} \]

\[ \leq \prod_{l=1}^{L} \frac{1}{1 + \text{SNR} \lambda_l^2 / 4} \]

• If all \(\lambda_l^2 > 0\) (only possible if \(N \geq L\)), we get
\[ \Pr(X_A \rightarrow X_B) \leq \prod_{l=1}^{L} \frac{1}{1 + \text{SNR} \lambda_l^2 / 4} \leq \frac{4^L}{\text{SNR}^L \prod_{l=1}^{L} \lambda_l^2} \]
\[ = \frac{1}{\text{SNR}^L \det[(X_A - X_B)(X_A - X_B)^*]} \]

\(\rightarrow\) Diversity gain of \(L\) is achieved.

\(\rightarrow\) Coding gain is determined by the determinant
\[ \det[(X_A - X_B)(X_A - X_B)^*] \text{ (determinant criterion).} \]
Antenna/Spatial Diversity
- 2 × 2 MIMO Example

Channel Model
- 2 transmit antennas, 2 receive antennas:
\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix} =
\begin{bmatrix}
  h_{11} & h_{12} \\
  h_{21} & h_{22}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} +
\begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}
\]
- Rayleigh distributed channel gains \( h_{ij} \) from transmit antenna \( j \) to receive antenna \( i \).
- Additive white complex Gaussian noise \( w_i \sim \mathcal{C}N(0, N_0) \).
- \( 4 \) independently faded signal paths, maximum diversity gain of 4.

\[ H = \begin{bmatrix}
  h_{11} & h_{12} \\
  h_{21} & h_{22}
\end{bmatrix} \]

Degrees of freedom
- Number of dimensions of the received signal space.
- MISO: one degree of freedom for every symbol time.
  - Repetition coding \((L = 2)\): \( 1 \) dimension over \( 2 \) time slots.
  - Alamouti scheme \((L = 2)\): \( 2 \) dimension over \( 2 \) time slots.
- SIMO: one degree of freedom for every symbol time.
  - Only one vector is used to transmit the data,
  \[
  y = hx + w.
  \]
- MIMO: potentially two degrees of freedom for every symbol time.
  - Two degrees of freedom if \( h_1 \) and \( h_2 \) are linearly independent.
  \[
  y = h_1x_1 + h_2x_2 + w.
  \]
Antenna/Spatial Diversity – 2 × 2 MIMO Example

Spatial multiplexing

- Motivation: Neither repetition coding nor the Alamouti scheme utilize all degrees of freedom of the channel.
- Spatial multiplexing (V-BLAST) utilizes all degrees of freedom.
  → Transmit independent uncoded symbols over the different antennas and the different symbol times.
- Pairwise error probability for transmit vectors \( x_1, x_2 \)
  \[
  \Pr(x_1 \rightarrow x_2) \leq \left[ \frac{1}{1 + \text{SNR} \|x_1 - x_2\|^2/4} \right]^2 \leq \frac{16}{\text{SNR}^2 \|x_1 - x_2\|^4}
  \]
  → Diversity gain of 2 (not 4) but higher coding gain as compared to the Alamouti scheme (see example in the book).
  → Spatial multiplexing is more efficient in exploiting the degrees of freedom.
- Optimal detector, joint ML detection: complexity grows exponentially with the number of antennas.
- Linear detection, e.g., decorrelator (zero forcing): \( \hat{y} = H^{-1}y \)

MIMO Capacity

- MIMO channel model with \( n_t \) transmit and \( n_r \) receive antennas:
  \[
  y = Hx + w, \quad \text{with} \quad w \sim \mathcal{CN}(0, \Lambda_0 I).
  \]
- \( x \in \mathbb{C}^{n_t}, \, y \in \mathbb{C}^{n_r}, \) and \( H \in \mathbb{C}^{n_r \times n_t} \).
- Channel matrix \( H \) is known at the transmitter and receiver.
- Power constraint \( E\{\|x\|^2\} = P \).
- Singular value decomposition (SVD): \( H = U \Lambda V^* \), where
  - \( U \in \mathbb{C}^{n_r \times n_r} \) and \( V \in \mathbb{C}^{n_t \times n_t} \) are unitary matrices;
  - \( \Lambda \in \mathbb{R}^{n_r \times n_t} \) is a matrix with diagonal elements \( \lambda_1, \ldots, \lambda_{n_{\text{min}}} \) and off-diagonal elements equal to zero;
  - \( \lambda_1, \ldots, \lambda_{n_{\text{min}}} \), with \( n_{\text{min}} = \min\{n_r, n_t\} \) are the ordered singular values of the matrix \( H \);
  - \( \lambda_1^2, \ldots, \lambda_{n_{\text{min}}}^2 \) are the eigenvalues of \( HH^* \) and \( H^*H \).
- Alternative formulation: \( H = \sum_{i=1}^{n_{\text{min}}} \lambda_i u_i v_i^* \).
  → Sum of rank-1 matrices \( \lambda_i u_i v_i^* \).
  → \( H \) has rank \( n_{\text{min}} \).
MIMO Capacity

- SVD can be used to decompose the MIMO channel into $n_{\text{min}}$ parallel SISO channels.
- \[
\begin{aligned}
\tilde{x} &= V^*x, \\
\tilde{y} &= U^*y, \\
\tilde{w} &= U^*w
\end{aligned}
\]
- \[
\tilde{y} = \Lambda \tilde{x} + \tilde{w}
\]
- with $\tilde{w} \sim \mathcal{CN}(0, N_0I_n)$ and $\|\tilde{x}\|^2 = ||x||^2$; i.e., the energy is preserved.
- MIMO capacity (with waterfilling)
- \[
C = \sum_{i=1}^{n_{\text{min}}} \log \left( 1 + \frac{P_i^* \lambda_i^2}{N_0} \right) \quad \text{with} \quad P_i^* = \left[ \mu - \frac{N_0}{\lambda_i^2} \right]^+
\]
- with $\mu$ chosen to satisfy the total power constraint $\sum P_i^* = P$. 

SVD architecture for MIMO communications

(D. Tse and P. Viswanath, Fundamentals of Wireless Communications.)
MIMO Capacity

Capacity at high SNR

- Uniform power allocation is asymptotically optimal; i.e., $P_i = P/k$.

$$C \approx \sum_{i=1}^{k} \log \left(1 + \frac{P \lambda_i^2}{kN_0}\right) \approx k \log \text{SNR} + \sum_{i=1}^{k} \log \left(\frac{\lambda_i^2}{k}\right)$$

$\rightarrow$ $k$ spatial degrees of freedom; if $H$ has full rank $k = n_{\min}$.

- With Jensen’s inequality

$$C \approx k \cdot \frac{1}{k} \sum_{i=1}^{k} \log \left(1 + \frac{P}{kN_0} \lambda_i^2\right) \leq k \log \left(1 + \frac{P}{kN_0} \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_i^2\right)\right)$$

$\rightarrow$ Maximum capacity in high SNR if all singular values are equal.

- Condition number: $\max_i \lambda_i / \min_i \lambda_i$, $H$ is well conditioned if $\text{CN} \approx 1$.

Capacity at low SNR

- Allocate power only to the strongest eigenmode

$$C \approx \frac{P}{N_0} (\max_i \lambda_i^2) \log_2 e$$