Distributed Design of Glocal Controllers via Hierarchical Model Decomposition

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***Abstract*— This paper proposes a distributed design method of controllers with a glocal (global/local) information structure for large-scale network systems. The glocal controller of interest has a hierarchical structure, wherein a global subcontroller coordinates a set of disjoint local subcontrollers. The global subcontroller regulates inter-area oscillations among subsystems, while local subcontrollers individually regulate intra-area oscillations of the respective subsystem. The distributed design of the glocal controller is addressed to enhance the scalability of controller synthesis, where the global subcontroller and all local subcontrollers are designed independently of each other. A design problem is formulated for subcontroller sets such that any combination of subcontrollers each of which belongs to its corresponding set guarantees stability of the closed-loop system. The core idea of the proposed method is to represent the original network system as a hierarchical cascaded system composed of reduced-order models representing the inter-area and intra-area dynamics, referred to as hierarchical model decomposition. Distributed design is achieved by virtue of the cascade structure. The primary findings of this study are two-fold: First, a tractable solution to the distributed design problem and an existence condition of the hierarchical model decomposition are presented. Second, a clustering method appropriate for the proposed framework and a robust extension are provided. Numerical examples of a power grid highlight the practical relevance of the proposed method.**

***Index Terms*— Distributed design, glocal control, large-scale systems, model reduction, network systems.**

I. Introduction

THE recent development of cyber and physical technologies facilitates the development of large-scale dynamical systems. Simultaneously, it increases the complexity of the network systems to be controlled [1], [2]. For large systems, it is crucial to deploy subcontrollers, each of which monitors and actuates a small network unit while communicating with the other subcontrollers. Owing to their sparse communication topology and distributed implementation, implementation of such structured controllers is scalable [3]–[5]. Nevertheless, for most conventional methods reported in the literature, design of such structured controllers is not necessarily scalable owing to its implicit philosophy of *centralized design*, where a unique authority is supposed to design the entire controller for a fixed network system. In practice, there are often multiple subcontroller designers, each of whom independently designs a subcontroller according to their control policy for modularity; this enhances scalability. For instance, a power grid is governed by multiple companies, each of whom is responsible for managing a subgrid. Accordingly, each company independently designs and operates each controller for frequency regulation [6].

Hence, in contrast to the centralized design, the notion of *distributed design*, where each subcontroller is designed independently of the others, has been proposed [7]. Despite its practical importance, only a few studies on distributed design are available in the literature owing to the technical difficulty associated with getting each subcontroller to allow variations of the other subcontrollers. To overcome this obstacle, several advanced distributed design methods have been proposed over the last decade. Retrofit control has been proposed as a distributed design method of decentralized controllers [8]–[10]. Distributed design methods of distributed controllers with a general communication topology have also been developed [7], [11]–[19].

Considering the aforementioned background and the research lacunae that exists in the field of distributed design, this study addresses the distributed design problem of controllers having a specific *glocal (global/local)* structure. Glocal control, originally proposed in [20], employs a structured controller inspired by the fact that the behaviors of a network system can typically be represented as a superposition of inter-area and intra-area oscillations. For example, the behavior of a power grid can be decomposed into global inter-area oscillations and local fluctuations [21]. Accordingly, a global coordinating subcontroller can be combined with local decentralized subcontrollers in the glocal control framework.

The objective of this study is to develop a distributed design method for glocal controllers. To this end, we introduce *hierarchical model decomposition*, a hierarchical cascaded representation whose upstream and downstream components represent local and global reduced-order models, respectively. Hierarchical model decomposition is an alternative representation of the original network system to be controlled. The fundamental idea is to design and implement subcontrollers

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for the reduced-order models while preserving the cascade structure of hierarchical model decomposition. Owing to the cascade structure, the stability of the entire closed-loop system can be guaranteed as long as each reduced-order model is stabilized by its corresponding subcontroller. Furthermore, the technical issues related to this idea are also resolved in this study.

The primary contributions of this paper are outlined as follows: First, we propose a systematic method for the distributed design of glocal controllers based on hierarchical model decomposition. Specifically, we provide a necessary and sufficient geometric condition for the existence of a hierarchical model decomposition. Subsequently, we derive an implicit representation of all hierarchical model decompositions using linear matrix equations and illustrate the implementation of the designed control policy based on a functional observer. Second, we develop a clustering algorithm that produces clusters appropriate for the proposed method based on a greedy approach. Third, we extend the framework to the case where exact hierarchical model decompositions are absent. To handle the situation, we introduce a robust hierarchical model decomposition with an approximation error, where the error dynamics is also decomposed into a hierarchical form. Preliminary versions of this work can be found in [22], [23], where detailed proofs, clustering algorithm, and the robust extension are not included.

Related Work

A few studies on distributed design of structured controllers can be found in the literature. Before considering the notion of distributed design in [7], similar problems have been discussed in [12], [13]. The main idea is to reduce the effect of interactions among the subsystems and guarantee the stability based on the small-gain theorem. Distributed design of distributed controllers guaranteeing bounded-input bounded-output stability has been proposed in [14] on the premise that interaction signals are bounded. In contrast to small-gain approaches, retrofit control has been proposed for the distributed design of decentralized controllers [8]–[10]. Regarding the distributed information structure, deadbeat control [7], [11], integral quadratic constraint approach [15], passivity-based approach [16], system-level synthesis [17], [18] approach, and dissipativity-based approach [19] have been proposed. These methods are developed on different ideas, providing various advantages depending on the system to be controlled.

Glocal control has been introduced in [20] based on [24], [25]. A key feature is the hierarchical structure with spatial multiple-resolutions. The idea of glocal control involves implementing multi-resolved subcontrollers. Especially in power system control, scholars have striven to designing controllers comprising hierarchical structure [26]. In the classical approach, referred to as multi-level control [27]–[29], the control signal is decomposed for each machine into two components generated by subcontrollers at global and local levels. More recently, with the advancement of wide-area measurement system technology with sophisticated phasor measurement units, wide-area control [30], [31] has attracted considerable attention. Accordingly, several applications based on wide-area control have been proposed [32]–[35]. However, distributed design of the hierarchical structured controllers has not been discussed in the power systems literature.

Finally, we emphasize that hierarchical model decomposition, which provides an equivalent system representation composed of reduced-order models, cannot be obtained with standard model reduction techniques, such as the projection-based model reduction [36] or the singular perturbation method via coordinate transformation [37]. The main concept of this study relies on modification of the state space, which can make the dimension of the proposed representation larger than that of the original system, although the models used for designing each subcontroller are decomposed.

Organization and Notation

In Sec. II, we present an illustration of the proposed method using an example of a second-order network system, and subsequently, the problem is formulated. In Sec. III, we present the proposed distributed design via hierarchical model decomposition, which provides the main technical results related to the proposed approach. Based on these findings, Sec. IV develops a clustering algorithm that produces clusters appropriate for the proposed design framework. In Sec. V, we propose a robust extension of the proposed method. Sec. VI verifies the theoretical findings and demonstrates its practical effectiveness via simulations for the 48-machine NPCC (Northeast Power Coordinating Council) system [38], which is a model of the power grid in New York and neighboring areas. Lastly, Sec. VII draws the main conclusion.

In this study, we denote the set of real numbers by $\mathbb{R}$, the n-dimensional identity matrix by I_n, $n \times m$ zero matrix by $0_{n\times m}$, n-dimensional all-ones vector by $\mathbb{1}_n$, matrices where X_i for $i \in \mathcal{I}$ are concatenated vertically and horizontally by $\mathrm{col}(X_i)_{i\in\mathcal{I}}$ and $\mathrm{row}(X_i)_{i\in\mathcal{I}}$, respectively, the block diagonal matrix whose diagonal blocks are composed of matrices M_i for $i \in \mathcal{I}$ by $\mathrm{D}(M_i)_{i\in\mathcal{I}}$, and the matrix whose (k,l)-th submatrix is given as $M_{[k,l]}$ for $k \in \mathcal{I}_i, j \in \mathcal{I}_l$ by $(M_{k,l})_{k\in\mathcal{I}_i, l\in\mathcal{I}_j}$. The subscript for the variables is omitted when the dimension is clear from the context. Moreover, we denote the Kronecker product by $\otimes$, transpose and a pseudoinverse of a matrix M by M^T and $M^\dagger$, respectively, direct sum and sum space of linear subspaces $\mathcal{X}$ and $\mathcal{Y}$ by $\mathcal{X} \oplus \mathcal{Y}$ and $\mathcal{X} + \mathcal{Y} := \{x + y : x \in \mathcal{X}, y \in \mathcal{Y}\}$, respectively, the image space of a matrix M by $\mathrm{im}\, M$, the set $\{y = Mx : x \in \mathcal{X}\}$ for a matrix M and a set $\mathcal{X}$ by $M\mathcal{X}$, and the controllable subspace with respect to the pair (A, B) by $\mathcal{R}(A, B)$. The proofs are given in Appendix.

II. Problem Formulation

A. Motivating Example

We introduce the proposed glocal control input structure through an example that motivated us to this study. We consider the network system illustrated in Fig. 1 representing a power grid [39] where the dynamics of each component

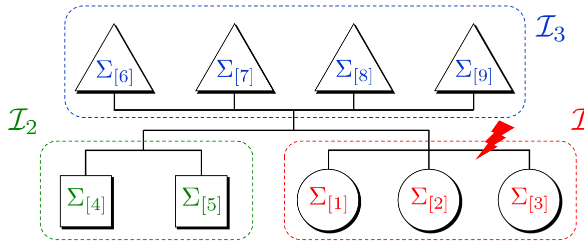

Fig. 1: Example: a network system where the dynamics of each component is given as a second-order system. It is assumed that the parameters of the components represented by the same shape are identical.

is given as a second-order system. For $k=1,\dots,9$, each component $\Sigma_{[k]}$ is given by

$$\Sigma_{[k]}: m_{[k]}\ddot{\theta}_{[k]} + d_{[k]}\dot{\theta}_{[k]} + v_{[k]} + u_{[k]} = 0, \quad y_{[k]} = \omega_{[k]} \tag{1}$$

where $\theta_{[k]} \in \mathbb{R}$ and $\omega_{[k]} := \dot{\theta}_{[k]} \in \mathbb{R}$ represent the state, $v_{[k]} := \sum_{l\in\mathcal{N}_{[k]}} \alpha_{[k,l]}(\theta_{[k]} - \theta_{[l]})$ represents an interaction signal, $u_{[k]} \in \mathbb{R}$ and $y_{[k]} \in \mathbb{R}$ indicate the control input and measurement output, respectively, and $\mathcal{N}_{[k]}$ represents the index set corresponding to the components connected to $\Sigma_{[k]}$. Let the strength of the interaction among the components be given by $\alpha_{[k,l]} = 1$, for any $l,k = 1,\dots,9$. The parameters of the components are given as

$$(m_{[k]}, d_{[k]}) = \begin{cases} (3, 0.4), & k = 1,2,3, \\ (2, 0.3), & k = 4,5, \\ (1, 0.2), & k = 6,7,8,9. \end{cases} \tag{2}$$

We arrange clusters containing homogeneous components as $\mathcal{I}_1 = \{1,2,3\}$, $\mathcal{I}_2 = \{4,5\}$, and $\mathcal{I}_3 = \{6,7,8,9\}$.

Fig. 2 depicts the free response following an impulsive disturbance to the initial state of $\mathcal{I}_1$, where the initial states in the other clusters are zero. The states in $\mathcal{I}_2$ exhibit identical trajectories, the property of which is well known in the literature and referred to as coherency [38], [40]. Similarly, coherency can be observed in $\mathcal{I}_3$. When oscillations of components in each cluster exhibit coherent behavior, they are referred to as inter-area oscillations, particularly in the power system literature [41]. These are quantitatively characterized as global behavior over the subspace $\operatorname{im} P_0$ with $P_0 := \mathrm{D}(\mathbb{1}_3 \otimes I_2, \mathbb{1}_2 \otimes I_2, \mathbb{1}_4 \otimes I_2)$. The observation suggests that the behavior can be interpreted as a superposition of "global behavior" and "local behavior," wherein the former refers to inter-area oscillations whereas the latter refers to intra-area oscillations. Note that the example is simplified to highlight the phenomenon; however, it is commonly observed in real systems where the parameters are non-identical [41].

Consider utilizing the interpretation for damping oscillations. Let the control input and measurement output for the ith cluster be given by $u_i := \mathrm{col}(u_{[k]})_{k\in\mathcal{I}_i}$, and $y_i := \mathrm{col}(y_{[k]})_{k\in\mathcal{I}_i}$ for $i = 1,2,3$. For global behavior regulation, it suffices to consider broadcast-type global control inputs that only excite coherent trajectories in the form $\hat{\mathbf{u}}_0 = \mathrm{D}(\mathbb{1}_3, \mathbb{1}_2, \mathbb{1}_4)\hat{u}_0$, referred to as aggregate control. With this type of control input, one can obtain a reduced-order model

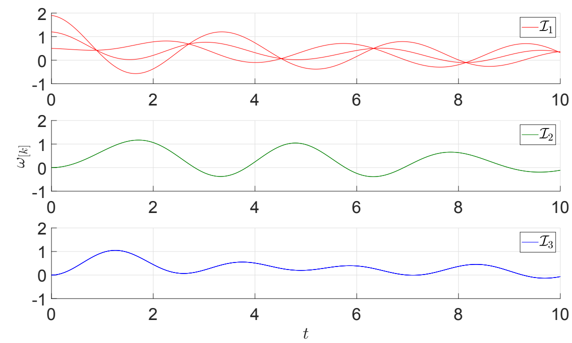

Fig. 2: Free response of the second-order system in Fig. 1 in response to an initial disturbance occurring inside $\mathcal{I}_1$. The top, middle, and bottom correspond to $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$, respectively. Notably, the curves associated with $\mathcal{I}_2$ are overlapped; similarly, the curves of $\mathcal{I}_3$ are also overlapped.

that represents the global behavior; hence, it can be designed in a scalable manner [42]. Accordingly, the overall control input is represented by $\mathrm{col}(u_i)_{i=1,2,3} = \hat{\mathbf{u}}_0 + \mathrm{col}(\hat{u}_i)_{i=1,2,3}$ with local control inputs $\hat{u}_i$. Similarly, the global measurement signal reads $y_0 := [\mathbb{1}_3^\top y_1 \ \mathbb{1}_2^\top y_2 \ \mathbb{1}_4^\top y_3]^\top$. We design a glocal controller that contains global and local subcontrollers associated with these signals.

B. System Description

Let us now introduce the general system description. Consider a linear time-invariant interconnected system with N_0 components

$$\Sigma_{[k]}: \begin{cases} \dot{x}_{[k]} = A_{[k]}x_{[k]} + \sum_{l\neq k} A_{[kl]}x_{[l]} + B_{[k]}u_{[k]} \\ y_{[k]} = C_{[k]}x_{[k]} \end{cases}$$

for $k = 1,\dots,N_0$. For simplicity, we assume that $u_{[k]}$ and $y_{[k]}$ are one-dimensional for simplicity. The aim is to design a glocal controller through appropriate clustering.

Let $\mathcal{I}_i \subset \{1,\dots,N_0\}$ for $i = 1,\dots,N$ be disjoint clusters satisfying $\bigcup_{i=1}^N \mathcal{I}_i = \{1,\dots,N_0\}$. The subsystem regarding the ith cluster can be written as

$$\Sigma_i: \begin{cases} \dot{x}_i = A_{ii}x_i + \sum_{j\neq i} A_{ij}x_j + B_i u_i \\ y_i = C_i x_i \end{cases}$$

where the state is defined by $x_i := \mathrm{col}(x_{[k]})_{k\in\mathcal{I}_i}$ and the other signals are defined in a similar manner and the matrices are given by $A_{ij} := (A_{[kl]})_{k\in\mathcal{I}_i, l\in\mathcal{I}_j}$, $B_i := \mathrm{D}(B_{[k]})_{k\in\mathcal{I}_i}$, and $C_i := \mathrm{D}(C_{[k]})_{k\in\mathcal{I}_i}$. We denote the dimension of the state and the input in $\mathcal{I}_i$ by n_i and r_i, respectively.

As in the motivating example, we let the control input be composed of global and local control inputs with the given clusters. The following assumption is made.

Assumption 1 The input and output matrices in each cluster are identical, i.e., $B_{[k]} = B_{[l]}$ and $C_{[k]} = C_{[l]}$ for any $k,l \in \mathcal{I}_i$ for $i = 1,\dots,N$.

Accordingly, we form the control input as

$$\mathrm{col}(u_i)_{i=1}^N = \hat{\mathbf{u}}_0 + \mathrm{col}(\hat{u}_i)_{i=1}^N, \quad \hat{\mathbf{u}}_0 := E_0\hat{u}_0 \tag{3}$$

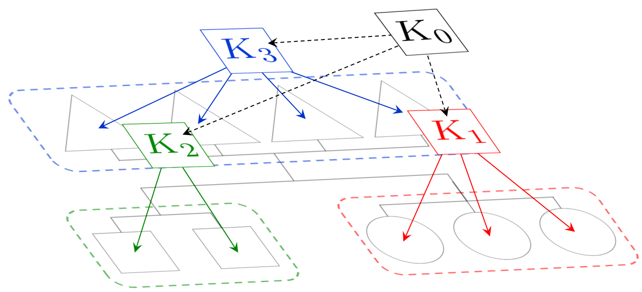

Fig. 3: Information structure of the glocal controller to be designed, where local subcontrollers K_1, K_2, and K_3 receive the global control input from the global subcontroller K_0 and all local subcontrollers do not directly communicate with each other.

with $E_0 := \mathrm{D}(\mathbb{1}_{r_i})_{i=1}^N$. While $\hat{u}_0$ represents the global control input, $\hat{u}_i$ for $i = 1, \ldots, N$ represent local control inputs for the subsystems Σ_i. Similarly, the global measurement signal is defined by $y_0 := E_0^\mathsf{T} y$.

The clustered interconnected system is described by

$$\begin{cases} \dot{x} = Ax + P_0 B_0 \hat{u}_0 + \sum_{i=1}^N P_i B_i \hat{u}_i \\ y_0 = C_0 P_0^\mathsf{T} x \\ y_i = C_i P_i^\mathsf{T} x. \end{cases} \tag{4}$$

The broadcasting matrix is given by $P_0 := \mathrm{D}(\mathbb{1}_{r_i} \otimes I_{n_{0,i}})_{i=1}^N$, where $n_{0,i} := n_i / r_i$ indicates the dimension of the state of a subsystem in $\mathcal{I}_i$. The embedding matrix is given by $P_i := [0_{n_1 \times n_i} \ \cdots \ I_{n_i} \ \cdots \ 0_{n_N \times n_i}]^\mathsf{T}$. The matrices B_0 and C_0 are chosen such that $P_0 B_0 = \mathrm{D}(B_i) E_0$, and $C_0 P_0^\mathsf{T} = E_0^\mathsf{T} \mathrm{D}(C_i)$. Note that there always exist such B_0 and C_0 owing to Assumption 1.

C. Problem Formulation

Based on the system description, we consider the distributed design of a glocal controller, where each subcontroller can be designed independently of the others. The information structure of glocal controllers to be designed is given as:

$$\hat{u}_0 = K_0(y_0), \quad \hat{u}_i = K_i(y_i, \hat{u}_0), \quad i = 1, \ldots, N \tag{5}$$

where K_i for $i = 0, \ldots, N$ indicate linear time-invariant dynamical controllers. The structure for the motivating example in Sec. II-A is illustrated in Fig. 3, where the entire controller has a star topology. The subcontroller K_0 is referred to as the global subcontroller, whereas the subcontrollers K_i for $i = 1, \ldots, N$ are referred to as the local subcontrollers. The global subcontroller transmits its control input to all local subcontrollers, whereas the local subcontrollers do not directly communicate with each other. In Sec. III-D, we discuss our motivation for this specific information structure.

Let us introduce subcontroller sets $\mathcal{K}_i$ such that the entire closed-loop system is internally stable for any combination of subcontrollers in $\mathcal{K}_i$. Distributed design of the subcontrollers is achieved by designing the ith subcontroller K_i to be an element of $\mathcal{K}_i$ in the sense that K_i can be chosen independently of the other subcontrollers.

Problem 1 Design a collection of subcontroller sets $\{\mathcal{K}_i\}_{i=0}^N$ such that the clustered interconnected system (4) with the subcontrollers $K_0, K_1, \ldots, K_N$ is internally stable for any choice of a tuple

$$(K_0, K_1, \ldots, K_N) \in \mathcal{K}_0 \times \mathcal{K}_1 \times \cdots \times \mathcal{K}_N.$$

In Problem 1, the subcontroller sets $\mathcal{K}_i$ are designed instead of subcontrollers K_i. Note that a trivial solution can be given as singletons $\mathcal{K}_i = \{K_i\}$ where the unique controller stabilizes the entire system. However, this choice is undesirable for a distributed design as the designed controller has no flexibility; hence we seek larger subcontroller sets. It should be noted that we do not put any restriction on the information structure of each local controller. Consequently, communication inside each cluster can be required in the resulting controller.

In Sec. III, we solve Problem 1 with the given clusters. Subsequently, in Sec. IV, we develop a clustering method that generates clusters appropriate for the proposed distributed design method.

III. Distributed Design via Hierarchical Model Decomposition

A. Motivating Example Revisited

This subsection describes the fundamental idea for the distributed design of a glocal controller through the motivating example in Sec. II-A. The core idea is to derive a *hierarchical* representation that explicitly describes global and local behaviors. Consider describing the state variables as a superposition of global and local states, e.g.,

$$\underbrace{\begin{bmatrix} \omega_{[1]} \\ \omega_{[2]} \\ \omega_{[3]} \end{bmatrix}}_{=:\omega_1} = \underbrace{\begin{bmatrix} \hat{\omega}_{[1]} \\ \hat{\omega}_{[2]} \\ \hat{\omega}_{[3]} \end{bmatrix}}_{=:\hat{\omega}_1} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \hat{\omega}_{0,1} \tag{6}$$

where $\hat{\omega}_{[k]}$ for $k = 1, 2, 3$ and $\hat{\omega}_{0,1}$ represent the local and global behaviors of $\omega_{[k]}$ in the first cluster, respectively. Similarly, we consider $\hat{\omega}_i$ and $\hat{\omega}_{0,i}$ for the other clusters. The global variable with respect to all clusters is denoted by $\hat{\omega}_0 := \mathrm{col}(\hat{\omega}_{0,i})_{i=1,2,3}$. Similarly, the variables with respect to θ_i, denoted by $\hat{\theta}_0$ and $\hat{\theta}_i$, are also defined.

Now we consider the dynamics that the global and local variables should follow in compliance with the superposition representation in (6). As such dynamics is not necessarily unique, we can possibly choose a representation that has a desirable property for controller design. We impose a *hierarchical* structure into the dynamics. Accordingly, it can be shown that there exists a hierarchical system

$$\begin{cases} \Xi_i : \begin{bmatrix} \dot{\hat{\theta}}_i \\ \dot{\hat{\omega}}_i \end{bmatrix} = \hat{A}_i \begin{bmatrix} \hat{\theta}_i \\ \hat{\omega}_i \end{bmatrix} + B_i \hat{u}_i, \quad i = 1, 2, 3 \\ \Xi_0 : \begin{bmatrix} \dot{\hat{\theta}}_0 \\ \dot{\hat{\omega}}_0 \end{bmatrix} = \hat{A}_0 \begin{bmatrix} \hat{\theta}_0 \\ \hat{\omega}_0 \end{bmatrix} + \sum_{i=1}^N \hat{R}_i \begin{bmatrix} \hat{\theta}_i \\ \hat{\omega}_i \end{bmatrix} + B_0 \hat{u}_0 \end{cases}$$

with certain system matrices such that the original states θ_i and ω_i can be reproduced as a superposition for any control inputs, as long as the initial condition is consistent. Although the specific matrices are omitted here, their structure is discussed

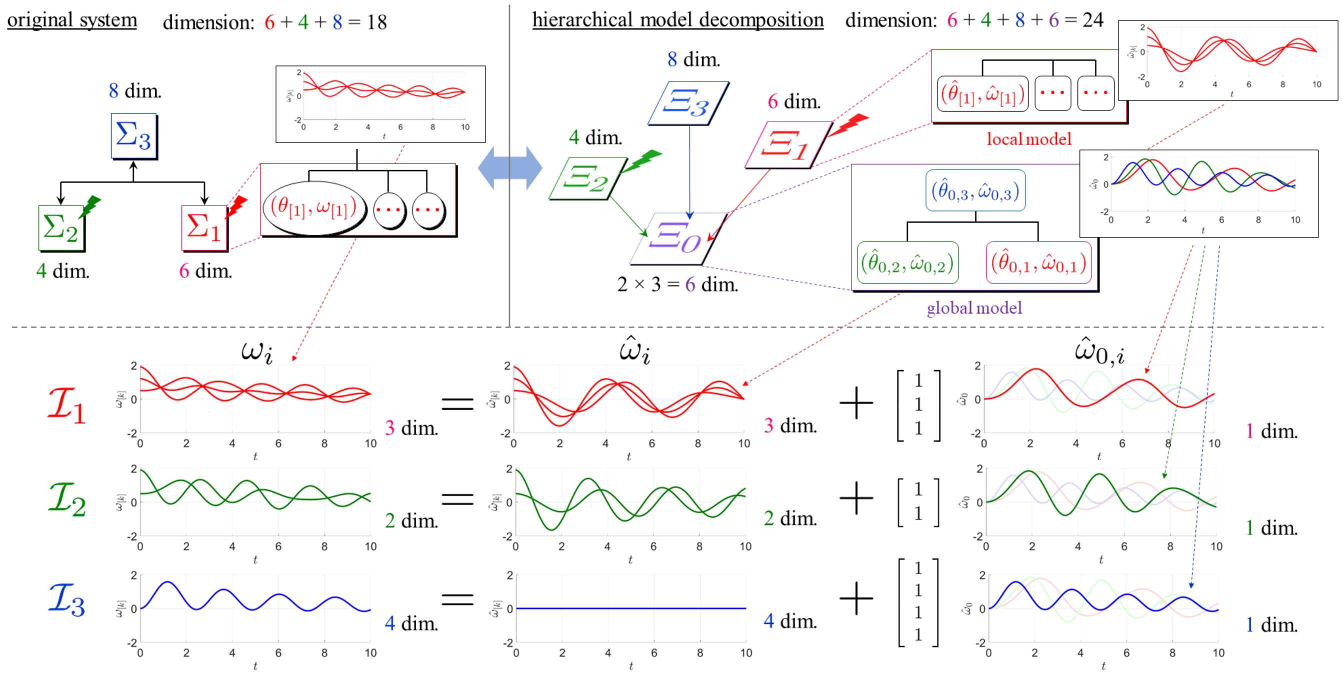

Fig. 4: Hierarchical model decomposition of the second-order network system of the motivating example. Top: block diagrams of the original network system and the hierarchical system. Bottom: responses of ω_i in Σ_i and their representations as a superposition of $\hat{\omega}_i$ in Ξ_i and $\hat{\omega}_{0,i}$ in Ξ_0 for $i = 1, 2, 3$.

in Sec. III-B. Block diagrams of the original network system and the hierarchical system are illustrated at the top of Fig. 4.

The hierarchical representation can be interpreted as dynamics in response to multiple local disturbances. The dynamics Ξ_0 stands for a reduced-order global model, whereas the dynamics of Ξ_1, Ξ_2, and Ξ_3 represent local models. Accordingly, the hierarchical representation decomposes the entire model into global and local models. The bottom of Fig. 4 depicts the state trajectories without control in response to a disturbance occurring in the first two clusters. Accordingly, the local state $\hat{\omega}_3$ in the third cluster show no sign of excitation. This hierarchical decomposition can reproduce the original state as a superposition of global and local states for any disturbance and control input.

Consider utilizing this decomposition for a distributed design. A block diagram of Ξ with control inputs is illustrated in Fig. 5a. From the hierarchical structure, it is evident that there are no feedback paths around the reduced-order models. Thus, we can guarantee stability of the entire system by attaching subcontrollers, as in Fig. 5b, provided that each subcontroller stabilizes the corresponding reduced-order system. It should be emphasized that although the dimension of Ξ (24) is larger than that of the original system (18) the models used for respective controller designs are reduced as shown in Fig. 4. Particularly, the dimensions of Ξ_i for $i = 0, 1, 2, 3$ are $6, 6, 4, 8$, which are less than 18. As observed, the dimension of each model is reduced via the decomposition, which enables scalable design of subcontrollers.

In the remainder of this section, we adopt the aforementioned scheme to general systems and then present some

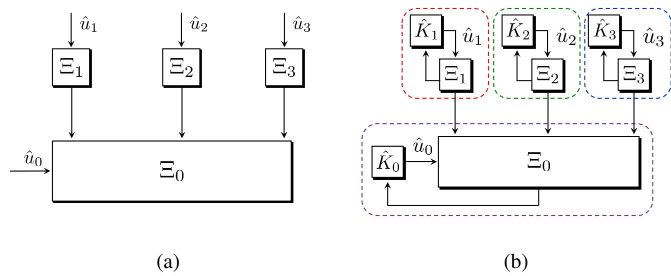

Fig. 5: Block diagrams of the hierarchical model decomposition with control signals. (a): With control inputs. (b): With feedback control.

crucial properties for control design and implementation.

B. Definition of Hierarchical Model Decomposition

The principal idea of the proposed method is to represent the entire network system as a hierarchical system comprising reduced-order models. We define the hierarchical model, referred to as *hierarchical model decomposition,* as follows.

Definition 1 (Hierarchical Model Decomposition) Consider the hierarchical system Ξ composed of

$$\begin{cases} \Xi_i : \dot{\xi}_i = \hat{A}_i \xi_i + B_i \hat{u}_i, \quad i = 1, \ldots, N \\ \Xi_0 : \dot{\xi}_0 = \hat{A}_0 \xi_0 + \sum_{i=1}^{N} \hat{R}_i \xi_i + B_0 \hat{u}_0. \end{cases} \tag{7}$$

The system Ξ is said to be a hierarchical model decomposition of the clustered interconnected system in (4) if

$$x(t) = \sum_{i=1}^{N} P_i \xi_i(t) + P_0 \xi_0(t), \quad \forall t \geq 0 \tag{8}$$

holds for arbitrary initial conditions and control inputs provided that $x(0) = \sum_{i=1}^{N} P_i \xi_i(0) + P_0 \xi_0(0)$.

A distributed design can be achieved by obtaining a hierarchical model decomposition (7). The technical questions related to the decomposition are as follows:

1) Does there exist a hierarchical model decomposition for the given system and clusters?
2) How to obtain a specific representation of a hierarchical model decomposition if it exists?
3) How to implement the designed controller preserving the cascade structure?

These questions are addressed in the remainder of this section.

C. Existence Condition and Implicit Representation

First, we give a necessary and sufficient condition for the existence of hierarchical model decomposition.

Theorem 1 (Existence Condition) Under Assumption 1, a hierarchical model decomposition of (4) exists if and only if the condition

$$\begin{cases} \mathcal{R}(A, P_i) \subset \operatorname{im} P_i + \operatorname{im} P_0, \quad i = 1, \dots, N & \text{(9a)} \\ \mathcal{R}(A, P_0) \subset \operatorname{im} P_0 & \text{(9b)} \end{cases}$$

holds.

The condition in Theorem 1 can be interpreted as follows. Condition (9a) indicates that state trajectories excited by external signals inside the ith cluster are restricted to the sum space. Consequently, the overall behavior can be represented as a sum of inter-area and intra-area behaviors. Condition (9b) implies that $\operatorname{im} P_0$ is an invariant subspace of A. Accordingly, the global control input $\hat{u}_0$ in (4) can only excite global inter-area behaviors restricted to $\operatorname{im} P_0$.

Based on Theorem 1, we derive an implicit representation of all hierarchical model decompositions via the linear matrix equations.

Theorem 2 (Implicit Representation) Under Assumption 1, the system Ξ in (7) is a hierarchical model decomposition of (4) if and only if $\hat{A}_i$ and $\hat{R}_i$ satisfy

$$\begin{cases} AP_i - P_0 \hat{R}_i - P_i \hat{A}_i = 0, \quad i = 1, \dots, N \\ AP_0 - P_0 \hat{A}_0 = 0. \end{cases} \tag{10}$$

The hierarchical system representation can be regarded as an extension of the hierarchical state-space expansion in [8], [9], which consider only local controllers and correspond to the case where $P_0 = I$.

Remark: In the proposed framework, the inter-area behavior and the intra-area behavior to be captured are determined by the matrices P_0 and P_i for $i = 1, \dots, N$, respectively. Using general P_0, the existence of hierarchical model decomposition is characterized through Theorem 1. Hence, the framework adopted in this study encompasses a general notion of inter-area behaviors by choosing an appropriate P_0. However, in practice, synchronization can be a representative inter-area behavior, and an intuitive interpretation can be obtained by choosing the particular P_0. In this study, we proceed with the discussion by considering synchronization as a specific inter-area behavior.

D. Controller Implementation

Next, we next consider an implementation issue. Let $\hat{K}_0, \dots, \hat{K}_N$ be subcontrollers that independently stabilize the closed-loop systems

$$\begin{cases} \dot{\xi}_i = \hat{A}_i \xi_i + B_i \hat{u}_i \\ \hat{u}_i = \hat{K}_i (C_i \xi_i) \end{cases}, \quad i = 0, \dots, N. \tag{11}$$

The cascade structure and linearity implies the internal stability of (7). Thus, the stability of the original system is also guaranteed as the original state trajectory for any $x(0)$ can be reproduced by appropriately choosing $\xi_i(0)$ for $i = 0, \dots, N$. However, since the virtual variable ξ_i is inaccessible, the control inputs in (11) cannot be created directly. The aim of this subsection is to develop an implementation method to circumvent this problem.

First, the global measurement signal y_0 can be used instead of $C_0 \xi_0$ in (11) for stabilization. Accordingly, the control input with a linear controller can be represented by

$$\hat{u}_0 = \hat{K}_0(y_0) = \hat{K}_0(C_0 \xi_0) + \hat{K}_0 \left(C_0 P_0^{\mathsf{T}} \sum_{i=1}^{N} P_i \xi_i \right).$$

The second term can be regarded as an external input signal from the upstream parts. As there is no feedback path from Ξ_0 to Ξ_i for $i = 1, \dots, N$, the stability of the downstream part can be guaranteed even with y_0. Hence, we confine our attention only to the upstream parts associated with $\xi_1, \dots, \xi_N$.

The idea is to estimate $C_i \xi_i$ for $i = 1, \dots, N$ using functional observers [43], [44]. The following theorem holds.

Theorem 3 (Stabilization through Functional Observers) Assume that Ξ is a hierarchical model decomposition of (4) and that the dynamical systems

$$\Phi_i : \begin{cases} \dot{\phi}_i = \mathbf{A}_i \phi_i + \mathbf{B}_i \hat{u}_i + \mathbf{D}_i \hat{u}_0 + \mathbf{E}_i y_i \\ \psi_i = \mathbf{C}_i \phi_i + \mathbf{F}_i y_i \end{cases} \tag{12}$$

for $i = 1, \dots, N$ are functional observers of $C_i \xi_i$, i.e., $\lim_{t \to +\infty} (C_i \xi_i(t) - \psi_i(t)) = 0$ holds for any initial condition and inputs. Design the subcontroller sets $\mathcal{K}_i$ for $i = 0, \dots, N$ to be composed of the subcontrollers

$$\begin{cases} \hat{u}_i = \hat{K}_i \psi_i, \quad i = 1, \dots, N & \text{(13a)} \\ \hat{u}_0 = \hat{K}_0 y_0 & \text{(13b)} \end{cases}$$

that stabilizes Ξ_i with (12). Then the clustered interconnected system (4) is internally stable for any choice of subcontrollers in $\mathcal{K}_i$ for $i = 0, \dots, N$.

Theorem 3 implies that the stability of the original system can be guaranteed using the estimated signal ψ_i instead of $C_i \xi_i$ itself. The actual control inputs u_i are determined from $\hat{u}_0$ and $\hat{u}_i$ according to (3).

Next, we provide a specific functional observer for (12). For simplicity, we consider the case where A_{ij} is decomposed by $A_{ij} = L_i \Gamma_{ij}$ and the interaction signal $v_i := \sum_{j \neq i} \Gamma_{ij} x_j$ can be measured by the ith local subcontroller in addition to the local measurement signal y_i for $i = 1, \dots, N$.

Proposition 1 Assume that A_i and $\hat{A}_i$ are stable. Then

$$\Phi_i : \begin{cases} \dot{\phi}_i = \hat{A}_i\phi_i + (A_i - \hat{A}_i)\hat{x}_i + L_i v_i + P_i^{\mathsf{T}} P_0 B_0 \hat{u}_0 \\ \dot{\hat{x}}_i = A_i\hat{x}_i + B_i\hat{u}_i + L_i v_i + P_i^{\mathsf{T}} P_0 B_0 \hat{u}_0 \\ \psi_i = -C_i\phi_i + y_i \end{cases} \tag{14}$$

is a functional observer of $C_i\xi_i$.

The idea to construct the observer in Proposition 1 is as follows. To extract $C_i\xi_i$ alone from $y_i = C_i\xi_i + C_iP_i^{\mathsf{T}}P_0\xi_0$, consider estimating $P_i^{\mathsf{T}}P_0\xi_0$, primarily because the dynamics of $P_i^{\mathsf{T}}P_0\xi_0$ can be represented using the first differential equation in (14). Accordingly, we use ϕ_i as a replacement of $P_i^{\mathsf{T}}P_0\xi_0$, which induces $\psi_i = y_i - C_i\phi_i$ as an estimation of $C_i\xi_i = y_i - C_iP_i^{\mathsf{T}}P_0\xi_0$. We emphasize that the functional observers can be designed in a distributed manner. Notably, a similar observer can be designed by introducing error feedback even when A_i is unstable. Extension of the result to the case with unstable $\hat{A}_i$ is proposed as future work.

Our solution to Problem 1 can be summarized as follows.

1) Determine the existence of a hierarchical model decomposition for a given system and clusters based on Theorem 1.
2) Construct a hierarchical model decomposition based on Theorem 2.
3) Design $\mathcal{K}_i$ to be the set whose elements are the controllers composed of the functional observer (14) and an internal controller that stabilizes (11) for $i = 0, \ldots, N$.

Theorem 3 implies that the proposed procedure provides a distributed design method of glocal controllers giving a solution to Problem 1

In the aforementioned discussion, it is assumed that the clusters are given in advance and satisfy the existence condition in Theorem 1. In Sec. IV, we develop a clustering method based on the results in this section. Furthermore, an extension of the proposed approach to the case where the existence condition is *not* satisfied is discussed in Sec. V.

IV. Clustering Method

A. Clustering Algorithm

We find a cluster set for which there exists a hierarchical model decomposition of the clustered interconnected system (4). Before proceeding, we state an assumption alternative to Assumption 1.

Assumption 2 The input and output matrices of all components are identical, i.e., $B_{[k]} = B_{[l]}$ and $C_{[k]} = C_{[l]}$ for any $k, l = 1, \ldots, N_0$.

The desirable clusters are characterized through (9a) and (9b). A trivial cluster set that fulfills them is provided by $\mathcal{I}_i = \{i\}$ for $i = 1, \ldots, N_0$. As mentioned in the remark in Sec. III-C, this choice does not reduce the complexity of designing a global subcontroller. Considering that the number of the clusters is maximized by choosing the trivial cluster set, we aim at minimizing the number of clusters. A critical observation is that the condition (9a) becomes a sufficient

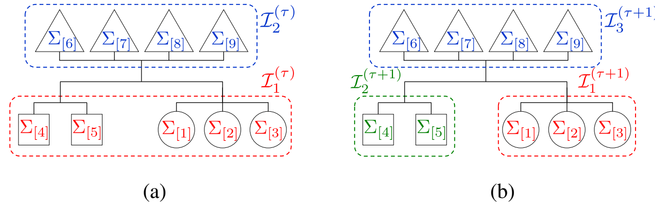

Fig. 6: Illustrative example of the clusters during the algorithm. (a): Clusters at the τth step. (b): Clusters at the $(\tau+1)$th step.

condition of (9b) under a mild condition specified in the following lemma.

Proposition 2 Assume that $(A, \{P_i\}_{i=0}^N)$ satisfies

$$\operatorname{im} P_0^{(i)} \subset \mathcal{R}(A, [P_1 \ \cdots P_{i-1} \ P_{i+1} \ \cdots \ P_N]), \quad i = 1, \ldots, N \tag{15}$$

where $P_0^{(i)} := \pi_{\operatorname{im} P_i} P_0$ with $\pi_{\operatorname{im} P_i} := P_iP_i^{\mathsf{T}}$, a projection matrix onto $\operatorname{im} P_i$. Then, if (9a) holds, (9b) holds as well.

In Proposition 2, condition (15) means that the subspace $\operatorname{im} P_0^{(i)}$ is reachable from one of the other clusters. Since this condition is not strictly restrictive, we first develop an algorithm producing clusters that satisfy (9a) disregarding (9b), and subsequently, we extend the algorithm to satisfy (15) as well.

We exemplify the proposed greedy algorithm using the motivating example in Sec. II-A. We begin with an initial cluster set, partition one of the clusters such that (9a) is satisfied, and repeat the process until all partitioned clusters satisfy (9a). Suppose that the current step is the τth step at which the temporary cluster set is given as the one in Fig. 6a. We check if the clusters satisfy (9a). Observe that

$$P_2^{(\tau)} = \begin{bmatrix} 0_{10\times 8} \\ I_8 \end{bmatrix}, \quad P_0^{(\tau)} = \begin{bmatrix} \mathbb{1}_5 \otimes I_2 & 0 \\ 0 & \mathbb{1}_4 \otimes I_2 \end{bmatrix}$$

and the controllable subspace from the second cluster

$$\mathcal{R}(A, P_2^{(\tau)}) = \operatorname{im} P_2^{(\tau)} \oplus \operatorname{im} \begin{bmatrix} \mathbb{1}_3 \otimes I_2 \\ 0_{4\times 2} \\ 0_{8\times 2} \end{bmatrix} \oplus \operatorname{im} \begin{bmatrix} 0_{6\times 2} \\ \mathbb{1}_2 \otimes I_2 \\ 0_{8\times 2} \end{bmatrix}. \tag{16}$$

Thus, (9a) does not hold. Accordingly, we choose $\mathcal{I}_1^{(\tau)}$ to be partitioned into multiple clusters at the next step. From (16), the components $\Sigma_{[k]}$ for $k = 1, 2, 3$ can be lumped together; $\Sigma_{[4]}$ and $\Sigma_{[5]}$ can also be lumped together. This procedure can be performed systematically by comparing rows of the controllability matrix. This partition results in the clusters illustrated in Fig. 6b at the $(\tau + 1)$th step. Then, we can terminate the algorithm by confirming that the condition (9a) is satisfied for $i = 1, 2, 3$. Note that (15) is satisfied in this case, and hence, (9b) also holds.

The detailed description is outlined in Algorithm 1, where $f_{P_i}(\mathcal{I}_i)$ generates the corresponding matrix P_i. $f_{\mathbf{R}}(A, P_i)$ generates the controllability matrix $\mathbf{R}_i$ with respect to the pair (A, P_i), $\operatorname{par}(\mathbf{R}_i, P_i)$ generates the minimal clusters that satisfy the condition (9a) for P_i based on $\mathbf{R}_i$ via the optimization problem

$$\min_{\{\mathcal{I}_j\}_{j=1}^N} N \text{ s.t. } \pi_{\mathrm{im}\, P_i^\perp} \mathcal{R}(A, P_i) \subset \pi_{\mathrm{im}\, P_i^\perp} \mathrm{im}\, f_{P_0}(\{\mathcal{I}_j\}_{j=1}^N), \tag{17}$$

with $\pi_{\mathrm{im}\, P_i^\perp} := I - P_i P_i^{\mathsf{T}}$, a projection matrix onto the orthogonal subspace of $\mathrm{im}\, P_i$, with which the state of the components in the ith cluster is disregarded, and $f_{P_0}(\{\mathcal{I}_i\})$ generates the corresponding matrix P_0. The subproblem, which can be regarded as a greedy part, can be solved by comparing rows of $\pi_{\mathrm{im}\, P_i^\perp} \mathbf{R}_i$ as mentioned earlier.

Clearly, Algorithm 1 produces a cluster set that satisfies (9a). Remarkably, if we use the single largest cluster $\mathcal{I}_1^{(0)} = \{1, \ldots, N_0\}$ as the initial cluster set, then condition (9a) holds for any systems, and Algorithm 1 does not produce any beneficial cluster set. As a heuristic, the initial cluster set should be composed only of two clusters, primarily since the clusters should be taken as large as possible. Prior information on the behavior, such as simulation results, would be helpful to the designer.

Algorithm 1 Clustering Algorithm

Input: $A, \{\mathcal{I}_i^{(0)}\}_{i=1}^{N^{(0)}}$
Output: $\{\mathcal{I}_i\}_{i=1}^{N}$
$\tau \leftarrow 0$
repeat
  $\tau \leftarrow \tau + 1$
  for $i = 1, \ldots, N^{(\tau-1)}$ **do**
    $P_i^{(\tau-1)} \leftarrow f_{P_i}(\mathcal{I}_i^{(\tau-1)})$
    $\mathbf{R}_i^{(\tau-1)} \leftarrow f_{\mathbf{R}}(A, P_i^{(\tau-1)})$
    if the condition (9a) is not satisfied for i **then**
      $\{\mathcal{I}\}_{i=1}^{N^{(\tau)}} \leftarrow \mathrm{par}(\mathbf{R}_i^{(\tau-1)}, P_i^{(\tau-1)})$
      break
    end if
  end for
until $\{\mathcal{I}_i^{(\tau)}\}_{i=1}^{N^{(\tau)}} = \{\mathcal{I}_i^{(\tau-1)}\}_{i=1}^{N^{(\tau-1)}}$
$\{\mathcal{I}_i\}_{i=1}^{N} = \{\mathcal{I}_i^{(\tau)}\}_{i=1}^{N^{(\tau)}}$

B. Algorithm Property

We show the optimality of Algorithm 1. The following notion is required.

Definition 2 (Partition of Clusters) A cluster set $\{\mathcal{I}_i\}_{i=1}^N$ is said to be a *partition* of $\{\mathcal{I}_{i'}\}_{i'=1}^{N'}$ when for any $i \in \{1, \ldots, N\}$ there exists $i' \in \{1, \ldots, N'\}$ such that $I_i \subset I_{i'}$.

Let $\mathfrak{F}(\{\mathcal{I}_i\}_{i=1}^N)$ denote the family of all cluster sets that are partitions of $\{\mathcal{I}_i\}_{i=1}^N$. Moreover, let $\mathfrak{G}(\{\mathcal{I}_i\}_{i=1}^N) \subset \mathfrak{F}(\{\mathcal{I}_i\}_{i=1}^N)$ denote the family of all cluster sets in $\mathfrak{F}(\{\mathcal{I}_i\}_{i=1}^N)$ such that (9a) is satisfied.

We say a cluster set $\{\mathcal{I}_i\} \in \mathfrak{I}$ to be a minimum cluster set when the number of elements of $\{\mathcal{I}_i\}$ is minimal in $\mathfrak{I}$. The following theorem shows minimality of the resulting cluster set.

Theorem 4 (Algorithm Property) The cluster set produced by Algorithm 1 is the minimum cluster set in $\mathfrak{G}(\{\mathcal{I}_i^{(0)}\}_{i=1}^{N^{(0)}})$.

Essentially, Theorem 4 is a direct consequence of the following lemma.

Lemma 1 The relation $\mathfrak{G}(\{\mathcal{I}_i^{(0)}\}_{i=1}^{N^{(0)}}) = \mathfrak{G}(\{\mathcal{I}_i^{(\tau)}\}_{i=1}^{N^{(\tau)}})$ holds for any $\tau \geq 0$.

Lemma 1 implies that Algorithm 1 preserves the admissible clusters. Therefore, we can guarantee minimality of the resulting cluster set in $\mathfrak{G}(\{\mathcal{I}_i^{(0)}\}_{i=1}^{N^{(0)}})$.

C. Extended Clustering Algorithm

We next extend Algorithm 1 to the case when the resulting cluster set does not satisfy (9b). In this case, (15) is not satisfied. The idea of the extension is to apply Algorithm 1 only to a subset of clusters that do not satisfy (15) inspired by the following proposition.

Proposition 3 Let $\{\mathcal{I}_i\}_{i=1}^N$ be a cluster set such that (9a) is satisfied for any $i \in \{1, \ldots, N\}$ and (15) is not satisfied for $j \in \mathcal{J} \subset \{1, \ldots, N\}$. Then for any cluster set $\{\mathcal{I}'_{j'}\}_{j' \in \mathcal{J}'}$ in $\mathfrak{F}(\{\mathcal{I}_j\}_{j \in \mathcal{J}})$, the cluster set $\{\mathcal{I}_i\}_{i \notin \mathcal{J}} \cup \{\mathcal{I}_{j'}\}_{j' \in \mathcal{J}'}$, which is obtained by partitioning $\{\mathcal{I}_j\}_{j \in \mathcal{J}}$ into $\{\mathcal{I}'_{j'}\}_{j' \in \mathcal{J}'}$, satisfies (9a) as well as (15) for $i \notin \mathcal{J}$.

Proposition 3 implies that once (9a) and (15) are satisfied for some clusters, this property is preserved even under partition of the other clusters. Owing to Proposition 3, we can reduce the clustering problem into a subproblem for the subclusters that do not satisfy (15).

The proposed clustering algorithm is described in Algorithm 2. Obviously, the resulting clusters satisfy the conditions in Theorem 1 from Propositions 2 and 3.

Proposition 4 Consider the clusters produced by Algorithm 2. Under Assumption 2, there exists a hierarchical model decomposition of the resulting clustered interconnected system.

Algorithm 2 Extended Clustering Algorithm

Input: $A, \{\mathcal{I}_i\}^{(0)}$
Output: $\{\mathcal{I}_i\}_{i=1}^{N}$
$\tau' \leftarrow 0$
$\{\mathcal{I}_i\}^{(\tau')} \leftarrow \{\mathcal{I}_i\}^{(0)}$
repeat
  $\tau' \leftarrow \tau' + 1$
  $\{\mathcal{I}_i\}^{(\tau')} \leftarrow$ Algorithm 1 with $\{\mathcal{I}_i\}^{(\tau'-1)}$
  if (15) is not satisfied for $j \in \mathcal{J} \subset \{1, \ldots, N\}$ **then**
    provide $\{\mathcal{I}'_{j'}\}_{j' \in \mathcal{J}'}$ in $\mathfrak{F}(\{\mathcal{I}_j\}_{j \in \mathcal{J}})$
    $\{\mathcal{I}_i\}^{(\tau')} \leftarrow \{\mathcal{I}_i\}_{i \notin \mathcal{J}} \cup \{\mathcal{I}'_{j'}\}_{j' \in \mathcal{J}'}$
    break
  end if
until the condition (15) is satisfied for all clusters
$\{\mathcal{I}_i\}_{i=1}^{N} = \{\mathcal{I}_i\}^{(\tau')}$

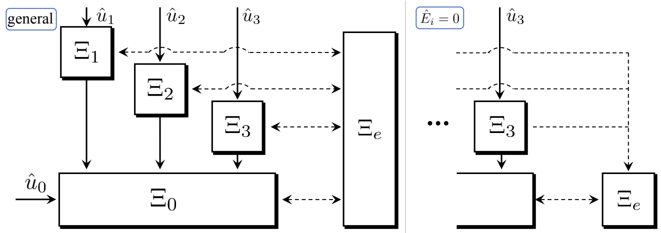

Fig. 7: Block diagrams of the hierarchical model decomposition with error dynamics in (18) when $N = 3$. Left: general case. Right: the case under choice (20), where the system has a hierarchical structure including the error dynamics.

V. Extension to Indecomposable Systems

A. Robust Hierarchical Model Decomposition

This section extends the proposed method to systems for which there are no exact hierarchical model decompositions. We consider the following system

$$\begin{cases} \dot{\xi}_i = \hat{A}_i\xi_i + \hat{E}_i e + B_i\hat{u}_i, \quad i = 1,\dots,N \\ \dot{\xi}_0 = \hat{A}_0\xi_0 + \sum_{i=1}^N \hat{R}_i\xi_i + \hat{E}_0 e + B_0\hat{u}_0 \\ \dot{e} = \hat{A}_e e + \hat{F}_0\xi_0 + \sum_{i=1}^N \hat{F}_i\xi_i, \end{cases} \tag{18}$$

where an approximation error dynamics is introduced. It is clear that

$$x(t) = \textstyle\sum_{i=1}^N P_i\xi_i(t) + P_0\xi_0(t) + e(t), \quad \forall t \geq 0$$

holds for any control inputs and initial states provided that $x(0) = \sum_{i=1}^N P_i\xi_i(0) + P_0\xi_0(0) + e(0)$ if and only if

$$\begin{aligned} &\hat{A}_e = A - P_0\hat{E}_0 - \textstyle\sum_{i=1}^N P_i\hat{E}_i, \quad \hat{F}_0 = AP_0 - P_0\hat{A}_0, \\ &\hat{F}_i = AP_i - P_i\hat{A}_i - P_0\hat{R}_i, \quad i = 1,\dots,N \end{aligned} \tag{19}$$

where $\hat{A}_0, \hat{A}_i, \hat{R}_i, \hat{E}_0$, and $\hat{E}_i$ are free parameters. The block diagram of the system (18) for the case $N = 3$ is depicted in the left of Fig. 7, where Ξ_e represents the dynamics relevant to e. Because there exist feedback paths from Ξ_e to Ξ_0 and Ξ_i for $i = 1,\dots,N$ as shown in this figure, the system (18) no longer has a cascade structure. Hence, the overall stability cannot be guaranteed even if we attach subcontrollers each of which stabilizes the corresponding subloop.

A crucial observation from (18) is that the hierarchical cascade structure can be recovered by choosing the free parameters appropriately. In particular, when the condition

$$\hat{E}_i = 0, \quad \forall i = 1,\dots,N \tag{20}$$

or $(\hat{E}_0, \hat{F}_0) = (0, 0)$ is satisfied, the cascade structure is preserved. In the former case, the error dynamics forms a feedback loop only with the downstream part Ξ_0, a block diagram of which is illustrated in the right of Fig. 7. Then, the system (18) has a cascade structure for any $\hat{E}_0$. A reasonable approach is to set $\hat{E}_0 = 0$, which results in $\hat{A}_e = A$. However, in the latter case, there exists a free parameter $\hat{A}_0$ that satisfies $\hat{F}_0 = 0$ only when $\operatorname{im} P_0$ is A-invariant, which is a restrictive requirement. Therefore, we only consider the former case.

When (20) is satisfied, the other parameters should be chosen so as to reduce the norm of the transfer matrix from $\xi_0, \xi_1, \dots, \xi_N$ to e given by $G_e := (sI - \hat{A}_e)^{-1}[\hat{F}_0\ \hat{F}_1\ \cdots\ \hat{F}_N]$ while satisfying (19). A reasonable policy to determine the parameters is to minimize the spectrum norm $\|\cdot\|_2$ of the input matrices through

$$\begin{aligned} \hat{A}_0 \quad &\in \arg\min_X \|\hat{F}_0(X)\|_2, \\ (\hat{A}_i, \hat{R}_i) &\in \arg\min_{(X,Y)} \|\hat{F}_i(X, Y)\|_2, \quad i = 1,\dots,N \end{aligned} \tag{21}$$

where $\hat{F}_0(X) := AP_0 - P_0X$, and $\hat{F}_i(X, Y) := AP_i - P_iX - P_0Y$. Here, problem (21) can be written as a semidefinite programming problem [45].

Note that this choice yields the exact hierarchical model decomposition with (10) in Theorem 2 if the conditions (9a) and (9b) in Theorem 1 hold. In this sense, the system (18) with parameters (20) and (21) can be regarded as a generalization of hierarchical model decomposition. Accordingly, we define *robust hierarchical model decomposition.*

Definition 3 (Robust Hierarchical Model Decomposition) The system in (18) with the parameters (19), (20), and (21) is said to be a robust hierarchical model decomposition of (4).

Distributed design can be achieved with the robust hierarchical model decomposition as long as the global subcontroller, which corresponds to the downstream part, can cope with the error signal. This fact is described by the following theorem.

Theorem 5 (Stabilization under Approximation Error) Consider a robust hierarchical model decomposition (18). Let $\hat{K}_1, \dots, \hat{K}_N$ be controllers such that the closed-loop systems (11) are internally stable. Moreover, let $\hat{K}_0$ be a controller such that the closed-loop system

$$\begin{cases} \dot{\xi}_0 = \hat{A}_0\xi_0 + \hat{E}_0 e + B_0\hat{u}_0 \\ \dot{e}\ = \hat{A}_e e + \hat{F}_0\xi_0 \\ \hat{u}_0 = \hat{K}_0(C_0\xi_0 + C_0P_0^{\mathsf{T}} e) \end{cases}$$

is internally stable. Then the controller composed of (13a) and (13b) with functional observers (12) stabilizes the clustered interconnected system (4).

Theorem 5 implies that the local subcontrollers can be designed without any concern about the approximation error as long as the downstream part is stabilized by the global subcontroller, for the design of which robust control [46] can apply.

Furthermore, the following proposition shows that the functional observer (14) still works for the robust version.

Proposition 5 Consider a robust hierarchical model decomposition (18). Then the system (14) is a functional observer of $C_i\xi_i$ for (18) as well.

The idea of the construction is almost the same as that in Proposition 1. The only difference is that ϕ_i is an estimation of $P_i^{\mathsf{T}}P_0\xi_0 + P_i^{\mathsf{T}}e$ instead of $P_i^{\mathsf{T}}P_0\xi_0$. Thus, we can estimate $C_i\xi_i$ using ϕ_i even when approximation errors are present.

VI. Numerical Examples

A. Motivating Example Revisited

Consider the motivating example. Each local internal controller $\hat{K}_i$ in (13b) for $i = 1, 2, 3$ is designed as a linear quadratic regulator (LQR) under the state weight $Q_i = I_{r_i} \otimes \mathrm{D}(q_\theta, q_\omega)$ with $(q_\theta, q_\omega) = (1, 10^4)$ and the input weight $R_i = 10^2 I_{r_i}$ with a state observer. Similarly, the global subcontroller $\hat{K}_0$ in (13a) is designed as a LQR under $Q_0 = I_N \otimes \mathrm{D}(q_\theta, q_\omega)$ and $R_0 = 10^2 I_N$ with the state observer.

The responses under the same initial condition as that of Fig. 4 are illustrated in Fig. 8, where Fig. 8a, Fig. 8b, Fig. 8c, and Fig. 8d correspond to the cases in which no controllers, only the local subcontrollers K_i for $i = 1, \ldots, N$, only the global subcontroller K_0, and the glocal controller K_0 with K_i for $i = 1, \ldots, N$ is implemented, respectively. It is observed in Fig. 8b that stationary inter-area oscillation remains. In Fig. 8c, local oscillation inside $\mathcal{I}_1$ and $\mathcal{I}_2$ cannot be suppressed only with the global subcontroller, although inter-area oscillation is removed. In contrast to the aforementioned two cases, both inter-area and intra-area behaviors can be suppressed using the glocal controller, as depicted in Fig. 8d. The result evidences the effectiveness of the glocal structure.

Next, we confirm scalability of the controller design by comparing the computation times for designing a glocal controller and a centralized controller. Consider increasing the number of subsystems within each cluster in Fig. 1. Set the number of components to $N_0 = 9n_0$ with a scale index n_0. Consider the three clusters constructed as

$$\begin{aligned} &\mathcal{I}_1 = \{1, \ldots, 3n_0\}, \quad \mathcal{I}_2 = \{3n_0 + 1, \ldots, 5n_0\}, \\ &\mathcal{I}_3 = \{5n_0 + 1, \ldots, 9n_0\} \end{aligned} \tag{22}$$

and let the parameters of the components be the same as those in Sec. II-A. We consider the centralized controller $\mathrm{col}(u_i)_{i=1}^N = K_\mathrm{c}\mathrm{col}(y_i)_{i=1}^N$ with a dense information structure designed by LQR. Moreover, we also suppose that the glocal controller is designed in accordance with the previous ones. The average computation times for design with varied n_0 ranging from 10 to 50 are depicted in Fig. 9 on a logarithmic scale. Evidently, the computation time is significantly reduced through hierarchical model decomposition, which indicates the potential of the scalability of the proposed distributed design.

B. NPCC system

To illustrate the practical relevance of the proposed control structure, we consider the 48-machine NPCC system [38, Chap. 3], a model of the power grid in New York and the neighboring areas. The NPCC 140-bus, 48-machine, 233-branch model can be found in the Power System Toolbox [47]. There is no exact hierarchical model decomposition for any non-trivial cluster set. Accordingly, as a given cluster set, we employ the nine clusters depicted in Fig. 3.5 in [38, Chap. 3], which was obtained via coherency-based aggregation. We apply the robust version of hierarchical model decomposition proposed in Sec. V. Each internal subcontroller is designed as an LQR controller with a state observer.

The frequency deviations of all generators are depicted in Fig. 10, where Figs. 10a, 10b, 10c, and 10d illustrate

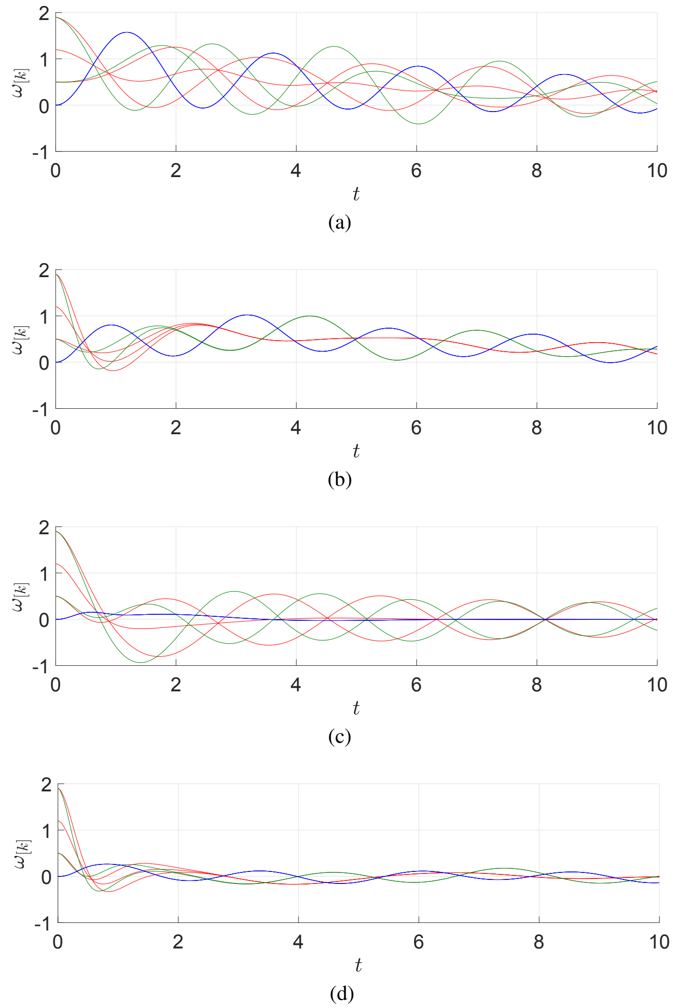

Fig. 8: Responses of the second-order network system with different control policies. (a): Free response. (b): Response only with the local subcontrollers. (c): Response only with the global subcontroller. (d): Response with the glocal controller.

the responses without any controller, only with the local subcontrollers, only with the global subcontroller, and with the proposed glocal controller, respectively. Evidently, local oscillations are efficiently suppressed with local subcontrollers. Moreover, it can be observed that the excitation of the slow global dynamics, which remains in Fig. 10b, is suppressed by the global subcontroller. The result highlights the potential effectiveness of the proposed glocal control for practical systems.

VII. Conclusion

In this paper, a distributed design of glocal controllers has been proposed for large-scale network systems. The proposed idea involves transforming the original system into a cascade structured system, called hierarchical model decomposition. Owing to this structure, stability of the overall system can be guaranteed by designing subcontrollers, each of which stabilizes the corresponding subsystem. We have provided a condition for the existence of the hierarchical model decomposition, a specific representation, clustering method, and a

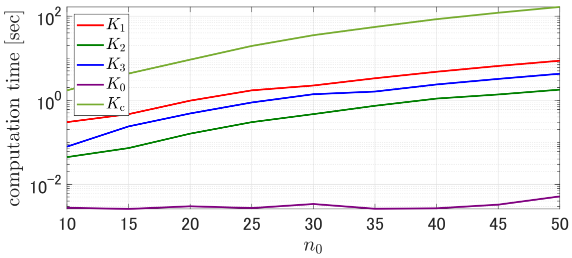

Fig. 9: Average computation times for designing the local subcontrollers K_1, K_2, and K_3, the global subcontroller K_0, and the centralized controller $K_{\rm c}$ for varied n_0 ranging from 10 to 50 on a logarithmic scale.

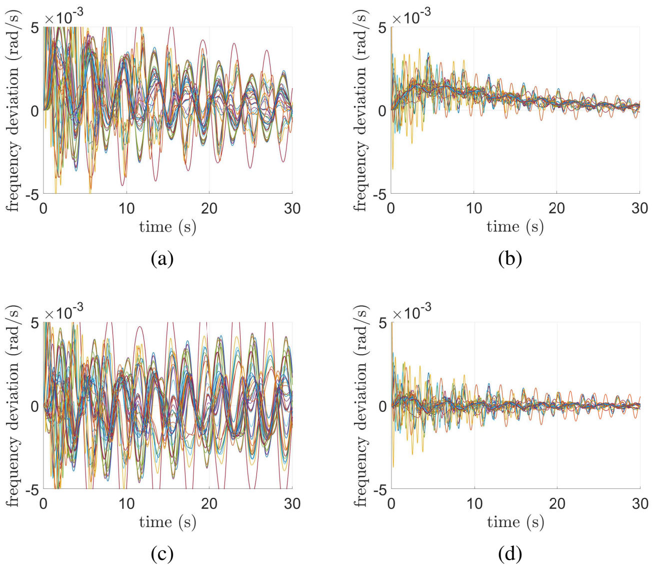

Fig. 10: Responses of the NPCC testbed with different controllers. (a): Free response. (b): Only with local subcontrollers. (c): Only with the global subcontroller. (d): Glocal controller.

robust extension.

Future research directions on the proposed framework include the development of a clustering method that can handle approximation errors. In this case, the proposed algorithm can result in a conservative decomposition. Furthermore, although we have focused on the glocal structure, distributed design of controllers with other particular information structures is another open problem.

Appendix
Proof of Theorems

Proof of Theorem 1: Sufficiency is shown in the proof of Theorem 2 by construction. We here show necessity. Assume that Ξ is a hierarchical model decomposition. Let $x_0 \in \operatorname{im} P_0$ and $x(0) = x_0, \xi_0(0) = P_0^\dagger x_0, \xi_i(0) = 0$ for $i = 1, \ldots, N$. Then $P_0 P_0^\dagger x_0 = x_0$ and $x(0) = P_0\xi_0(0) + \sum_{i=1}^N P_i\xi_i(0)$. Because Ξ is a hierarchical model decomposition, $x(t) = P_0\xi_0(t)$ holds for any $t \geq 0$ with $\hat{u}_i = 0$ for any i. Thus $x(t) = e^{At}x_0 \in \operatorname{im} P_0$ for any $t \geq 0$. Since $\operatorname{im} P_0$ is a closed subspace, $\lim_{t\to 0}(e^{At}x_0 - x_0)/t = Ax_0 \in \operatorname{im} P_0$. Because x_0 is arbitrary in $\operatorname{im} P_0$, $\operatorname{im} P_0$ is an invariant subspace of A. Hence $\mathcal{R}(A, P_0) \subset \operatorname{im} P_0$. Similarly, it can be shown that $\mathcal{R}(A, P_i) \subset \operatorname{im} P_0 + \operatorname{im} P_i$ for $i = 1, \ldots, N$ by taking $x(0) \in \operatorname{im} P_i$. □

Proof of Theorem 2: We first show sufficiency. As a preparation, we show that there exist matrices $\hat{A}_i$ and $\hat{R}_i$ such that (10) holds when the conditions (9a) and (9b) are satisfied. From (9a), $A \operatorname{im} P_i \subset \operatorname{im} P_i + \operatorname{im} P_0$ for $i = 1, \ldots, N$. Hence there exist X_i and X_{0i} such that $AP_i = P_iX_i + P_0X_{0i}$. Moreover, from (9b), $\operatorname{im} P_0$ is A-invariant and hence there exists X_0 such that $AP_0 = P_0X_0$. Thus the condition (10) holds with $\hat{A}_i = X_i, \hat{R}_i = X_{0i}, \hat{A}_0 = X_0$.

Let us assume that the condition (10) holds. Define the error signal $e := x - P_0\xi_0 - \sum_{i=1}^N P_i\xi_i$ and then

$$\begin{aligned} \dot{e} &= Ax - P_0(\hat{A}_0\xi_0 + \textstyle\sum_{i=1}^N \hat{R}_i\xi_i) - \sum_{i=1}^N P_i\hat{A}_i\xi_i \\ &= Ax - P_0\hat{A}_0\xi_0 - \textstyle\sum_{i=1}^N (P_0\hat{R}_i + P_i\hat{A}_i)\xi_i \\ &= Ax - AP_0\xi_0 - \textstyle\sum_{i=1}^N AP_i\xi_i = Ae. \end{aligned}$$

When $x(0) = \sum_{i=1}^N P_i\xi_i(0) + P_0\xi_0(0)$ holds, $e(0) = 0$ and hence $e(t) = 0$ for any $t \geq 0$ and $\hat{u}_i$. Thus Ξ is a hierarchical model decomposition.

We next show the necessity part. Note that, from Theorem 1 $\mathcal{R}(A, P_0) \subset \operatorname{im} P_0$ holds. We first show that $AP_0 - P_0\hat{A}_0 = 0$, which is equivalent to $\hat{A}_0 = P_0^\dagger AP_0$ under $\mathcal{R}(A, P_0) \subset \operatorname{im} P_0$. When $\hat{u}_i = 0$ and $\xi_i(0) = 0$, $\xi_i(t) = 0$ for any $t \geq 0$. Then $x(t) = P_0\xi_0(t)$ for any $t \geq 0$ for arbitrary $\hat{u}_0$ provided that $x(0) = P_0\xi_0(0)$. Define $e_0 := P_0^\dagger x - \xi_0$ and then $\dot{e}_0 = P_0^\dagger AP_0 e_0 + (P_0^\dagger AP_0 - \hat{A}_0)\xi_0$. Since $e_0(t) = 0$ for any $t \geq 0$, $\dot{e}_0(0) = 0$ for any ξ_0. Therefore the kernel of $P_0^\dagger AP_0 - \hat{A}_0$ contains the entire space, which leads to $\hat{A}_0 = P_0^\dagger AP_0$. Similarly, it can be shown that $AP_i - P_0\hat{R}_i - P_i\hat{A}_i = 0$ for $i = 1, \ldots, N$. □

Proof of Theorem 3: From the necessary and sufficient condition of functional observers [44, Lemma 2], $\mathbf{A}_i$ is stable and there exists a matrix U_{i0} such that

$$\begin{aligned} &U_{i0}\begin{bmatrix} \hat{A}_i & 0 \\ \hat{R}_i & \hat{A}_0 \end{bmatrix} - \mathbf{A}_iU_{i0} = \mathbf{E}_iC_i[I \;\; P_i^{\mathsf{T}}P_0], \\ &[\mathbf{B}_i \;\; \mathbf{D}_i] = U_{i0}\begin{bmatrix} B_i & 0 \\ 0 & B_0 \end{bmatrix}, \;\; [C_i \;\; 0] = \mathbf{C}_iU_{i0} + \mathbf{F}_iC_i[I \;\; P_i^{\mathsf{T}}P_0] \end{aligned} \tag{23}$$

for any $i = 1, \ldots, N$. Let $U_{i0} = [U_i \;\; U_0]$ and define $\epsilon_i := U_i\xi_i + U_0\xi_0 - \phi_i$. Then because Φ_i is a functional observer, the dynamics of ϵ_i can be represented as $\dot{\epsilon}_i = \mathbf{A}_i\epsilon_i$, which is stable. Moreover, we have

$$\begin{aligned} \psi_i &= \mathbf{C}_i\phi_i + \mathbf{F}_iC_i[I \;\; P_i^{\mathsf{T}}P_0][\xi_i^{\mathsf{T}} \;\; \xi_0^{\mathsf{T}}]^{\mathsf{T}} \\ &= \mathbf{C}_i([U_i \;\; U_0][\xi_i^{\mathsf{T}} \;\; \xi_0^{\mathsf{T}}]^{\mathsf{T}} - \epsilon_i) + \mathbf{F}_iC_i[I \;\; P_i^{\mathsf{T}}P_0][\xi_i^{\mathsf{T}} \;\; \xi_0^{\mathsf{T}}]^{\mathsf{T}} \\ &= C_i\xi_i - \mathbf{C}_i\epsilon_i \end{aligned}$$

in view of the third identity in (23). Thus the entire closed-loop system composed of Ξ and $\{K_i\}_{i=0}^N$ can be described by

$$\begin{cases} \begin{cases} \dot{\epsilon}_i = \mathbf{A}_i\epsilon_i \\ \dot{\xi}_i = \hat{A}_i\xi_i + B_i\hat{u}_i \\ \hat{u}_i = \hat{K}_i(C_i\xi_i - \mathbf{C}_i\epsilon_i) \end{cases}, \quad i = 1, \ldots, N \\ \dot{\xi}_0 = \hat{A}_0\xi_0 + \sum_{i=1}^N \hat{R}_i\xi_i + B_0u_0 \\ \hat{u}_0 = \hat{K}_0C_0(\xi_0 + \sum_{i=1}^N P_0^{\mathsf{T}}P_i\xi_i). \end{cases} \tag{24}$$

From the cascade structure of (24) and the assumption on stability of every closed-loop system, the entire system is internally stable. Because the original state can be represented by superposition of the states in (24), the original system with the controllers is also internally stable. □

Proof of Proposition 1: The whole observer composed of $\Phi_1,\ldots,\Phi_N$ can be represented by

$$\begin{cases} \dot{\phi} = \mathrm{D}(\hat{A}_i)\phi + \mathrm{D}(A_i-\hat{A}_i)\hat{x} + \mathrm{D}(L_i)v + P_0B_0\hat{u}_0 \\ \dot{\hat{x}} = \mathrm{D}(A_i)\hat{x} + \mathrm{D}(B_i)\mathrm{col}(\hat{u}_i) + \mathrm{D}(L_i)v + P_0B_0\hat{u}_0 \\ \psi = -\mathrm{D}(C_i)\phi + y \end{cases}$$

with $\phi := \mathrm{col}(\phi_i), \hat{x} := \mathrm{col}(\hat{x}_i), \psi := \mathrm{col}(\psi_i)$ where the measurement signals are represented by

$$\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} \mathrm{D}(C_i) & \mathrm{D}(C_i)P_0 \\ M & MP_0 \end{bmatrix} \begin{bmatrix} \mathrm{col}(\xi_i)_{i=1}^N \\ \xi_0 \end{bmatrix}.$$

It suffices to show that the matrix

$$U := \begin{bmatrix} 0 & P_0 \\ I & P_0 \end{bmatrix}$$

satisfies the conditions

$$U\begin{bmatrix} \mathrm{D}(\hat{A}_i) & 0 \\ \begin{bmatrix}\hat{R}_1 & \cdots \hat{R}_N\end{bmatrix} & \hat{A}_0 \end{bmatrix} - \begin{bmatrix} \mathrm{D}(\hat{A}_i) & \mathrm{D}(A_i-\hat{A}_i) \\ 0 & \mathrm{D}(A_i) \end{bmatrix} U = \begin{bmatrix} 0 & \mathrm{D}(L_i) \\ 0 & \mathrm{D}(L_i) \end{bmatrix} \begin{bmatrix} \mathrm{D}(C_i) & \mathrm{D}(C_i)P_0 \\ M & MP_0 \end{bmatrix}, \tag{25}$$

and

$$\begin{bmatrix} 0 & P_0B_0 \\ \mathrm{D}(B_i) & P_0B_0 \end{bmatrix} = U \begin{bmatrix} \mathrm{D}(B_i) & 0 \\ 0 & B_0 \end{bmatrix},$$
$$[\mathrm{D}(C_i)\ 0] = -[\mathrm{D}(C_i)\ 0]U + [\mathrm{D}(C_i)\ \mathrm{D}(C_i)P_0].$$

The second and third identities obviously hold. Regarding the first condition, the right-hand side of (25) is described by

$$\begin{aligned} (\mathrm{RHS}) &= [I\ I]^{\mathsf{T}}\mathrm{D}(L_i)M\,[I\ P_0] \\ &= [I\ I]^{\mathsf{T}}(A-\mathrm{D}(A_i))\,[I\ P_0] \end{aligned} \tag{26}$$

and the left-hand side is described by

$$(\mathrm{LHS}) = \begin{bmatrix} I \\ I \end{bmatrix} \begin{bmatrix} P_0\begin{bmatrix}\hat{R}_1 & \cdots & \hat{R}_N\end{bmatrix} - \mathrm{D}(A_i-\hat{A}_i) & P_0\hat{A}_0 - \mathrm{D}(A_i)P_0 \end{bmatrix}.$$

Since Ξ is a hierarchical model decomposition, $P_0\hat{R}_i = AP_i - P_i\hat{A}_i$ for $i=1,\ldots,N$ and $P_0\hat{A}_0 = AP_0$. Therefore $(\mathrm{LHS}) = [I\ I]^{\mathsf{T}}\left[A-\mathrm{D}(A_i)\ \ AP_0-\mathrm{D}(A_i)P_0\right]$, which is equal to (26). □

Proof of Proposition 2: We show that $A\,\mathrm{im}\,P_0^{(1)} \subset \mathrm{im}\,P_0$. Noting that $\mathcal{R}(A,[P_2\ \cdots\ P_N]) \subset \mathrm{im}\,P_0^{(1)} + \mathrm{im}\,[P_2\ \cdots\ P_N]$ from (9a), we have

$$\begin{aligned} A\,\mathrm{im}\,P_0^{(1)} &\subset A\,\mathrm{im}\,P_0 \\ &\subset A\,\mathcal{R}(A,[P_2\ \cdots\ P_N]) \\ &\subset \mathcal{R}(A,[P_2\ \cdots\ P_N]) \\ &\subset \mathrm{im}\,P_0^{(1)} + \mathrm{im}\,[P_2\ \cdots\ P_N] \end{aligned}$$

because of (15) and A-invariance of the controllable subspace. Moreover, since $\mathrm{im}\,P_0^{(1)} \subset \mathrm{im}\,P_1$, we have

$$\begin{aligned} A\,\mathrm{im}\,P_0^{(1)} &\subset A\,\mathrm{im}\,P_1 \\ &\subset \mathcal{R}(A,P_1) \\ &\subset \mathrm{im}\,P_1 + \mathrm{im}\,P_0 \end{aligned}$$

from (9a). Therefore, it follows that

$$\begin{aligned} A\,\mathrm{im}\,P_0^{(1)} &\subset (\mathrm{im}\,P_0^{(1)} + \mathrm{im}\,[P_2\ \cdots\ P_N]) \cap (\mathrm{im}\,P_1 + \mathrm{im}\,P_0) \\ &= (\mathrm{im}\,P_0^{(1)} \cap \mathrm{im}\,P_1) \oplus (\mathrm{im}\,[P_2\ \cdots\ P_N] \cap \mathrm{im}\,P_0) \\ &= \mathrm{im}\,P_0. \end{aligned}$$

Similarly, we can show the same inclusion property for the other clusters. Hence, $\mathrm{im}\,P_0$ is A-invariant and the condition (9b) holds. □

Proof of Theorem 4: Let $\{\mathcal{I}_i\}$ be the cluster set produced by Algorithm 1. From Lemma 1, any cluster set in $\mathfrak{G}(\{\mathcal{I}_i^{(0)}\}_{i=1}^{N^{(0)}})$ can be generated from $\{\mathcal{I}_i\}$. Since the number of clusters increases by partition, the claim holds. □

Proof of Lemma 1: We prove the claim by induction. It suffices to show $\mathfrak{G}(\{\mathcal{I}_i^{(\tau)}\}) \subset \mathfrak{G}(\{\mathcal{I}_i^{(\tau+1)}\})$ for any τ. Take a cluster set $\{\mathcal{I}_i\}_{i=1}^N$ that belongs to $\mathfrak{G}(\{\mathcal{I}_i^{(\tau)}\})$. Because the clusters satisfy (9a), it suffices to show that $\{\mathcal{I}_i\}$ belongs to $\mathfrak{F}(\{\mathcal{I}_i^{(\tau+1)}\})$. Let $k \in \{1,\ldots,N^{(\tau)}\}$ be the minimum index such that

$$\mathcal{R}(A,P_k^{(\tau)}) \not\subset \mathrm{im}\,P_k^{(\tau)} + \mathrm{im}\,P_0^{(\tau)}$$

where $P_k^{(\tau)} := f_{P_i}(\mathcal{I}_k^{(\tau)})$ and $P_0^{(\tau)} := f_{P_0}(\{\mathcal{I}_i^{(\tau)}\})$. Because $\mathcal{I}_k^{(\tau)}$ is not partitioned at the τth step, there exists $j \in \{1,\ldots,N^{(\tau+1)}\}$ such that $\mathcal{I}_j^{(\tau+1)} = \mathcal{I}_k^{(\tau)}$. Hence

$$\mathcal{R}(A,P_j^{(\tau+1)}) = \mathcal{R}(A,P_k^{(\tau)}) \subset \mathrm{im}\,P_k^{(\tau)} + \mathrm{im}\,P_0^{(\tau+1)}$$

since (9a) holds for j. Because $\{\mathcal{I}_i\} \subset \mathfrak{F}(\{\mathcal{I}_i^{\tau}\})$, there exists an index set $\mathcal{L} \subset \{1,\ldots,N\}$ such that $\cup_{l\in\mathcal{L}}\mathcal{I}_l = \mathcal{I}_k^{(\tau)} = \mathcal{I}_j^{(\tau+1)}$. Because $\{\mathcal{I}_i\}$ satisfies (9a), we have

$$\mathcal{R}(A,P_{\mathcal{L}}) = \mathcal{R}(A,P_k^{(\tau)}) \subset \mathrm{im}\,P_k^{(\tau)} + \mathrm{im}\,P_0$$

where $P_{\mathcal{L}}$ is composed of $P_l := f_{P_i}(\mathcal{I}_l)$ for $l \in \mathcal{L}$ and $P_0 := f_{P_0}(\{\mathcal{I}_i\})$. By taking the intersection, we have

$$\mathcal{R}(A,P_k^{(\tau)}) \subset \mathrm{im}\,P_k^{(\tau)} + (\mathrm{im}\,P_0^{(\tau+1)} \cap \mathrm{im}\,P_0).$$

Taking the projection onto the orthogonal subspace of $\mathrm{im}\,{P_k}^{(\tau)}$, we have

$$\pi_{\mathrm{im}\,{P_k}^{(\tau)\perp}}\mathcal{R}(A,P_k^{(\tau)}) \subset \pi_{\mathrm{im}\,{P_k}^{(\tau)\perp}}(\mathrm{im}\,P_0^{(\tau+1)} \cap \mathrm{im}\,P_0).$$

The optimality of $\{\mathcal{I}_i^{(\tau+1)}\}$ in the problem (17) yields

$$\pi_{\mathrm{im}\,{P_k}^{(\tau)\perp}}(\mathrm{im}\,P_0^{(\tau+1)} \cap \mathrm{im}\,P_0) = \pi_{\mathrm{im}\,{P_k}^{(\tau)\perp}}\mathrm{im}\,P_0^{(\tau+1)}.$$

This identity and the relation $\cup_{l\in\mathcal{L}}\mathcal{I}_l = \mathcal{I}_k^{(\tau)} = \mathcal{I}_j^{(\tau+1)}$ imply that $\{\mathcal{I}_i\}$ is a partition of $\{\mathcal{I}_i^{(\tau+1)}\}$. □

Proof of Proposition 3: Let P_i and P_0 be the corresponding matrices of the initial cluster set and P_i' and P_0' be the ones corresponding to the expanded cluster set. Because $\operatorname{im} P_0 \subset \operatorname{im} P_0'$, we have

$$\mathcal{R}(A, P_i) \subset \operatorname{im} P_i + \operatorname{im} P_0 \subset \operatorname{im} P_i + \operatorname{im} P_0', \quad i \notin \mathcal{J}.$$

Similarly, because $\operatorname{im} P_{\mathcal{J}} \subset \operatorname{im} P_{\mathcal{J}'}$, where $P_{\mathcal{J}}$ is composed of P_j for $j \in \mathcal{J}$, the condition (15) holds for $i \notin \mathcal{J}$. □

Proof of Proposition 4: From Assumption 2, the condition of Assumption 1 holds. Thus Theorem 1 leads to the claim. □

Proof of Theorem 5: As in the proof of Theorem 3, we can obtain an equivalent system

$$\begin{cases} \begin{cases} \dot{\epsilon}_i = \mathbf{A}_i \epsilon_i \\ \dot{\xi}_i = \hat{A}_i \xi_i + B_i \hat{u}_i \qquad , \quad i = 1, \ldots, N \\ \hat{u}_i = \hat{K}_i (C_i \xi_i - \mathbf{C}_i \epsilon_i) \end{cases} \\ \dot{\xi}_0 = \hat{A}_0 \xi_0 + \sum_{i=1}^N \hat{R}_i \xi_i + B_0 u_0 \\ \dot{e} = \hat{A}_e e + \hat{F}_0 \xi_0 \\ \hat{u}_0 = \hat{K}_0 C_0 (\xi_0 + \sum_{i=1}^N P_0^{\mathsf{T}} P_i \xi_i + P_0^{\mathsf{T}} e). \end{cases}$$

From the cascade structure and the assumption on internal stability, the claim holds. □

Proof of Proposition 5: By simple algebra, it can be confirmed that the matrix

$$U_e := \begin{bmatrix} 0 & P_0 & I \\ I & P_0 & I \end{bmatrix}$$

satisfies the conditions for the functional observers of $\operatorname{col}(C_i \xi_i)$. □

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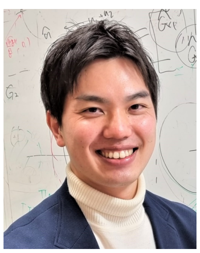

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