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Data-Driven Set-Based Estimation using Matrix Zonotopes with Set Containment Guarantees

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Abstract— **We propose a method to perform set-based state estimation of an unknown dynamical linear system using a data-driven set propagation function. Our method comes with set-containment guarantees, making it applicable to safety-critical systems. The method consists of two phases: (1) an offline learning phase where we collect noisy input-output data to determine a function to propagate the state-set ahead in time; and (2) an online estimation phase consisting of a time update and a measurement update. It is assumed that known finite sets bound measurement noise and disturbances, but we assume no knowledge of their statistical properties. These sets are described using zonotopes, allowing efficient propagation and intersection operations. We propose a new approach to compute a set of models consistent with the data and noise-bound, given input-output data in the offline phase. The set of models is utilized in replacing the unknown dynamics in the data-driven set propagation function in the online phase. Then, we propose two approaches to perform the measurement update. Simulations show that the proposed estimator yields state sets comparable in volume to the 3σ confidence bounds obtained by a Kalman filter approach, but with the addition of state set-containment guarantees. We observe that using constrained zonotopes yields smaller sets but with higher computational costs than unconstrained ones.**

I. Introduction

Set-based estimation involves the computation of a set, which is guaranteed to contain the system's true state at each time step given bounded uncertainties [1]. Existing set-based observers require a system model to propagate the state set at each time step [2], [3]. We address the problem of propagating the state set using only noisy offline input-output data and merging this with online measurements to obtain a time-varying state set which is guaranteed to contain the true system's state at each time-step. This problem is essential in safety-critical applications [4].

Two popular set-based estimators are interval observers and set-membership observers. Interval-based observers generally generate state estimates by utilizing an observer gain to fuse a model-based time update of the state with current measurements. For example, the authors in [5] propose an exponentially stable interval-based observer for time-invariant linear systems. Set-membership observers generally follow a geometrical approach by intersecting the state-space regions consistent with the model with those from the measurements to obtain the current state set [6]. This approach has been extended to sensor networks with event-based communication in [7] and multi-rate systems in [8]. Various set representations have been used for set-membership observers such as ellipsoids [9], polytopes [10] and zonotopes [11]. Zonotopes are a special class of polytopes for which one can efficiently compute linear maps, and Minkowski sums – both frequent operations performed by set-based observers.

All the aforementioned observers use a model of the underlying system to propagate the state set. However, identifying a system model is often time-consuming, and the identified model is not necessarily well-suited for estimation or control. Recent works based on Willems' fundamental lemma [12] have shown that system trajectories can be used directly to synthesize controllers. The authors in [13] present an extended Kalman filter and model predictive control (MPC) scheme computed directly from system trajectories. Stability and robustness guarantees for such a data-driven control scheme are presented in [14], and for an MPC scheme in [15]. An alternative approach is to find a set of models that is consistent with data and use this set of models to propagate a state set [16].

Our contribution is a novel method to perform set-based state estimation with set-containment guarantees given bounded, noisy measurements and known inputs. The algorithm, summarized in Fig. 1, consists of an *offline learning phase* to determine a state-propagation function $f(\cdot)$ directly from data, and an *online estimation phase* to perform a time update using $f(\cdot)$ and measurements iteratively to track the system state. A new approach to compute the set of models consistent with the data and noise bound from input-output data is proposed different from input-state data in [16], [17]. Then, we present two approaches to perform the measurement update utilizing either the singular value decomposition (SVD) of the observation matrix or an optimization formulation. We compare the approaches in simulation. Our method is shown to yield set-based state estimates similar in size to 3σ confidence bounds of an approach based on system identification and a Kalman filter, but with the addition of set-containment guarantees. The code to recreate our findings is publicly available[1].

The rest of this paper is outlined as follows. Sec. II introduces the preliminaries and problem statement. We present our method in Sec. III and evaluate it in Sec. IV. Finally, Sec. V concludes the paper.

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[1]https://github.com/alexberndt/data-driven-set-based-estimation-zonotopes

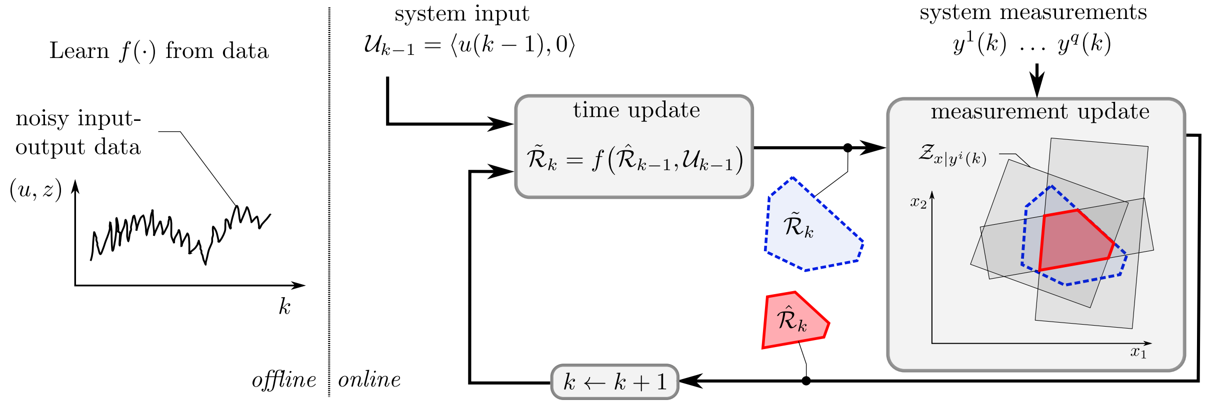


Fig. 1: The proposed method showing the offline learning phase yielding $f(\cdot)$ and the online estimation phase which utilizes $f(\cdot)$ to perform the time update, followed by a measurement update yielding the set $\hat{\mathcal{R}}_k$ at time-step k.

II. Preliminaries and Problem Statement

We denote the i-th element of a vector or list A by $A^{(i)}$. We first introduce some set representations.

Definition 1. (Zonotope [18]) *Given a center $c \in \mathbb{R}^n$ and a number $\xi \in \mathbb{N}$ of generator vectors in a generator matrix $G = [g^{(1)}, ..., g^{(\xi)}] \in \mathbb{R}^{n\times\xi}$, a zonotope is a set*

$$\mathcal{Z} = \Big\{ x \in \mathbb{R}^n \;\Big|\; x = c + \sum_{i=1}^{\xi} \beta^{(i)} g^{(i)}, -1 \leq \beta^{(i)} \leq 1 \Big\}. \quad (1)$$

We use the shorthand notation $\mathcal{Z} = \langle c, G\rangle$.

Given two zonotopes $\mathcal{Z}_1$ and $\mathcal{Z}_2$, we use the notation $+$ for the Minkowski sum, and $\mathcal{Z}_1 - \mathcal{Z}_2$ to denote $\mathcal{Z}_1 + (-\mathcal{Z}_2)$ not the Minkowski difference.

Definition 2. (Matrix zonotope [4, p.52]) *Given a center matrix $C \in \mathbb{R}^{n\times k}$ and $\xi \in \mathbb{N}$ generator matrices $G^{(i)} \in \mathbb{R}^{n\times k}$ where $i \in \{1, \dots, \xi\}$, a matrix zonotope is the set*

$$\mathcal{M} = \Big\{ X \in \mathbb{R}^{n\times k} \;\Big|\; X = C + \sum_{i=1}^{\xi} \beta^{(i)} G^{(i)}, -1 \leq \beta^{(i)} \leq 1 \Big\}.$$

We use the notation $\mathcal{M} = \langle C, G^{(1:\xi)}\rangle$, where $G^{(1:\xi)} = [G^{(1)}, \dots, G^{(\xi)}]$.

Definition 3. *(*Interval matrix [4, p. 42]) An interval matrix $\mathcal{I}$ specifies the interval of all possible values for each matrix element between the left limit $\underline{I}$ and right limit $\bar{I}$:

$$\mathcal{I} = \left[\underline{I}, \bar{I}\right], \quad \underline{I}, \bar{I} \in \mathbb{R}^{r\times c} \quad (2)$$

We consider estimating the set of all possible system states using an array of q sensors. Our system is described as

$$x(k+1) = A_{\text{tr}} x(k) + B_{\text{tr}} u(k) + w(k), \quad (3a)$$

$$y^i(k) = C^i x(k) + v^i(k), \;\; i \in \{1, \dots, q\}, \quad (3b)$$

where $x(k) \in \mathbb{R}^n$ is the system state, $u(k) \in \mathbb{R}^m$ the input, $y^i(k) \in \mathbb{R}^{p_i}$ the measurement of sensor i, $x(0) \in \mathcal{X}_0$ the initial condition where $\mathcal{X}_0$ is the initial bounding zonotope. Furthermore, the system matrices $A_{\text{tr}} \in \mathbb{R}^{n\times n}$ and $B_{\text{tr}} \in \mathbb{R}^{n\times m}$ are *unknown* whereas $C^i \in \mathbb{R}^{p_i\times n}$ is *known* for all $i \in \{1, \dots, q\}$. The noise $w(k) \in \mathcal{Z}_w$ and $v^i(k) \in \mathcal{Z}_{v,i}$ are assumed to belong to the bounding zonotopes $\mathcal{Z}_w = \langle c_w, G_w\rangle \subset \mathbb{R}^n$ and $\mathcal{Z}_{v,i} = \langle c_{v,i}, G_{v,i}\rangle \subset \mathbb{R}^{p_i}$ for $i \in \{1, \dots, q\}$, respectively. We denote the Frobenius norm by $\|.\|_F$ and the null space of a matrix A by $\texttt{ker}(A)$. We compute the pseudoinverse of an interval matrix by adapting [19, Thm 2.40]. The pseudoinverse of an interval matrix is denoted by $\dagger$.

Let $\mathcal{R}_k$ denote a set containing $x(k)$ given the *exact* system model and bounded, but *unknown*, process and measurement noise. The problem addressed in this paper is to develop an algorithm that returns a set $\hat{\mathcal{R}}_k \supseteq \mathcal{R}_k$, which is *guaranteed* to contain the true state $x(k)$ at each time instance k, i.e., $x(k) \in \hat{\mathcal{R}}_k$ for all k, given input-output data and bounds for model uncertainties and measurement noise without knowledge of the model $\begin{bmatrix} A_{\text{tr}} & B_{\text{tr}} \end{bmatrix}$.

III. Data-driven Set-based Estimation

Our proposed data-driven set estimator consists of two phases: an *offline learning phase* and an *online estimation phase*. In the offline phase, we compute the function to perform the time update. The online phase consists of iteratively performing a time update and a measurement update. We denote the time and measurement updated sets at k by $\tilde{\mathcal{R}}_k \subset \mathbb{R}^n$ and $\hat{\mathcal{R}}_k \subset \mathbb{R}^n$, respectively.

A. Offline Learning Phase

The objective of this phase is to compute a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, such that $\tilde{\mathcal{R}}_{k+1} = f(\hat{\mathcal{R}}_k, \mathcal{U}_k)$, i.e., f returns $\tilde{\mathcal{R}}_{k+1}$ given a known input zonotope $\mathcal{U}_k$ and the measurement updated set $\hat{\mathcal{R}}_k$ at time-step k such that we can guarantee $x(k+1) \in \tilde{\mathcal{R}}_{k+1}$ for all k. During this phase, we assume that we have offline an access to an input sequence $u(k)$ and noisy output $z^i(k)$ such that

$$z^i(k) = C^i x(k) + \gamma^i(k), \quad (4)$$

where the noise $\gamma^i(k)$ is bounded by the zonotope $\mathcal{Z}_{\gamma,i} = \langle c_{\gamma,i}, G_{\gamma,i}\rangle$, i.e., $\gamma^i(k) \in \mathcal{Z}_{\gamma,i}, \forall k$. We have for all sensors vertically combined noisy output $z(k) = \begin{bmatrix} {z^1}^T(k) & \dots & {z^q}^T(k) \end{bmatrix}^T$ and similarly for γ and C. For the sake of clarity, we differentiate the notation of the offline noisy output $z^i(k)$ from the online noisy output $y^i(k)$ and similarly for the measurement noise. Given an experiment yielding a sequence of noisy data of length T, we can construct the following sequences

$$\begin{aligned} Z^+ &= \begin{bmatrix} z(1) & \dots & z(T) \end{bmatrix}, \\ Z^- &= \begin{bmatrix} z(0) & \dots & z(T-1) \end{bmatrix}, \\ U^- &= \begin{bmatrix} u(0) & \dots & u(T-1) \end{bmatrix}. \end{aligned} \tag{5}$$

We further construct

$$Z = \begin{bmatrix} z(0) & \dots & z(T) \end{bmatrix},$$

and similarly for other signals. The data $D = \begin{bmatrix} U^- & Z \end{bmatrix}$ can be from one sensor or multiple sensors. Furthermore, we denote the sequence of *unknown* process noise $w(k)$ as $W^- = \begin{bmatrix} w(0) & \dots & w(T-1) \end{bmatrix}$. Here, $W^- \in \mathcal{M}_w$ where $\mathcal{M}_w = \langle C_{\mathcal{M},w}, G^{(1:\xi T)}_{\mathcal{M},w} \rangle$ is the matrix zonotope resulting from the concatenation of multiple noise zonotopes $\mathcal{Z}_w = \langle c_w, [g_w^{(1)}, \dots, g_w^{(\xi)}] \rangle$ as

$$\begin{aligned} C_{\mathcal{M},w} &= \begin{bmatrix} c_w & \dots & c_w \end{bmatrix}, \\ G^{(1+(i-1)T)}_{\mathcal{M},w} &= \begin{bmatrix} g_w^{(i)} & 0_{n\times(T-1)} \end{bmatrix}, \\ G^{(j+(i-1)T)}_{\mathcal{M},w} &= \begin{bmatrix} 0_{n\times(j-1)} & g_w^{(i)} & 0_{n\times(T-j)} \end{bmatrix}, \\ G^{(T+(i-1)T)}_{\mathcal{M},w} &= \begin{bmatrix} 0_{n\times(T-1)} & g_w^{(i)} \end{bmatrix}, \end{aligned}$$

for all $i = \{1, \dots, \xi\}$, $j = \{2, \dots, T-1\}$ [16]. In a similar fashion, we describe the unknown noise and matrix zonotope of $\gamma(k)$ as $\Gamma^+, \Gamma^- \in \mathcal{M}_\gamma = \langle C_{\mathcal{M},\gamma}, G^{(1:\xi T)}_{\mathcal{M},\gamma} \rangle$. We denote all system matrices $\begin{bmatrix} A & B \end{bmatrix}$ that are consistent with the data:

$$\begin{aligned} \mathcal{N}_\Sigma = \{ \begin{bmatrix} A & B \end{bmatrix} \mid\ & X^+ = AX^- + BU^- + W^-, \\ & Z^- = CX^- + \Gamma^-, W^- \in \mathcal{M}_w, \Gamma^+ \in \mathcal{M}_\gamma, \\ & \Gamma^- \in \mathcal{M}_\gamma \}. \end{aligned}$$

By definition, $\begin{bmatrix} A_{\text{tr}} & B_{\text{tr}} \end{bmatrix} \in \mathcal{N}_\Sigma$ as $\begin{bmatrix} A_{\text{tr}} & B_{\text{tr}} \end{bmatrix}$ is one of the systems that are consistent with the data. The following theorem finds a set of models $\mathcal{M}_\Sigma$ that over-approximates $\mathcal{N}_\Sigma$, i.e., $\mathcal{N}_\Sigma \subseteq \mathcal{M}_\Sigma$, which defines $f(\cdot)$ introduced above. For this, we aim to determine the mapping of the observation Z^+ and Z^- to the corresponding state-space region. Specifically, we construct a zonotope $\mathcal{Z}_{x|z^i(k)} \subset \mathbb{R}^n$ that contains all *possible* $x \in \mathbb{R}^n$ given $z^i(k)$, C^i and bounded noise $\gamma^i(k) \in \mathcal{Z}_{\gamma,i}$ satisfying (4), for each i. This can be written as

$$\mathcal{Z}_{x|z^i(k)} = \left\{ x \in \mathbb{R}^n \;\middle|\; C^i x = z^i(k) - \mathcal{Z}_{\gamma,i} \right\}. \tag{6}$$

Extending (6) to a matrix zonotope allows to find the mapping of Z^+ and Z^- to the state space which is utilized to compute the $\mathcal{M}_\Sigma$. We omit the time index k and sensor index i when possible for simplicity. We assume a prior known upper bound M on the state trajectory, i.e., $M \geq \|x\|_2$.

Lemma 1. *Given input-output trajectories $D = \begin{bmatrix} U^- & Z \end{bmatrix}$ of the system (3). Then, the matrix zonotope*

$$\mathcal{M}_\Sigma = (\mathcal{M}^+_{x|z} - \mathcal{M}_w) \begin{bmatrix} \mathcal{M}^-_{x|z} \\ U^- \end{bmatrix}^\dagger \tag{7}$$

contains all matrices $\begin{bmatrix} A & B \end{bmatrix}$ that are consistent with the data D and the noise bounds, i.e., $\mathcal{N}_\Sigma \subseteq \mathcal{M}_\Sigma$, with $\mathcal{M}^+_{x|z} = \langle C^+_{\mathcal{M},x|z}, G^{(1:\xi T+1)}_{\mathcal{M},x|z} \rangle$ and $\mathcal{M}^-_{x|z} = \langle C^-_{\mathcal{M},x|z}, G^{(1:\xi T+1)}_{\mathcal{M},x|z} \rangle$ where

$$C^+_{\mathcal{M},x|z} = V_1 \Sigma^{-1}_{r\times r} P_1^\top \left(Z^+ - C_{\mathcal{M},\gamma} \right), \tag{8}$$

$$C^-_{\mathcal{M},x|z} = V_1 \Sigma^{-1}_{r\times r} P_1^\top \left(Z^- - C_{\mathcal{M},\gamma} \right), \tag{9}$$

$$G^{(i)}_{\mathcal{M},x|z} = V_1 \Sigma^{-1}_{r\times r} P_1^\top G^{(i)}_{\mathcal{M},\gamma}, \quad i = \{1, \dots, \xi T\}, \tag{10}$$

$$G^{(\xi T+1)}_{\mathcal{M},x|z} = M V_2 1_{(n-r)\times T}, \tag{11}$$

for all $M \geq \|x\|_2$, with P_1, V_1, Σ and V_2 obtained from the SVD of C. Assuming C has rank r, then

$$C = \begin{bmatrix} P_1 & P_2 \end{bmatrix} \begin{bmatrix} \Sigma_{r\times r} & 0_{r\times(n-r)} \\ 0_{(p-r)\times r} & 0_{(p-r)\times(n-r)} \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}, \tag{12}$$

where a matrix with non-positive index is an empty matrix.

Proof. From (12), we rewrite (4) as $P_1 \Sigma V_1^\top x = z - \gamma$, so $x = V_1 \Sigma^{-1} P_1^\top (z - \gamma)$. Since γ is bounded by $\mathcal{Z}_\gamma = \langle c_\gamma, G_\gamma \rangle$, we can write

$$x = \underbrace{V_1 \Sigma^{-1} P_1^\top \left(z - c_\gamma \right)}_{c_{x|z}} - \underbrace{V_1 \Sigma^{-1} P_1^\top G_\gamma}_{G'_{x|z}} \beta, \quad |\beta| \leq 1.$$

This set corresponds to all possible x values within the range space of C satisfying (4). By definition, if $r = n$, then $V_2 = \emptyset$, V_1 spans the domain of x, and $\langle c_{x|z}, G'_{x|z} \rangle$ sufficiently defines all possible x satisfying (4). However, if $r < n$, V_1 only spans a subset of the domain of x. To ensure $\mathcal{Z}_{x|z}$ contains all possible x we include a basis for $\texttt{ker}(C)$ in $G_{x|z}$ by appending the generator $V_2 M$ to $G_{x|z}$, and ensuring $M \geq \|x\|_2$ such that $V_2 M$ includes all x values in the directions of V_2. In both cases for r, the generator matrix can be written as

$$G_{x|z} = \begin{bmatrix} G'_{x|z} & V_2 M \end{bmatrix} = \begin{bmatrix} V_1 \Sigma^{-1} P_1^\top G_\gamma & V_2 M \end{bmatrix},$$

and the set $\mathcal{Z}_{x|z} = \langle c_{x|z}, G_{x|z} \rangle$. This result extends to the case when $r < p$ using similar argumentation in the respective cases $r = n$ and $r < n$. Considering the matrix version of $\mathcal{Z}_{x|z}$ results in proving $\mathcal{M}^+_{x|z}$ and $\mathcal{M}^-_{x|z}$. Then, we extend the proof of [17, Lem.1] for input-output data: For any $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{N}_\Sigma$, we know that there exists a $W^- \in \mathcal{M}_w$ such that

$$AX^- + BU^- = X^+ - W^-. \tag{13}$$

Every $W^- \in \mathcal{M}_w$ can be represented by a specific choice $\hat{\beta}^{(i)}_{\mathcal{M},w}$, $-1 \leq \hat{\beta}^{(i)}_{\mathcal{M},w} \leq 1$, $i = 1, \dots, \xi_{\mathcal{M},w}$, that results in a matrix inside the matrix zonotope $\mathcal{M}_w$:

$$W^- = C_{\mathcal{M},w} + \sum_{i=1}^{\xi_{\mathcal{M},w}} \hat{\beta}^{(i)}_{\mathcal{M},w} G^{(i)}_{\mathcal{M},w}.$$

Rearranging (13) and considering $\mathcal{M}^+_{x|z}$ and $\mathcal{M}^-_{x|z}$ as an over-approximation of X^+ and X^-, respectively, yields

$$\begin{bmatrix} A & B \end{bmatrix} = \left(\mathcal{M}^+_{x|z} - C_{\mathcal{M},w} - \sum_{i=1}^{\xi_{\mathcal{M},w}} \hat{\beta}^{(i)}_{\mathcal{M},w} G^{(i)}_{\mathcal{M},w} \right) \begin{bmatrix} \mathcal{M}^-_{x|z} \\ U^- \end{bmatrix}^\dagger \tag{14}$$

Hence, for all $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{N}_\Sigma$, there exists $\hat{\beta}^{(i)}_{\mathcal{M},w}$, $-1 \leq \hat{\beta}^{(i)}_{\mathcal{M},w} \leq 1$, $i = 1, \dots, \xi_{\mathcal{M},w}$, such that (14) holds. Therefore, for all $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{N}_\Sigma$, it also holds that $\begin{bmatrix} A & B \end{bmatrix} \in \mathcal{M}_\Sigma$ as defined in (7), which concludes the proof. □

Given that we have found a matrix zonotope $\mathcal{M}_\Sigma$ that contains the true system dynamics $\begin{bmatrix}A_{\text{tr}} & B_{\text{tr}}\end{bmatrix} \in \mathcal{M}_\Sigma$, we can utilize it in computing the time update reachable set $\tilde{\mathcal{R}}_k$ in the following theorem.

Theorem 1. *The set $\tilde{\mathcal{R}}_k$ over-approximates the exact reachable set, i.e., $\tilde{\mathcal{R}} \supseteq \mathcal{R}_k$ where*

$$\tilde{\mathcal{R}}_{k+1} = \mathcal{M}_\Sigma(\tilde{\mathcal{R}}_k \times \mathcal{U}_k) + \mathcal{Z}_w, \tag{15}$$

and $\tilde{\mathcal{R}}_0 = \mathcal{X}_0$.

Proof. As $\begin{bmatrix}A_{\text{tr}} & B_{\text{tr}}\end{bmatrix} \in \mathcal{M}_\Sigma$ according to Lemma 1 and starting from the same initial set $\mathcal{X}_0$, it follows that $\tilde{\mathcal{R}}_k \supseteq \mathcal{R}_k$. □

B. Online Estimation Phase using Zonotopes

In this subsection, we present the *online estimation phase*. We are now considering the system (3a) with observations (3b). This phase consists of a time update and a measurement update. In Sec. III-A, we derived the function $f(\cdot)$ for the time update. We next present two approaches to perform the measurement update.

1) Approach 1 - Reverse-Mapping: For this approach, we aim to determine the mapping of an observation $y^i(k)$ to the corresponding state-space region. Similar to Lemma 1, we construct a zonotope $\mathcal{Z}_{x|y^i(k)} \subset \mathbb{R}^n$ that contains all *possible* $x \in \mathbb{R}^n$ given $y^i(k)$, C^i and bounded noise $v^i(k) \in \mathcal{Z}_{v,i}$ satisfying (3b), for each i.

Proposition 1. *Assume $\|x\|_2 \leq K$. Given a measurement $y^i(k)$ with noise $v^i(k) \in \mathcal{Z}_{v,i} = \langle c_{v,i}, G_{v,i}\rangle$ satisfying (3b), the possible states x that correspond to this measurement are contained within the zonotope $\mathcal{Z}_{x|y^i} = \langle c_{x|y^i}, G_{x|y^i}\rangle$, where*

$$\begin{aligned} c_{x|y^i} &= V_1 \Sigma^{-1}_{r^i \times r^i} P_1^\top \big(y^i(k) - c_{v,i}\big), \\ G_{x|y^i} &= \begin{bmatrix} V_1 \Sigma^{-1}_{r^i \times r^i} P_1^\top G_{v,i} & V_2 M \end{bmatrix}, \end{aligned} \tag{16}$$

for all $M \geq K$, with P_1, V_1, Σ and V_2 obtained from the SVD of C^i as in (12).

Proof. The proof follows immediately from Lemma 1. □

Remark 1. *In our case, $\mathcal{Z}_{x|y^i(k)}$ will eventually be intersected with $\tilde{\mathcal{R}}_k = \langle \tilde{c}_k, \tilde{G}_k\rangle$. It is therefore sufficient to set $M \geq \texttt{radius}(\tilde{\mathcal{R}}_k) + \|V_2^\top \tilde{c}_k\|_2$ instead of the more conservative $M \geq \|x\|_2$, where $\texttt{radius}(\tilde{\mathcal{R}}_k)$ returns the radius of a minimal hyper-sphere containing $\tilde{\mathcal{R}}_k$ [20].*

Having determined the sets $\mathcal{Z}_{x|y^i(k)}$ for all $i \in \{1,\dots,q\}$, we can compute the measurement updated set $\hat{\mathcal{R}}_k$ given the predicted set $\tilde{\mathcal{R}}_k$ and each measurement set $\mathcal{Z}_{x|y^i(k)}$ as

$$\hat{\mathcal{R}}_k = \tilde{\mathcal{R}}_k \cap_{i=1}^q \mathcal{Z}_{x|y^i(k)}, \tag{17}$$

which can be performed using the standard intersection operations presented in [11], [20].

2) Approach 2 - Implicit Intersection: Contrary to Approach 1, here, we do not *explicitly* determine the sets $\mathcal{Z}_{x|y^i(k)}$. Instead, $\hat{\mathcal{R}}_k$ is determined directly from the set $\tilde{\mathcal{R}}_k$, the measurements $y^i(k)$ and some weights λ_k^i for $i \in \{1,\dots,q\}$. We then optimize over the weights to minimize the volume of $\hat{\mathcal{R}}_k$.

Proposition 2. *The intersection of $\tilde{\mathcal{R}}_k = \langle \tilde{c}_k, \tilde{G}_k\rangle$ and the q regions for x corresponding to $y^i(k)$ with noise $v^i(k) \in \mathcal{Z}_{v,i} = \langle c_{v,i}, G_{v,i}\rangle$ satisfying (3b) can be over-approximated by the zonotope $\hat{\mathcal{R}}_k = \langle \hat{c}_k, \hat{G}_k\rangle$ with*

$$\hat{c}_k = \tilde{c}_k + \sum_{i=1}^q \lambda_k^i \Big(y^i(k) - C^i \tilde{c}_k - c_{v,i}\Big), \tag{18}$$

$$\hat{G}_k = \Big[(I - \sum_{i=1}^q \lambda_k^i C^i)\tilde{G}_k \quad -\lambda_k^1 G_{v,1} \quad \dots \quad -\lambda_k^q G_{v,q}\Big], \tag{19}$$

where $\lambda_k^i \in \mathbb{R}^{n \times p_i}$ for $i \in \{1,\dots,q\}$ are weights.

Proof. The proof is based on [21, Prop.1] but with zonotopes as measurements instead of strips. Let $x \in \tilde{\mathcal{R}}_k \cap \mathcal{Z}_{x|y^1} \cap \dots \cap \mathcal{Z}_{x|y^q}$. Then there exists a z such that $x = \tilde{c}_k + \tilde{G}_k z$. Adding and subtracting $\sum_{i=1}^q \lambda_k^i C^i \tilde{G}_k z$ yields

$$x = \tilde{c}_k + \sum_{i=1}^q \lambda_k^i C^i \tilde{G}_k z + (I - \sum_{i=1}^q \lambda_k^i C^i)\tilde{G}_k z. \tag{20}$$

From (3b), we obtain $C^i x = y^i - c_{v,i} - G_{v,i} d^i$. Using $x = \tilde{c}_k + \tilde{G}_k z$ yields $C^i \tilde{G}_k z = y^i(k) - C^i \tilde{c}_k - c_{v,i} - G_{v,i} d^i$, which we insert into (20) to obtain

$$\begin{aligned} x &= \tilde{c}_k + \sum_{i=1}^q \lambda_k^i \Big(y^i(k) - C^i \tilde{c}_k - c_{v,i} - G_{v,i} d^i\Big) \\ &\quad + \Big(I - \sum_{i=1}^q \lambda_k^i C^i\Big)\tilde{G}_k z, \\ &= \underbrace{\Big[(I - \sum_{i=1}^q \lambda_k^i C^i)\tilde{G}_k \quad -\lambda_k^1 G_{v,1} \quad \dots \quad -\lambda_k^q G_{v,q}\Big]}_{\hat{G}_k} \underbrace{\begin{bmatrix} z \\ d^1 \\ \vdots \\ d^q \end{bmatrix}}_{z^b} \\ &\quad \underbrace{+ \tilde{c}_k + \sum_{i=1}^q \lambda_k^i (y^i(k) - C^i \tilde{c}_k - c_{v,i})}_{\hat{c}_k} = \hat{G}_k z^b + \hat{c}_k. \end{aligned}$$

Note that $z^b \in [-1, 1]$ since $d^i \in [-1, 1]$ and $z \in [-1, 1]$. $\hat{R}_k$ adheres to Definition 1 with center $\hat{c}_k$ and generators $\hat{G}_k$. □

As in [11], we find the optimal weights $\lambda_k^i \in \mathbb{R}^{n \times p_i}$ from

$$\bar{\lambda}_k^* = \arg\min_{\bar{\lambda}_k} \|\hat{G}_k\|_F^2, \tag{21}$$

where $\bar{\lambda}_k = [\lambda_k^1 \dots \lambda_k^q]$.

The online estimation phase is illustrated in the block diagram of Fig. 1. The detailed estimation phase is presented in Algorithm 1. The function *measZon()* executes Proposition 1, and *optZon()* Proposition 2. The function *reduce*($\tilde{\mathcal{R}}_{k+1}$) reduces the order of $\tilde{\mathcal{R}}_{k+1}$ using the method proposed in [22], which ensures the number of generators in $\tilde{\mathcal{R}}_{k+1}$ remains relatively low, avoiding potential tractability issues after multiple iterations.

C. Online Estimation Phase using Constrained Zonotopes

When intersecting zonotopes, the result is an over-approximation of the true intersection. However, it is possible

Algorithm 1 *Online Estimation Phase*

$\hat{\mathcal{R}}_0 = \mathcal{X}_0$
$k = 1$
while *True* **do**
  $\tilde{\mathcal{R}}_k = f(\hat{\mathcal{R}}_{k-1}, \langle u(k-1), 0\rangle)$ using (15)
  if *Approach 1* **then**
    foreach $i \in \{1, \dots, q\}$ **do**
      $\mathcal{Z}_{x|y^i(k)} = \textit{measZon}\big(y^i(k), \mathcal{Z}_{v,i}, C^i\big)$ using (16)
    end
    $\hat{\mathcal{R}}_k = \tilde{\mathcal{R}}_k \bigcap_{i=1}^{q} \mathcal{Z}_{x|y^i(k)}$
  if *Approach 2* **then**
    $\langle \hat{c}_k, \hat{G}_k\rangle = \textit{optZon}(\tilde{\mathcal{R}}_k, y(k), C, \mathcal{Z}_v)$
    $\hat{G}_k^*,\ \bar{\lambda}^* \leftarrow$ Solve (21)
    $\hat{\mathcal{R}}_k = \langle \hat{c}_k, \hat{G}_k^*\rangle$
  $\hat{\mathcal{R}}_k = \textit{reduce}(\hat{\mathcal{R}}_k)$ using [22]
  $k \leftarrow k+1$
end

to determine the *exact* intersection of constrained zonotopes.

Definition 4. *(Constrained zonotope [23]) An n-dimensional constrained zonotope is*

$$\mathcal{C} = \{x \in \mathbb{R}^n \mid x = c_{\mathcal{C}} + G_{\mathcal{C}}\beta,\ A_{\mathcal{C}}\beta = b_{\mathcal{C}},\ \|\beta\|_\infty \le 1\}, \tag{22}$$

where $c_{\mathcal{C}} \in \mathbb{R}^n$ is the center, $G_{\mathcal{C}} \in \mathbb{R}^{n\times n_g}$ the generator matrix and $A_{\mathcal{C}} \in \mathbb{R}^{n_c \times n_g}$ and $b_{\mathcal{C}} \in \mathbb{R}^{n_c}$ the constraints. In short, we write $\mathcal{C} = \langle c_{\mathcal{C}}, G_{\mathcal{C}}, A_{\mathcal{C}}, b_{\mathcal{C}}\rangle$.

When using constrained zonotopes, we replace the time and measurement updated sets $\tilde{\mathcal{R}}_k$ and $\hat{\mathcal{R}}_k$ by the constrained zonotopes $\tilde{\mathcal{C}}_k$ and $\hat{\mathcal{C}}_k$, respectively.

1) Approach 1 - Reverse-Mapping: This approach works directly with constrained zonotopes. The sets $\mathcal{Z}_{x|y^i(k)}$ of Proposition 1 are constrained zonotopes with no $A_{\mathcal{C}}, b_{\mathcal{C}}$ constraints. The intersection in (17) becomes $\hat{\mathcal{C}}_k = \tilde{\mathcal{C}}_k \cap_{i=1}^{q} \mathcal{Z}_{x|y^i(k)}$ which can be performed as described in [23].

2) Approach 2 - Implicit Intersection: We adapt Proposition 2 to use constrained zonotopes.

Proposition 3. *The intersection of $\tilde{\mathcal{C}}_k = \langle \tilde{c}_k, \tilde{G}_k, \tilde{A}_k, \tilde{b}_k\rangle$ and q regions for x corresponding to $y^i(k)$ as in (3b) can be described by the constrained zonotope $\hat{\mathcal{C}}_k = \langle \hat{c}_k, \hat{G}_k, \hat{A}_k, \hat{b}_k\rangle$ with weights $\lambda_k^i \in \mathbb{R}^{n\times p_i}$ for $i \in \{1, \dots, q\}$ where*

$$\hat{c}_k = \tilde{c}_k + \sum_{i=1}^{q} \lambda_k^i\big(y^i(k) - C^i\tilde{c}_k - c_{v,i}\big),$$

$$\hat{G}_k = \begin{bmatrix}(I - \sum_{i=1}^{q}\lambda_k^i C^i)\tilde{G}_k & -\lambda_k^1 G_{v,1} & \dots & -\lambda_k^q G_{v,q}\end{bmatrix}, \tag{23}$$

$$\hat{A}_k = \begin{bmatrix}\tilde{A}_k & 0 & \dots & 0\\ C^1\tilde{G}_k & G_{v,1} & \dots & 0\\ \vdots & & \ddots & \\ C^q\tilde{G}_k & 0 & \dots & G_{v,q}\end{bmatrix}, \tag{24}$$

$$\hat{b}_k = \begin{bmatrix}\tilde{b}_k\\ y^1(k) - C^1 c_k - c_{v,1}\\ \vdots\\ y^q(k) - C^q c_k - c_{v,q}\end{bmatrix}. \tag{25}$$

Proof. We follow a similar approach to [24, Thm. 6.3] and [23], but extend the proof by defining measurement sets as zonotopes instead of strips. $\mathcal{Z}_{x|y^i}$ refers to $\mathcal{Z}_{x|y^i(k)}$ unless specified otherwise. Let $x_k \in \tilde{\mathcal{C}}_k \cap \mathcal{Z}_{x|y^1} \cap \dots \cap \mathcal{Z}_{x|y^q}$, then there exists a $z_k \in [-1, 1]$ such that

$$x_k = \tilde{c}_k + \tilde{G}_k z_k, \qquad \tilde{A}_k z_k = \tilde{b}_k. \tag{26}$$

Using (3b) and the measurement noise $\langle c_{v,i}, G_{v,i}\rangle$, we write

$$C^i x = y^i(k) - c_{v,i} - G_{v,i} d^i, \tag{27}$$

where $d^i \in [-1, 1]$. Inserting (26) into (27) yields

$$C^i\tilde{G}_k z_k = y^i(k) - C^i\tilde{c}_k - c_{v,i} - G_{v,i}d^i, \tag{28}$$

which, combined with (26), yields

$$\underbrace{\begin{bmatrix}\tilde{A}_k & 0 & \dots & 0\\ C^1 G_k & G_{v,1} & \dots & 0\\ \vdots & & \ddots & \\ C^q G_k & 0 & \dots & G_{v,q}\end{bmatrix}}_{\hat{A}_k}\underbrace{\begin{bmatrix}z_k\\ d^1\\ \vdots\\ d^q\end{bmatrix}}_{z_b} = \underbrace{\begin{bmatrix}\tilde{b}_k\\ y^1(k) - C^1 c_k - c_{v,1}\\ \vdots\\ y^q(k) - C^q c_k - c_{v,q}\end{bmatrix}}_{\hat{b}_k}. \tag{29}$$

Adding and subtracting $\sum_{i=1}^{q}\lambda_{i,k}C^i\tilde{G}_k z_k$ to (26) yields

$$x_k = \tilde{c}_k + \sum_{i=1}^{q}\lambda_k^i C^i\tilde{G}_k z_k + (I - \sum_{i=1}^{q}\lambda_k^i C^i)\tilde{G}_k z_k. \tag{30}$$

If we now insert (28) into (30), we obtain

$$x = \underbrace{\begin{bmatrix}(I - \sum_{i=1}^{q}\lambda_k^i C^i)\tilde{G}_k & -\lambda_k^1 G_{v,1} & \dots & -\lambda_k^{m_i} G_{v,q}\end{bmatrix}}_{\hat{G}_k} z_b$$

$$+ \underbrace{\hat{c}_{k-1} + \sum_{i=1}^{q}\lambda_k^j\big(y^i(k) - C^i\tilde{c}_k - c_{v,i}\big)}_{\hat{c}_k} = \hat{G}_k z_b + \hat{c}_k.$$

Hence, $x(k) \in \hat{\mathcal{C}}_k$ and $(\tilde{\mathcal{C}} \cap \mathcal{Z}_{x|y^1} \cap \dots \cap \mathcal{Z}_{x|y^q}) \subseteq \hat{\mathcal{C}}_k$. Conversely, let $x(k) \in \hat{\mathcal{C}}_k$. Then, there exists a z_b such that (22) in Definition 4 is satisfied. Partitioning z_b into $z_b = [z_k, d^1 \dots, d^q]^T$, it follows that we can construct a constrained zonotope $\tilde{\mathcal{C}}_k = \{\tilde{c}_k, \tilde{G}_k, \tilde{A}_k, \tilde{b}_k\}$ given that $\|z_k\|_\infty \le 1$. Thus, $x(k) \in \tilde{\mathcal{C}}$. Similarly, we can get the constraints in (27). Inserting (26) in (28) results in obtaining all the equations in (27). Therefore, $x(k) \in \mathcal{Z}_{x|y^i(k)}$, $\forall i \in \{1, \dots, q\}$. Thus, $x(k) \in (\tilde{\mathcal{C}}_k \cap \mathcal{Z}_{x|y^1} \cap \dots \cap \mathcal{Z}_{x|y^q})$ and $\hat{\mathcal{C}}_k \subseteq (\tilde{\mathcal{C}}_k \cap \mathcal{Z}_{x|y^1} \cap \dots \cap \mathcal{Z}_{x|y^q})$, which concludes the proof. □

IV. Evaluation

We evaluate our method by considering an input-driven variant of the rotating target described in [11]. We set

$$A_{\text{tr}} = \begin{bmatrix}0.9455 & -0.2426\\ 0.2486 & 0.9455\end{bmatrix}, \qquad B_{\text{tr}} = \begin{bmatrix}0.1\\ 0\end{bmatrix} \tag{31}$$

with $q = 3$ measurements parameterized as follows

$$C^1 = \begin{bmatrix}1 & 0.4\end{bmatrix}, C^2 = \begin{bmatrix}0.9 & -1.2\end{bmatrix}, C^3 = \begin{bmatrix}-0.8 & 0.2 \\ 0 & 0.7\end{bmatrix},$$

$$\mathcal{Z}_{v,1} = \langle 0, 1\rangle, \mathcal{Z}_{v,2} = \langle 0, 1\rangle, \mathcal{Z}_{v,3} = \langle [0 \;\; 0]^\top, I_2\rangle.$$

The noise signals are characterized by the zonotopes $\mathcal{Z}_\gamma = \langle [0 \;\; 0]^\top, 0.02I_2\rangle$ and $\mathcal{Z}_w = \langle [0 \;\; 0]^\top, 0.02I_2\rangle$. We run the *offline learning phase* with $T = 500$ and inputs sampled uniformly from the set $\mathcal{U} = \langle 0, 10\rangle$. The noise signals $v^i(k)$, $w(k)$ and $\gamma(k)$ are sampled uniformly from their respective zonotope sets using the command *randPoint*$(\mathcal{Z})$ as described in [20].

After learning $f(\cdot)$, we run the *online estimation phase*. The initial state set is $\mathcal{X}_0 = \langle [0 \;\; 0]^\top, 15I_2\rangle$ and the true initial state is $x(0) = \begin{bmatrix}-10 & 10\end{bmatrix}^\top$. Once again, we sample the inputs uniformly from $\mathcal{U}$. We evaluate both the zonotope and constrained zonotope methods, each time using either of the two proposed measurement update approaches. Fig. 2a shows the bounds of $\hat{\mathcal{R}}_k$ in the x_1 state dimension for both approaches. Fig. 2b shows the equivalent results when our method uses constrained zonotopes. As expected, $x(k)$ is always contained within $\hat{\mathcal{R}}_k$ (or $\hat{\mathcal{C}}_k$) at each time step. Although both measurement update approaches yield similar set sizes on average, the set evolution of Approach 2 is comparatively smoother.

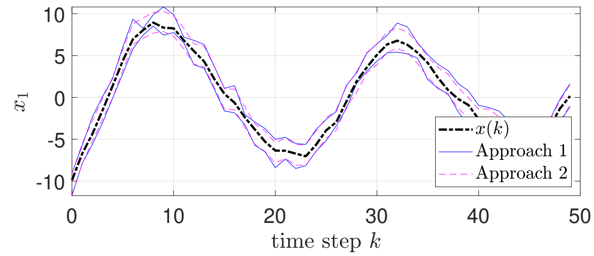
Furthermore, we compare our results with *N4SID* subspace identification [25] combined with a Kalman filter (KF). In Fig. 3, we show the sets $\hat{\mathcal{R}}_k$ and $\hat{\mathcal{C}}_k$, using either measurement update approach, using zonotopes or constrained zonotopes. We also show the ellipse corresponding to the 3σ uncertainty bound of the KF estimate, indicating that our estimator provides state sets comparable in size to that of the KF. We should mention that KF bounds come without any guarantees.

Referring to both Fig. 2 and Fig. 3, it is clear that the constrained zonotopes yield smaller state sets at each time step. However, this comes at the cost of increased computational load. Running our simulations on a Dell laptop with an 8-core i5-8365U processor at 1.6GHz, the average computation time per iteration for Approach 1 increased from 0.656sec to 1.267sec. when using constrained zonotopes; for Approach 2, the corresponding times were 0.221sec and 0.971sec, respectively. For all our approaches, we observed that reducing the order of the sets to 5, which reduces the number of generators in $\hat{\mathcal{R}}$ (or $\hat{\mathcal{C}}$), was critical to keep the computational load low.

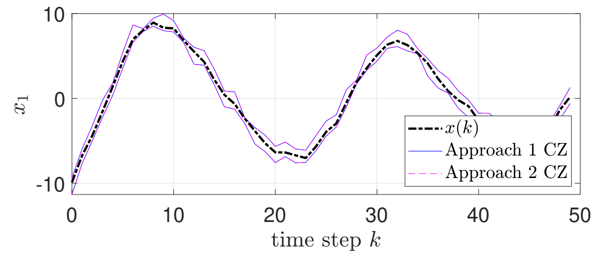
V. Conclusions and Recommendations

In this paper, we introduced a novel zonotope-based method to perform set-based state estimation with set containment guarantees using a data-driven set propagation function. We presented an approach to compute the set of model that is consistent with the data and noise bounds given input-output data. Then, we presented two approaches to perform the measurement update which merges the time updated state set with the observed measurements. We extended our method to use constrained zonotopes, which yielded smaller state sets at the cost of increased computational load. Our results show state sets comparable in size to the 3σ uncertainty bounds obtained when running *N4SID* subspace identification and a Kalman filter, but with the added feature of set-containment guarantees and without requiring any knowledge of the statistical properties of the noise.

Future work includes evaluating our proposed estimator on real-world examples as well as gaining more insight into the limitations of our method when applied to more complex dynamical systems. Additionally, improving the zonotope intersection operation to lessen the degree of over-approximation of the resultant state set would yield tighter state set estimates at each time step.



(a) Using zonotopes showing bounds of $\hat{\mathcal{R}}_k$ in x_1



(b) Using constrained zonotopes showing bounds of $\hat{\mathcal{C}}_k$ in x_1

Fig. 2: Bounds of the set $\hat{\mathcal{R}}_k$ in (a), and $\hat{\mathcal{C}}_k$ in (b), projected onto the first state dimension x_1 of $x(k)$ using measurement update approaches 1 and 2.

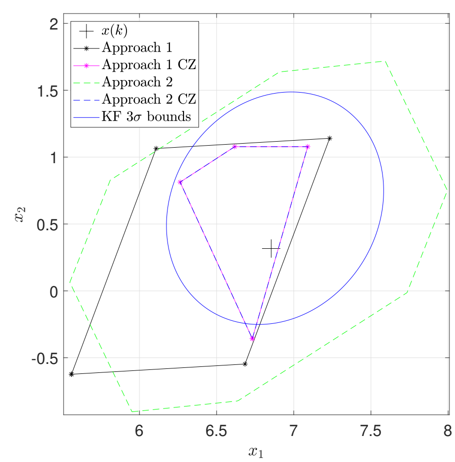


Fig. 3: Sets $\hat{\mathcal{R}}_k$ using measurement update approaches 1 and 2, and the equivalent sets $\hat{\mathcal{C}}_k$ using constrained zonotopes (*CZ*), compared to the KF's 3σ confidence bounds.

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