SIR – an efficient root solver for systems of nonlinear equations

- a semi-implicit approach with simple geometrical interpretation

Jan Scheffel, e-mail: jans@kth.se (8 July 2010)
Fusion Plasma Physics, Royal Institute of Technology, Stockholm, Sweden

The semi-implicit root solver (SIR) is an efficient iterative method for solving nonlinear equations and systems of nonlinear equations. Convergence is quasi-monotonous and approaches second order in the proximity of the real roots. The algorithm, being straight-forward to code, is related to semi-implicit methods earlier being applied to partial differential equations. The Newton-Raphson and Newton methods are limiting cases of the method. Convergence is controlled by the semi-implicit parameters. In contrast to Newton's method with linesearch, SIR does not end on local, non-zero, minima.

When applied to *a single equation*, efficient global convergence and convergence to a nearby root makes the method attractive in comparison with methods as those of Newton-Raphson and van Wijngaarden-Dekker-Brent.

When applied to a large test set of *systems of equations*, a novel type of convergence diagrams show the robustness, efficiency and simplicity of the method as compared to Newton's method using linesearch.

An extensive description and tests of the method are given in Ref. [1]. Here a brief account of the main characteristics of the method is given.

1) For a **single equation**, convergence to a nearby root is guaranteed. Thus the method is ideal for successively finding all roots to an equation. The risk for overshooting to a non-nearby root is eliminated through a sub-iteration procedure that produces strictly monotonic convergence (see Fig. 1, where "*i*" denotes i:th iteration). The equation $x = \varphi(x)$ is transformed to the equation $x = \Phi(x;\alpha)$, which has exactly the same roots, but produces monotonic convergence. As the root is approached, the slope of $x = \Phi(x;\alpha)$ is adjusted close to zero, using the α parameter, and convergence becomes rapid (second order). If the iterations would lead away from the root, "mirroring" of the $x = \Phi(x;\alpha)$ function is applied (Fig.2).

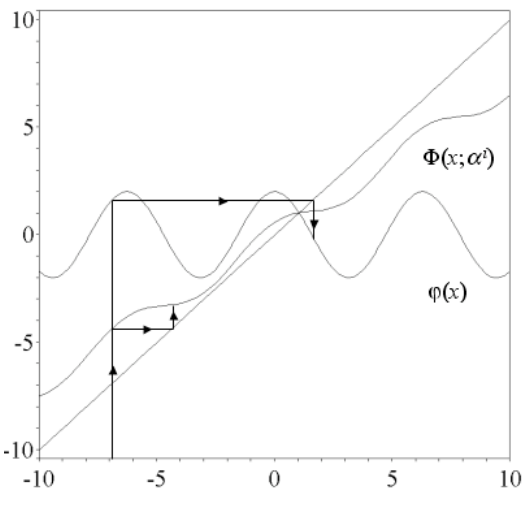

Fig. 1

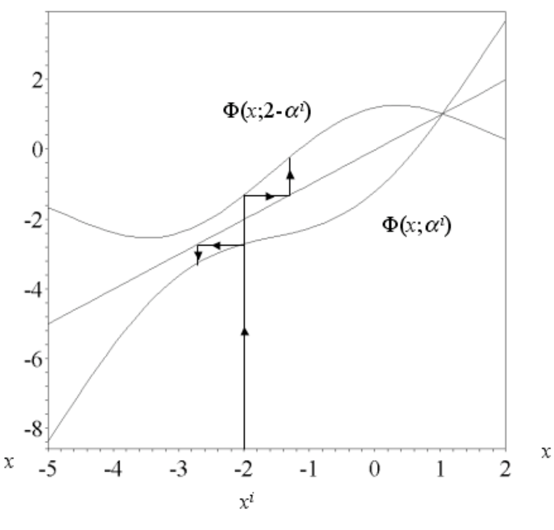

Fig. 2

2) For solving **systems of equations**, our fixed point method can be generalized to higher dimensions. Since the approximate root locations cannot be determined by root bracketing, the mirroring procedure is omitted. In essence, the method works as follows.

Given the system $\mathbf{x} = \varphi(\mathbf{x})$ to be solved, the system $\mathbf{x} = \Phi(\mathbf{x};\mathbf{A}) \equiv \mathbf{A}(\mathbf{x} - \varphi(\mathbf{x})) + \varphi(\mathbf{x})$ is constructed. It has the same solutions as the original system, but contains free parameters in the form of the **A** matrix components α_{mn}. The parameters are chosen to govern $\partial\Phi_m / \partial x_n$, the gradients of the hypersurfaces given by Φ_m. Adjusting these parameters, global, quasi-monotonous and superlinear convergence can be attained.

The method efficiently avoids the tendency of Newton's method to take too great strides, and it *avoids the risk associated with Newton's method using linesearch to end up on a local non-zero minimum* of $f^2(\mathbf{x}) = (\mathbf{x} - \varphi(\mathbf{x}))^2$. Since Newton's method is a limiting case of the present method, namely when $\partial\Phi_m / \partial x_n \to 0$, second order convergence is usually approached after some iteration steps. The formal relationship to Newton's method leads to approximately similar numerical work for systems of equations.

SIR can be shown to provide convergence properties beyond those of Newton's method using linesearch for a number of cases. Below we show a 2D example, using a novel type of convergence diagrams, illustrating the sensitivity to the choice of starting points. The computations use 61 by 61 uniformly distributed starting points (x_1^0, x_2^0) in the domain $-5 \le x_i^0 \le 5$, $i = 1,2$ for the functions to be zeroed; $f_1 = x_1 - \cos(x_2)$ and $f_2 = x_2 - 3\cos(x_1)$. The function $f^2/2 = (f_1^2 + f_2^2)/2$ is shown for Fig. 2 a) SIR and b) Newton's method with linesearch. The root position x* is indicated. SIR usually converges within a few iterations, whereas Newton's method using linesearch often ends on a local minimum (orange colour).

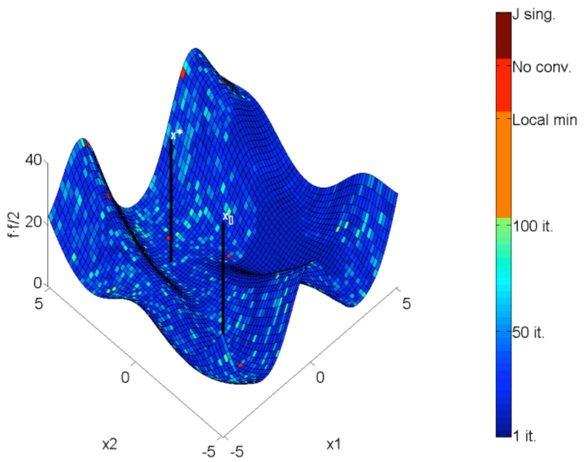

Fig. 2 a)

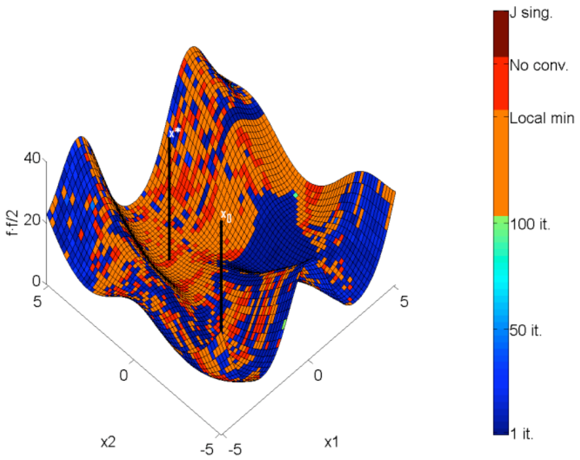

Fig. 2 b)

[1] J. Scheffel, C. Håkansson, *Solution of systems of equations – a semi-implicit approach*, Applied Numerical Mathematics (2009), doi:10.1016/j.apnum.2009.05.002