6 Optimal Model Order Reduction in the Hankel Norm

In this lecture, we discuss optimal model reduction in the Hankel norm. There is no known computationally efficient solution to the optimal model reduction problem in the H_∞-norm. Since there is a connection between the Hankel norm and the H_∞-norm, the presented method is still interesting to us, and gives a lot of insight. The analysis also establishes a link to SVD.

6.1 The Hankel Norm and the Hankel Operator

The Hankel norm of a system $G = (A, B, C, D) \in H_\infty$ is defined by

$$\|G\|_H^2 := \sup_{u \in L_2(\infty,0]} \frac{\int_0^\infty y(t)^2 dt}{\int_{-\infty}^0 u(t)^2 dt}, \quad \text{where} \quad y(t) = \int_{-\infty}^0 Ce^{A(t-s)}Bu(s)ds.$$

The Hankel norm tells how much energy can be transferred from past inputs into future outputs through the system G. In Exercise 3.4 b), we showed that $\|G\|_H = \sqrt{\lambda_{\max}(PQ)} = \sigma_1$. One defines the Hankel operator Γ_G of the system G by,

$$\Gamma_G : L_2(-\infty, 0] \to L_2[0, \infty) : \quad (\Gamma_G u)(t) = \int_{-\infty}^0 Ce^{A(t-s)}Bu(s)ds, \quad t > 0.$$

The induced norm of Γ_G is equal to the Hankel norm of G, $\|\Gamma_G\| = \|G\|_H$.

A nice property of Γ_G is that it has finite rank, with the rank being equal to the minimal number of states required to realize the input-output map G,

$$\text{Rank } \Gamma_G = n.$$

Furthermore, one can make an SVD of Γ_G, with the dyadic expansion

$$(\Gamma_G u)(t) = \sum_{i=1}^n \sigma_i u_i(t)(v_i, u)_{L_2(-\infty,0]}, \tag{6.1}$$

where σ_i are the Hankel singular values of G, and the singular vectors are $v_i \in L_2(-\infty, 0]$, $u_i \in L_2[0, \infty)$. Maybe one would think it is more natural to consider the operator G directly,

$$G : L_2(-\infty, \infty) \to L_2(-\infty, \infty) : \quad (Gu)(t) = \int_{-\infty}^t Ce^{A(t-s)}Bu(s)ds + Du(t).$$

This operator is not of finite rank, however, and has no finite dyadic expansion such as (6.1). This make further analysis difficult.

Since one can make an SVD of Γ_G, one can essentially apply the Schmidt-Mirsky theorem to prove that

$$\|\Gamma_G - \Gamma_{G_r}\| = \|G - G_r\|_H \geq \sigma_{r+1}, \tag{6.2}$$

for all G_r of order r. From the definition of the Hankel norm it should also be clear that $\|G\|_H \leq \|G\|_\infty$ for all $G \in H_\infty$. Hence, the bound (6.2) also holds for the H_∞-norm, as has already been stated in earlier lectures.

Relation between the Hankel norm and the L_∞-norm

To proceed, we need to introduce more system spaces, namely L_∞, H_∞^-, and $H_\infty^-(r)$. A transfer function $G(s)$ belongs to L_∞ if, and only if,

$$\|G\|_\infty := \sup_\omega \|G(j\omega)\|$$

is finite. Hence $G(s)$ may have poles in both $\mathbb{C}_+$ and $\mathbb{C}_-$, but not on the imaginary axis. A transfer function $G(s)$ belongs to H_∞^- if, and only if, $G(-s) \in H_\infty$. Hence, unstable systems with no stable poles belong to H_∞^-. Such systems are called *anti-stable*. A transfer function belongs to $H_\infty^-(r)$ if it belongs to L_∞ and it has at most r poles in the complex left half plane, $\mathbb{C}_-$.

We can now state the following fundamental theorems.

Theorem 12 (Nehari)**.** *Suppose that $G \in H_\infty$, and $F \in H_\infty^-$. Then $G - F \in L_\infty$, and*

$$\min_{F \in H_\infty^-} \|G - F\|_\infty = \|G\|_H (= \sigma_1).$$

Hence, one interpretation of $\|G\|_H$ is that it is the minimum distance between G and an anti-stable system, measured in the L_∞-norm.

The following important extension to Nehari's theorem is available.

Theorem 13 (Adamjan-Arov-Krein)**.** *Suppose that $G \in H_\infty$ and $Q \in H_\infty^-(r)$. Then $G - Q \in L_\infty$, and*

$$\min_{Q \in H_\infty^-(r)} \|G - Q\|_\infty = \sigma_{r+1}, \tag{6.3}$$

where σ_{r+1} is the $(r+1)$-th largest Hankel singular value of G.

Denote an optimal $Q \in H_\infty^-(r)$ by Q^*, i.e., $\|G - Q^*\|_\infty = \sigma_{r+1}$. A stable/anti-stable decomposition of Q^* can be performed such that $Q^* = G_r + F$, where $G_r \in H_\infty$, $\deg G_r \leq r$, and $F \in H_\infty^-$. It follows that

$$\|G - G_r\|_\infty = \|G - Q^* + F\|_\infty \leq \|G - Q^*\|_\infty + \|F\|_\infty = \sigma_{r+1} + \|F\|_\infty. \tag{6.4}$$

If we can show that the L_∞-norm of the unstable term F is small, then the stable part of Q^* can be a good candidate for H_∞ model reduction. A bound on $\|F\|_\infty$ is derived in Section 6.3.

Note that the stable part G_r of Q^* solves the optimal Hankel norm approximation problem, since

$$\|G - G_r\|_H = \min_{F \in H_\infty^-} \|G - G_r - F\|_\infty = \min_{Q \in H_\infty^-(r)} \|G - Q\|_\infty = \sigma_{r+1},$$

and we know that a lower bound of the problem is σ_{r+1}, see (6.2). Note also that the direct term D_r of G_r is not determined by Hankel norm approximation since it does not appear in the Hankel operator.

6.2 State-Space Formulas for Construction of Q^*

In this section, we give state-space formulas for the computation of an optimal Hankel approximation Q^* whose existence is guaranteed by Theorem 13.

Suppose the system $G = (A, B, C, D) \in H_\infty$ is of order n, and that it is square ($m = p$). The formulas for non-square systems are slightly more complicated, and are given in *Linear Robust Control*. Assume the realization of G is such that the Gramians take the form

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & \sigma_{r+1} I_l \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & \sigma_{r+1} I_l \end{bmatrix},$$

where l is the multiplicity of the singular value σ_{r+1}. One can choose a permuted balanced realization of G, for example. Conformably to P, Q, partition the state-space realization of G,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

One can prove (Exercise 6.3) that under the given assumptions, there is a unitary matrix $U \in \mathbb{R}^{p \times m}$, $U^T U = I$, such that

$$B_2 = -C_2^T U. \tag{6.5}$$

Also define

$$E_1 := Q_1 P_1 - \sigma_{r+1}^2 I.$$

A realization of the optimal approximation Q^* is now given by

$$\begin{aligned}
\hat{A} &= E_1^{-1}(\sigma_{r+1}^2 A_{11}^T + Q_1 A_{11} P_1 - \sigma_{r+1} C_1^T U B_1^T) \\
\hat{B} &= E_1^{-1}(Q_1 B_1 + \sigma_{r+1} C_1^T U) \\
\hat{C} &= C_1 P_1 + \sigma_{r+1} U B_1^T \\
\hat{D} &= D - \sigma_{r+1} U.
\end{aligned}$$

The transfer function $Q^*(s) = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is of order $n - l$ and belongs to $H_\infty^-(r)$. In fact, it has exactly r stable poles and $n - r - l$ unstable poles. We verify below that $E(s) := G(s) - Q^*(s)$ satisfies

$$E(s)^\sim E(s) = \sigma_{r+1}^2 I,$$

where $E(s)^\sim = E(-s)^T$, and thus $\|E\|_\infty = \sigma_{r+1}$. Note that $\|E(j\omega)\| = \sigma_{r+1}$ for all ω.

The given Q^* solves the optimization problem in Theorem 13. This is generally not the unique solution, however. All the solutions (also in the non-square G case) are parameterized in *Linear Robust Control* using a linear fractional transformation.

To verify that $E(s)^\sim E(s) = \sigma_{r+1}^2 I$, we use the following lemma.

Lemma 1. *Let $G = (A, B, C, D)$. If there is a symmetric matrix P such that*

$$\begin{aligned}
PA^T + AP + BB^T &= 0 \\
PC^T + BD^T &= 0 \\
D^T D &= \gamma^2 I,
\end{aligned}$$

then $G(s)^\sim G(s) = \gamma^2 I$ and $G \in L_\infty$. Conversely, if $G = (A, B, C, D)$ is minimal and $G(s)^\sim G(s) = I$, then there is a symmetric matrix P that satisfies the above conditions.

If we realize $E(s) = G(s) - Q^*(s)$ with

$$A_e = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B_e = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C_e = \begin{bmatrix} C & -\hat{C} \end{bmatrix}, \quad D_e = D - \hat{D},$$

the assumptions of Lemma 1 are satisfied using

$$P_e = \begin{bmatrix} P_1 & 0 & I \\ 0 & \sigma_{r+1} I_l & 0 \\ I & 0 & E_1^{-1} Q_1 \end{bmatrix}$$

as P, and $\gamma = \sigma_{r+1}$.

6.3 H_∞ Error Bounds

In this section, we make the standing assumption that the Hankel singular values of G are distinct. This simplifies the notation. The results can be strengthened if some singular values are identical, just as in balanced truncation.

Using the error bound for balanced truncation, the H_∞-norm of G can be bounded using the Hankel singular values. If Theorem 6 is applied and all states are truncated, we obtain

$$\|G - G(\infty)\|_\infty \leq 2 \sum_{i=1}^{n} \sigma_i.$$

Optimal Hankel norm approximation can be used to improve this general bound.

If we construct the optimal Hankel approximation G_{n-1}, we obtain $\|G - G_{n-1}\|_\infty = \sigma_n$. The Gramians of the system G_{n-1} are given by $P_1 E_1$ and $E_1^{-1} Q_1$, and hence the Hankel singular values of G_{n-1} are $\sqrt{\lambda_i(P_1 E_1 E_1^{-1} Q_1)} = \sqrt{\lambda_i(P_1 Q_1)} = \sigma_i(G)$, $i = 1, \ldots, n-1$ (the remaining Hankel singular values). We can repeat this procedure on G_{n-1}, and remove Hankel singular values one by one. In the end only a direct term $\tilde{D} = G_0$ remains. Applying the triangle inequality, we have

$$\|G - \tilde{D}\|_\infty = \|(G - G_{n-1}) + (G_{n-1} - G_{n-2}) + \ldots + (G_1 - \tilde{D})\|_\infty \leq \sum_{i=1}^{n} \sigma_i. \tag{6.6}$$

Hence, there always exists a constant $\tilde{D}$ that will shift the Nyquist diagram of G so that it is contained in a circle of radius equal to the sum of the distinct Hankel singular values. If we use $G(\infty)$ as the direct term, then we need to make the radius a factor two larger, in general.

Let us now return to the bound (6.4). We want an a priori bound on $\|F\|_\infty$ since that yields a bound on $\|G - G_r\|_\infty$. We have that $F \in H_\infty^-$ is of order $n - r - 1$ (remember the multiplicity $l = 1$). Furthermore, $F^\sim \in H_\infty$ and $\|F\|_\infty = \|F^\sim\|_\infty$. The following lemma can be derived.

Lemma 2. *Assume that $\sigma_i(G)$ are distinct, and let the optimal approximation $Q \in H_\infty^-(r)$ of G be $Q^* = G_r + F$, such that $G_r, F^\sim \in H_\infty$. Then*

$$\sigma_i(F^\sim) \leq \sigma_{i+r+1}(G), \quad i = 1, \ldots, n - r - 1.$$

Using (6.6) and Lemma 2 on $F^\sim$, there is a $\tilde{D}$ such that $\|F - \tilde{D}\|_\infty \leq \sum_{i=r+2}^{n} \sigma_i(G)$. It follows that this $\tilde{D}$ yields

$$\|G - G_r - \tilde{D}\|_\infty = \|G - G_r - F + F - \tilde{D}\|_\infty \leq \|G - Q^*\|_\infty + \|F - \tilde{D}\|_\infty = \sigma_{r+1} + \|F - \tilde{D}\|_\infty \leq \sum_{i=r+1}^{n} \sigma_i.$$

We sum the results up in the following theorem.

Theorem 14. *Suppose $G \in H_\infty$ is of order n, and has distinct Hankel singular values σ_i. Then for all approximations $G_r \in H_\infty$ of order r, it holds that*

$$\sigma_{r+1} \leq \|G - G_r\|_H \leq \|G - G_r\|_\infty.$$

Furthermore, there exists $G_r \in H_\infty$ of order r, given by the stable part of the transfer function $Q^* = (\hat{A}, \hat{B}, \hat{C}, \hat{D}) \in H_\infty^-(r)$, and $\tilde{D}$ such that

$$\|G - G_r\|_H = \|\Gamma_G - \Gamma_{G_r}\| = \sigma_{r+1}$$
$$\|G - G_r - \tilde{D}\|_\infty \leq \sum_{i=r+1}^{n} \sigma_i.$$

In particular, when $r = n - 1$, a solution to the optimal H_∞-norm approximation problem is obtained.

6.4 Suggested Reading

Chapter 10 in *Linear Robust Control* gives a thorough and readable presentation of Hankel norm approximation. All the stated results above are proved there.

6.5 Exercises

EXERCISE 6.1 (Antoulas [2004])

Let the model G be given by

$$G(s) = \frac{-s+1}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}.$$

Compute optimal Hankel norm approximations G_r of G, and the corresponding Q^* for $r = 3$ and $r = 5$. Also plot the Bode diagrams of the approximation errors $G - Q^*$ and $G - G_r$, and compute the L_∞/H_∞-norm of the errors.

EXERCISE 6.2

Derive a state-space algorithm that performs a stable/anti-stable decomposition of state-space models $G = (A, B, C, D) \in L_\infty$.

EXERCISE 6.3

Prove that there is always a unitary U that satisfies (6.5). (Hint: Show first that $B_2 B_2^T = C_2^T C_2$.)

EXERCISE 6.4

Prove that all transfer functions $G \in H_\infty$ of degree n (with distinct Hankel singular values) can be expanded as

$$G(s) = \tilde{D} + \sigma_1 E_1(s) + \sigma_2 E_2(s) + \ldots + \sigma_n E_n(s),$$

where $E_i \in H_\infty$ are all-pass, i.e., $E_i(s)^\sim E_i(s) = I$, and $G_r(s) = \tilde{D} + \sigma_1 E_1(s) + \ldots + \sigma_r E_r(s)$ is of order r.

EXERCISE 6.5 (Extra)

Prove that the following lower bound on the weighted approximation criterion holds,

$$\min_{G_r \in H_\infty,\, \deg G_r \le r} \|W_o(G - G_r)W_i\|_\infty \ge \sigma_{r+1}([M_o G M_i]_+),$$

where $\sigma_{r+1}([M_o G M_i]_+)$ is computed as follows: Compute the (unstable) spectral factors M_o and M_i,

$$M_o(s)^\sim M_o(s) = W_o(s)^\sim W_o(s), \quad M_i(s) M_i(s)^\sim = W_i(s) W_i(s)^\sim,$$

such that $M_o(s), M_o(s)^{-1}, M_i(s), M_i(s)^{-1}$ have their poles in the the open right half half plane $\mathbb{C}_+$. Here $[P]_+$ denotes the stable part of P.