KUNGL TEKNISKA HÖGSKOLAN

International Master Program in System-on-Chip Design

L4: Binary Decision Diagrams

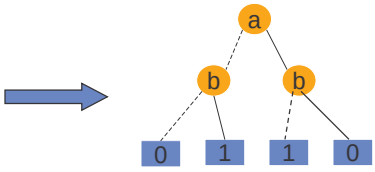
Reading material

- de Micheli pp. 75 - 85
- R. Bryant, “Graph-based algorithms for Boolean function manipulation”, IEEE Transactions on computers, C-35, No 8, August 1986; can be downloaded from
  - http://www.cs.cmu.edu/~bryant/
- CUDD-package manual
  - http://vlsi.colorado.edu/~fabio/

Binary Decision Diagram

- Graph-based representation for Boolean functions f: $B^n \rightarrow B^m$
  - directed acyclic graph
  - one root node, terminal nodes 0, 1
  - each non-terminal node has two children and is labeled by a variable

a	b	f
0	0	0
0	1	1
1	0	1
1	1	0



Binary Decision Diagram

- Mathematical foundation for BDDs is the Shannon decomposition theorem:

$$f(x_1,x_2, ...,x_n) = x'_1 \cdot f|_{x_1=0} + x_1 \cdot f|_{x_1=1}$$

where

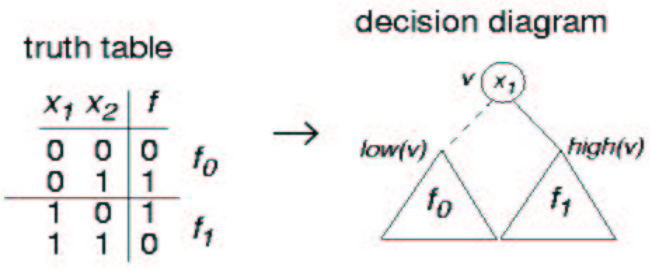
$f|_{x_1=0} := f(0,x_2, ...,x_n)$, $f|_{x_1=1} := f(1,x_2, ...,x_n)$

are subfunctions of f, called co-factors

Binary Decision Diagram

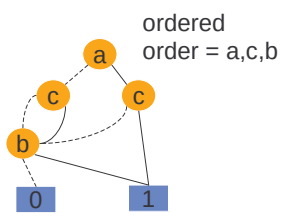
- Let v be a node of a ROBDD labeled by some variable x_i, then
  - low(v) is the subghraph pointed by the edge corresponding to $x_i = 0$
    - this subgraph represents $f|_{x_1=0}$
  - high(v) is the subghraph pointed by the edge corresponding to $x_i = 1$
    - this subgraph represents $f|_{x_1=1}$
  - index(v) = i

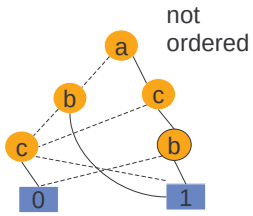
Binary Decision Diagram



Ordering rules

- no variable appears more than once along a path
- in all paths variables appear in the same order





Reduction rules

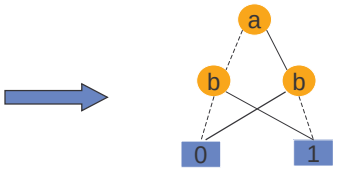
1) no vertex v with low(v) = high(v)

2) no distinct vertices v and u such that the subgraphs rooted by v and u are isomorphic

- these two rules guarantee that each node represents a distinct logic function

Reduced Ordered Binary Decision Diagram (ROBDD)

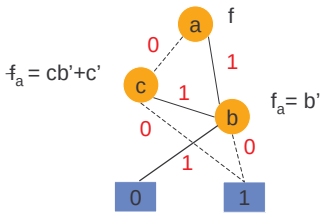
a	b	f
0	0	0
0	1	1
1	0	1
1	1	0



- canonical representation (fixed ordering)
- easy manipulation algorithms
- compact for many practical functions

Properties of ROBDD

- function is given by tracing all the paths to 1 terminal node: f = b'+a'c' = ab'+a'cb'+a'c'



•cubes given by paths are pair-wise disjoint

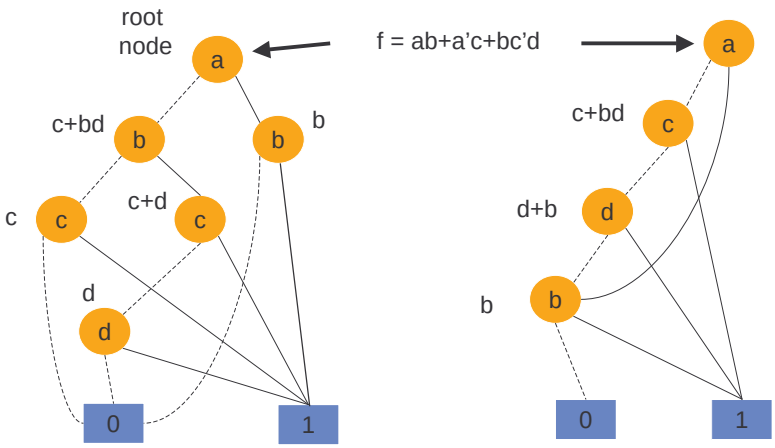
Advantages of ROBDDs

- The number of cubes we represent might be exponential
- But the size of a ROBDD is measured by its nodes, not by its paths
- a ROBDD can represent an exponential number of paths with a linear number of nodes

Problems with ROBDDs

- Some functions have exponential-size ROBDDs
  - multipliers, HWB
- size of a ROBDD depends critically on variable ordering
  - adders - exponential for bad orderings, linear for good orderings
- finding a best variable ordering requires exponential time

Example



Two different orderings, same function

Algorithms for BDD manipulation

- Basic algorithms for BDD manipulation:
  - ***Reduce*** : BDD G reduced to canonical form, complexity O(|G|log |G|)
  - ***Apply*** : performes $f_1 \bullet f_2$ for some operation "•", complexity $O(|G_1| \cdot |G_2|)$
  - ***Compose*** : substitues f_2 in a variable x of f_1, complexity $O(|G_1|^2 \cdot |G_2|)$
  - ***Satisfy-one***: checkes whether $f(x_1,.., x_n)=1$ for some assignment $x_1,.., x_n$, O(n)

A simple reduction algorithm:

1) Merge all identical terminal nodes and appropriately redirect their incoming edges
2) Proceeding from bottom to top:
   - if a node v is found such that low(v) = high(v), remove v and redirect its incoming edges to low(v) (reduction rule 1)
   - if two nodes v and u with index(v) = index(u) are found such that low(v) = low(u) and high(v) = high(u), remove v and redirect its incoming edges to u (reduction rule 2)

Apply

- The procedure *Apply* provides the basic method for creating the BDD representation of a function given by a Boolean expression
- It takes BDDs for functions f_1, f_2 and a binary operation $\bullet$, and produces a BDD for the function $f_1 \bullet f_2$ defined as

$$[f_1 \bullet f_2](x_1,\ldots,x_n) = f_1(x_1,\ldots,x_n) \bullet f_2(x_1,\ldots,x_n)$$

Apply

- Apply can be used to complement a function (compute f ⊕ 1)
- to compute intersection (• = ·)
- to compute union (• = +)
- Note: in cube representations we needed 3 different procedures for that

ITE operator

- If-then-else: ite(f,g,h) equals to g when f is 1 and equals to h otherwise:

$$ite(f,g,h) = f \cdot g + f' \cdot h$$

- Each node is written as a triple: f = (v,g,h) where $g = f_v$ and $h = f_{\bar{v}}$, where v is the root node of f
- We read this triple as:

  f = if v then g else h = ite (v,g,h) = vg+v ' h

ITE operator

- Ite can be used to implement any binary operation, like AND, OR

  $f \cdot g$ = ite(f,g,0)

  $f + g$ = ite(f,1,g)

- Ite can be implemented by extending Apply procedure to ternary operations

ITE operator

Table	Subset	Expression	Equivalent Form
0000	0	0	0
0001	AND(f, g)	fg	ite(f, g, 0)
0010	f > g	$f\bar{g}$	ite(f, $\bar{g}$, 0)
0011	f	f	f
0100	f < g	$\bar{f}g$	ite(f, 0, g)
0101	g	g	g
0110	XOR(f, g)	$f \oplus g$	ite(f, $\bar{g}$, g)
0111	OR(f, g)	$f + g$	ite(f, 1, g)
1000	NOR(f, g)	$f + g$	ite(f, 0, $\bar{g}$)
1001	XNOR(f, g)	$f \oplus g$	ite(f, g, $\bar{g}$)
1010	NOT(g)	$\bar{g}$	ite(g, 0, 1)
1011	f ≥ g	$f + \bar{g}$	ite(f, 1, $\bar{g}$)
1100	NOT(f)	$\bar{f}$	ite(f, 0, 1)
1101	f ≤ g	$\bar{f} + g$	ite(f, g, 1)
1110	NAND(f, g)	fg	ite(f, $\bar{g}$, 1)
1111	1	1	1

Variable ordering problem

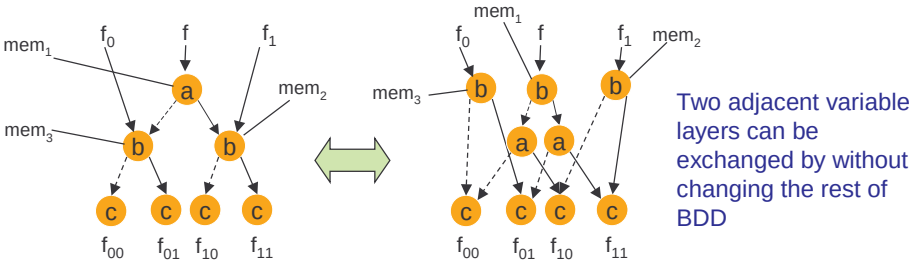
- Exact algorithms are intractable
- Heuristics are used:
  - static: use properties to group the variables together (e.g. keep symmetric variable together)
  - dynamic sifting: use the property that exchanging adjacent variables has only a local effect on the BDD; the variables are sifted from top to bottom one by one and the best position is remembered

Static variable ordering

- variable ordering is computed up-front based on the problem structure
- works very well for many combinational functions that come from circuits we actually build
  - general scheme: control variables first
- works bad for unstructured problems
  - e.g. using BDDs to represent arbitrary sets
- lots of research in ordering algorithms
  - simulated annealing, genetic algorithms
  - give better results but extremely costly

Dynamic variable ordering

Theorem (Friedman): Permuting any top part of the variable order has no effect on the nodes labeled by variables in the bottom part. Permuting any bottom part of the variable order has no effect on the nodes labeled by variables in the top part.



Two adjacent variable layers can be exchanged by without changing the rest of BDD

Dynamic variable ordering

- Dynamic variable ordering is done as follows:
  - shift a BDD variable to the top and then to the bottom and see which position had minimal number of BDD nodes
  - fix the variable to this position
  - repeat for all variables

BDD package (CUDD)

- A BDD package is a software program that can manipulate ROBDDs:
  - It can store multiple Boolean functions, sharing all vertices that can be shared.
  - It can create new functions by combining existing ones (e.g. $f = g \cdot h$)
  - It can convert the internal representation back to an external one

Basic data structures

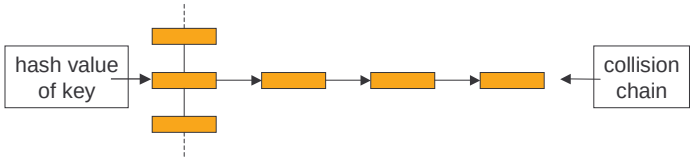
- The basic component is DdNode, which is a structure with several fields

```
typedef struct DdNode {
   struct DdNode *low, *high; /* pt to low, high child */
   int index;          /* index of the variable that */
                       /* labels the node, in 0...n  */
  char value;          /* range in {0,1} of terminal nodes */
                       /* x for non-terminal nodes */
   int id;             /* reference count - identifier */
   ...                 /* unique to that node   */
}
```

Efficient Implementation

Unique Table:

- avoids duplication of existing nodes
  - Implemented by a hash-table



Computed Table:

- avoids re-computation of existing results
  - Implemented by a software cache

Building a BDD

- It is normally more efficient to build BDDs "bottom-up"
- ITE-procedure *Cudd_bddIteVar* can be used
- Other procedures for binary operations, like *Cudd_bddAnd or Cudd_bddOr* can also be used

Example of building a BDD using CUDD package

- Building BDD for $f = x'_0 x'_1 x'_2 x'_3$:

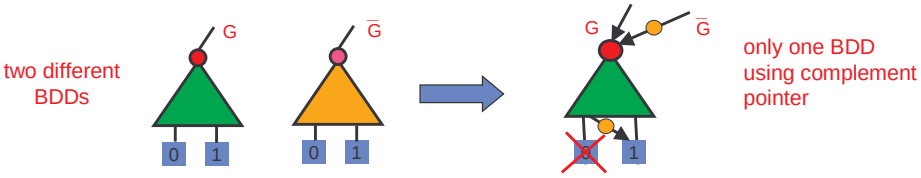
```
DdManager *manager;
DdNode *f, *var, *tmp;
int i;
    f = Cudd_readOne(manager); /* start from const 1*/
    Cudd_ref(f);               /* referencing f */
    for(i=3; i >= 0; i--) {    /* building bottom-up */
      var = Cudd_bddIthVar(manager,i);
      tmp = Cudd_bddAnd(manager, Cudd_Not(var),f); /*assigning to a
      tmp variable, old f should be kept to free its nodes */
      Cudd_ref(tmp);
      Cudd_recursiveDeref(manager,f);
      f = tmp; }
```


Garbage Collection

- It is very important to free and reuse memory of unused BDD nodes. This is done by garbage collection.
- Timing is very crucial because garbage collection is expensive. There are two approaches:
  1. do immediately when a node gets freed
     - bad because dead nodes get often reincarnated in next operation
  2. regular garbage collections based on statistics collected during BDD operations
     - better

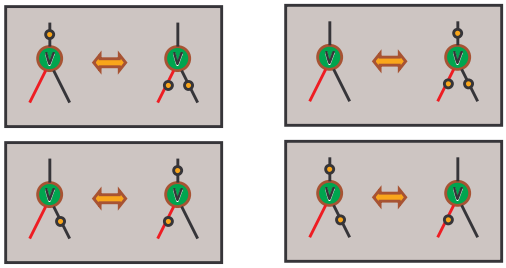
Extension - complemented edges

- Complemented edge indicates that the function associated with it is the complement of the function being pointed by the edge
  - reduces memory requirements
  - BUT MORE IMPORTANT:
    - makes some operations more efficient (NOT, ITE)



Extension - complement edges

- To maintain strong canonical form, need to resolve 4 equivalences:



Solution: Always choose one on left, i.e. the "then" edge must have no complement edge.

Other extensions

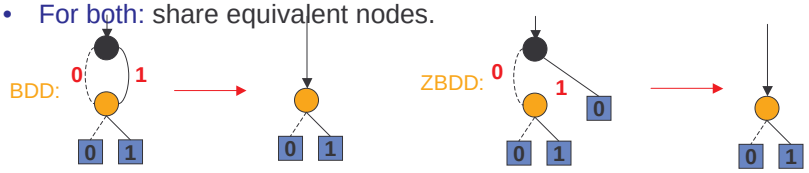
- Many different extentions of BDDs are possible
  - algebraic DDs: for $B^n \rightarrow M$ functions
    - multi-terminal BDDs
    - decision tree is binary
    - multiple leafs, including real numbers, sets or arbitrary objects
    - efficient for matrix computations and other non-integer applications
  - MDDs: for $M^n \rightarrow M$ functions
    - can be implemented using a regular BDD package with binary encoding
    - advantage that binary BDD variables for one MV variable do not have to stay together -> potentially better ordering
  - FDDs: Free Bdds
    - variable ordering differs; not canonical anymore

Zero Suppressed BDD's - ZBDD's

ZBDD's were invented by Minato to efficiently represent sparse sets. They have turned out to be useful in implicit methods for representing primes (which usually are a sparse subset of all cubes).

Different reduction rules:

- BDD: eliminate all nodes where dotted edge and solid edge point to the same node.
- ZBDD: eliminate all nodes where the solid edge points to 0. Connect incoming edges to node pointed by dotted edge.
- For both: share equivalent nodes.



Summary of BDDs

- BDDs give a compact representation for many practical functions as well as fast manipulation algorithms
  - widely used in CAD-tools nowadays
- They are very convenient for formal verification (equivalence checking)
  - comparing of two functions is done by comparing two pointers to the root nodes (constant-time operation)