Receding Horizon Control with Online Barrier Function Design under Signal Temporal Logic Specifications

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***Abstract*—Signal Temporal Logic (STL) has been found an expressive language for describing complex, time-constrained tasks in several robotic applications. Existing methods encode such specifications by either using integer constraints or by employing set invariance techniques. While in the first case this results in MILP control problems, in the latter case designer-specific choices may induce conservatism in the robot's performance and the satisfaction of the task. In this paper a continuous-time receding horizon control scheme (RHS) is proposed that exploits the trade-off between task satisfaction and performance costs such as actuation and state costs, traditionally considered in RHS schemes. The satisfaction of the STL tasks is encoded using time-varying control barrier functions (CBFs) that are designed online, thus avoiding the integer expressions that are often used in literature. The recursive feasibility of the proposed scheme is guaranteed by the satisfaction of a time-varying terminal constraint that ensures the satisfaction of the task with pre-determined robustness. The effectiveness of the method is illustrated in a multi-robot simulation scenario.**

***Index Terms*—Autonomous systems, control barrier functions, formal-methods control synthesis, receding horizon control, signal temporal logic**

I. Introduction

OVER the last decades, multiple robots have been considered in a variety of tasks, examples of which are object transportation [1], coverage control [2] and search and rescue missions [3]. Literature is rich in application-specific solution approaches in the majority of which problems like task-allocation, multi-robot coordination and planning are addressed independently. Recently, trajectory planning methods were proposed for the satisfaction of a general class of local or global, complex tasks described by Linear Temporal Logic formulas (LTL) [4]–[6]. In these works the dynamical systems describing the motion of the agents are abstracted into *Finite Transition Systems* and discrete plans are obtained using graph-based methods.

An important limitation of LTL is its inability to express tasks with strict deadlines. Signal Temporal Logic (STL) [7], on the other hand, provides an appropriate framework for introducing time-constrained tasks. In STL planning the satisfaction of such tasks is examined over continuous time signals and evaluated by robust semantics examples of which are introduced in [8], [9]. In the majority of the works [10]–[14] the STL formulas are encoded by integer variables and the agents' plans are obtained as solutions to MILP problems. Although suggestions towards reducing the computational burden have been made [15], these problems are known for the scalability issues arising as the optimization horizon or number of robots increases [12]. Addressing this problem, authors in [10] make use of the agents' past actions during planning increasing conservatism in the overall performance of the task. Other approaches [11], [16] consider an arbitrarily small optimization horizon without however providing guarantees on the recursive feasibility of the proposed RHS scheme while recently learning-based controllers have been designed for satisfaction of STL tasks [17], [18] or maximization of the STL robustness [19] in discrete time.

An important limitation of the aforementioned works is the lack of guarantees for the satisfaction of the task in continuous time. This problem is addressed in [13], [14] where in the former case a high-rate trajectory is designed while in the latter case the differential flatness property of the continuous-time nonlinear system is exploited.

Closer to our work is the control scheme proposed in [20]–[22] for continuous-time, input-affine, nonlinear systems under STL tasks. Here, a desired temporal behavior of the system is introduced guaranteeing the satisfaction of the STL task with predetermined robustness. Based on these temporal behaviors, a time-varying *Control Barrier Function (CBF)* is defined as a function of the error between the actual and desired behavior of the system, and feedback control laws are designed rendering a desired superlevel set of the barrier function forward invariant. In that way, satisfaction of the STL formula is ensured and a lower bound on the robustness is obtained. Although computationally efficient, this method does not consider input constraints while the performance of the task is highly dependent on the user-defined temporal behavior of the agents.

In this work a continuous-time receding horizon control scheme is proposed for the satisfaction of a set of STL tasks by a team of dynamically-coupled robots under state and input constraints. Motivated by [21], a time-varying barrier function is designed encapsulating a desired, temporal behavior for

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the system guaranteeing the satisfaction of the STL task. In our previous work [23], the desired temporal behavior of the system is designed offline ensuring the satisfaction of the task with a pre-determined robustness. Here, contrary to [20], [23], the desired behavior of the system is shaped online with the robustness value found as a solution to an RHS in which the trade off between an increased performance and state and input costs is explored. Based on [22], we design a least-violating control law for the cases when the satisfaction of a task is not possible due to state and input limitations. Given an arbitrarily small optimization horizon, the recursive feasibility of the proposed scheme is guaranteed by the design of an appropriate terminal controller, initially introduced in [23], that ensures the satisfaction of the global task with a minimum, pre-determined robustness provided that the initial problem is feasible. Extending [23], in this paper we provide a detailed proof of existence of the proposed terminal controller.

The remainder of the paper is organized as follows: Section II introduces the preliminaries and Section III the problem formulation. Section IV describes the proposed control barrier functions and Section V presents the proposed RHS scheme. Simulations are shown in Section VI and conclusions are summarized in Section VII.

II. Preliminaries

In this paper $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers. True and false are denoted by $\top, \perp$ respectively. Scalars and vectors are denoted by non-bold and bold letters respectively. The partial derivative of a function $\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta})$ with respect to $\mathbf{x}$ and t evaluated at $(\mathbf{x}', t')$ is abbreviated by $\frac{\partial \mathfrak{b}}{\partial \mathbf{x}} = \frac{\partial \mathfrak{b}(\mathbf{x},t;\boldsymbol{\theta})}{\partial \mathbf{x}}\big|_{\substack{\mathbf{x}=\mathbf{x}' \\ t=t'}}$ and $\frac{\partial \mathfrak{b}}{\partial t} = \frac{\partial \mathfrak{b}(\mathbf{x},t;\boldsymbol{\theta})}{\partial t}\big|_{\substack{\mathbf{x}=\mathbf{x}' \\ t=t'}}$ respectively. Here, $\frac{\partial \mathfrak{b}}{\partial \mathbf{x}}$ is considered to be a row vector. We denote by $\boldsymbol{\theta} = [\boldsymbol{\theta}_i]_{i\in\mathcal{I}} = \begin{bmatrix} \boldsymbol{\theta}_1^T & \dots & \boldsymbol{\theta}_{|\mathcal{I}|}^T \end{bmatrix}^T$ the stacked vector of $\boldsymbol{\theta}_i,\ i \in \mathcal{I}$. An extended class $\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a locally Lipschitz continuous and strictly increasing function with $\alpha(0) = 0$. The function $\mathbf{u} : [t_1, t_2] \to \mathbb{R}^m$ has a property a.e. (almost everywhere) if the property holds everywhere in $[t_1, t_2]$ except from a set of points of measure zero. The Euclidean norm of a vector $\boldsymbol{\zeta} \in \mathbb{R}^n$ is given by $\|\boldsymbol{\zeta}\| = \sqrt{\boldsymbol{\zeta}^T \boldsymbol{\zeta}}$. The induced 2-norm of a rectangular matrix C is defined as: $\|C\| = \sigma_{\max}(C)$, where $\sigma_{\max}(C)$ is the maximum singular value of C. The minimum singular value of the matrix C is denoted by $\sigma_{\min}(C)$. The Moore-Penroose matrix of a full row-rank matrix $B \in M_{n\times m}(\mathbb{R})$ is defined as: $B^\dagger = B^T(BB^T)^{-1}$. Given $a, b \in \mathbb{R}$, a divides b, denoted by $a|b$ if there exists an integer k such that $b = ka$. The Cartesian product of the sets $X_1, \dots, X_n$ is denoted by $X = \prod_{i=1}^n X_i$.

A. Signal Temporal Logic (STL)

Signal Temporal Logic (STL) determines whether a predicate μ is true or false. The validity of each predicate μ is evaluated based on a continuously differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$\mu = \begin{cases} \top, & h(\boldsymbol{\zeta}) \geq 0 \\ \perp, & h(\boldsymbol{\zeta}) < 0 \end{cases}.$$

The basic STL formulas are given by the grammar:

$$\phi ::= \top \mid \mu \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \mathcal{G}_{[a,b]}\phi \mid \mathcal{F}_{[a,b]}\phi \mid \phi_1\ \mathcal{U}_{[a,b]}\ \phi_2,$$

where ϕ_1, ϕ_2 are STL formulas and $\mathcal{G}_{[a,b]},\ \mathcal{F}_{[a,b]},\ \mathcal{U}_{[a,b]}$ is the always, eventually and until operator defined over the interval $[a, b]$ with $0 \leq a \leq b$. Let $\boldsymbol{\zeta}' \models \phi$ denote the satisfaction of the formula ϕ by a signal $\boldsymbol{\zeta}' : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. The formula ϕ is satisfiable if $\exists\, \boldsymbol{\zeta}' : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\boldsymbol{\zeta}' \models \phi$.

The STL semantics for a signal $\boldsymbol{\zeta}' : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ are recursively given by [24]: $(\boldsymbol{\zeta}', t) \models \mu \Leftrightarrow h(\boldsymbol{\zeta}'(t)) \geq 0$, $(\boldsymbol{\zeta}', t) \models \neg\phi \Leftrightarrow \neg((\boldsymbol{\zeta}', t) \models \phi)$, $(\boldsymbol{\zeta}', t) \models \phi_1 \wedge \phi_2 \Leftrightarrow (\boldsymbol{\zeta}', t) \models \phi_1 \wedge (\boldsymbol{\zeta}', t) \models \phi_2$, $(\boldsymbol{\zeta}', t) \models \mathcal{G}_{[a,b]}\phi \Leftrightarrow \forall t_1 \in [t+a, t+b]$, $(\boldsymbol{\zeta}', t_1) \models \phi$, $(\boldsymbol{\zeta}', t) \models \mathcal{F}_{[a,b]}\phi \Leftrightarrow \exists t_1 \in [t+a, t+b]$ s.t $(\boldsymbol{\zeta}', t_1) \models \phi$ and $(\boldsymbol{\zeta}', t) \models \phi_1\ \mathcal{U}_{[a,b]}\ \phi_2 \Leftrightarrow \exists t_1 \in [t+a, t+b]$ s.t $(\boldsymbol{\zeta}', t_1) \models \phi_2 \wedge (\boldsymbol{\zeta}', t_2) \models \phi_1,\ \forall t_2 \in [t, t_1]$.

STL is equipped with robustness metrics determining how robustly an STL formula ϕ is satisfied at time t by a signal $\boldsymbol{\zeta}'$. These semantics are defined as follows [8], [9]:

$$\begin{aligned}
\rho^{\mu}(\boldsymbol{\zeta}', t) &= h(\boldsymbol{\zeta}'(t)) \\
\rho^{\neg\phi}(\boldsymbol{\zeta}', t) &= -\rho^{\phi}(\boldsymbol{\zeta}', t) \\
\rho^{\phi_1 \wedge \phi_2}(\boldsymbol{\zeta}', t) &= \min(\rho^{\phi_1}(\boldsymbol{\zeta}', t), \rho^{\phi_2}(\boldsymbol{\zeta}', t)) \\
\rho^{\phi_1\ \mathcal{U}_{[a,b]}\ \phi_2} &= \max_{t_1 \in [t+a, t+b]} \min(\rho^{\phi_2}(\boldsymbol{\zeta}', t_1), \min_{t_2 \in [t, t_1]} \rho^{\phi_1}(\boldsymbol{\zeta}', t_2)) \\
\rho^{\mathcal{F}_{[a,b]}\phi}(\boldsymbol{\zeta}', t) &= \max_{t_1 \in [t+a, t+b]} \rho^{\phi}(\boldsymbol{\zeta}', t_1) \\
\rho^{\mathcal{G}_{[a,b]}\phi}(\boldsymbol{\zeta}', t) &= \min_{t_1 \in [t+a, t+b]} \rho^{\phi}(\boldsymbol{\zeta}', t_1)
\end{aligned}$$

Finally, it should be noted that $\boldsymbol{\zeta}' \models \phi$ if $\rho^{\phi}(\boldsymbol{\zeta}', 0) > 0$.

B. Control Barrier Functions for STL satisfaction

In this Section we summarize the basic steps towards designing a control barrier function (CBF) for STL satisfaction as described in [20], [21]. Consider the STL fragment:

$$\psi = \top \mid \mu \mid \neg\mu, \tag{1a}$$

$$\bar{\varphi} = \mathcal{G}_{[\bar{a},\bar{b}]}\psi \mid \mathcal{F}_{[\bar{a},\bar{b}]}\psi \mid \psi_1\ \mathcal{U}_{[\bar{a},\bar{b}]}\ \psi_2, \tag{1b}$$

$$\phi = \bigwedge_{l=1}^{n_\phi} \bar{\varphi}_l, \tag{1c}$$

where $\psi_1,\ \psi_2$ are STL formulas of the form (1a), $\bar{\varphi}_l,\ l = 1, \dots, n_\phi$ are STL formulas of the form (1b), $n_\phi \geq 1$ and $0 \leq \bar{a} \leq \bar{b} < \infty$. By definition of the STL semantics, the satisfaction of any until formula $\psi_1\ \mathcal{U}_{[\bar{a},\bar{b}]}\ \psi_2$ of (1b) is ensured by the satisfaction of a formula written as a conjunction of an always and an eventually formula, i.e., as $\mathcal{G}_{[\bar{a},t']}\ \psi_1 \wedge \mathcal{F}_{[t',t']}\ \psi_2$ where $t' \in [\bar{a}, \bar{b}]$ is a-priori chosen in [20], [21]. Hence, it is sufficient to ensure the satisfaction of the formula ϕ that is defined as a conjunction of eventually and always formulas φ_i as follows:

$$\phi = \bigwedge_{i\in\mathcal{I}} \varphi_i, \tag{2}$$

where $|\mathcal{I}| = p = n_\phi + n_u$ and n_ϕ is the total number of STL tasks in (1c) and n_u the number of until operators in (1c). Let $\bar{\varphi}_{l_i}$ be the l_i-th formula in (1c). The new formula φ_i is identical to a formula $\bar{\varphi}_{l_i}$ in (1c), if $\bar{\varphi}_{l_i} = \mathcal{G}_{[\bar{a}_{l_i},\bar{b}_{l_i}]}\psi_i$

or $\bar{\varphi}_{l_i} = \mathcal{F}_{[\bar{a}_{l_i},\bar{b}_{l_i}]}\psi_i$. If $\bar{\varphi}_{l_i}$ is an until formula, i.e., $\bar{\varphi}_{l_i} = \psi_{1,i}\ \mathcal{U}_{[\bar{a}_{l_i},\bar{b}_{l_i}]}\psi_{2,i}$, then $\varphi_i = \mathcal{G}_{[\bar{a}_{l_i},t_i']}\psi_{1,i}$ or $\varphi_i = \mathcal{F}_{[0,t_i']}\psi_{2,i}$ where $t_i' \in [\bar{a}_{l_i},\bar{b}_{l_i}]$. We denote the time interval associated with the temporal operator of φ_i as $[a_i, b_i]$. For each subformula $\varphi_i, i \in \mathcal{I}$, let $\mathfrak{b}^i(\mathbf{x}, t; \boldsymbol{\theta}^i) = -\gamma^i(t; \boldsymbol{\theta}^i) + h_i(\mathbf{x})$, where $h_i : \mathbb{R}^n \to \mathbb{R}$ is the predicate function corresponding to φ_i, assumed to be continuously differentiable and $\gamma^i : \mathbb{R}_{\geq 0} \times \Theta^i \to \mathbb{R}$ is a function describing a desired temporal behavior of the system that ensures satisfaction of φ_i with a minimum robustness r. In [21] the performance functions $\gamma^i(t; \boldsymbol{\theta}^i)$ are defined as piecewise linear functions, whose values depend on a set of parameters $\boldsymbol{\theta}^i \in \Theta^i \subset \mathbb{R} \times \mathbb{R}^2_{\geq 0}$ that are chosen offline. Based on the functions $\mathfrak{b}^i(\mathbf{x}, t; \boldsymbol{\theta}^i)$, the CBF function $\mathfrak{b} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \Theta \to \mathbb{R}$ corresponding to ϕ is defined as:

$$\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta}) = -\ln\left(\sum_{i\in\mathcal{I}} o^i(t) \exp\left(-\mathfrak{b}^i(\mathbf{x}, t; \boldsymbol{\theta}^i)\right)\right),$$

where $\boldsymbol{\theta} = \left[\boldsymbol{\theta}^i\right]_{i\in\mathcal{I}}$, $\Theta = \prod_{i\in\mathcal{I}} \Theta^i$ and $o^i : \mathbb{R}_{\geq 0} \to \{0, 1\}$ is an integer valued function introduced to ensure that the barrier function corresponding to subtask φ_i stops contributing to $\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta})$, when φ_i is satisfied. Note that due to the existence of the integer variables, $\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta})$ is differentiable only at $\mathbb{R}^n \times (\sigma^d, \sigma^{d+1}) \times \Theta$, where $\sigma^d \leq \sigma^{d+1}$ and $\sigma^d \in \{0, \infty\} \cup \bar{\Sigma}$ where $\bar{\Sigma} = \{a_i, b_i : i \in \mathcal{I}^{\mathcal{G}}, a_i \neq 0\} \cup \{b_i : i \in \mathcal{I}^{\mathcal{G}}, a_i = 0\} \cup \{t^{i*} : i \in \mathcal{I}^{\mathcal{F}}\}$, and where $\mathcal{I}^{\mathcal{G}}, \mathcal{I}^{\mathcal{F}} \subset \mathcal{I}$ are the sets of always and eventually formulas respectively. For this particular choice of $\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta})$, it can be shown [24, Lemma 2] that: $\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta}) \leq \min_{i\in\mathcal{A}(t)} \mathfrak{b}^i(\mathbf{x}, t; \boldsymbol{\theta}^i)$, where $\mathcal{A}(t) = \{i \in \mathcal{I} : o^i(t) \neq 0\}$. Therefore, if there exists $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\mathfrak{b}(\mathbf{x}, t; \boldsymbol{\theta}) \geq 0$ for every $t \geq 0$, then each subtask $\varphi_i, i \in \mathcal{I}$, is satisfied with a minimum robustness r.

III. Multi-Agent Dynamics and Problem Formulation

In this work we consider a team of R agents with each agent identified by its index $k \in \{1, \ldots, R\}$. For every agent k let $\mathbf{x}_k \in \mathbb{R}^{\bar{n}}$, $\mathbf{u}_k \in \mathbb{R}^{\bar{m}}$ denote its state and input vector respectively. The states of agent k evolve over time based on the following equation:

$$\dot{\mathbf{x}}_k = A_{kk}\mathbf{x}_k + \sum_{k'\neq k} A_{kk'}\mathbf{x}_{k'} + B_k\mathbf{u}_k$$

where $A_{kk},\ A_{kk'} \in M_{\bar{n}}(\mathbb{R}),\ B_k \in M_{\bar{n}\times\bar{m}}(\mathbb{R})$. Here, the matrix $A_{kk'}$ describes possible *dynamical couplings* between the states of agents k, k' and is a-priori known by agent k. Examples of dynamically coupled systems include networked systems, platoons, energy systems and mobile manipulators.

Let $\mathbf{x} = \begin{bmatrix}\mathbf{x}_1^T & \ldots & \mathbf{x}_R^T\end{bmatrix}^T \in \mathbb{R}^n, \mathbf{u} = \begin{bmatrix}\mathbf{u}_1^T & \ldots & \mathbf{u}_R^T\end{bmatrix}^T \in \mathbb{R}^m$ be the stacked vector of the states and inputs of all agents in the team respectively with $n = R\bar{n}$ and $m = R\bar{m}$. Then, the dynamics of the agents can be written in stacked form as:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{3}$$

where $A = \begin{bmatrix} A_{11} & \ldots & A_{1R} \\ \vdots & \ddots & \vdots \\ A_{R1} & \ldots & A_{RR}\end{bmatrix}$, $B = \operatorname{diag}\{B_1, \ldots, B_R\}$.

Assumption 1. *The matrices $B_k,\ k = 1, \ldots, R$ have full row rank equal to $\bar{n}$ ($\bar{n} \leq \bar{m}$).*

In this paper each agent is restricted to work within a pre-specified area and is subject to actuation limitations. These constraints could be expressed as $\mathbf{x}_k \in \mathbb{X}_k,\ \mathbf{u}_k \in \mathbb{U}_k,\ k = 1, \ldots, R$ where $\mathbb{X}_k = \{\mathbf{x}_k \in \mathbb{R}^{\bar{n}} :\ \|\mathbf{x}_k\| \leq d_x^k\}$ and $\mathbb{U}_k = \{\mathbf{u}_k \in \mathbb{R}^{\bar{m}} :\ \|\mathbf{u}_k\| \leq d_u^k\}$ with $d_x^k, d_u^k > 0$ known constants for any $k = 1, \ldots, R$. Let $\mathbb{X} = \prod_{k=1}^R \mathbb{X}_k$, $\mathbb{U}' = \prod_{k=1}^R \mathbb{U}_k$ and $d_x = \sum_{k=1}^R d_x^k$. Then, we may write the state and input constraints of the centralized system as $\mathbf{x} \in \mathbb{X},\ \mathbf{u} \in \mathbb{U}$ with $\mathbb{X}, \mathbb{U}$ satisfying:

$$\mathbb{X} \subseteq \{\mathbf{x} \in \mathbb{R}^n :\ \|\mathbf{x}\| \leq d_x\} \tag{4a}$$
$$\mathbb{U} = \{\mathbf{u} \in \mathbb{R}^m :\ \|\mathbf{u}\| \leq d_u\} \subseteq \mathbb{U}'. \tag{4b}$$

Definition 1. *Given a control signal $\mathbf{u} : [t_1, t_2] \to \mathbb{U}$ a solution $\mathbf{x} : [t_1, t_2] \to \mathbb{X}$ of* (3) *with $\mathbf{x}(t_1) = \mathbf{x}_1$ is an absolutely continuous function such that:*

$$\mathbf{x}(t) = \mathbf{x}_1 + \int_{t_1}^{t} (A\mathbf{x}(\tau) + B\mathbf{u}(\tau))d\tau$$

holds a.e. in $[t_1, t_2]$.

Assumption 2. *There exist sets $\mathbb{X}, \mathbb{U}$ satisfying* (4a)-(4b)*, such that $d_u\sigma_{\min}(B) > \sigma_{\max}(A)d_x$ is true for the system dynamics* (3) *subject to state and input constraints of the form $\mathbf{x} \in \mathbb{X},\ \mathbf{u} \in \mathbb{U}$.*

Intuitively, Assumption 2 guarantees that there exists enough control input to prevent the multi-agent system from leaving the workspace in the most "aggressive" way based on its dynamics. As will be shown in Section V-A this argument is necessary for the design of a terminal controller that ensures the satisfaction of task with a pre-determined robustness.

Remark 1. *Assumptions 1 and 2 can be easily generalized for input-affine systems $\dot{\mathbf{x}}_k = f_k(\mathbf{x}) + g_k(\mathbf{x})\mathbf{u}_k, k = 1, \ldots, R$, where $f_k : \mathbb{R}^n \to \mathbb{R}^{\bar{n}}$ and $g_k : \mathbb{R}^n \to \mathbb{R}^{\bar{m}}$ are locally Lipschitz functions, as follows: 1) $g_k(\mathbf{x})$ is full-row rank for every $\mathbf{x} \in \mathbb{R}^n$, $k \in \{1, \ldots, R\}$ and 2) $\|f(\mathbf{x})\| < \sigma_{\min}(g(\mathbf{x}))d_u, \forall \mathbf{x} \in \mathbb{X}$, where $\max_{\mathbf{x}\in\mathbb{X}} \|f(\mathbf{x})\| < \infty$, $\max_{\mathbf{x}\in\mathbb{X}} \|g(\mathbf{x})\| < \infty$, $f(\mathbf{x}) = \begin{bmatrix} f_1^T(\mathbf{x}) & \ldots & f_R^T(\mathbf{x})\end{bmatrix}^T$ and $g(\mathbf{x})$ is the block diagonal matrix of $g_1(\mathbf{x}), \ldots, g_R(\mathbf{x})$. Hence, the RHS scheme proposed in Section V-B can be easily applied to more general, nonlinear dynamics.*

Given the STL fragment defined by (1a)-(1c) let a finite sequence of time instants $\{\tau_j\}_{j=0}^J$ with $\tau_j = j\Delta\tau,\ j \in \mathcal{J} = \{0, \ldots, J\}$ and $\tau_J = \max_l b_l$, where $\Delta\tau$ is a given, positive constant satisfying $\Delta\tau | \max_l b_l$ and b_l is the upper bound of the interval of satisfaction corresponding to the temporal operator of $\bar{\varphi}_l$ in (1c). Based on the above we are in position to define the Problem considered in this paper as follows:

Problem 1: Consider the dynamical system (3) subject to state and input constraints $\mathbf{x} \in \mathbb{X},\ \mathbf{u} \in \mathbb{U}$ with $\mathbb{X}, \mathbb{U}$ known, compact sets satisfying (4a)-(4b). Given an STL formula ϕ as in (1c), a positive prediction horizon length N and a sampling rate $\Delta\tau$ satisfying $\Delta\tau | \max_l b_l$, design a control input $\mathbf{u}$ such that any solution $\mathbf{x} : [0, \tau_J] \to \mathbb{X}$ of (3) with initial condition $\mathbf{x}(0)$ guarantees $\rho^\phi(\mathbf{x}, 0) \geq \bar{\rho}$, where $\bar{\rho}$ is maximized.

IV. Barrier Functions for Task Satisfaction in the RHS

In this Section we begin by designing the control barrier functions (CBFs) encoding the STL constraints induced by ϕ in (2). Motivated by the work in [20], [21], we introduce two piece-wise differentiable functions $\mathfrak{b}_w : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \times \Theta_w \to \mathbb{R}, w \in \{H, F\}$, defined as in Section II-B. The barrier $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_{j,H})$ encodes the satisfaction of the STL formula ϕ, defined in (2), and is designed online at each time interval $[\tau_j, \tau_j + N]$. On the contrary, $\mathfrak{b}_F(\mathbf{x}, t; \boldsymbol{\theta}_F)$, called the *terminal barrier function*, is designed offline and encodes the satisfaction of $\phi' = \bigwedge_{i \in \mathcal{I}_F} \varphi_i = \phi \wedge \varphi_{p+1} \wedge \ldots \wedge \varphi_{p+R}$, where:

$$\varphi_{p+k} = \mathcal{G}_{[0, b_{p+k}]} \psi_{p+k}, \tag{5a}$$

$$\psi_{p+k} = \begin{cases} \top, & h_{p+k}(\mathbf{x}) \geq 0 \\ \bot, & h_{p+k}(\mathbf{x}) < 0 \end{cases}, \tag{5b}$$

with $b_{p+k} = \tau_J + N$ and $h_{p+k}(\mathbf{x}) = d_x^{k\,2} - \|\mathbf{x}_k\|^2$ for $k = 1, \ldots, R$. The extra tasks $\varphi_{p+1}, \ldots, \varphi_{p+R}$ are introduced to ensure that the agents will stay within the workspace, thus the state constraints $\mathbf{x} \in \mathbb{X}$ are always satisfied. Since the terminal barrier function is designed offline, for brevity, we will sometimes omit the dependence on $\boldsymbol{\theta}_F$ and use the notation $\mathfrak{b}_F(\mathbf{x}, t)$ instead.

Definition 2. *A sub-formula φ_i, $i \in \mathcal{I} = \{1, \ldots, p\}$ is called active in the interval $[\tau_j, \tau_j + N]$, $j \in \mathcal{J}$, if either the intersection of the interval of satisfaction $[\bar{a}_{l_i}, \bar{b}_{l_i}]$ of the formula $\bar{\varphi}_{l_i}$ with $[\tau_j, \tau_j + N]$ is non-empty, i.e., $[\bar{a}_{l_i}, \bar{b}_{l_i}] \cap [\tau_j, \tau_j + N] \neq \emptyset$ or if $\tau_j + N \leq \bar{a}_{l_i}$ holds.*

Let $\mathcal{I}_H^j$ denote the set of indices of the sub-formulas φ_i, $i \in \mathcal{I}$, that are active at $[\tau_j, \tau_j + N]$ for every $j \in \mathcal{J}$. Note that by definition of (2), $\mathcal{I}_H^j = \mathcal{I}_H^{j,\mathcal{G}} \cup \mathcal{I}_H^{j,\mathcal{F}}$, where $\mathcal{I}_H^{j,\mathcal{F}} = \{i \in \mathcal{I}_H^j : \varphi_i = \mathcal{F}_{[a_i, b_i]} \psi_i\}$ and $\mathcal{I}_H^{j,\mathcal{G}} = \{i \in \mathcal{I}_H^j : \varphi_i = \mathcal{G}_{[a_i, b_i]} \psi_i\}$. Given the active sub-formulas, we may define the barrier function at each time interval $[\tau_j, \tau_j + N]$ as:

$$\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_{j,H}) = -\ln\left(\sum_{i \in \mathcal{I}_H^j} o_H^i(t) \exp\left(-\mathfrak{b}_H^i(\mathbf{x}, t; \boldsymbol{\theta}_{j,H}^i)\right) \right),$$

where $\boldsymbol{\theta}_{j,H} \in \Theta_H$ is a set of parameters to be optimized online, $\Theta_H = \prod_{i \in \mathcal{I}_H^j} \Theta_H^i$ and $\boldsymbol{\theta}_{j,H}^i \in \Theta_H^i$ is a set of parameters defined shortly below. To simplify notation, we may omit the subscript j from the elements defining $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_{j,H})$, when it is clear from context. Next, we may define the terminal barrier function $\mathfrak{b}_F(\mathbf{x}, t; \boldsymbol{\theta}_F)$ as:

$$\mathfrak{b}_F(\mathbf{x}, t; \boldsymbol{\theta}_F) = -\ln\left(\sum_{i \in \mathcal{I}_F} o_F^i(t) \exp\left(-\mathfrak{b}_F^i(\mathbf{x}, t; \boldsymbol{\theta}_F^i)\right) \right), \tag{6}$$

where $\boldsymbol{\theta}_F \in \Theta_F$ is a set of parameters to be chosen offline, $\Theta_F = \prod_{i \in \mathcal{I}_F} \Theta_F^i$ and $\boldsymbol{\theta}_F^i \in \Theta_F^i$ is also given below. We compactly denote the barrier functions considered here by $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w), w \in \{H, F\}$. For every $w \in \{H, F\}$ the functions $\mathfrak{b}_w^i(\mathbf{x}, t; \boldsymbol{\theta}_w^i), i \in \mathcal{I}_w$ are defined as:

$$\mathfrak{b}_w^i(\mathbf{x}, t; \boldsymbol{\theta}_w^i) = -\gamma_w^i(t; \boldsymbol{\theta}_w^i) + h_i(\mathbf{x}),$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ is the predicate function corresponding to $\varphi_i, i \in \mathcal{I}_w, w \in \{H, F\}$ and $\gamma_w^i : \mathbb{R}_{\geq 0} \times \Theta_w^i \to \mathbb{R}$ is the performance function ensuring satisfaction of φ_i with robustness r_w defined as follows [21]:

$$\gamma_w^i(t; \boldsymbol{\theta}_w^i) = \begin{cases} \frac{\gamma_{w,\infty}^i - \gamma_{w,0}^i}{t_w^{i*}} t + \gamma_{w,0}^i, & \text{if } t < t_w^{i*} \\ \gamma_{w,\infty}^i, & \text{if } t \geq t_w^{i*} \end{cases}, \tag{7}$$

where $\boldsymbol{\theta}_w^i = \begin{bmatrix} \gamma_{w,0}^i & \gamma_{w,\infty}^i & t_w^{i*} \end{bmatrix}^T \in \Theta_w^i, w \in \{H, F\}$ is a set of parameters depending on the robustness value r_w and satisfy the following:

$$\gamma_{w,0}^i \in (-\infty, h_i(\mathbf{x}(0))), \tag{8a}$$

$$\gamma_{w,\infty}^i \in (\max(r_w, \gamma_{w,0}^i), h_i^{\max}), \tag{8b}$$

$$t_w^{i*} \in \begin{cases} \{\bar{a}_{l_i}\}, & i \in \mathcal{I}_w^{j,\mathcal{G}} \\ [\bar{a}_{l_i}, \bar{b}_{l_i}], & i \in \mathcal{I}_w^{j,\mathcal{F}} \end{cases}, \tag{8c}$$

$$t_H^{i*} \geq \tau_{d(i)}, \qquad i \in \mathcal{I}_H^{j,\mathcal{F}}, \tag{8d}$$

$$r_w \in \begin{cases} (0, h_i(\mathbf{x}(0)), & t_w^{i*} = 0 \\ (0, h_i^{\max}), & t_w^{i*} \neq 0, \end{cases} \tag{8e}$$

where $h_i^{\max} = \sup_{\mathbf{x} \in \mathbb{R}^n} h_i(\mathbf{x}) < \infty$, $d(i) = \max\{j \in \mathcal{J} : [\bar{a}_{l_i}, \bar{b}_{l_i}] \cap [\tau_j, \tau_{j+1}) \neq \emptyset\}$, where $\tau_{J+1} = (J+1)\Delta\tau$. From (8a), $\mathfrak{b}_w(\mathbf{x}(0), 0; \boldsymbol{\theta}_w) > 0$, for every $w \in \{H, F\}$. In addition, due to (8b), for every $t \geq t_w^{i*}$ we have that $\mathfrak{b}_w^i(\mathbf{x}, t; \boldsymbol{\theta}_w^i) \leq -r_w + h_i(\mathbf{x})$. Thus, $\mathfrak{b}_w^i(\mathbf{x}, t; \boldsymbol{\theta}_w^i) \geq 0$ implies $h_i(\mathbf{x}) \geq r_w$ for all $t \geq t_w^{i*}$. Constraint (8c) ensures that t_w^{i*} takes values within the time interval of satisfaction of $\varphi_i, i \in \mathcal{I}_w$ while (8d) is introduced to ensure that cases where $\varphi_i, i \in \mathcal{I}_H^{j,\mathcal{F}}$ is deactivated prematurely without being satisfied are avoided. Based on the above, we define $\Theta_H^i = \{\boldsymbol{\theta} \in \mathbb{R} \times \mathbb{R}_{\geq 0}^2 : \boldsymbol{\theta} \text{ satisfies } (8a){-}(8d)\}$ and $\Theta_F^i = \{\boldsymbol{\theta} \in \mathbb{R} \times \mathbb{R}_{\geq 0}^2 : \boldsymbol{\theta} \text{ satisfies } (8a) - (8c)\}$, for every $i \in \mathcal{I}_H$ and $i \in \mathcal{I}_F$, respectively.

When a formula φ_i is satisfied, the contribution of $\mathfrak{b}_w^i(\mathbf{x}, t; \boldsymbol{\theta}_w^i)$ to $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w)$ is deactivated. The deactivation policy is introduced using an integer variable $o_w^i : \mathbb{R}_{\geq 0} \to \{0, 1\}$ defined as follows:

$$o_w^i(t) = \begin{cases} 1, & t \in T_w^i \\ 0, & t \notin T_w^i \end{cases}, \tag{9}$$

where $T_w^i = [0, t_w^{i*})$, if $i \in \mathcal{I}_w^{\mathcal{F}}$, or $T_w^i = [0, b_i)$, if $i \in \{i' \in \mathcal{I}_w^{\mathcal{G}} : a_{i'} = 0\}$, or $T_w^i = [0, a_i) \cup (a_i, b_i)$ if $i \in \{i' \in \mathcal{I}_w^{\mathcal{G}} : a_{i'} \neq 0\}$. In addition, in order to ensure that $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w), w \in \{H, F\}$ are well defined at every time interval $[\tau_j, \tau_j + N], j \in \mathcal{J}$, we set $o_w^{i_w}(t) = 1, \forall t \geq 0$, where $i_F \in \{p+1, \ldots, p+R\}$ and $i_H = \arg\max_{i \in \mathcal{I}} \bar{b}_{l_i}$. Based on $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w)$, $w \in \{H, F\}$, we may define the δ-level sets of $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w)$ as:

$$\mathcal{C}_w^\delta(t; \boldsymbol{\theta}_w) = \{\mathbf{x} \in \mathbb{R}^n |\ \mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w) \geq \delta\}, \tag{10}$$

where $\boldsymbol{\theta}_w \in \Theta_w$ is a set of parameters on which the value of $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w)$ depends at each $(\mathbf{x}, t)$. If $\delta = 0$, we will omit the superscript and write $\mathcal{C}_H(t; \boldsymbol{\theta}_H)$ for $w = H$ and $\mathcal{C}_F(t)$, for $w = F$.

V. Control Approach

As discussed in Section II-B maintaining a non-negative value of the barrier function $\mathfrak{b}_w(\mathbf{x}, t; \boldsymbol{\theta}_w), w \in \{H, F\}$ for any $t \geq 0$ implies that there exists a time instant t_w^{i*} such that $h_i(\mathbf{x}(t)) > r_w, t \in T_w^i$, i.e., the satisfaction of φ_i. In [20] authors construct a barrier function $\mathfrak{b}(\mathbf{x},t)$ for determining the satisfaction of ϕ and design a feedback control law $\mathbf{u}(\mathbf{x},t)$ satisfying:

$$\frac{\partial \mathfrak{b}}{\partial \mathbf{x}}(A\mathbf{x}+B\mathbf{u})+\frac{\partial \mathfrak{b}}{\partial t} \geq -\alpha(\mathfrak{b}(\mathbf{x},t)), \quad t \geq 0, \tag{11}$$

where $\alpha(\cdot)$ is an appropriately chosen extended class $\mathcal{K}$ function. The constraint above guarantees that $\mathfrak{b}(\mathbf{x},t) \geq 0$ for all $t \geq 0$ when $\mathfrak{b}(\mathbf{x}(0),0) \geq 0$. Specifically, as long as $\mathbf{x} \in \{\boldsymbol{\zeta} \in \mathbb{R}^n : \mathfrak{b}(\boldsymbol{\zeta},t) > 0\}$, (11) forces the state of the multi-agent system to keep the value of $\mathfrak{b}(\mathbf{x},t)$ non-negative without necessarily increasing it. However, when $\mathbf{x} \in \{\boldsymbol{\zeta} \in \mathbb{R}^n : \mathfrak{b}(\boldsymbol{\zeta},t)=0\}$ is true, (11) becomes $\dot{\mathfrak{b}}(\mathbf{x},t) \geq 0$ forcing the state of the system to move towards maximizing $\mathfrak{b}(\mathbf{x},t)$.

A. Terminal Controller

Motivated by the work in [20], we design a feedback control law satisfying an equivalent constraint to (11) for $\mathfrak{b}_F(\mathbf{x},t)$. The resulting control law will work as a *terminal controller* guaranteeing $\mathbf{x}(t) \in \mathcal{C}_F(t)$ for any $t > \tau_j + N$ if $\mathbf{x}(\tau_j + N) \in \mathcal{C}_F(\tau_j+N)$ holds for any $j \in \mathcal{J}$. To further simplify notation, let $\Sigma_F = \{0,\infty\} \cup \Sigma'_F$, where $\Sigma'_F = \bar{\Sigma} \backslash \{b_{i_F} : i_F \in \{p+1,\dots,p+R\}\}$ is the set of points at which $\mathfrak{b}_F(\mathbf{x},t)$ is discontinuous with $\bar{\Sigma}$ as defined in Section II-B. We begin by formally introducing control barrier functions as follows:

Definition 3. *The function $\mathfrak{b}_F(\mathbf{x},t)$ is a control barrier function (CBF) within each time interval $(\sigma_F^s,\sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$, if there exists an extended class $\mathcal{K}$ function $\alpha_F(\cdot)$ and an open, connected set $\Omega \subset \mathbb{R}^n$, where $\mathcal{C}_F(t) \subset \Omega \subset \mathbb{X}$ is assumed to be satisfied for every $t \geq 0$, such that for all $(\mathbf{x},t) \in \Omega \times (\sigma_F^s,\sigma_F^{s+1})$ holds the following:*

$$\sup_{\mathbf{u}\in\mathbb{U}} \Big\{ \frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x}+B\mathbf{u})+\frac{\partial \mathfrak{b}_F}{\partial t}+\alpha_F(\mathfrak{b}_F(x,t)) \Big\} \geq 0. \tag{12}$$

Assumption 3. *Consider a differentiable function $\mathfrak{b}_F(\mathbf{x},t)$ on $\Omega \times (\sigma_F^s,\sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$, defined as in (6). Let Assumption 2 hold. Consider further an extended class $\mathcal{K}$ function $\alpha_F(\cdot)$ and a given, positive constant δ_1 satisfying $\delta_1 > \frac{L_t+|\alpha_F(\chi)|}{d_u\sigma_{\min}(B)-\sigma_{\max}(A)d_x}$, where $L_t = \max_{i\in\mathcal{I}_F} \frac{d\gamma_F^i}{dt}|_{t=0}$ and $\chi < \inf_{(\mathbf{x},t)\in\Omega\times\mathbb{R}_{\geq 0}} \mathfrak{b}_F(\mathbf{x},t)$. Then, the barrier function $\mathfrak{b}_F(\mathbf{x},t)$ is designed such that $\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}A\mathbf{x}+\frac{\partial \mathfrak{b}_F}{\partial t}+\alpha_F(\mathfrak{b}_F(\mathbf{x},t)) \geq 0$ holds, for every $(\mathbf{x},t) \in \Omega \times (\sigma_F^s,\sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| \leq \delta_1$.*

Assumption 3 ensures that a constraint similar to (11) is satisfied when applying $\mathbf{u} = \mathbf{0}$ for any $(\mathbf{x},t) \in \Omega \times (\sigma_F^s,\sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| \leq \delta_1$. A high value of δ_1 e.g., due to actuation limitations and/or increased performance expectations, may introduce conservatism on the choice of $\boldsymbol{\theta}_F \in \Theta_F$ for the design of $\mathfrak{b}_F(\mathbf{x},t)$. Hence, a trade-off should be considered between the performance of the multi-agent system and the size of $\{(\mathbf{x},t) \in \Omega \times (\sigma_F^s,\sigma_F^{s+1}) : \left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| \leq \delta_1\}$.

Theorem 1. *Consider the multi-agent system dynamics* (3) *subject to input constraints $\mathbf{u} \in \mathbb{U}$ with $\mathbb{U}$ defined by* (4b) *and an STL formula $\phi' = \bigwedge_{i\in\mathcal{I}_F} \varphi_i$, defined by* (2) *and* (5a)-(5b). *Assume that $\mathfrak{b}_F(\mathbf{x},t)$ is a differentiable function on $\Omega \times (\sigma_F^s,\sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$, defined as in* (6). *Let Assumptions 1-3 hold. Consider an extended class $\mathcal{K}$ function $\alpha_F(\cdot)$ and a control law $\bar{\mathbf{u}}(\mathbf{x},t) := \bar{\mathbf{u}}$ with $\bar{\mathbf{u}}$ given by:*

$$\bar{\mathbf{u}} = \arg\min_{\mathbf{u}\in\mathbb{U}} \mathbf{u}^T\mathbf{u}, \tag{13}$$

subject to:

$$\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x}+B\mathbf{u})+\frac{\partial \mathfrak{b}_F}{\partial t} \geq -\alpha_F(\mathfrak{b}_F(\mathbf{x},t)). \tag{13a}$$

Then, there exists a function $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{X}$ satisfying (3) *a.e. guaranteeing $\rho^{\phi'}(\mathbf{x},0) \geq r_F > 0$, where r_F is a designing parameter of the terminal barrier function $\mathfrak{b}_F(\mathbf{x},t)$, provided that $\mathbf{x}(0) \in \mathcal{C}_F(0)$.*

Proof. The proof of Theorem 1 is given in Appendix I. ■

Remark 2. *The proposed terminal barrier function $\mathfrak{b}_F(\mathbf{x},t)$ ensures the satisfaction of the STL formula ϕ' with a minimum robustness $r_F > 0$ for systems subject to input constraints. Not surprisingly, the limited actuation capabilities of the systems require a stronger assumption than [20, As. 3], [21, As. 3], where no input limitations are imposed. To that end, Assumption 3 is introduced to ensure the existence and continuity of a terminal controller for every $(\mathbf{x},t) \in \Omega \times (\sigma_F^s,\sigma_F^{s+1})$ that respects the input constraints $\mathbf{u} \in \mathbb{U}$. Although this condition is sufficient for the existence of the terminal controller, our simulations show it is not necessary. Relaxing Assumption 3 while ensuring existence and continuity of the optimal controller will be a subject of future research.*

B. Receding Horizon Control Problem

A basic assumption on CBF based control under STL tasks [20] is the existence of an appropriate CBF function $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H)$, where $\boldsymbol{\theta}_H \in \Theta_H$ are chosen offline. This requirement may potentially introduce conservatism and limit the performance of the agents towards satisfying the task, as feasible solutions of (3) may be excluded from $\mathcal{C}_H(t;\boldsymbol{\theta}_H)$. Towards increasing the size of $\mathcal{C}_H(t;\boldsymbol{\theta}_H)$, a novel RHS problem is proposed in which the CBF function $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H)$ is designed online while state and input costs, often considered in RHS problems, are minimized. The proposed RHS is solved at pre-determined, equidistant time instants τ_j based on the current state of the system $\mathbf{x}(\tau_j)$. The resulting control law is applied over a finite time interval $[\tau_j,\tau_{j+1})$ until the next state measurement $\mathbf{x}(\tau_{j+1})$ becomes available at τ_{j+1}. The aforementioned procedure is repeated for a finite number of times $J+1$ with $J = \max_l b_l/\Delta\tau$.

Given the actuation limitations of the agents satisfaction of ϕ might not be possible at all times as this decision might lead to excessive state and input costs. Therefore, a modified version of (11) is considered for the barrier function $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H)$ and applied over the time interval $[\tau_j,\tau_j+N]$, $j \in \mathcal{J}$. More specifically, motivated by [22], we propose the relaxation of (11) by a factor ϵ with $\epsilon : [\tau_j,\tau_j+N] \to \mathbb{R}_{\geq 0}$ allowing the violation of the task when necessary. Hence, the modified version of (11) can be written as:

$$\frac{\partial \mathfrak{b}_H}{\partial \mathbf{x}}(A\mathbf{x}+B\mathbf{u})+\frac{\partial \mathfrak{b}_H}{\partial t} \geq -\alpha_H(\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H))-\epsilon, \quad (14)$$

where $\alpha_H(\cdot)$ is an extended class $\mathcal{K}$ function. The relaxation factor ϵ is considered as a variable of the RHS and the goal is to minimize its value within $[\tau_j, \tau_j+N],\ j \in \mathcal{J}$. Based on (14), we may impose the following constraint in the RHS:

$$(\mathbf{x},\mathbf{u},\epsilon,\boldsymbol{\theta}_H) \in K_H(t), \quad \text{a.e. } [\tau_j, \tau_j+N) \quad (15)$$

where $K_H(t) = \{(\mathbf{x},\mathbf{u},\epsilon,\boldsymbol{\theta}_H) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \times \Theta_H : \frac{\partial \mathfrak{b}_H}{\partial \mathbf{x}}(A\mathbf{x}+B\mathbf{u}) + \frac{\partial \mathfrak{b}_H}{\partial t} + \alpha_H(\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H)) + \epsilon \geq 0\}$. Considering the above, we may define our problem at each $[\tau_j, \tau_j+N]$ as follows:

$$\min_{\substack{\mathbf{u},\epsilon\\ \boldsymbol{\theta}_H, r_H}} J(\mathbf{x},\mathbf{u},\epsilon,r_H,\boldsymbol{\theta}_H) \quad (16)$$

subject to:

$$\dot{\mathbf{x}} = A\mathbf{x}+B\mathbf{u}, \quad \text{a.e. } [\tau_j, \tau_j+N] \quad (16a)$$
$$(\mathbf{x},\mathbf{u},\epsilon,\boldsymbol{\theta}_H) \in K_H(t), \quad \text{a.e. } [\tau_j, \tau_j+N) \quad (16b)$$
$$\boldsymbol{\theta}_H^i \in \Theta_H^i,\ i \in \mathcal{I}_H^j \quad (16c)$$
$$r_H \text{ satisfying (8e) for every } i \in \mathcal{I}_H^j \quad (16d)$$
$$\mathbf{x}(0) \in \mathcal{C}_H^{\delta_2}(0;\boldsymbol{\theta}_H), \quad \text{if } \mathcal{I}_H^j = \mathcal{I}_H^0 \quad (16e)$$
$$\mathbf{x}(\tau_j+N) \in \mathcal{C}_F(\tau_j+N), \quad (16f)$$
$$\mathbf{x}(\tau_j) = \mathbf{x}_{\tau_j}, \quad (16g)$$
$$\mathbf{x} \in \mathbb{X}, \quad t \in [\tau_j, \tau_j+N] \quad (16h)$$
$$\mathbf{u} \in \mathbb{U}, \quad t \in [\tau_j, \tau_j+N] \quad (16i)$$
$$\epsilon \in \mathbb{R}_{\geq 0}, \quad t \in [\tau_j, \tau_j+N] \quad (16j)$$

where δ_2 is a strictly positive tuning parameter such that $\mathfrak{b}_H(\mathbf{x}(0),0;\boldsymbol{\theta}_H) \geq \delta_2$. The performance criterion of (16) is defined as the sum of two cost functions as follows:

$$J(\mathbf{x},\mathbf{u},\epsilon,r_H,\boldsymbol{\theta}_H) = \int_{\tau_j}^{\tau_j+N}\left(\|\mathbf{u}\|^2+\|\mathbf{x}\|^2+\|\epsilon\|^2\right)dt + \sum_{i\in\mathcal{I}_H^j} t_H^{i*} - r_H. \quad (17)$$

The first function expresses the state, input and task violation costs over the horizon while the second is introduced as a function of the parameters of the barrier $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H)$. The goal of this problem is hence threefold: 1) to minimize operational costs usually considered in RHS problems while 2) maximizing the robustness r_H of ϕ and 3) minimizing the time instants t_H^{i*} at which each task φ_i is satisfied with robustness r_H.

Equation (16a) defines the dynamics of the multi-agent system. Assuming that $\mathfrak{b}_H(\mathbf{x}(\tau_j),\tau_j;\boldsymbol{\theta}_H) \geq 0$ and $\epsilon(t)=0$ is true for every $t \in [\tau_j, \tau_j+N)$, constraint (16b) guarantees that the states of the agents will stay in $\mathcal{C}_H(t;\boldsymbol{\theta}_H)$ for all $t \in [\tau_j, \tau_j+N)$. However, when this is not possible, e.g., due to input limitations, agents' states are allowed to lie outside the set $\mathcal{C}_H(t;\boldsymbol{\theta}_H)$. This is encoded by allowing $\epsilon(t)$ to take a positive value. One of the main goals of this RHS framework is to minimize the value of $\epsilon(t)$ so as agents stay at the closest possible distance from $\mathcal{C}_H(t;\boldsymbol{\theta}_H)$. Constraints (16c), (16d) impose conditions for the choice of the parameters of the barriers and the robustness value respectively. Constraint (16e) forces agents to lie in the interior of $\mathcal{C}_H(0;\boldsymbol{\theta}_H)$. Note that (16i) is omitted when $\mathcal{I}_H^j \subset \mathcal{I}_H^0$ increasing the flexibility in the choice of the parameters of $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_H)$ allowing the design of gamma functions with an increased robustness value r_H. Constraint (16f) guarantees that the final state of the system lies inside the set $\mathcal{C}_F(\tau_j+N)$ and (16g) determines the initial condition of the system. Finally, (16h)-(16i) define the allowable values of the system's state and input, respectively while (16j) constrains the violating factor to be non-negative.

Due to the deactivation policy and since the time instants $t_{j,H}^{i*}$ are decision variables of (16), the proposed optimization problem becomes a Mixed Integer Nonlinear problem (MINLP) which can be solved by global optimization solvers or other solvers like BARON [25] and SCIP [26]. Nevertheless, if the sampling rate $\Delta\tau$ is chosen such that $\Delta\tau|\bar{b}_{l_i}$ for every $i \in \mathcal{I}_H^{j,\mathcal{F}}$, then (16) becomes a Nonlinear Program (NLP) with continuous variables.

C. Theoretical Analysis

For the optimal control problem (16) we make the following assumption on the regularity of $\mathbf{u}(t)$ on any time interval $[\tau_j, \tau_j+N],\ j \in \mathcal{J}$:

Assumption 4. *Any control input $\mathbf{u} : [\tau_j, \tau_j+N] \to \mathbb{U},\ j \in \mathcal{J}$ satisfying (16a)-(16j) is continuous a.e. in $[\tau_j, \tau_j+N]$.*

Assumption 4 is a common assumption in continuous time model predictive control schemes, introduced to ensure the existence of solutions of (3). Let $\{\eta_1, \ldots, \eta_{s(\mathbf{u})}\}$ denote the points of discontinuity of the control input $\mathbf{u}(t) \in \mathbb{U}$ in the interval $[\tau_j, \tau_j+N]$. For brevity we will omit the dependency of s on $\mathbf{u}$ when $\mathbf{u}$ is clear from context. From now on we restrict our analysis to linear extended class $\mathcal{K}$ functions $\alpha_H(\xi) = \alpha_H\xi$.

Proposition 1. *Consider the dynamical system (3) under the STL task ϕ defined by (2). Let Assumption 4 hold. Assume further that (16) is feasible over the time interval $[\tau_j, \tau_j+N],\ j \in \mathcal{J}$ with $\alpha_H(\xi) = \alpha_H\xi$ a linear, extended class $\mathcal{K}$ function and let $(\mathbf{u}_j, \epsilon_j, \boldsymbol{\theta}_{j,H}, r_{j,H})$ be a feasible solution of (16) over $[\tau_j, \tau_j+N]$. In addition, let $\mathbf{x}_j : [\tau_j, \tau_j+N] \to \mathbb{X}$ be a solution to (3) with $\mathbf{x}(\tau_j) = \mathbf{x}_{\tau_j}$ when $\mathbf{u}_j(t)$ is applied a.e. in $[\tau_j, \tau_j+N]$. Then, for any $t \in [\tau_j, \tau_j+N)$ it holds:*

$$\mathbf{x}_j(t) \in \mathcal{C}_{wc}^j(t;\boldsymbol{\theta}_{j,H}), \quad (18)$$

where $\mathcal{C}_{wc}^j(t;\boldsymbol{\theta}_{j,H}) = \left\{\mathbf{x} \in \mathbb{R}^n : \mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_{j,H}) \geq \alpha_H^{-1}(-\epsilon_{wc}^j)+c\right\}$, $c = \min\left(0, \mathfrak{b}_H(\mathbf{x}_j(\tau_j),\tau_j;\boldsymbol{\theta}_{j,H})\right)$ and $\epsilon_{wc}^j = \sup_{t\in[\tau_j,\tau_j+N)} \epsilon_j(t)$.

Proof. The proof of Proposition 1 is given in Appendix II. ■

For any feasible solution of (16) at $[\tau_j, \tau_j+N]$, Proposition 1 ensures a worst-case lower bound of the barrier function that depends on the violating factor $\epsilon(t), t \in [\tau_j, \tau_j+N)$ and the initial value of the barrier function at τ_j. This proposition will be used later in Theorem 3 to determine a lower bound on the robustness of the STL formula ϕ. Next, assuming the initial feasibility of (16), we show the recursive feasibility property of the proposed RHS scheme.

Theorem 2. *Consider the system* (3) *and the STL formula* ϕ *defined by* (2)*. Let Assumptions 1-4 hold. Assume further that* (16) *is feasible at* τ_0*, and* $\alpha_H(\xi)$ *is a linear, extended class* $\mathcal{K}$ *function with* $\alpha_H(\xi) = \alpha_H \xi$*. Then,* (16) *is recursively feasible for the same class* $\mathcal{K}$ *function* $\alpha_H(\xi)$*.*

Proof. The proof of Theorem 2 is given in Appendix III. ∎

Having shown the recursive feasibility property of the RHS scheme and the existence of a worst case lower bound of the online designed barrier function $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H)$ at each time interval $[\tau_j, \tau_j + N)$, we can now determine a lower bound on the robustness of the STL formula ϕ as follows:

Theorem 3. *Let the Assumptions of Theorem 2 hold. Let* $\mathbf{x}_{CL} : [0, \tau_J] \to \mathbb{X}$ *be a solution of* (3) *under the control law:*

$$\kappa(\mathbf{x}(t)) = \begin{cases} \mathbf{u}_j(t), & t \in [\tau_j, \tau_{j+1}),\ j \in \mathcal{J} \backslash \{J\} \\ \mathbf{u}_J(t), & t = \tau_J \end{cases}$$

Then, $\rho^\phi(\mathbf{x}_{CL}, 0) \geq \min_{j\in\mathcal{J}} \bar{\rho}_j$ *with* $\bar{\rho}_j \geq \inf_{t\in[\tau_j,\tau_j+N)} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) \geq c + \alpha_H^{-1}(-\epsilon_{wc}),\ j \in \mathcal{J}$*, where* $\mathbf{x}_j : [\tau_j, \tau_j + N] \to \mathbb{X}$ *is the solution to* (3) *when applying the optimal control input* $\mathbf{u}_j(t), t \in [\tau_j, \tau_j + N]$*,* $c = \min\big(0, \mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H})\big)$*,* $\epsilon_{wc} = \max_{j\in\mathcal{J}} \epsilon_{wc}^j$*,* $\epsilon_{wc}^j = \sup_{t\in[\tau_j,\tau_j+N)} \epsilon(t)$ *and* $\bar{\rho}_j$ *is maximized within* $[\tau_j, \tau_j + N)$*.*

Proof. The proof of Theorem 3 is given in Appendix IV. ∎

VI. Simulation Results

In this Section we present a simulation scenario for $R = 4$ agents. All simulations are performed in an Intel Core i7-8665U with 16GB RAM using MATLAB.

The multi-robot dynamics are given by:

$$\dot{\mathbf{x}} = b \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ -2 & 0 & -2 & 4 \end{bmatrix} \otimes I_2 + \mathbf{u}$$

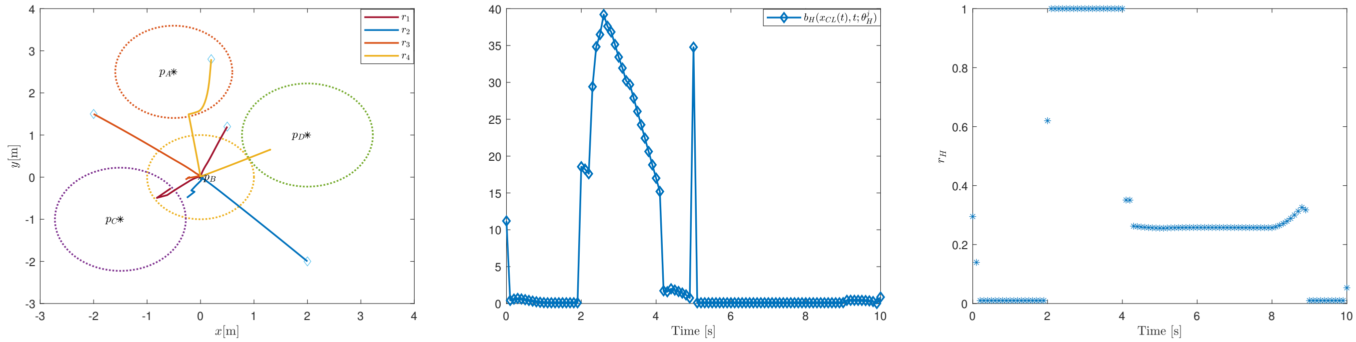
where $\mathbf{x}^T = \begin{bmatrix} \mathbf{x}_1^T & \dots & \mathbf{x}_4^T \end{bmatrix}$, $\mathbf{x}_k^T = \begin{bmatrix} x_k & y_k \end{bmatrix}$, $k = 1, \dots, 4$, $b = 0.025$ and I_n is the $n \times n$ identity matrix. The agents are under state and inputs constraints with $d_x^k = 4,\ d_u = 20,\ k = 1, \dots, 4$. Next consider the formula $\phi = \bigwedge_{i=1}^4 \varphi_i$ with the sub-formulas $\varphi_i, i = 1, \dots, 4$ defined as: $\varphi_1 = \mathcal{G}_{[0,2]}\psi_1$, $\varphi_2 = \mathcal{F}_{[2,5]}\psi_2$, $\varphi_3 = (\psi_3 \wedge \psi_4)\, \mathcal{U}_{[5,10]} \psi_5$ and $\varphi_4 = \mathcal{F}_{[5,10]}\psi_6$. The predicate functions corresponding to $\psi_i,\ i = 1, \dots 6$ are defined as: $h_1(\mathbf{x}) = 1.2 - \|\mathbf{x}_4 - p_A\|^2$, $h_2(\mathbf{x}) = 1 - \|\mathbf{x}_1 - p_B\|^2$, $h_3(\mathbf{x}) = 1.2 - \|\mathbf{x}_2 - \mathbf{x}_1 - p_{\text{up}}\|^2$, $h_4(\mathbf{x}) = 1.2 - \|\mathbf{x}_3 - \mathbf{x}_1 - p_{\text{down}}\|^2$, $h_5(\mathbf{x}) = 2 - \|\mathbf{x}_1 - p_C\|^2$ and $h_6(\mathbf{x}) = 1.5 - \|\mathbf{x}_4 - p_D\|^2$ where $p_A = \begin{bmatrix} -0.5 & 2.5 \end{bmatrix}^T$, $p_B = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$, $p_C = \begin{bmatrix} -1.5 & -1 \end{bmatrix}^T$, $p_D = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$, $p_{\text{up}} = \begin{bmatrix} -0.3 & 0.3 \end{bmatrix}^T$ and $p_{\text{down}} = \begin{bmatrix} -0.3 & -0.3 \end{bmatrix}^T$. Based on ϕ agent 4 needs to stay close to p_A for 2 sec and agent 1 should approach p_B between 2 and 5 sec. Then, agents 1,2 and 3 move as a formation until agent 1 approaches p_C between 5 and 10 sec while agent 4 eventually approaches p_D within the same time interval. The optimization horizon and sampling rate are chosen as $N = 1$ and $\Delta\tau = 0.1$ respectively. Observe that for any $i \in \mathcal{I}_H^{j,\mathcal{F}}$ it holds that $\frac{\bar{b}_{l_i}}{\Delta\tau} \in \mathcal{J}$, hence due to (8d), $t_H^{i*} = \bar{b}_{l_i}, \forall i \in \mathcal{I}_H^{j,\mathcal{F}}$, and thus the problem becomes a nonlinear problem with continuous variables. Observe that Assumption 1 is satisfied since $B = I_8$ while $d_u - \sigma_{\max}(A) d_x = 19.4737$.

In Figure 1b the evolution of $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H)$ is shown when $\boldsymbol{\theta}_H \in \Theta_H$ is found as a solution to (16). The jumps shown at time instants $t = 0, 2, 5$ and 10 are a result of the deactivation policy defined in (9). Observe that $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H) \geq 0.0059$ for any $t \in [0, 10]$. Hence, considering the fact that $\epsilon_j(t) = 0$ for any $j \in \mathcal{J}$ and $t \in [\tau_j, \tau_j + N]$, by Theorem 3 we may conclude that $\rho^\phi(\mathbf{x}_{\text{CL}}, 0) \geq \min_j \bar{\rho}_j \geq 0$ where $\bar{\rho}_j \geq r_{j,H}$. In Figure 1c it is shown that $r_{j,H} \geq 0.01$ for any $j \in \mathcal{J}$. Combining the results above, $\rho^\phi(\mathbf{x}_{\text{CL}}, 0) \geq 0.01$. The agents' trajectories found by the proposed RHS framework are shown in Figure 1a.

For comparison we design the feedback controller proposed in [22], found as a solution to a quadratic program (QP) where $\mathbb{U} = \mathbb{R}^8$ when the performance functions $\gamma_H^i(t; \boldsymbol{\theta}_H^i)$ are linearly defined as in (7). The parameters $\boldsymbol{\theta}_H^i \in \Theta_H^i$ are those guaranteeing the initial feasibility of the RHS with the robustness value r chosen equal to 0.009. In Figure 2 the agents' trajectories are shown for the QP-based feedback controller of [22]. Notice that agent 4 after satisfying ϕ_1, being affected by its dynamics tries to stay away from agent 2 and 3 before heading towards p_D. The evolution of the barrier function $\mathfrak{b}(\mathbf{x}_{\text{fdbk}}, t)$ when applying the QP-based controller is presented in Figure 3. Observe that $\mathfrak{b}(\mathbf{x}_{\text{fdbk}}, t) \geq 0.007$ implying $\rho^\phi(\mathbf{x}_{\text{fdbk}}, 0) \geq 0.009$ [22, Thm 1]. Based on the above, we can conclude that the proposed controller ensures a higher robustness of the STL task while respecting the input and state constraints of the system at all times. In addition, the actuation costs are considerably low when compared to those of the feedback controller with $\sup_{t\in[0,10]} \|\kappa(\mathbf{x}_{\text{CL}}(t))\|$ being 53.91% of the corresponding value of the QP-based controller. On the downside is the average computational time required for solving the RHS online, being 5.98 sec contrary to 12.3 msec for the QP. In this work our focus was on the design of a RHS scheme that ensures STL satisfaction under input constraints. As part of our future work, we plan to work towards decreasing the computational time of the problem.

VII. Discussion-Future Work

A RHS framework is proposed for a linear system under Signal Temporal Logic Tasks. The STL specifications are encoded using time-varying shaped online towards maximizing the robustness of the STL task. The recursive feasibility of the proposed scheme is proven by introducing a time-varying terminal constraint that ensures a worst-case temporal behavior of the system towards the satisfaction of the task with a predetermined robustness. The proposed framework is centralized, thus amenable to "curse of dimensionality" problems when a large number of robots is considered. Additionally, the actions of the agents are chosen such that the robustness of the



(a) Agents' Trajectories (b) Barrier Function Evolution (c) Robustness r_H

Fig. 1: The closed loop trajectory $\mathbf{x}_{\text{CL}}(t)$, the evolution of the barrier function $\mathfrak{b}_H(\mathbf{x}_{\text{CL}}(t), t; \boldsymbol{\theta}_H)$ and the robustness r_H found as a solution to the proposed RHS framework.

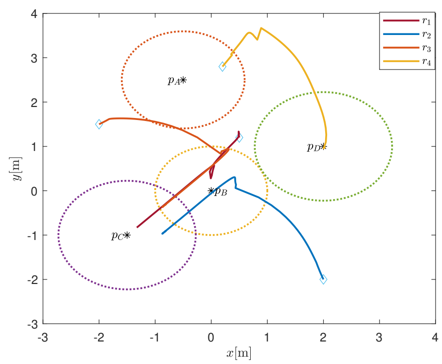


Fig. 2: Agent's Trajectories with the feedback controller proposed in [22].

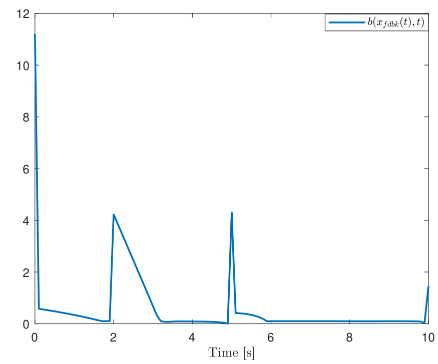


Fig. 3: Barrier Function Evolution with the feedback controller proposed in [22].

STL tasks is maximized with respect to $(\mathbf{x}, t)$. As a result the complexity of the problem is significantly increased compared to existing CBF-based STL feedback approaches. Future work will focus on the design of a decentralized framework as also on reducing the complexity of the problem by for example exploring alternative ways for encoding the satisfaction of ϕ.

Appendix I
Proof of Theorem 1

For the proof of Theorem 1, we consider the following lemma:

Lemma 1. *For any matrix $B \in M_{n\times m}(\mathbb{R})$ with $B \neq 0$ it holds: $\|B^\dagger\| = \frac{1}{\sigma_{\min}(B)}$, where $\sigma_{\min}(B)$ is the minimum, non-zero singular value of B.*

Proof. Consider the SVD decomposition of B, i.e., $B = U\tilde{\Sigma}V^T$. Then, $B^\dagger$ can be written as $B^\dagger = V\tilde{\Sigma}^\dagger U^T$ where $\tilde{\Sigma}^\dagger$ is defined by taking the reciprocals of the non-zero diagonal elements of $\tilde{\Sigma}$, leaving the zero elements at place and transposing the resulting matrix [27, Rem. 2.2]. By [28, Thm. 2.1.4] the Euclidean norm is unitarily invariant. Therefore, $\|B^\dagger\| = \|V\tilde{\Sigma}^\dagger U^T\| = \|\tilde{\Sigma}^\dagger\| = \sigma_{\max}(\tilde{\Sigma}^\dagger) = \frac{1}{\sigma_{\min}(\tilde{\Sigma})} = \frac{1}{\sigma_{\min}(B)}$. ■

The proof of Theorem 1 is divided in 2 parts. In part 1 we the existence and continuity of the terminal controller for every $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$ and in part 2 we deduce $\rho^{\phi'}(\mathbf{x}, 0) \geq r_F > 0$.

Part 1: Let δ_1 a known constant chosen as in Assumption 3. By Assumption 2 and the choice of the parameters $\boldsymbol{\theta}_F \in \Theta_F$ it holds that δ_1 is strictly positive. We may split the analysis of part 1 considering the following two cases: 1) $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| \leq \delta_1$ and 2) $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| > \delta_1$. By Assumption 3, the control law $\bar{\mathbf{u}} = \mathbf{0}$ ensures that $\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}} A\mathbf{x} + \frac{\partial \mathfrak{b}_F}{\partial t} \geq -\alpha_F(\mathfrak{b}_F(x, t))$ for every $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| \leq \delta_1$. Next consider the case when $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| > \delta_1$. First we show that (13) has a feasible solution $\mathbf{u}_{\text{feas}}$ that lies in the interior of the solution space defined by (13a) and the constraint $\|\mathbf{u}\| \leq d_u$. This is an essential requirement for guaranteeing the existence of the Lagrange multipliers in the KKT conditions (22) when the linear independent constraint qualification [29, Thm. 8] is not guaranteed. By assumption 1, B is of full row rank. Hence, we may define the Moore-Penrose matrix of B as $B^\dagger = B^T(BB^T)^{-1}$. A candidate solution of (13) could be the following: $\mathbf{u}_{\text{feas}} = B^\dagger(-A\mathbf{x} + \mathbf{v}_{\text{feas}})$ where $\mathbf{v}_{\text{feas}} = (L_t - \alpha_F(\chi))\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\|^{-2}\frac{\partial \mathfrak{b}_F^T}{\partial \mathbf{x}}$. Next we show that $\mathbf{u}_{\text{feas}}$ lies in the interior of the solution space of (13). Substituting

$\mathbf{u} = \mathbf{u}_{\text{feas}}$ to $\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x} + B\mathbf{u})$ we get:

$$\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x} + B\mathbf{u}_{\text{feas}}) = \frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\mathbf{v}_{\text{feas}} = L_t - \alpha_F(\chi). \tag{19}$$

For any $(\mathbf{x}, t) \in \Omega \times \mathbb{R}_{\geq 0}$ the following is true: $\mathfrak{b}_F(\mathbf{x}, t) \geq \inf_{(\mathbf{x},t)\in\Omega\times\mathbb{R}_{\geq 0}} \mathfrak{b}_F(\mathbf{x}, t) > \chi$. Hence, for the class $\mathcal{K}$ function $\alpha_F(\cdot)$ it holds:

$$\alpha_F(\mathfrak{b}_F(\mathbf{x}, t)) - \alpha_F(\chi) > 0. \tag{20}$$

By the choice of $\gamma_F^i(t)$ in (7) we have:

$$\frac{d\gamma_F^i(t)}{dt} = \begin{cases} \frac{\gamma_{F,\infty}^i - \gamma_{F,0}^i}{t_F^{i*}}, & t < t_F^{i*} \\ 0, & t \geq t_F^{i*} \end{cases}.$$

For every $t \geq 0$ and due to the particular choice of $\boldsymbol{\theta}_F$ (i.e., $\boldsymbol{\theta}_F$ satisfies (8a)-(8e)) it holds that $0 \leq \frac{d\gamma_F^i(t)}{dt} \leq \frac{d\gamma_F^i(t)}{dt}|_{t=0}$. Let $L_t = L_t(\boldsymbol{\theta}_F) = \max_{i\in\mathcal{I}_F} \frac{d\gamma_F^i(t)}{dt}|_{t=0}$. Then, for every t the following is true:

$$\begin{aligned} \frac{\partial \mathfrak{b}_F}{\partial t} &= \frac{-\sum_{i\in\mathcal{I}_F} o_F^i(t)\exp\left(-\mathfrak{b}_F^i(\mathbf{x}, t)\right)\frac{d\gamma_F^i(t)}{dt}}{\sum_{i\in\mathcal{I}_F} o_F^i(t)\exp\left(-\mathfrak{b}_F^i(\mathbf{x}, t)\right)} \\ &\geq -L_t \frac{\sum_{i\in\mathcal{I}_F} o_F^i(t)\exp\left(-\mathfrak{b}_F^i(\mathbf{x}, t)\right)}{\sum_{i\in\mathcal{I}_F} o_F^i(t)\exp\left(-\mathfrak{b}_F^i(\mathbf{x}, t)\right)} \\ &\geq -L_t. \end{aligned} \tag{21}$$

Based on (19)-(21) we may conclude that:

$$\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x} + B\mathbf{u}_{\text{feas}}) + \frac{\partial \mathfrak{b}_F}{\partial t} + \alpha_F(\mathfrak{b}_F(\mathbf{x}, t)) > 0,$$

for any $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| > \delta_1$. Furthermore, considering Lemma 1 and Assumption 2 we have:

$$\begin{aligned} \|\mathbf{u}_{\text{feas}}\| &\leq \frac{1}{\sigma_{\min}(B)}\left(\sigma_{\max}(A)d_x + \frac{L_t + |\alpha_F(\chi)|}{\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\|}\right) \\ &< \frac{1}{\sigma_{\min}(B)}\left(\sigma_{\max}(A)d_x + \frac{L_t + |\alpha_F(\chi)|}{\delta_1}\right) \\ &< d_u. \end{aligned}$$

Since $\mathbf{u}_{\text{feas}}$ belongs in the interior of the solution space defined by (13a) and the constraint $\mathbf{u} \in \mathbb{U}$, Slater's constraint qualification is satisfied. Hence, by [30, Prp 3.3.9] the KKT optimality conditions imply that for any $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| > \delta_1$ there exist $\lambda_{\bar{w}}(\mathbf{x}, t) \geq 0,\ \bar{w} = 1, 2$ such that $\bar{\mathbf{u}}'$ and $\lambda_{\bar{w}}(\mathbf{x}, t) \geq 0,\ \bar{w} = 1, 2$ satisfy:

$$\begin{cases} 2(\lambda_2(\mathbf{x}, t) + 1)\bar{\mathbf{u}}' - \lambda_1(\mathbf{x}, t)B^T\frac{\partial \mathfrak{b}_F^T}{\partial \mathbf{x}} = \mathbf{0}, \\ \frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x} + B\bar{\mathbf{u}}') + \frac{\partial \mathfrak{b}_F}{\partial t} + \alpha_F(\mathfrak{b}_F(\mathbf{x}, t)) \geq 0, \\ \|\bar{\mathbf{u}}'\| - d_u \leq 0, \\ \lambda_1(\mathbf{x}, t) = 0, \text{ if } \frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}(A\mathbf{x} + B\bar{\mathbf{u}}') + \frac{\partial \mathfrak{b}_F}{\partial t} + \alpha_F(\mathfrak{b}_F(\mathbf{x}, t)) > 0, \\ \lambda_2(\mathbf{x}, t) = 0, \text{ if } \|\bar{\mathbf{u}}'\| - d_u < 0. \end{cases} \tag{22}$$

Since the problem (13) is convex, these conditions are also sufficient [31]. Therefore, for any $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| > \delta_1$ we may define $\bar{\mathbf{u}}'$ in closed form as follows:

$$\bar{\mathbf{u}}' = \begin{cases} \mathbf{0}, & C(\mathbf{x}, t) > 0 \\ \mathbf{u}_1, & C(\mathbf{x}, t) \in \left[-d_u\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}B\right\|, 0\right] \end{cases},$$

where $C(\mathbf{x}, t) = \frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}A\mathbf{x} + \frac{\partial \mathfrak{b}_F}{\partial t} + \alpha_F(\mathfrak{b}_F(\mathbf{x}, t))$ and $\mathbf{u}_1 = -\frac{C(\mathbf{x},t)}{\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}B\|^2}B^T\frac{\partial \mathfrak{b}_F^T}{\partial \mathbf{x}}$. Next, we denote by $\bar{\mathbf{u}} = \bar{\mathbf{u}}(\mathbf{x}, t)$ the control input for any $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$. Consider any $(\mathbf{x}, t) \in \Omega \times (\sigma_F^s, \sigma_F^{s+1})$, $\sigma_F^s \in \Sigma_F$ with $\left\|\frac{\partial \mathfrak{b}_F}{\partial \mathbf{x}}\right\| = \delta_1$. By continuity of $C(\mathbf{x}, t)$ and for any $(\mathbf{x}', t')$ in a neighborhood $\mathcal{U}$ around $(\mathbf{x}, t)$ it holds $C(\mathbf{x}', t') \geq 0$. Hence, $\bar{\mathbf{u}}(\mathbf{x}', t') = \mathbf{0}$ holds, implying continuity of $\bar{\mathbf{u}}(\mathbf{x}, t)$ in $\mathbf{x}$.

Part 2: By continuity of $\bar{\mathbf{u}}$ in $\mathbf{x}$ there exist solutions $\mathbf{x} : [0, \tau_{\max}) \to \Omega$ with $\tau_{\max} > 0$. Constraint (13a) is equivalent to $\dot{\mathfrak{b}}_F(\mathbf{x}, t) \geq -\alpha_F(\mathfrak{b}_F(\mathbf{x}, t))$ with $\mathfrak{b}_F(\mathbf{x}(0), 0) \geq 0$. Then, by [32, Lem. 4.4] and the Comparison Lemma [32, Ch. 3.4] it follows $\mathfrak{b}_F(\mathbf{x}, t) \geq 0$ for all $t \in (0, \min(\tau_{\max}, \sigma_F^1))$. Assuming that $\tau_{\max} \geq \sigma_F^1$ and using similar arguments we may conclude that $\mathfrak{b}_F(\mathbf{x}, t) \geq 0$ for all $t \in (\sigma_F^1, \min(\tau_{\max}, \sigma_F^2))$. Note that for $\mathbf{x}$ satisfying $\mathfrak{b}_F(\mathbf{x}, t) \geq 0$ as $t \to \sigma_F^{1-}$ it holds $\lim_{t\to\sigma_F^{1-}} \sum_{i\in\mathcal{I}_F} o_F^i(t)\exp\left(-\mathfrak{b}_F^i(\mathbf{x}, t)\right) \geq \sum_{i\in\mathcal{I}_F} o_F^i(\sigma_F^1)\exp\left(-\mathfrak{b}_F^i(\mathbf{x}, \sigma_F^1)\right)$. As a result, we have $0 \leq \lim_{t\to\sigma_F^{1-}} \mathfrak{b}_F(\mathbf{x}, t) \leq \mathfrak{b}_F(\mathbf{x}, \sigma_F^1)$. This implies that $\mathbf{x}(t) \in \mathcal{C}_F(t)$ for any $t \in [\sigma_F^1, \min(\tau_{\max}, \sigma_F^2))$. The previous arguments may be repeated unless there exists a $\sigma_F^s \in \Sigma_F$ such that $\tau_{\max} < \sigma_F^s$. If $\mathfrak{b}_F(\mathbf{x}, t) \geq 0$ is true for any $t \in (\sigma_F^s, \sigma_F^{s+1})$, then, as discussed in Section II-B, $\mathfrak{b}_F^i(\mathbf{x}, t) \geq 0$ holds for any $i \in \mathcal{I}_F$ with $o_F^i(t) \neq 0$ implying $h_i(\mathbf{x}(t)) \geq \gamma_F^i(t)$. Since $\boldsymbol{\theta}_F \in \Theta$ for any $i = p + k$, $k = 1, \ldots, R$ the relation $h_i(\mathbf{x}(t)) \geq \gamma_F^i(t)$ implies that $\|\mathbf{x}_k\|^2 \leq d_x^{k\,2} - r_F$. Hence, any solution $\mathbf{x}$ lies within the compact set $\mathbb{X}$. If $\tau_{\max} < \infty$ by [33, Ch. 2, Thm 1.3] there exists a $\bar{\beta} > 0$ such that $\mathbf{x}$ is continued on $[0, \tau_{\max} + \bar{\beta})$ which leads to contradiction. Hence, the solutions $\mathbf{x}$ of (3) are defined for every $t \geq 0$.

Next we show $\rho^{\phi'}(\mathbf{x}, 0) \geq r_F >$ with r_F a design parameter of $\mathfrak{b}_F(\mathbf{x}, t)$. Note that the STL formula ϕ' is a conjunction of always and eventually formulas. By definition of the robust semantics we have: $\rho^{\phi'}(\mathbf{x}, 0) = \min_{i\in\mathcal{I}_F} \rho^{\varphi_i}(\mathbf{x}, 0)$. If the formula φ_i is an always formula by the robustness semantics, the choice of $\boldsymbol{\theta}_F$, and the non-negativity of $\mathfrak{b}_F^i(\mathbf{x}, t)$ we have:

$$\rho^{\varphi_i}(\mathbf{x}, 0) \geq \min_{t_1\in[a_i, b_i]} \gamma_F^i(t) \geq \gamma_F^i(a_i) \geq r_F.$$

The above is true since by design of $\gamma_F^i(t)$ it holds that $\gamma_F^i(t) \geq r_F$ for any $t \geq t_F^{i*} = a_i$. If the formula i is an eventually formula, it holds:

$$\rho^{\varphi_i}(\mathbf{x}, 0) \geq \max_{t_1\in[a_i, t_F^{i*}]} h_i(\mathbf{x}(t_1)) \geq \max_{t_1\in[a_i, t_F^{i*}]} \gamma_F^i(t) \geq r_F.$$

Based on the above we may conclude that $\rho^{\phi'}(\mathbf{x}, 0) = \min_{i\in\mathcal{I}_F} \rho^{\varphi_i}(\mathbf{x}, 0) \geq r_F$. This concludes the proof.

Appendix II
Proof of Proposition 1

For the proof of Proposition 1 the following lemma will be useful:

Lemma 2. *Consider the initial value problem:* $\dot{y} = -\alpha(y) - \epsilon$, $y(t_1) = y_1$ *where* $\alpha(\xi) = \alpha\xi$ *is a linear, extended class* $\mathcal{K}$ *function and* $\epsilon \geq 0$*. Then, the solution to this initial value problem is given by:* $y(t) = e^{-\alpha(t-t_1)}(y_1 + \alpha^{-1}(\epsilon)) + \alpha^{-1}(-\epsilon)$, $t \geq t_1$.

Proof. The proof follows standard arguments for solving first-order differential equations with given initial conditions and is thus omitted. ■

Next, we continue with the proof of Proposition 1. By Assumption 4 and the feasibility of (16) there exists an absolute continuous function $\mathbf{x}_j : [\tau_j, \tau_j + N] \to \mathbb{X}$ satisfying (3) when $\mathbf{u}_j(t)$ is applied a.e. in $[\tau_j, \tau_j + N]$. Let $\Sigma_H = (\{t^{i*}_{j,H} : i \in \mathcal{I}^{j,\mathcal{F}}_H\} \cup \{a_i, b_i : i \in \mathcal{I}^{j,\mathcal{G}}_H, a_i \neq 0\} \cup \{b_i : i \in \mathcal{I}^{j,\mathcal{G}}_H, a_i = 0\}) \backslash \{i_H\}$ and $H = \{\eta_\omega : \omega = 1, \ldots, \bar{s}\}$ the points of discontinuity of $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_{j,H})$ and $\mathbf{u}_j(t)$ respectively. Let $\Sigma = \Sigma_H \cap [\tau_j, \tau_j + N)$. We next consider two cases: 1) $t \in [\tau_j, \tau_j + N) \backslash (\Sigma \cup H)$ and 2) $t \in \Sigma \cup H$, i.e., t is a time instant at which either the derivative of $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H)$ with respect to $(\mathbf{x}, t)$ does not exist or $\dot{\mathbf{x}}$ is not continuous with respect to t.

Case 1: By feasibility of (16), $\mathbf{x}_j(t)$ satisfies (16b) for any $t \in [\tau_j, \tau_j + N) \backslash (\Sigma \cup H)$. Let $\epsilon^j_{\mathrm{wc}} = \sup_{t \in [\tau_j, \tau_j + N)} \epsilon_j(t)$. Since α_H is a linear, class $\mathcal{K}$ function we have $\alpha_H^{-1}(-\epsilon_j(t)) \geq \alpha_H^{-1}(-\epsilon^j_{\mathrm{wc}})$ for any $t \in [\tau_j, \tau_j + N)$. As a result, the satisfaction of (16b) implies the satisfaction of the following inequality:

$$\frac{\partial \mathfrak{b}_H}{\partial \mathbf{x}}(A\mathbf{x}_j + B\mathbf{u}_j) + \frac{\partial \mathfrak{b}_H}{\partial t} \geq -\alpha_H(\mathfrak{b}_H(\mathbf{x}_j, t; \boldsymbol{\theta}_{j,H})) - \epsilon^j_{\mathrm{wc}}. \tag{23}$$

Applying the Comparison Lemma [32, Ch. 3.4] to (23) and due to Lemma 2 the following holds:

$$\begin{aligned} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) \geq e^{-\alpha_H(t-\tau_j)} \big(&\mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H}) \\ &+ \alpha_H^{-1}(\epsilon^j_{\mathrm{wc}}) \big) + \alpha_H^{-1}(-\epsilon^j_{\mathrm{wc}}). \end{aligned} \tag{24}$$

By (16j) and for the linear, class $\mathcal{K}$ function $\alpha_H(\xi) = \alpha_H \xi$ we have $\alpha_H^{-1}(\epsilon^j_{\mathrm{wc}}) \geq 0$. Hence, it holds that $\mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H}) + \alpha_H^{-1}(\epsilon^j_{\mathrm{wc}}) \geq \mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H})$. This implies the following:

$$\begin{aligned} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) \geq e^{-\alpha_H(t-\tau_j)} &\mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H}) \\ &+ \alpha_H^{-1}(-\epsilon^j_{\mathrm{wc}}). \end{aligned} \tag{25}$$

If $\mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H}) \geq 0$, the inequality above implies $\mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) \geq \alpha_H^{-1}(-\epsilon^j_{\mathrm{wc}})$. If $\mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H}) < 0$, given that $t \in [\tau_j, \tau_j + N)$, (25) becomes:

$$\begin{aligned} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) \geq \mathfrak{b}_H(&\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H}) \\ &+ \alpha_H^{-1}(-\epsilon^j_{\mathrm{wc}}). \end{aligned}$$

Setting $c = \min\big(0, \mathfrak{b}_H(\mathbf{x}_j(\tau_j), \tau_j; \boldsymbol{\theta}_{j,H})\big)$ the result follows.

Case 2: By design of $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H)$ for any $\sigma_H \in \Sigma$ it holds that: $\lim_{t \to \sigma_H^-} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) \leq \mathfrak{b}_H(\mathbf{x}_j(\sigma_H), \sigma_H; \boldsymbol{\theta}_{j,H})$. Additionally, for any $t \in H \backslash \Sigma$ the barrier function $\mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H})$ is continuous in $(\mathbf{x}_j, t)$ and $\mathbf{x}_j(t)$ is by Definition 1 absolutely continuous in t, thus $\lim_{t \to \eta_\omega^-} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H}) = \mathfrak{b}_H(\mathbf{x}_j(\eta_\omega), \eta_\omega; \boldsymbol{\theta}_{j,H})$. Note that the quantities $\lim_{t \to \sigma_H^-} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H})$ and $\lim_{t \to \eta_\omega^-} \mathfrak{b}_H(\mathbf{x}_j(t), t; \boldsymbol{\theta}_{j,H})$ satisfy the conditions of Case 1, thus are bounded from below by $c + \alpha_H^{-1}(-\epsilon^j_{\mathrm{wc}})$. Hence, the result follows also for $\mathfrak{b}_H(\mathbf{x}_j(\sigma_H), \sigma_H; \boldsymbol{\theta}_{j,H})$ and $\mathfrak{b}_H(\mathbf{x}_j(\eta_\omega), \eta_\omega; \boldsymbol{\theta}_{j,H})$.

Appendix III
Proof of Theorem 2

The recursive feasibility of (16) is proven by induction. Assume that (16) is feasible at τ_j, $j \geq 1$. To prove the feasibility of (16) at τ_{j+1} given its feasibility at τ_j, we propose a candidate solution $(\mathbf{u}_{j+1}, \epsilon_{j+1}, \boldsymbol{\theta}_{j+1,H}, r_{j+1,H})$ over $[\tau_{j+1}, \tau_{j+1} + N]$ and show it satisfies (16a)-(16j).

Let $\mathbf{u}_j : [\tau_j, \tau_j + N] \to \mathbb{U}$, $\epsilon_j : [\tau_j, \tau_j + N] \to \mathbb{R}_{\geq 0}$, $\boldsymbol{\theta}_{j,H} = \big[\boldsymbol{\theta}^i_{j,H}\big]_{i \in \mathcal{I}^j_H}$, $r_{j,H}$ denote the control input, the violating factor, the vector of parameters and the robustness value respectively, found as a solution of (16) over the time interval $[\tau_j, \tau_j + N]$. Consider $\boldsymbol{\theta}_{j+1,H} = \big[\boldsymbol{\theta}^i_{j,H}\big]_{i \in \mathcal{I}^j_H \cap \mathcal{I}^{j+1}_H}$ and $r_{j+1,H} = r_{j,H}$ the vector of parameters and the robustness value used in the design of $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H)$ over the time interval $[\tau_{j+1}, \tau_{j+1} + N]$. Note that by feasibility of (16) at $[\tau_j, \tau_j + N]$, $(\boldsymbol{\theta}_{j+1,H}, r_{j+1,H})$ satisfy the constraints (16c)-(16d). Furthermore, let the candidate control signal:

$$\mathbf{u}_{j+1}(t) = \begin{cases} \mathbf{u}_j(t), & t \in [\tau_{j+1}, \tau_j + N] \\ \bar{\mathbf{u}}(t), & t \in (\tau_j + N, \tau_{j+1} + N] \end{cases},$$

where $\bar{\mathbf{u}} : (\tau_j + N, \tau_{j+1} + N] \to \mathbb{U}$ is the optimal solution of (13). By Theorem 1, (13) is always feasible with the resulting control law $\bar{\mathbf{u}}(\bar{\mathbf{x}}(t), t)$ being continuous in $(\bar{\mathbf{x}}(t), t) \in \Omega \times (\sigma^s_F, \sigma^{s+1}_F)$, $\sigma^s_F \in \Sigma_F$, where $\bar{\mathbf{x}} : [\tau_j + N, \tau_{j+1} + N] \to \mathbb{X}$ is a solution of (3) when $\bar{\mathbf{u}}(\bar{\mathbf{x}}(t), t)$is applied to the system. By Definition 1, $\bar{\mathbf{x}}(t)$ is an absolutely continuous function in t. Hence, the feedback control law $\bar{\mathbf{u}}(\bar{\mathbf{x}}(t), t) = \bar{\mathbf{u}}(t)$ is piecewise continuous over $(\tau_j + N, \tau_{j+1} + N]$. Additionally, by feasibility of (16) at $[\tau_j, \tau_j + N]$, the feasible control $\mathbf{u}_j(t)$ satisfies Assumption 4. As a result the proposed control signal $\mathbf{u}_{j+1}(t)$ satisfies Assumption 4 guaranteeing the existence of solutions of (3) over the interval $[\tau_{j+1}, \tau_{j+1} + N]$.

By Theorem 1, $\mathfrak{b}_F(\bar{\mathbf{x}}(t), t) \geq 0$ is true for all $t \in [\tau_j + N, \tau_{j+1} + N]$. Therefore, (16f) is satisfied. In addition, since $\mathbf{u}_j(t), t \in [\tau_{j+1}, \tau_j + N]$ is a feasible input for the RHS at τ_j, any solution $\mathbf{x}_j$ of (3) satisfies the state constraints, i.e., $\mathbf{x}_j(t) \in \mathbb{X}$ for any $t \in [\tau_{j+1}, \tau_j + N]$. By design of the terminal barrier function, if $\mathfrak{b}_F(\bar{\mathbf{x}}(t), t) \geq 0$, then $\bar{\mathbf{x}}(t) \in \mathbb{X}$ implying that $\mathbf{x}_{j+1}(t) = \bar{\mathbf{x}}(t) \in \mathbb{X}$, for every $t \in (\tau_j + N, \tau_{j+1} + N]$. Note also that $\mathbf{u}_{j+1}(t) \in \mathbb{U}$ for $t \in [\tau_{j+1}, \tau_{j+1} + N]$. Hence, (16h)-(16i) are satisfied.

Let $\mathfrak{b}^i_H(\mathbf{x}, t; \boldsymbol{\theta}^i_{j,H})$, $\mathfrak{b}^i_H(\mathbf{x}, t; \boldsymbol{\theta}^i_{j+1,H})$ be the barrier functions of φ_i designed over the time interval $[\tau_j, \tau_j + N]$ and $[\tau_{j+1}, \tau_{j+1} + N]$ respectively. At the time interval $[\tau_{j+1}, \tau_{j+1} + N]$ it is possible that some formulas φ_i do not contribute to the construction of $\mathfrak{b}_H(\mathbf{x}, t; \boldsymbol{\theta}_H)$ because $b_i < \tau_{j+1}$ holds. This implies that $\mathcal{I}^{j+1}_H \subseteq \mathcal{I}^j_H$. As a result, $\boldsymbol{\theta}^i_{j+1,H} = \boldsymbol{\theta}^i_{j,H}$ is true for any $i \in \mathcal{I}^j_H \cap \mathcal{I}^{j+1}_H$. If $\mathcal{I}^{j+1}_H = \mathcal{I}^j_H = \mathcal{I}^0_H$, then for any $(\mathbf{x}, t) \in \mathbb{X} \times \mathbb{R}_{\geq 0}$ it holds that $\sum_{i \in \mathcal{I}^j_H} o^i_H(t) \exp\left(-\mathfrak{b}^i_H(\mathbf{x}, t; \boldsymbol{\theta}^i_{j,H})\right) = \sum_{i \in \mathcal{I}^{j+1}_H} o^i_H(t) \exp\left(-\mathfrak{b}^i_H(\mathbf{x}, t; \boldsymbol{\theta}^i_{j+1,H})\right)$. The latter implies that $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_{j,H}) = \mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_{j+1,H})$ where $\mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_{j,H}),\ \mathfrak{b}_H(\mathbf{x},t;\boldsymbol{\theta}_{j+1,H})$ denote the barrier functions designed at $[\tau_j,\tau_j+N],\ [\tau_{j+1},\tau_{j+1}+N]$, respectively. Since (16) is feasible at τ_j, it holds that: $\mathfrak{b}_H(\mathbf{x}(0),0;\boldsymbol{\theta}_{j,H}) \geq \delta_2$. Hence, (16e) is satisfied at τ_{j+1}.

The proof is completed by introducing a candidate violation factor $\epsilon_{j+1}(t)$ a.e. in $[\tau_{j+1},\tau_{j+1}+N]$ and showing that (16b), (16j) are satisfied. Let $\beta_H(t') = \mathfrak{b}_H(\bar{\mathbf{x}}(t'),t';\boldsymbol{\theta}_{j+1,H})$ denote the barrier function $\mathfrak{b}_H$ as a function of time with $\bar{\mathbf{x}} : [\tau_j+N,\tau_{j+1}+N] \to \mathbb{X}$ the solution of (3) when applying $\bar{\mathbf{u}}(\mathbf{x},t)$. Additionally, let $\dot{\beta}_H(t') = \frac{d\beta_H}{dt}|_{t=t'} = \frac{\partial \mathfrak{b}_H(\bar{\mathbf{x}}(t'),t';\boldsymbol{\theta}_{j+1,H})}{\partial \mathbf{x}}\dot{\bar{\mathbf{x}}}(t') + \frac{\partial \mathfrak{b}_H(\bar{\mathbf{x}}(t'),t';\boldsymbol{\theta}_{j+1,H})}{\partial t}$ a.e. in $[\tau_j+N,\tau_{j+1}+N]$. Note that $(\beta_H(t),\dot{\beta}_H(t))$ can be computed a.e. in $[\tau_j+N,\tau_{j+1}+N]$ since $\bar{\mathbf{x}}(t),\boldsymbol{\theta}_{j+1,H}$ are known. To simplify notation we will omit the subscript H and the dependence of $\epsilon(t),\dot{\beta}_H(t),\beta_H(t)$ on t when necessary.

Consider the violation factor $\epsilon_{j+1}(t)$ a.e. in $[\tau_{j+1},\tau_{j+1}+N]$:

$$\epsilon_{j+1}(t) = \begin{cases}\epsilon_j(t), & t\in[\tau_{j+1},\tau_j+N]\\ \bar{\epsilon}(t), & t\in(\tau_j+N,\tau_{j+1}+N]\end{cases},$$

where $\bar{\epsilon} := \bar{\epsilon}(t)$ is defined as:

$$\bar{\epsilon} = \begin{cases}0, & (\beta,\dot{\beta})\in\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3\\ -(\dot{\beta}+\alpha_H(\beta)), & (\beta,\dot{\beta})\in\mathcal{B}_4\cup\mathcal{B}_5\cup\mathcal{B}_6\end{cases},$$

where the sets $\mathcal{B}_q, q=1,\dots,6$ are defined as:

$$\begin{aligned}
\mathcal{B}_1 &= \{(\beta,\dot{\beta})\in\mathbb{R}^2 : \dot{\beta}\leq 0,\ \beta\geq 0,\ \alpha_H(\beta)\geq -\dot{\beta}\},\\
\mathcal{B}_2 &= \{(\beta,\dot{\beta})\in\mathbb{R}^2 : \dot{\beta}> 0,\ \beta\geq 0\},\\
\mathcal{B}_3 &= \{(\beta,\dot{\beta})\in\mathbb{R}^2 : \dot{\beta}> 0,\ \beta< 0,\ \dot{\beta}\geq -\alpha_H(\beta)\},\\
\mathcal{B}_4 &= \{(\beta,\dot{\beta})\in\mathbb{R}^2 : \dot{\beta}\leq 0,\ \beta\geq 0,\ \alpha_H(\beta)< -\dot{\beta}\},\\
\mathcal{B}_5 &= \{(\beta,\dot{\beta})\in\mathbb{R}^2 : \dot{\beta}\leq 0,\ \beta< 0\},\\
\mathcal{B}_6 &= \{(\beta,\dot{\beta})\in\mathbb{R}^2 : \dot{\beta}> 0,\ \beta< 0,\ \dot{\beta}< -\alpha_H(\beta)\}.
\end{aligned}$$

By feasibility of (16) at τ_j, it holds that: 1) $\epsilon_j(t)\geq 0$ is true in $[\tau_j,\tau_j+N]$ and 2) (16b) is satisfied for any solution $\mathbf{x}_j(t)$ of (3) when applying $\mathbf{u}_j(t)$ a.e. in $[\tau_j,\tau_j+N]$. Observe further $\dot{\beta}+\alpha_H(\beta)<0$ is true for any $(\beta,\dot{\beta})\in\mathcal{B}_4\cup\mathcal{B}_5\cup\mathcal{B}_6$. Hence, $\bar{\epsilon}\geq 0$. For any $(\beta,\dot{\beta})\in\mathcal{B}_1\cup\mathcal{B}_2\cup\mathcal{B}_3$ we have $\dot{\beta}+\alpha_H(\beta)\geq 0$. Therefore, choosing $\epsilon_{j+1}=0$ implies the satisfaction of (16b). Finally, setting $\epsilon_{j+1} = -(\dot{\beta}+\alpha_H(\beta))$ when $(\beta,\dot{\beta})\in\mathcal{B}_4\cup\mathcal{B}_5\cup\mathcal{B}_6$ results in $\epsilon_{j+1}+\dot{\beta}+\alpha_H(\beta)=0$. Therefore, (16b) is satisfied as equality. Based on the analysis above the candidate solution $(\mathbf{u}_{j+1},\epsilon_{j+1},\boldsymbol{\theta}_{j+1,H},r_{j+1,H})$ satisfies the constraints (16a)-(16j). As a result, (16) is feasible over $[\tau_{j+1},\tau_{j+1}+N]$. Since the above hold for any $j\geq 1$, (16) is recursively feasible when $\alpha_H(\xi)=\alpha_H\xi$.

Appendix IV
Proof of Theorem 3

By Theorem 2, (16) is feasible for every $j\in\mathcal{J}$, i.e., there always exists a continuous function $\mathbf{u}_j(t)$ a.e. in $[\tau_j,\tau_j+N]$ for every j. This implies that $\kappa(\mathbf{x}(t))$ is always defined and due to Assumption 4 it is a continuous function a.e. in $[0,\tau_J]$. Let $\mathbf{x}_{\text{CL}} : [0,\tau_J]\to\mathbb{X}$ a solution of (3), when $\kappa(\mathbf{x}(t))$ is applied for $t\in[0,\tau_J]$.

The formula ϕ, defined by (2) is a conjunction of always and eventually formulas $\varphi_i,\ i\in\mathcal{I}_H^0$. Hence, by definition of the robust semantics we have: $\rho^{\phi}(\mathbf{x}_{\text{CL}},0) = \min_{i\in\mathcal{I}_H^0}\rho^{\varphi_i}(\mathbf{x}_{\text{CL}},0)$. By construction, for any $(\mathbf{x}_j,t)\in\mathbb{X}\times[\tau_j,\tau_j+N]$ it holds that $\mathfrak{b}_H(\mathbf{x}_j,t;\boldsymbol{\theta}_{j,H}) \leq \min_{i\in\mathcal{I}_H^j}\mathfrak{b}_H^i(\mathbf{x}_j,t;\boldsymbol{\theta}_{j,H}^i)$, where $\boldsymbol{\theta}_{j,H} = \left[\boldsymbol{\theta}_{j,H}^i\right]_{i\in\mathcal{I}_H^j}$ is found as the solution of (16) at $[\tau_j,\tau_j+N]$. As a result, $h_i(\mathbf{x}_{\text{CL}}(t)) \geq \gamma_H^i(t;\boldsymbol{\theta}_{j,H}^i) + \mathfrak{b}_H(\mathbf{x}_{\text{CL}}(t),t;\boldsymbol{\theta}_{j,H}),\ t\in[a_i,b_i]\cap[\tau_j,\tau_{j+1}), j\in J_i^{\mathcal{G}}$, where $J_i^{\mathcal{G}} = \{j\in\mathcal{J} : [a_i,b_i]\cap[\tau_j,\tau_{j+1})\neq\emptyset\}$. If φ_i is an always formula, by design of $\gamma_H^i(t;\boldsymbol{\theta}_{j,H}^i)$, at every interval $[\tau_j,\tau_j+N], j\in J_i^{\mathcal{G}}$ and (8c) we have:

$$\gamma_H^i(t;\boldsymbol{\theta}_{j,H}^i) \geq \gamma_H^i(a_i;\boldsymbol{\theta}_{j,H}^i) \geq r_{j,H}, \quad t\in[a_i,b_i]\cap[\tau_j,\tau_{j+1})$$

where $r_{j,H}$ is the robustness value found as a solution to (16) at $[\tau_j,\tau_j+N]$. The inequality above implies that for every $j\in J_i^{\mathcal{G}}$ it holds that $h_i(\mathbf{x}_{\text{CL}}(t)) \geq r_{j,H} + \mathfrak{b}_H(\mathbf{x}_{\text{CL}}(t),t;\boldsymbol{\theta}_{j,H}) \geq r_{j,H} + \inf_{t'\in[\tau_j,\tau_j+N)}\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H}),\quad t\in[a_i,b_i]\cap[\tau_j,\tau_{j+1})$. Let $\bar{\rho}_j = r_{j,H} + \inf_{t'\in[\tau_j,\tau_j+N)}\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H})$. Then, we may conclude the following for the robustness value $\rho^{\varphi_i}(\mathbf{x}_{\text{CL}},0)$ of an always formula φ_i:

$$\rho^{\varphi_i}(\mathbf{x}_{\text{CL}},0) = \min_{t_1\in[a_i,b_i]} h_i(\mathbf{x}_{\text{CL}}(t_1)) \geq \min_{j\in J_i^{\mathcal{G}}}\bar{\rho}_j \geq \min_{j\in\mathcal{J}}\bar{\rho}_j. \tag{26}$$

If the formula φ_i is an eventually formula, the following is always true for its robustness value:

$$\rho^{\varphi_i}(\mathbf{x}_{\text{CL}},0) \geq \sup_{t_1\in[\max(\tau_j,a_i),t_{j,H}^{i*})} h_i(\mathbf{x}_{\text{CL}}(t_1)),$$

for any $j\in\mathcal{J}$ such that the eventually formula φ_i is active at $[\tau_j,\tau_j+N]$. Due to (8d), there exists $\iota = d(i)\in\mathcal{J}$ such that $t_{\iota,H}^{i*}\geq\tau_\iota$. By the choice of $\gamma_H^i(t;\boldsymbol{\theta}_{\iota,H}^i)$ and its design at the time interval $[\tau_\iota,\tau_\iota+N]$ we have:

$$\gamma_H^i(t_1;\boldsymbol{\theta}_{H,\iota}^i) \leq \sup_{t_1\in I}\gamma_H^i(t_1;\boldsymbol{\theta}_{\iota,H}^i) = r_{\iota,H}, \tag{27}$$

for any $t_1\in I = [\max(\tau_\iota,a_i),t_{\iota,H}^{i*})$. By definition of $\iota = d(i)$ and (8d), $I\subset[\tau_\iota,\tau_{\iota+1})$. Hence, for any closed loop trajectory $\mathbf{x}_{\text{CL}}(t_1)$ with $t_1\in I$ it holds that: $h_i(\mathbf{x}_{\text{CL}}(t_1)) \geq \gamma_H^i(t_1;\boldsymbol{\theta}_{\iota,H}^i) + \mathfrak{b}_H(\mathbf{x}_{\text{CL}}(t_1),t_1;\boldsymbol{\theta}_{\iota,H})$. This in addition to (27) implies:

$$\rho^{\varphi_i}(\mathbf{x}_{\text{CL}},0) \geq \sup_{t_1\in I} h_i(\mathbf{x}_{\text{CL}}(t_1)) \geq \bar{\rho}_\iota \geq \min_{j\in\mathcal{J}}\bar{\rho}_j,$$

where $\bar{\rho}_j = r_{j,H} + \inf_{t'\in[\tau_j,\tau_j+N)}\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H}),\ j\in\mathcal{J}$. Considering the aforementioned results for the always and eventually formulas, we may conclude that $\rho^{\phi}(\mathbf{x}_{\text{CL}},0) = \min_{i\in\mathcal{I}_H^0}\rho^{\varphi_i}(\mathbf{x}_{\text{CL}},0) \geq \min_{j\in\mathcal{J}}\bar{\rho}_j$. Note that $\bar{\rho}_j = r_{j,H} + \inf_{t'\in[\tau_j,\tau_j+N)}\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H})$ holds, irrespective of the choice of the temporal operator. Due to (16d), $r_{j,H}>0$ is true, thus $\bar{\rho}_j \geq \inf_{t'\in[\tau_j,\tau_j+N)}\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H})$. By Proposition 1, it is known that $\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H}) \geq c + \alpha_H^{-1}(-\epsilon_{\text{wc}}^j),\ t'\in[\tau_j,\tau_j+N)$. Considering this result and for $\epsilon_{\text{wc}} = \max_{j\in\mathcal{J}}\epsilon_{\text{wc}}^j$ we have that $\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H}) \geq c + \alpha_H^{-1}(-\epsilon_{\text{wc}})$. Hence, $\bar{\rho}_j \geq \inf_{t'\in[\tau_j,\tau_j+N)}\mathfrak{b}_H(\mathbf{x}_j(t'),t';\boldsymbol{\theta}_{j,H}) \geq c + \alpha_H^{-1}(-\epsilon_{\text{wc}})$ is concluded.

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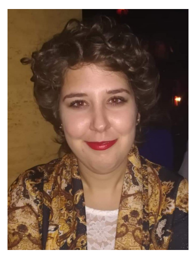

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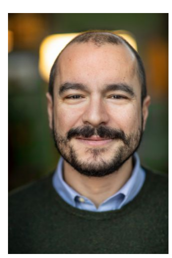

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