Compositional Synthesis of Signal Temporal Logic Tasks via Assume-Guarantee Contracts

Siyuan Liu[1,2], Adnane Saoud[3], Pushpak Jagtap[4], Dimos V. Dimarogonas[5], Majid Zamani[6,2]

***Abstract*— In this paper, we focus on the problem of compositional synthesis of controllers enforcing signal temporal logic (STL) tasks over a class of continuous-time nonlinear interconnected systems. By leveraging the idea of funnel-based control, we show that a fragment of STL specifications can be formulated as assume-guarantee contracts. A new concept of contract satisfaction is then defined to establish our compositionality result, which allows us to guarantee the satisfaction of a global contract by the interconnected system when all subsystems satisfy their local contracts. Based on this compositional framework, we then design closed-form continuous-time feedback controllers to enforce local contracts over subsystems in a decentralized manner. Finally, we demonstrate the effectiveness of our results on a numerical example.**

I. Introduction

In the last few decades, the world has witnessed rapid progresses in the development and deployment of cyber-physical systems (CPSs). Typical examples of real-world CPSs include smart grids and multi-robot systems. Nowadays, these systems are often large-scale interconnected resulting from tight interactions between computational components and physical entities, subjecting to complex specifications that are difficult to handle using classical control design approaches.

To address the emerging challenges in dealing with modern CPSs, various approaches [1], [2] have been developed to formally verify or synthesize certifiable controllers against rich specifications given by temporal logic formulae. Despite considerable development and progress in this field, when encountering large-scale CPSs, existing methods suffer severely from the *curse of dimensionality*, which limits their applications to systems of moderate size. To tackle this complexity issue, one can resort to compositional approaches which allow to tackle large-scale complex systems in a divide and conquer manner, by breaking down complex large design problems into sub-problems of manageable sizes. This compositional strategy can be implemented in terms of assume-guarantee contracts (AGCs) [3]–[6]. Specifically, the notion of AGCs prescribes properties that a component must guarantee under assumptions on the behavior of its environment (or its neighboring subsystems) [7].

The main aim of this paper is to develop a compositional controller synthesis scheme to enforce signal temporal logic (STL) tasks on continuous-time interconnected systems via AGCs. STL [8] entails space robustness [9], which determines how robust is the satisfaction of a task. Despite the advantages of STL formulae, the design of control systems under STL specifications is known to be a challenging task. In [10], the problem of synthesizing STL tasks on discrete-time systems is handled using model predictive control where space robustness is encoded as mixed-integer linear programs. The results in [11] established a connection between funnel-based control and the robust semantics of STL formulae, based on which a feedback control law is derived for continuous-time systems. This work is then extended to handle coupled multi-agent systems by providing a least violating solution for conflicting STL tasks [12].

In this paper, we consider a fragment of STL specifications which is first formulated as funnel-based control problems. By leveraging the derived funnels, we formalize the desired STL tasks as AGCs at the subsystem's level. A new concept of contract satisfaction, namely *uniform strong satisfaction* (cf. Definition 3.1), is introduced, which is critical for the compositional reasoning by making it possible to ensure the global satisfaction of STL tasks. Our main compositionality result is then presented using assume-guarantee reasoning, based on which the control of STL tasks can be conducted in a decentralized fashion. Finally, we derive continuous-time feedback controllers for subsystems in the spirit of funnel-based control, which ensures the satisfaction of local assume-guarantee contracts. To the best of our knowledge, this paper is the first to handle STL specifications on continuous-time systems using assume-guarantee contracts. Thanks to the derived closed-form control strategy and the decentralized framework, our approach requires very low computational complexity compared to existing results in the literature which mostly rely on discretizations in state space or time.

Related work: While AGCs have been extensively used in computer science community [7], [13], new frameworks of AGCs for dynamical systems with continuous state variables have been proposed recently in [4], [6] for continuous-time systems, and [3], [14, Chapter 2] for discrete-time systems. In this paper, we follow the same behavioural framework of AGCs for continuous-time systems as in [4]. In the following, we provide a comparison with the approach proposed in [4], [6]. A detailed comparison between the framework in [4], the one in [3], and existing approaches from the computer

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[1]Department of Electrical and Computer Engineering, Technical University of Munich, Germany. [2]Computer Science Department, LMU Munich, Germany. [3]Laboratoire des Signaux et Systèmes, CentraleSupélec, Univ. Paris Saclay, Gif sur Yvette, France. [4]Robert Bosch Center for Cyber-Physical Systems in Indian Institute of Science, Bangalore, India. [5]Division of Decision and Control Systems, KTH Royal Institute of Technology, Stockholm, Sweden. [6]Computer Science Department, University of Colorado Boulder, USA.

science community [7], [13] can be found in [4, Section 1].

The contribution of the paper is twofold:

• At the level of compositionality rules: The authors in [4] rely on a notion of *strong contract satisfaction* to provide a compositionality result (i.e., how to go from the satisfaction of local contracts at the component's level to the satisfaction of the global specification for the interconnected system) under the condition of the set of guarantees (of the contracts) being closed. In this paper, we are dealing with STL specifications, which are encoded as AGCs made of open sets of assumptions and guarantees. The non-closedness of the set of guarantees makes the concept of contract satisfaction proposed in [4] not sufficient to establish a compositionality result. For this reason, in this paper, we introduce a new concept of *uniform strong contract satisfaction* and show how the proposed concept makes it possible to go from the local satisfaction of the contracts at the component's level to the satisfaction of the global STL specification at the interconnected system's level.

• At the level of controller synthesis: When the objective is to synthesize controllers to enforce the satisfaction of AGCs for continuous-time systems, to the best of our knowledge, existing approaches in the literature can only deal with the particular class of invariance AGCs[1] in [6], where the authors used symbolic control techniques to synthesize controllers. In this paper, we present a new approach to synthesize controllers for a more general class of AGCs, where the set of assumptions and guarantees are described by STL formulas, by leveraging tools in the spirit of funnel-based control.

Due to lack of space, we provide the proofs of all statements in an arXiv version of the paper [15].

II. Preliminaries and Problem Formulation

Notation: We denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real and natural numbers, respectively. These symbols are annotated with subscripts to restrict them in the usual way, e.g., $\mathbb{R}_{>0}$ denotes the positive real numbers. We denote by $\mathbb{R}^n$ an n-dimensional Euclidean space and by $\mathbb{R}^{n\times m}$ a space of real matrices with n rows and m columns. We denote by I_n the identity matrix of size n, by $\mathbf{1}_n = [1,\dots,1]^\mathsf{T}$ the vector of all ones of size n, and by $\text{diag}(a_1,\dots,a_n)$ the diagonal matrix with diagonal elements being $a_1,\dots,a_n$.

A. Signal Temporal Logic (STL)

Signal temporal logic (STL) is a predicate logic based on continuous-time signals, which consists of predicates μ that are obtained by evaluating a continuously differentiable predicate function $\mathcal{P}:\mathbb{R}^n\to\mathbb{R}$ as $\mu := \begin{cases}\top & \text{if } \mathcal{P}(x)\ge 0\\ \bot & \text{if } \mathcal{P}(x)<0,\end{cases}$ for $x\in\mathbb{R}^n$. The STL syntax is given by

$$\phi ::= \top \mid \mu \mid \neg\phi \mid \phi_1\wedge\phi_2 \mid \phi_1\mathcal{U}_{[a,b]}\phi_2,$$

where ϕ_1, ϕ_2 are STL formulae, and $\mathcal{U}_{[a,b]}$ denotes the temporal until-operator with time interval $[a,b]$, where $a\le b<\infty$. Given a state trajectory $\mathbf{x}:\mathbb{R}_{\ge 0}\to X\subseteq\mathbb{R}^n$, we use $(\mathbf{x},t)\models\phi$ to denote that $\mathbf{x}$ satisfies ϕ at time t, and use $(\mathbf{x},0)\models\phi$ to denote that $\mathbf{x}$ satisfies formula ϕ. The semantics of STL for a state trajectory $\mathbf{x}$ are recursively given and can be found in [8, Def. 1]. Note that the disjunction-, eventually-, and always-operator can be derived as $\phi_1\vee\phi_2 = \neg(\neg\phi_1\wedge\neg\phi_2)$, $F_{[a,b]}\phi = \top\mathcal{U}_{[a,b]}\phi$, and $G_{[a,b]}\phi = \neg F_{[a,b]}\neg\phi$, respectively. Next, we introduce the robust semantics for STL (also referred to as space robustness) [9, Def. 3], which determines how robustly a signal $\mathbf{x}$ satisfies the STL formula ϕ: $\rho^{\mu}(\mathbf{x},t):=\mathcal{P}(\mathbf{x}(t))$, $\rho^{\neg\phi}(\mathbf{x},t) := -\rho^{\phi}(\mathbf{x},t)$, $\rho^{\phi_1\wedge\phi_2}(\mathbf{x},t) := \min(\rho^{\phi_1}(\mathbf{x},t),\rho^{\phi_2}(\mathbf{x},t))$, $\rho^{F_{[a,b]}\phi}(\mathbf{x},t) := \max_{t_1\in[t+a,t+b]}\rho^{\phi}(\mathbf{x},t_1)$, $\rho^{G_{[a,b]}\phi}(\mathbf{x},t) := \min_{t_1\in[t+a,t+b]}\rho^{\phi}(\mathbf{x},t_1)$. Note that $(\mathbf{x},t)\models\phi$ if $\rho^{\phi}(\mathbf{x},t)>0$ holds [16, Prop. 16]. We abuse the notation as $\rho^{\phi}(\mathbf{x}(t)) := \rho^{\phi}(\mathbf{x},t)$ if t is not explicitly contained in $\rho^{\phi}(\mathbf{x},t)$. However, t is explicitly contained in $\rho^{\phi}(\mathbf{x},t)$ if temporal operators (eventually, always, or until) are used. Similarly as in [17], throughout the paper, the non-smooth conjunction is approximated by smooth functions as $\rho^{\phi_1\wedge\phi_2}(\mathbf{x},t)\approx -\ln(\exp(-\rho^{\phi_1}(\mathbf{x},t))+\exp(-\rho^{\phi_2}(\mathbf{x},t)))$.

In the remainer of the paper, we will focus on a fragment of STL as introduced below. Consider

$$\psi ::= \top \mid \mu \mid \neg\mu \mid \psi_1\wedge\psi_2, \tag{1}$$

$$\phi ::= G_{[a,b]}\psi \mid F_{[a,b]}\psi \mid F_{[\underline{a},\underline{b}]}G_{[\bar{a},\bar{b}]}\psi, \tag{2}$$

where μ is the predicate, ψ in (2) and ψ_1,ψ_2 in (1) are formulae of class ψ given in (1). Formulae of class ψ in (1) are non-temporal (Boolean) atomic formulae, whereas formulae of class ϕ in (2) are temporal formulae. This STL fragment allows us to encode concave temporal tasks, which is a necessary assumption used later for the design of closed-form, continuous feedback controllers (cf. Assumption 4.1). However, by leveraging the results in e.g., [18], it is possible to expand our results to full STL semantics.

B. Interconnected control systems

In this paper, we study the interconnection of finitely many continuous-time control subsystems. Consider a network consisting of $N\in\mathbb{N}$ control subsystems Σ_i, $i\in I=\{1,\dots,N\}$. For each $i\in I$, the set of *in-neighbors* of Σ_i is denoted by $\mathcal{N}_i\subseteq I\setminus\{i\}$, i.e., the set of subsystems Σ_j, $j\in\mathcal{N}_i$, directly influencing subsystem Σ_i.

A continuous-time control subsystem is formalized in the following definition.

Definition 2.1: (Continuous-time control subsystem) A continuous-time control subsystem Σ_i is a tuple $\Sigma_i = (X_i,U_i,W_i,f_i,g_i,h_i)$, where

• $X_i=\mathbb{R}^{n_i}$, $U_i=\mathbb{R}^{m_i}$ and $W_i=\mathbb{R}^{p_i}$ are the state, external input, and internal input spaces, respectively;

• $f_i:\mathbb{R}^{n_i}\to\mathbb{R}^{n_i}$ is the flow drift, $g_i:\mathbb{R}^{n_i}\to\mathbb{R}^{n_i\times m_i}$ is the external input matrix, and $h_i:\mathbb{R}^{p_i}\to\mathbb{R}^{n_i}$ is the internal input map.

A trajectory of Σ_i is an absolutely continuous map $(\mathbf{x}_i,\mathbf{u}_i,\mathbf{w}_i):\mathbb{R}_{\ge 0}\to X_i\times U_i\times W_i$ such that for all $t\ge 0$

$$\dot{\mathbf{x}}_i(t) = f_i(\mathbf{x}_i(t)) + g_i(\mathbf{x}_i(t))\mathbf{u}_i(t) + h_i(\mathbf{w}_i(t)), \tag{3}$$

[1] where the set of assumptions and guarantees of the contract are described by invariants.

where $\mathbf{u}_i : \mathbb{R}_{\geq 0} \to U_i$ is the external input trajectory, and $\mathbf{w}_i : \mathbb{R}_{\geq 0} \to W_i$ is the internal input trajectory.

Note that $w_i \in W_i$ are termed as "internal" inputs describing the interaction between subsystems and $u_i \in U_i$ are called "external" inputs served as interfaces for controllers.

An interconnected control system is defined as follows.

Definition 2.2: (Continuous-time interconnected control system) Consider $N \in \mathbb{N}$ control subsystems Σ_i as in Definition 2.1. An interconnected control system denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ is a tuple $\Sigma = (X, U, f, g)$ where

- $X = \prod_{i \in I} X_i$ and $U = \prod_{i \in I} U_i$ are the state and external input spaces, respectively;
- $f : \mathbb{R}^n \to \mathbb{R}^n$ is the flow drift and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ is the external input matrix defined as: $f(\mathbf{x}) = [f_1(\mathbf{x}_1) + h_1(\mathbf{w}_1); \ldots; f_N(\mathbf{x}_N) + h_N(\mathbf{w}_N)]$, $g(\mathbf{x}) = \mathrm{diag}(g_1(\mathbf{x}_1), \ldots, g_N(\mathbf{x}_N))$, where $\mathbf{x} = [\mathbf{x}_1; \ldots; \mathbf{x}_N]$, $\mathbf{w}_i = [\mathbf{x}_{j_1}; \ldots; \mathbf{x}_{j_{|\mathcal{N}_i|}}]$, for all $i \in I$, $n = \sum_{i \in I} n_i$, $m = \sum_{i \in I} m_i$.

A trajectory of Σ is an absolutely continuous map $(\mathbf{x}, \mathbf{u}) : \mathbb{R}_{\geq 0} \to X \times U$ such that for all $t \geq 0$

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t), \tag{4}$$

where $\mathbf{u} : \mathbb{R}_{\geq 0} \to U$ is the external input trajectory.

In the above definition, the interconnection structure implies that all the internal inputs of a subsystem are states of its neighboring subsystems. Therefore, the definition of an interconnected control system boils down to the tuple $\Sigma = (X, U, f, g)$ since it has trivial null internal inputs.

We have now all the ingredients to provide a formal statement of the problem considered in the paper:

Problem 2.3: Given an interconnected system $\Sigma = (X, U, f, g)$, consisting of subsystems $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$, $i \in I$, and given an STL specification ϕ as in (1)–(2), where $\phi = \wedge_{i=1}^{N} \phi_i$ and ϕ_i is the local STL task assigned to Σ_i, synthesize local controllers $\mathbf{u}_i : X_i \times \mathbb{R}_{\geq 0} \to U_i$ for subsystems Σ_i such that Σ satisfies the specification ϕ.

In the remainder of the paper, to provide a solution to Problem 2.3, the desired STL tasks will be first casted as funnel functions in Section III-A. Then, we present our main compositionality result based on a notion of assume-guarantee contracts as in Section III-B, which allows us to tackle the synthesis problem in a decentralized fashion. We will further explain in Section III-C on how to assign assume-guarantee contracts tailored to the funnel-based formulation of STL tasks. A closed-form continuous-time control law will be derived in Section IV to enforce local contracts over subsystems individually.

III. Assume-Guarantee Contracts and Compositional Reasoning

In this section, we present a compositional approach based on a notion of assume-guarantee contracts (AGCs), which enables us to reason about the properties of a continuous-time interconnected system based on the properties of its components. Before introducing the compositionality result, we first show how to cast STL formulae into time-varying funnel functions which will be leveraged later to design continuous-time AGCs. Note that the idea of casting STL as funnel functions was originally proposed in [11].

A. Casting STL as funnel functions

First, let us define a funnel function $\gamma_i(t) = (\gamma_i^0 - \gamma_i^\infty)\exp(-l_i t) + \gamma_i^\infty$, where $l_i, t \in \mathbb{R}_{\geq 0}$, $\gamma_i^0, \gamma_i^\infty \in \mathbb{R}_{>0}$ with $\gamma_i^0 \geq \gamma_i^\infty$. Consider the robust semantics of STL introduced in Subsection II-A. For each Σ_i, one can ensure the satisfaction of a STL task ϕ_i as in (2) (with corresponding ψ_i) if $0 < r_i \leq \rho_i^{\phi_i}(\mathbf{x}_i, 0) \leq \rho_i^{max}$ holds, where r_i is a robustness measure and ρ_i^{max} is a robustness delimiter [11]. This can be achieved by prescribing a temporal behavior to $\rho_i^{\psi_i}(\mathbf{x}_i(t))$ through properly designed parameters γ_i and ρ_i^{max} as

$$-\gamma_i(t) < \rho_i^{\psi_i}(\mathbf{x}_i(t)) - \rho_i^{max} < 0. \tag{5}$$

Note that functions $\gamma_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $i \in I$, are positive, continuously differentiable, bounded, and non-increasing. The design of γ_i and ρ_i^{max} that leads to the satisfaction of $0 < r_i \leq \rho_i^{\phi_i}(\mathbf{x}_i, 0) \leq \rho_i^{max}$ through (5) will be discussed in Section IV-A. We kindly refer interested readers to [15, Example 3.1] for an intuitive example that illustrates the satisfaction of STL tasks using funnel-based strategy.

In the sequel, STL tasks will be formulated as contracts by leveraging the above-presented funnel-based framework. We will then design local controllers enforcing the local contracts over the subsystems (cf. Section IV, Theorem 4.6).

B. Compositional reasoning via assume-guarantee contracts

In this subsection, we introduce a notion of continuous-time assume-guarantee contracts to establish our compositional framework. A new concept of contract satisfaction is defined which is tailored to the funnel-based formulation of STL specifications as discussed in Subsection III-A.

Definition 3.1: (Assume-guarantee contracts) Consider a subsystem $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$. An assume-guarantee contract for Σ_i is a tuple $\mathcal{C}_i = (A_i, G_i)$ where

- $A_i : \mathbb{R}_{\geq 0} \to W_i$ is a set of assumptions on internal input trajectories;
- $G_i : \mathbb{R}_{\geq 0} \to X_i$ is a set of guarantees on state trajectories.

We say that Σ_i (*weakly*) satisfies $\mathcal{C}_i$, denoted by $\Sigma_i \models \mathcal{C}_i$, if for any trajectory $(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times U_i \times W_i$ of Σ_i, the following holds: for all $t \in \mathbb{R}_{\geq 0}$ such that $\mathbf{w}_i(s) \in A_i(s)$ for all $s \in [0, t]$, we have $\mathbf{x}_i(s) \in G_i(s)$ for all $s \in [0, t]$.

We say that Σ_i *uniformly strongly* satisfies $\mathcal{C}_i$, denoted by $\Sigma_i \models_{us} \mathcal{C}_i$, if for any trajectory $(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times U_i \times W_i$ of Σ_i, the following holds: there exists $\delta_i > 0$ such that for all $t \in \mathbb{R}_{\geq 0}$ and for all $s \in [0, t]$ where $\mathbf{w}_i(s) \in A_i(s)$, we have $\mathbf{x}_i(s) \in G_i(s)$ for all $s \in [0, t + \delta_i]$.

Note that $\Sigma_i \models_{us} \mathcal{C}_i$ obviously implies $\Sigma_i \models \mathcal{C}_i$.

Remark 3.2: It should be mentioned that interconnected systems have no assumptions on internal inputs since they have a trivial null internal input set as in Definition 2.2. Hence, an AGC for an interconnected system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ will be denoted by $\mathcal{C} = (\emptyset, G)$. The concepts of contract satisfaction by Σ are similar as in the above definition by removing the conditions on internal inputs.

We are now ready to state the main result of this section providing conditions under which one can go from the satisfaction of local contracts at subsystem's level to the satisfaction of a global contract for an interconnected system.

Theorem 3.3: Consider an interconnected control system $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_N)$ as in Definition 2.2. To each subsystem Σ_i, $i \in I$, we associate a contract $\mathcal{C}_i = (A_i, G_i)$ and let $\mathcal{C} = (\emptyset, G) = (\emptyset, \prod_{i\in I} G_i)$ be the corresponding contract for Σ. Assume the following conditions hold:

(i) for all $i \in I$ and for any trajectory $(\mathbf{x}_i, \mathbf{u}_i, \mathbf{w}_i) : \mathbb{R}_{\geq 0} \to X_i \times U_i \times W_i$ of Σ_i, $\mathbf{x}_i(0) \in G_i(0)$;

(ii) for all $i \in I$, $\Sigma_i \models_{us} \mathcal{C}_i$;

(iii) for all $i \in I$, $\prod_{j\in\mathcal{N}_i} G_i \subseteq A_i$.

Then, $\Sigma \models \mathcal{C}$.

Remark 3.4: It is important to note that while in the definition of the strong contract satisfaction in [4] the parameter δ may depend on time, our definition of assume-guarantee contracts requires a uniform δ for all time. The reason for this choice is that the uniformity of δ is critical in our compositional reasoning, since we do not require the set of guarantees to be closed as in [4] (See [4, Example 9] for an example, showing that the compositionality result does not hold using the concept of strong satisfaction when the set of guarantees of the contract is open). Indeed, as it will be shown in the next section, the set of guarantees of the considered contracts are open and one will fail to provide a compositionality result based on the classical (non-uniform) notion of strong satisfaction in [4].

C. From STL tasks to assume-guarantee contracts

The objective of the paper is to synthesize local controllers $\mathbf{u}_i : X_i \times \mathbb{R}_{\geq 0} \to U_i$, $i \in I$, for subsystems Σ_i to achieve the STL specification ϕ, where $\phi = \wedge_{i=1}^{N} \phi_i$ and ϕ_i is the local STL task assigned to subsystem Σ_i. Hence, in view of the interconnection between the subsystems and the decentralized nature of the local controllers, one has to make some assumptions on the behaviour of the neighbouring components while synthesizing the local controllers. This property can be formalized in terms of contracts, where the contract should reflect the fact that the objective is to ensure that subsystem Σ_i satisfies "the guarantee" ϕ_i under "the assumption" that each of its neighbouring subsystems Σ_j satisfies its local task ϕ_j, $j \in \mathcal{N}_i$. In this context, by leveraging the concept of funnel function to cast local STL tasks ϕ_i, as presented in Section III-A, a natural assignment of the local assume-guarantee contract $\mathcal{C}_i = (A_i, G_i)$ for a subsystem Σ_i can be defined formally as follows:

- $A_i = \prod_{j\in\mathcal{N}_i} \{\mathbf{x}_j : \mathbb{R}_{\geq 0} \to X_j \mid -\gamma_j(t) + \rho_j^{max} < \rho_j^{\psi_j}(\mathbf{x}_j(t)) < \rho_j^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$,
- $G_i = \{\mathbf{x}_i : \mathbb{R}_{\geq 0} \to X_i \mid -\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_i(t)) < \rho_i^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$,

where $\mathbf{x}_j$ denotes the state trajectories of neighboring subsystem Σ_j, $j \in \mathcal{N}_i$, and $-\gamma_i, \rho_i^{\psi_i}, \rho_i^{max}$ are the functions discussed in Subsection III-A corresponding to STL task ϕ_i.

Once the specification ϕ is decomposed into local contracts[2] and in view of Theorem 3.3, Problem 2.3 can be resolved by considering local control problems for each subsystem Σ_i. These control problems can be solved in a decentralized manner and are formally defined as follows:

Problem 3.5: Given a subsystem $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$ and an assume-guarantee contract $\mathcal{C}_i = (A_i, G_i)$, where A_i and G_i are given by STL formulae by means of funnel functions, synthesize a local controller $\mathbf{u}_i : X_i \times \mathbb{R}_{\geq 0} \to U_i$ such that $\Sigma_i \models_{us} \mathcal{C}_i$.

IV. Decentralized Controller Design

Here, we first provide a solution to Problem 3.5 by designing controllers ensuring that local contracts for subsystems are uniformly strongly satisfied. Then, we show that based on our compositionality result proposed in the last section, the global STL task for the network is satisfied by applying the derived local controllers to subsystems individually.

A. Local controller design

As discussed in Subsection III-A, one can enforce STL tasks via funnel-based strategy by prescribing the temporal behavior of $\rho_i^{\psi_i}(\mathbf{x}_i(t))$ within the predefined region in (5), i.e., $-\gamma_i(t) < \rho_i^{\psi_i}(\mathbf{x}_i(t)) - \rho_i^{max} < 0$. In order to design feedback controllers to achieve this, we translate the funnel functions into notions of errors as follows. First, define a one-dimensional error as $e_i(\mathbf{x}_i(t)) = \rho_i^{\psi_i}(\mathbf{x}_i(t)) - \rho_i^{max}$. Now, by normalizing the error $e_i(\mathbf{x}_i(t))$ with respect to the funnel function γ_i, we define the modulated error as $\hat{e}_i(\mathbf{x}_i, t) = \frac{e_i(\mathbf{x}_i(t))}{\gamma_i(t)}$. Now, (5) can be rewritten as $-1 < \hat{e}_i(t) < 0$. We use $\hat{\mathcal{D}}_i := (-1, 0)$ to denote the performance region for $\hat{e}_i(t)$. Next, the modulated error is transformed through a transformation function $T_i : (-1, 0) \to \mathbb{R}$ defined as $T_i(\hat{e}_i(\mathbf{x}_i, t)) = \ln(-\frac{\hat{e}_i(\mathbf{x}_i,t)+1}{\hat{e}_i(\mathbf{x}_i,t)})$. Note that the transformation function $T_i : (-1, 0) \to \mathbb{R}$ is a strictly increasing function, bijective and hence admitting an inverse. By differentiating the transformed error $\epsilon_i := T_i(\hat{e}_i(\mathbf{x}_i, t))$ w.r.t time, we obtain

$$\dot{\epsilon}_i = \mathcal{J}_i(\hat{e}_i, t)[\dot{e}_i + \alpha_i(t) e_i], \tag{6}$$

where $\mathcal{J}_i(\hat{e}_i, t) = \frac{\partial T_i(\hat{e}_i)}{\partial \hat{e}_i} \frac{1}{\gamma_i(t)} = -\frac{1}{\gamma_i(t)\hat{e}_i(1+\hat{e}_i)} > 0$, for all $\hat{e}_i \in (-1, 0)$, is the normalized Jacobian of the transformation function, and $\alpha_i(t) = -\frac{\dot{\gamma}_i(t)}{\gamma_i(t)} > 0$ for all $t \in \mathbb{R}_{\geq 0}$ is the normalized derivative of the performance function γ_i.

Note that, if ϵ_i is bounded for all t, then $\hat{e}_i$ is constrained within the performance region $\hat{\mathcal{D}}_i$, which further implies that the error e_i evolves within the prescribed funnel bounds as desired in (5). We will derive feedback control law in Theorem 4.6 to achieve this.

Furthermore, we make the two following assumptions on functions $\rho_i^{\psi_i}$ for formulae ψ_i, which are required for the local controller design in the our main result of this section.

Assumption 4.1: Each formula within class ψ as in (1) has the following properties: (i) $\rho_i^{\psi_i} : \mathbb{R}^{n_i} \to \mathbb{R}$ is concave and (ii) the formula is well-posed in the sense that for all $C \in \mathbb{R}$ there exists $\bar{C} \geq 0$ such that for all $\mathbf{x}_i \in \mathbb{R}^{n_i}$ with $\rho_i^{\psi_i}(\mathbf{x}_i) \geq C$, one has $\|\mathbf{x}_i\| \leq \bar{C} < \infty$.

Define the global maximum of $\rho_i^{\psi_i}(\mathbf{x}_i)$ as $\rho_i^{opt} = \sup_{\mathbf{x}_i\in\mathbb{R}^{n_i}} \rho_i^{\psi_i}(\mathbf{x}_i)$. Note that ψ_i is feasible only if $\rho_i^{opt} > 0$, which leads to the following assumption.

[2] Note that the decomposition of a global STL formula is out of the scope of this paper. In this paper, we use a natural decomposition of the specification, where the assumptions of a component coincide with the guarantees of its neighbours. However, given a global STL for an interconnected system, one can utilize existing methods provided in recent literature, e.g., [19], to decompose the global STL task into local ones.

Assumption 4.2: The global maximum of $\rho_i^{\psi_i}(\mathbf{x}_i)$ is positive.

The following assumption is imposed on subsystems in order to design controllers enforcing local contracts.

Assumption 4.3: Consider subsystem Σ_i as in Definition 2.1. The functions $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i}$, $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i \times m_i}$, and $h_i : \mathbb{R}^{p_i} \to \mathbb{R}^{n_i}$ are locally Lipschitz continuous, and $g_i(\mathbf{x}_i)g_i(\mathbf{x}_i)^\mathsf{T}$ is positive definite for all $\mathbf{x}_i \in \mathbb{R}^{n_i}$.

Next, we provide an important result in Proposition 4.5 to be used to prove the main theorem, which shows how to go from *weak* to *uniform strong satisfaction* of AGCs by relaxing the assumptions. The following definition is needed to measure the distance between continuous-time trajectories.

Definition 4.4: (ε-closeness of trajectories) Let $Z \subseteq \mathbb{R}^n$. Consider $\varepsilon > 0$ and two continuous-time trajectories $z_1 : \mathbb{R}_{\geq 0} \to Z$ and $z_2 : \mathbb{R}_{\geq 0} \to Z$. z_2 is said to be ε-close to z_1, if for all $t_1 \in \mathbb{R}_{\geq 0}$, there exists $t_2 \in \mathbb{R}_{\geq 0}$ such that $|t_1 - t_2| \leq \varepsilon$ and $\|z_1(t_1) - z_2(t_2)\| \leq \varepsilon$. We define the ε-expansion of z_1 by : $\mathcal{B}_\varepsilon(z_1) = \{z' : \mathbb{R}_{\geq 0} \to Z \mid z' \text{ is } \varepsilon\text{-close to } z_1\}$. For set $A = \{z : \mathbb{R}_{\geq 0} \to Z\}$, $\mathcal{B}_\varepsilon(A) = \cup_{z \in A} \mathcal{B}_\varepsilon(z)$.

Proposition 4.5: (From weak to uniform strong satisfaction of AGCs) Consider a subsystem $\Sigma_i = (X_i, U_i, W_i, f_i, g_i, h_i)$ associated with a local AGC $\mathcal{C}_i = (A_i, G_i)$. If trajectories of Σ_i are uniformly continuous and $\Sigma_i \models \mathcal{C}_i^\varepsilon$ with $\mathcal{C}_i^\varepsilon = (\mathcal{B}_\varepsilon(A_i), G_i)$ for $\varepsilon > 0$, then $\Sigma_i \models_{us} \mathcal{C}_i$.

Now, we are ready to present the main result of this section solving Problem 3.5 for the local controller design.

Theorem 4.6: Consider subsystem Σ_i as in Definition 2.1 satisfying Assumption 4.3, with corresponding local assume-guarantee contract $\mathcal{C}_i = (A_i, G_i)$, where

- $A_i = \prod_{j \in \mathcal{N}_i} \{\mathbf{x}_j : \mathbb{R}_{\geq 0} \to X_j \mid -\gamma_j(t) + \rho_j^{max} < \rho_j^{\psi_j}(\mathbf{x}_j(t)) < \rho_j^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$,
- $G_i = \{\mathbf{x}_i : \mathbb{R}_{\geq 0} \to X_i \mid -\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_i(t)) < \rho_i^{max}, \forall t \in \mathbb{R}_{\geq 0}\}$,

where ψ_i is a non-temporal formula as in (1) satisfying Assumptions 4.1-4.2. If $-\gamma_i(0) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_i(0)) < \rho_i^{max} < \rho_i^{opt}$, then the controller

$$\mathbf{u}_i(\mathbf{x}_i, t) = - g_i(\mathbf{x}_i)^\mathsf{T}\Big(\frac{\partial \rho_i^{\psi_i}(\mathbf{x}_i)^\mathsf{T}}{\partial \mathbf{x}_i}\mathcal{J}_i(\hat{e}_i, t)\epsilon_i(\mathbf{x}_i, t) + h_i(d_i(t))\Big) \tag{7}$$

ensures $\Sigma_i \models_{us} \mathcal{C}_i$, where $d_i(t) = [\gamma_{j_1}(t)\mathbf{1}_{n_{j_1}}; \ldots; \gamma_{j_{|\mathcal{N}_i|}}(t)\mathbf{1}_{n_{|\mathcal{N}_i|}}]$.

Remark that the connection between atomic formulae $\rho_i^{\psi_i}(\mathbf{x}_i(t))$ and temporal formulae $\rho_i^{\phi_i}(\mathbf{x}_i, 0)$ is made by γ_i and ρ_i^{max} as in (5), which need to be designed as instructed in [11]. Specifically, if Assumption 4.2 holds, one can select

$$t_i^* \in \begin{cases} a_i & \text{if } \phi_i = G_{[a_i,b_i]}\psi_i \\ [a_i, b_i] & \text{if } \phi_i = F_{[a_i,b_i]}\psi_i \\ [\underline{a}_i + \bar{a}_i, \underline{b}_i + \bar{a}_i] & \text{if } \phi_i = F_{[\underline{a}_i,\underline{b}_i]}G_{[\bar{a}_i,\bar{b}_i]}\psi_i \end{cases} \tag{8}$$

$$\rho_i^{\max} \in \Big(\max(0, \rho_i^{\psi_i}(\mathbf{x}_i(0))), \rho_i^{\text{opt}}\Big) \tag{9}$$

$$r_i \in (0, \rho_i^{\max}) \tag{10}$$

$$\gamma_i^0 \in \begin{cases} (\rho_i^{\max} - \rho_i^{\psi_i}(\mathbf{x}_i(0)), \infty) & \text{if } t_i^* > 0 \\ (\rho_i^{\max} - \rho_i^{\psi_i}(\mathbf{x}_i(0)), \rho_i^{\max} - r_i] & \text{else} \end{cases} \tag{11}$$

$$\gamma_i^\infty \in \big(0, \min(\gamma_i^0, \rho_i^{\max} - r_i)\big] \tag{12}$$

$$l_i \in \begin{cases} \mathbb{R}_{\geq 0} & \text{if } -\gamma_i^0 + \rho_i^{\max} \geq r_i \\ \dfrac{-\ln\Big(\frac{r_i + \gamma_i^\infty - \rho_i^{\max}}{-\gamma_i^0 - \gamma_i^\infty}\Big)}{t_i^*} & \text{else.} \end{cases} \tag{13}$$

Now, with γ_i and ρ_i^{max} chosen properly, one can achieve $0 < r_i \leq \rho_i^{\phi_i}(\mathbf{x}_i, 0) \leq \rho_i^{max}$ by prescribing a temporal behavior to $\rho_i^{\psi_i}(\mathbf{x}_i(t))$ as in the set of guarantee G_i in Theorem 4.6, i.e., $-\gamma_i(t) + \rho_i^{max} < \rho_i^{\psi_i}(\mathbf{x}_i(t)) < \rho_i^{max}$ for all $t \geq 0$.

B. Global task satisfaction

In this subsection, we show that by applying the local controllers to the subsystems, the global STL task for the network is also satisfied based on our compositionality result.

Corollary 4.7: Consider an interconnected control system $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ as in Definition 2.2. If we apply the controllers as in (7) to all subsystems Σ_i, then we get $\Sigma \models \mathcal{C} = (\emptyset, \prod_{i \in I} G_i)$. This means that the control objective in Problem 2.3 is achieved, i.e., system Σ satisfies signal temporal logic task ϕ.

V. Case Study

We demonstrate the effectiveness of the proposed results on two case studies: a room temperature regulation and a mobile robot control problem. The second example can be found in [15] and is omitted here due to lack of space.

Here, we apply our results to the temperature regulation of a circular building with $N \geq 3$ rooms each equipped with a heater. The evolution of the temperature of the interconnected model is described by the differential equation:

$$\Sigma : \begin{cases} \dot{\mathbf{T}}(t) = & A\mathbf{T}(t) + \alpha_h T_h \nu(t) + \alpha_e T_e, \\ \mathbf{y}(t) = & \mathbf{T}(t), \end{cases} \tag{14}$$

adapted from [20], where $A \in \mathbb{R}^{N \times N}$ is a matrix with elements $\{A\}_{ii} = (-2\alpha - \alpha_e - \alpha_h \nu_i)$, $\{A\}_{i,i+1} = \{A\}_{i+1,i} = \{A\}_{1,N} = \{A\}_{N,1} = \alpha$, $\forall i \in \{1, \ldots, N-1\}$, and all other elements are identically zero, $\mathbf{T}(t) = [\mathbf{T}_1(t); \ldots; \mathbf{T}_N(t)]$, $T_e = [T_{e1}; \ldots; T_{eN}]$, $\nu(t) = [\nu_1(t); \ldots; \nu_N(t)]$, where $\nu_i(t) \in [0, 1]$, $\forall i \in \{1, \ldots, N\}$, represents the ratio of the heater valve being open in room i. Parameters $\alpha = 0.05$, $\alpha_e = 0.008$, and $\alpha_h = 0.0036$ are heat exchange coefficients, $T_{ei} = -1\,^\circ C$ is the external environment temperature, and $T_h = 50\,^\circ C$ is the heater temperature.

Now, by introducing the subsystem Σ_i, representing the evolution of the temperature in the room i, and described by

$$\Sigma_i : \begin{cases} \dot{\mathbf{T}}_i(t) = & a\mathbf{T}_i(t) + d\mathbf{w}_i(t) + \alpha_h T_h \nu_i(t) + \alpha_e T_{ei}, \\ \mathbf{y}_i(t) = & \mathbf{T}_i(t), \end{cases}$$

where $a = -2\alpha - \alpha_e - \alpha_h \nu_i$, $d = \alpha$, and $\mathbf{w}_i(t) = [\mathbf{y}_{i-1}(t); \mathbf{y}_{i+1}(t)]$ (with $\mathbf{y}_0 = \mathbf{y}_n$ and $\mathbf{y}_{n+1} = \mathbf{y}_1$), one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ as in Definition 2.2. The initial temperatures of these rooms are, respectively, $\mathbf{T}_i(0) = 19\,^\circ C$ if $i \in I_o = \{i \text{ is odd } | i \in \{1, \ldots, N\}\}$, and $\mathbf{T}_i(0) = 25\,^\circ C$ if $i \in I_e = \{i \text{ is even } | i \in \{1, \ldots, N\}\}$. The room temperatures are subject to the following STL tasks ϕ_i:

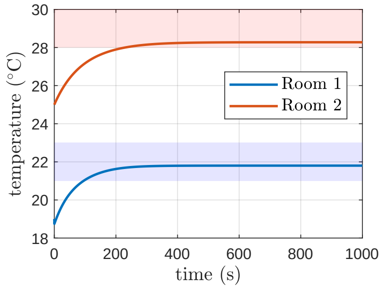
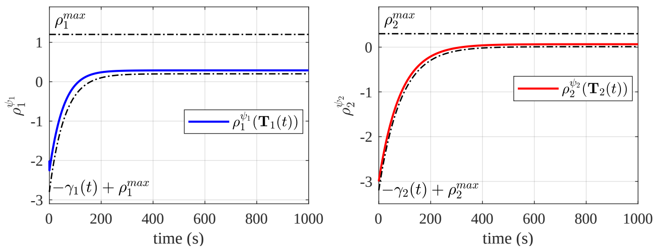


Fig. 1: Temperature evolution of the closed-loop subsystems Σ_1 and Σ_2 under control policy in (7).



(a) Funnel for Σ_1 with task ψ_1 (b) Funnel for Σ_2 with task ψ_2

Fig. 2: Funnels for the local STL tasks for subsystems Σ_1 and Σ_2. Performance bounds are indicated by dashed lines. Evolution of $\rho_i^{\psi_i}(\mathbf{T}_i(t))$ are depicted using solid lines.

$F_{[0,1000]}G_{[200,1000]}(\mathbf{T}_i \leq 25) \wedge (\mathbf{T}_i \geq 21)$, for $i \in I_o$, and ϕ_i: $F_{[0,1000]}G_{[500,1000]}(\mathbf{T}_i \leq 30) \wedge (\mathbf{T}_i \geq 28)$, for $i \in I_e$. Intuitively, the STL tasks ϕ_i requires that the controller (heater) should be synthesized such that the temperature of the rooms reaches the specified region ($[21, 25]$ for odd-numbered rooms or $[28, 30]$ for the even-numbered rooms) and remains there in the desired time slots.

Next, we apply the proposed funnel-based feedback controllers as in (7) to enforce the STL tasks on a circular building consisting of $N = 1000$ rooms. Numerical implementations were performed using MATLAB on a computer with a processor Intel Core i7 3.6 GHz CPU. Note that the computation of local controllers took on average 0.01 ms, which is negligible. The computation cost is very cheap since the local controller $\boldsymbol{u}_i$ is given by a closed-form expression and computed individually for the subsystems only. The simulation results for Σ_1 and Σ_2 are shown in Figs. 1 and 2. The state trajectories of the closed-loop subsystems are depicted as in Fig. 1. The shaded areas represent the desired temperature regions to be reached and stayed by the systems. In Fig. 2, we present the temporal behaviors of $\rho_i^{\psi_i}(\mathbf{T}_i(t))$ for the two rooms Σ_1 and Σ_2. It can be readily seen that the prescribed performances of $\rho_i^{\psi_i}(\mathbf{T}_i(t))$ are satisfied w.r.t. the error funnels, which shows that the time bounds are also respected. Remark that the design parameters of the funnels are chosen according to (8)-(13), which guarantees the satisfaction of temporal formulae $\rho_i^{\phi_i}(\mathbf{T}_i, 0)$ by prescribing temporal behaviors of atomic formulae $\rho_i^{\psi_i}(\mathbf{T}_i(t))$ as in Fig. 2. We can conclude that all STL tasks are satisfied within the desired time interval.

VI. Conclusions

We proposed a compositional approach for the synthesis of a fragment of STL tasks for continuous-time interconnected systems using assume-guarantee contracts. A new concept of contract satisfaction, i.e., uniform strong satisfaction, was introduced to establish our contract-based compositionality result. A continuous-time feedback controller was designed to enforce the uniform strong satisfaction of local contracts by all subsystems, while guaranteeing the satisfaction of global STL for the interconnected system based on the proposed compositionality result.

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