On Asymptotic Stability of Leader-Follower Multi-agent Systems Under Transient Constraints

Esteban Restrepo and Dimos V. Dimarogonas *Senior Member, IEEE*

***Abstract*—We address the agreement-based coordination of first-order multi-agent systems interconnected over arbitrary connected undirected graphs and under transient and steady-state constraints. The system is in a leader-follower configuration where only a part of the agents, the leaders, are directly controlled via an external control input, in addition to the agreement protocol. We propose a control law for the leaders, based on the gradient of a potential function, that achieves consensus and guarantees that the trajectories of the inter-agent distances of the entire system remain bounded by a performance function. Relying on the edge-agreement framework and Lyapunov's first method, we establish strong stability results in the sense of asymptotic stability of the consensus manifold and, in the leaderless case, nonuniform-in-time input-to-state stability with respect to additive disturbances. A numerical simulation illustrates the effectiveness of the proposed approach.**

***Index Terms*—Multi-agent systems, distributed control, Lyapunov methods.**

I. Introduction

MULTI-AGENT systems are commonly subject to inter-agent and output constraints such as, e.g., (local) connectivity maintenance, collision and obstacle avoidance [1]. Besides these, in many applications the transient and steady-state behavior of the trajectories of the multi-agent system must also satisfy some performance specifications in terms of time of convergence, overshoot, etc. For this purpose, since the seminal paper of [2] in which the evolution of the system's output is prescribed within some predefined region, numerous prescribed-performance- and funnel-control approaches have been developed for multi-agents systems. In [3] a controller is proposed that achieves agreement of second-order systems and guarantees that the evolution of the consensus error satisfies a performance bound. A similar result is presented in [4] for average consensus of agents modeled by single-integrator dynamics. Funnel-control approaches for multi-agent systems are proposed, e.g., in [5]–[7].

In these Lyapunov-based design methods, a nonlinear transformation is applied on the system's output or on the errors so that the constrained system is transformed into an unconstrained one. Then, the analysis of the closed-loop system reduces to showing that the trajectories of the unconstrained system are bounded, implying that the trajectories of the original (constrained) system satisfy the performance bounds. From the latter, it is then concluded that the trajectories of the system ultimately converge to a small region defined by the steady-sate values of the performance functions. Hence, most of the works in the literature only establish convergence of the multi-agent system to a small neighborhood of the agreement manifold rather than asymptotic stability.

A workaround to the latter has been achieved by making the performance bounds asymptotically converge to the origin [8]–[10]. However, this might result in undesirable large inputs. Indeed, the funnel- and prescribed-performance-control approaches rely on the magnitude of the input growing unbounded as the trajectories go near the bounds of the performance function. Moreover, although establishing boundedness of the trajectories is sufficient for guaranteeing practical convergence to consensus, in the presence of uncertainties or disturbances this property alone does *not* guarantee that, without violating the constraints, the trajectories do not oscillate or asymptotically converge to the performance bounds, making the control inputs grow unbounded at steady-state. Therefore, when considering more complex high-order and nonlinear systems and/or disturbances and uncertainties, stronger stability properties are usually needed in order to guarantee consensus while guaranteeing the fulfillment of the transient constraints. Asymptotic stability under prescribed performance constraints has been established in [11], [12], albeit for single-agent systems. In [13] asymptotic synchronization is established for heterogeneous multi-agent systems, with bounded inputs at steady state, using asymptotically convergent bounds.

From an implementation perspective, in the works mentioned above the funnel- and prescribed-performance-control laws are applied by all the agents in the system and a single leader, if existent, is given as reference to the followers. However, a more general and less costly approach is to consider that only one or more agents, selected as *leaders*, are directly controlled via external inputs in addition to the first-order agreement protocol; the remaining agents are *followers* evolving only under the agreement protocol. In this setting, the objective is to design the control inputs of the leaders so that the performance bounds for the entire system are satisfied. Leader-follower relative-position-based formation control under prescribed performance constraints is addressed in [14]

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for first- and second-order systems, using the edge-agreement transformation, albeit for tree graphs, and only convergence to a neighborhood of the consensus manifold is established.

In this letter we address the problem of leader-follower consensus-based control of multi-agent systems interacting over arbitrary connected (fixed) undirected graphs and under transient inter-agent constraints. For that purpose, our control design and analysis rely on the edge-agreement framework introduced in [15], which allows us to recast the consensus problem as one of stabilization of the origin and constitutes a more natural setting to consider inter-agent constraints.

With respect to the existing literature, our contribution is to propose a control design for the problem of leader-follower consensus-based coordination of multi-agent systems under transient and steady-state constraints, establishing strong stability results. Although restricted to first-order systems, we establish asymptotic stability of the consensus manifold in the leader-follower case, via Lyapunov's direct method. Then, in the leaderless (homogeneous) case, that is, when all the agents apply the funnel-based control law, we establish robustness in the sense of nonuniform-in-time input-to-state stability in addition to asymptotic stability. This differs from the existing results where only ultimate boundedness and practical convergence to a neighborhood of the consensus manifold is guaranteed. Establishing asymptotic stability and disposing of a strong Lyapunov function, even for just first-order linear systems, is an important step towards extending our results to networks of complex high-order nonlinear systems and considering disturbances or uncertainties.

The remainder of this letter is organized as follows. In Section II the model and the problem statement are presented. The main results are presented in Section III and are illustrated via numerical simulations in Section IV. Finally, some concluding remarks are given in Section V.

II. Model and problem statement

In this work we present an approach to the control design and stability analysis for the consensus-based control of a multi-agent system, in a leader-follower configuration, subject to transient and steady-state performance bounds.

The agents interact over a topology described by an *undirected* graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ where the set of nodes $\mathcal{V} := \{1,2,\ldots,N\}$ corresponds to the labels of the agents and the set of edges, $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$, of cardinality M, represents the communication between a pair of nodes, that is, an edge $e_k := (i,j) \in \mathcal{E}$, $k = \{1,\ldots,M\}$, is an unordered pair indicating that agent j has access to information from node i and vice-versa. Moreover, we consider that the multi-agent system is composed of *leader* and *follower* agents. Without loss of generality, suppose that the first $N_F < N$ agents are followers while the last $N_L < N$ are leaders with $\mathcal{V}_F := \{1,\ldots,N_F\} \subset \mathcal{V}$ and $\mathcal{V}_L := \{N_F+1,\ldots,N_L+N_F\} \subset \mathcal{V}$, with $N = N_F + N_L$.

All the agents are governed by a consensus-based protocol—cf. [16], but only the leader agents, i.e., $i \in \mathcal{V}_L$ can be directly controlled via an additional control input. More precisely, we have that for each agent $i \in \mathcal{V}$ its evolution is governed by the first-order system[1]

$$\dot{x}_i = -c\sum_{j\in\mathcal{N}_i}\left[(x_i - x_j) - (x_i^d - x_j^d)\right] + b_i u_i, \qquad (1)$$

where $(x_i^d - x_j^d)$ is a desired displacement between agents i and j, u_i is an external input and $b_i = 1$ if $i \in \mathcal{V}_L$ and $b_i = 0$ otherwise.

Let us define an edge state as

$$z_k := (x_i - x_j) - (x_i^d - x_j^d) \quad \forall k \leq M, \quad e_k \in \mathcal{E}. \qquad (2)$$

Then the coordination problem under transient and steady-state constraints consists in guaranteeing that the edge-variables defined in (2) remain bounded by a positive smooth performance function $\rho_k : [t_0,\infty) \to \mathbb{R}_{>0}$, $k \leq M$, i.e.,

$$|z_k(t)| < \rho_k(t), \quad \forall k \leq M, \qquad (3)$$

that satisfies the following assumption.

Assumption 1: For each edge $e_k \in \mathcal{E}$, there exists constants $\bar{\rho}_k > \underline{\rho}_k > 0$ and $\bar{\varphi}_k > 0$ such that the performance function $\rho_k : [t_0,\infty) \to \mathbb{R}_{>0}$ and its derivative satisfy

$$\underline{\rho}_k \leq \rho_k(t) \leq \bar{\rho}_k, \quad |\dot{\rho}_k(t)| \leq \bar{\varphi}_k, \quad \forall t \geq t_0. \qquad (4)$$

The goal is to design the inputs of the leaders u_i, $i \in \mathcal{V}_L$, so that the agents reach consensus and fulfill the inter-agent performance bounds given by (3). Mathematically, the consensus problem translates into making $(x_i - x_j) - (x_i^d - x_j^d) \to 0$, or equivalently, $z_k \to 0$ in the relative coordinates, while respecting the performance bounds (3).

To address the problem at hand we use a modified version of the control-design and analysis methodology for consensus under constraints presented in [17] based on the edge-agreement framework. In an edge-based representation we consider the states of the interconnection edges in the graph at each mode, instead of those of the nodes. Let us denote the so-called incidence matrix of the graph by $E \in \mathbb{R}^{N\times M}$, which is a matrix with rows indexed by the nodes and columns indexed by the edges. Its (i,k)-th entry is defined as follows: $[E]_{ik} := -1$ if i is the terminal node of edge e_k, $[E]_{ik} := 1$ if i is the initial node of edge e_k, and $[E]_{ik} := 0$ otherwise. Then, the edge states in (2) satisfy

$$z := E^\top(x - x^d) \qquad (5)$$

where $x^\top = [x_1 \cdots x_N] \in \mathbb{R}^N$, $x^{d\top} = \left[x_1^d \cdots x_N^d\right] \in \mathbb{R}^N$, and $z^\top := [z_1 \ldots z_M] \in \mathbb{R}^M$.

Now, let us decompose the incidence matrix into the rows corresponding to the followers and the rows corresponding to the leaders, i.e., $E = \left[E_F^\top \;\; E_L^\top\right]^\top$. Then, collecting the inputs of the leaders into the vector $u^\top = [u_{N_F+1} \cdots u_{N_L+N_F}] \in \mathbb{R}^{N_L}$ and taking the time derivative of (5), using (1), the edge dynamics take the form

$$\dot{z} = -cE^\top E z + E_L^\top u. \qquad (6)$$

As observed in [15], the dynamics of the whole system is captured by that of a spanning tree $\mathcal{G}_\mathcal{T} \subset \mathcal{G}$. Specifically, using

[1]Throughout this note we consider 1-dimensional agents. However we note that the results presented herein hold also for n-dimensional agents using the properties of the Kronecker product.

an appropriate labeling of the edges, the incidence matrix is expressed as $E = [E_{\mathcal{T}} \quad E_{\mathcal{C}}]$, where $E_{\mathcal{T}} \in \mathbb{R}^{N\times(N-1)}$ denotes the full-column-rank incidence matrix corresponding to an arbitrary spanning tree $\mathcal{G}_{\mathcal{T}}$ and $E_{\mathcal{C}} \in \mathbb{R}^{N\times(M-N+1)}$ represents the incidence matrix corresponding to the remaining edges not contained in $\mathcal{G}_{\mathcal{T}}$. Then, defining

$$R := [I_{N-1} \;\; T], \quad T := \left(E_{\mathcal{T}}^\top E_{\mathcal{T}}\right)^{-1} E_{\mathcal{T}}^\top E_{\mathcal{C}}, \tag{7}$$

one obtains a relation between the incidence matrix of $\mathcal{G}$ and that of the spanning tree $\mathcal{G}_{\mathcal{T}}$ as

$$E = E_{\mathcal{T}} R. \tag{8}$$

Moreover, from the row decomposition into leaders and followers, the incidence matrix is partitioned as

$$E = \begin{bmatrix} E_{F\mathcal{T}} & E_{F\mathcal{C}} \\ E_{L\mathcal{T}} & E_{L\mathcal{C}} \end{bmatrix}. \tag{9}$$

Then, from (8), the following identities hold:

$$E_F = E_{F\mathcal{T}} R, \quad E_L = E_{L\mathcal{T}} R. \tag{10}$$

The identities (8) and (10) are useful to derive a reduced-order dynamic model. Indeed, the edges' states may also be split as

$$z = \left[z_{\mathcal{T}}^\top \;\; z_{\mathcal{C}}^\top\right]^\top, \quad z_{\mathcal{T}} \in \mathbb{R}^{N-1}, \; z_{\mathcal{C}} \in \mathbb{R}^{M-N+1} \tag{11}$$

where $z_{\mathcal{T}}$ are the states corresponding to the edges of an arbitrary spanning tree $\mathcal{G}_{\mathcal{T}}$ and $z_{\mathcal{C}}$ denotes the states of the remaining edges, $e_k \in \mathcal{G}\backslash\mathcal{G}_{\mathcal{T}}$. Thus, from (7) and (11), we obtain

$$z = R^\top z_{\mathcal{T}}. \tag{12}$$

Using the identities (8) and (12) into (6) we obtain the reduced-order system

$$\dot{z}_{\mathcal{T}} = -cE_{\mathcal{T}}^\top E_{\mathcal{T}} R R^\top z_{\mathcal{T}} + E_{L\mathcal{T}}^\top u, \tag{13}$$

where $E_{\mathcal{T}}^\top E_{\mathcal{T}}$ is positive definite [15].

Note that, asymptotic stability for the origin of (13) implies asymptotic stability for the origin of (6). Therefore, the consensus objective is achieved if the origin is asymptotically stabilized for the reduced-order system (13).

III. Main result

A. Control approach

To address the problem at hand we propose a control law based on the gradient of a potential function encoding the performance constraints (3). For that purpose let us rewrite, for each $k \leq M$, the inter-agent constraints using the set

$$\mathcal{D}_k := \{z_k(t) \in \mathbb{R} \,:\, |z_k(t)| < \rho_k(t)\}. \tag{14}$$

Then, for each $k \leq M$, we define a positive function $W_k : \mathbb{R}_{\geq 0} \times \mathcal{D}_k \to \mathbb{R}_{\geq 0}$, of the form

$$W_k(t, z_k) = \frac{1}{2} \ln\left(\frac{\rho_k(t)^2}{\rho_k(t)^2 - z_k^2}\right), \tag{15}$$

that satisfies $W_k(t, 0) = 0$ and $W_k(t, z_k) \to \infty$ as $|z_k| \to \rho_k(t)$, and denote

$$\nabla W_k(t, z_k) := \frac{\partial W_k(t, z_k)}{\partial z_k} = \frac{z_k^2}{\rho_k(t)^2 - z_k^2}, \tag{16}$$

which satisfies $\nabla W_k(t, 0) = 0$ and $\nabla W_k(t, z_k) \to \infty$ as $|z_k| \to \rho_k(t)$, for all $t \geq t_0$. Moreover, from the properties of the logarithm function, the bounds

$$\frac{1}{2\bar{\rho}_k^2} z_k^2 \leq W_k(t, z_k) \leq \frac{\bar{\rho}_k^2}{2} |\nabla W_k(t, z_k)|^2, \tag{17}$$

hold for all $z_k(t) \in \mathcal{D}_k$ and for all $t \geq t_0$.

In order to satisfy the prescribed performance constraints (3), the leaders' inputs are set to

$$u_i := -c \sum_{k \leq M} [E_L]_{ik} \nabla W_k(t, z_k), \quad i \in \mathcal{V}_L, \tag{18}$$

where $c > 0$ is a control gain and $[E_L]_{ik}$ denotes the (i, k)-th entry of E_L. Now, denote $\mathcal{E}_L$ as the set of edges that are connected to at least a leader node. That is, $\mathcal{E}_L := \{e_k = (i, j) \in \mathcal{E} \;:\; i \in \mathcal{V}_L \vee j \in \mathcal{V}_L\}$. Moreover, denote $M_L \leq M$ the cardinality of $\mathcal{E}_L$. Similarly, denote $\mathcal{E}_F$ as the set of edges that are not connected to any leader node, and denote $M_F \leq M$ the cardinality of $\mathcal{E}_F$, where $M_L + M_F = M$. Then, in compact form, using (18), the leader inputs are rewritten as

$$u := -cE_L \nabla W_L(t, z), \tag{19}$$

where

$$W_L(t, z) = \sum_{k=M_F+1}^{M_F+M_L} W_k(t, z_k), \quad e_k \in \mathcal{E}_L. \tag{20}$$

Recalling the identity (12), denote $\nabla W_L(t, z_{\mathcal{T}}) := \partial W_L\left(t, R^\top z_{\mathcal{T}}\right)/\partial z_{\mathcal{T}}$. Using the chain rule, we have that

$$\nabla W_L(t, z_{\mathcal{T}}) = \left[\frac{\partial z}{\partial z_{\mathcal{T}}}\right]^\top \frac{\partial W_L\left(t, z\right)}{\partial z} = R \nabla W_L(t, z). \tag{21}$$

Replacing (19) into (13) and using (8) and (21) we obtain

$$\dot{z}_{\mathcal{T}} = -cE_{\mathcal{T}}^\top E_{\mathcal{T}} R R^\top z_{\mathcal{T}} - cE_{L\mathcal{T}}^\top E_{L\mathcal{T}} \nabla W_L(t, z_{\mathcal{T}}). \tag{22}$$

B. Main result

Theorem 1: Consider the leader-follower multi-agent system (1) interconnected over a connected undirected graph and in closed loop with the control law (18). Then, given performance functions $\rho_k(t)$, $k \leq M$, satisfying Assumption 1, for all initial conditions such that $z_k(t_0) \in \mathcal{D}_k$, $k \leq M$, where z_k and $\mathcal{D}_k$ are defined, respectively, in (2) and (14), if the following condition holds

$$\bar{\gamma} \geq \frac{\bar{\kappa}}{c} =: \frac{1}{c} \max_k \left\{\frac{\bar{\varphi}_k}{\underline{\rho}_k}\right\}, \tag{23}$$

where $\bar{\gamma}$ is the largest value of γ that ensures that the symmetric matrix

$$\Gamma := \begin{bmatrix} \gamma E_{\mathcal{T}}^\top E_{\mathcal{T}} & \frac{1}{2}\left(E_{\mathcal{T}}^\top E_{\mathcal{T}} - \gamma\left((RR^\top)^{-1} - E_{L\mathcal{T}}^\top E_{L\mathcal{T}}\right)\right) \\ \star & E_{L\mathcal{T}}^\top E_{L\mathcal{T}} \end{bmatrix} \tag{24}$$

is positive semi-definite, i.e., $\Gamma \geq 0$, then the trajectories of the closed-loop system satisfy the performance bounds (3), $\forall k \leq M$, $\forall t \geq t_0$, and the origin of the closed-loop system (22) is asymptotically stable with domain of attraction $\mathcal{D} := \bigcap_{k\leq M} \mathcal{D}_k$. □

Proof: First let us denote a potential function for all the edges $e_k \in \mathcal{E}$

$$W(t, z) = \sum_{k=1}^{M} W_k(t, z_k). \tag{25}$$

Let a candidate Lyapunov function be given by

$$V_1(t, z_\mathcal{T}) = \frac{\gamma}{2} z_\mathcal{T}^\top RR^\top z_\mathcal{T} + W(t, z_\mathcal{T}), \tag{26}$$

where, with a slight abuse of notation, we have written W as a function of $z_\mathcal{T}$ since $W(t, z) = W(t, R^\top z_\mathcal{T})$. From the definition of W in (25) and (17), V_1 satisfies the bounds

$$\alpha_1 |z_\mathcal{T}|^2 \leq V_1(t, z_\mathcal{T}) \leq \beta |\nabla W(t, z_\mathcal{T})|^2, \quad \alpha_1, \beta > 0. \tag{27}$$

Note that, from the structure of $E_{L\mathcal{T}}$, (22) may be equivalently written as

$$\dot{z}_\mathcal{T} = -cE_\mathcal{T}^\top E_\mathcal{T} RR^\top z_\mathcal{T} - cE_{L\mathcal{T}}^\top E_{L\mathcal{T}} \nabla W(t, z_\mathcal{T}). \tag{28}$$

Then, the derivative of (26) yields

$$\begin{aligned} \dot{V}_1(t, z_\mathcal{T}) = & - c\gamma z_\mathcal{T}^\top RR^\top E_\mathcal{T}^\top E_\mathcal{T} RR^\top z_\mathcal{T} \\ & - c\gamma z_\mathcal{T}^\top RR^\top E_{L\mathcal{T}}^\top E_{L\mathcal{T}} \nabla W(t, z_\mathcal{T}) \\ & - c\nabla W(t, z_\mathcal{T})^\top E_{L\mathcal{T}}^\top E_{L\mathcal{T}} \nabla W(t, z_\mathcal{T}) \\ & - c\nabla W(t, z_\mathcal{T})^\top E_\mathcal{T}^\top E_\mathcal{T} RR^\top z_\mathcal{T} \\ & + \sum_{k\leq M} z_k \nabla W_k(t, z_k) \phi_k(t), \end{aligned} \tag{29}$$

where

$$\phi_k(t) := -\frac{\dot{\rho}_k(t)}{\rho_k(t)} \quad \text{and} \quad |\phi_k(t)| \leq \frac{\bar{\varphi}_k}{\underline{\rho}_k}. \tag{30}$$

Adding and subtracting $c\gamma z_\mathcal{T}^\top RR^\top \nabla W(t, z_\mathcal{T})$ to the right-hand side of (29), choosing $\gamma = \bar{\gamma}$ satisfying (23), and defining $y^\top = \begin{bmatrix} z_\mathcal{T}^\top RR^\top & \nabla W(t, z_\mathcal{T})^\top \end{bmatrix}$, (29) becomes

$$\dot{V}_1(t, y) = -cy^\top \Gamma y \leq 0, \tag{31}$$

with Γ given in (24).

Now, we establish invariance of the set $\mathcal{D}$. We proceed by contradiction. Assume that there exists $T > t_0$ such that for all $t \in [t_0, T)$, $z_k(t) \in \mathcal{D}_k$ and $z_k(T) \notin \mathcal{D}_k$. More precisely, we have $|z_k(t)| \to \rho_k(t)$ as $t \to T$ for at least one $k \leq M$. From the definition of W_k in (15), this implies that $V_1(t, z_\mathcal{T}(t)) \to \infty$ as $t \to T$ which is in contradiction with (31). Indeed, (31) implies that $V_1(t, z_\mathcal{T}(t))$ is bounded, i.e., $V_1(t, z_\mathcal{T}(t)) \leq V(t_0, z_\mathcal{T}(t_0)) < \infty$ for all $t \geq t_0$. Therefore, $\mathcal{D}_k$, and consequently, $\mathcal{D}$ is forward invariant, which in turn implies that (3) is satisfied for all $t \geq t_0$ and all $k \leq M$.

Now, from the forward invariance of $\mathcal{D}$ we know that $|z_k(t)| \leq \rho_k(t)$ for all $t \geq t_0$ and all $k \leq M$, hence, it is possible to deduce the boundedness of $\dot{z}_\mathcal{T}(t)$. Then, from the boundedness of $\rho_k(t)$, $\dot{\rho}_k(t)$ we have that $\ddot{V}_1(t, z_\mathcal{T}(t))$ is bounded. The latter implies uniform continuity of $\dot{V}_1(t, z_\mathcal{T}(t))$, which in turn, applying Barbălat's Lemma, implies that $V_1(t, z_\mathcal{T}) \to 0$ as $t \to \infty$. Hence, convergence to consensus is achieved.

In order to show that the origin of (22) is asymptotic stable, let us define the Lyapunov function

$$V_2(t, z_\mathcal{T}) = \frac{1}{2} z_\mathcal{T}^\top RR^\top z_\mathcal{T}, \tag{32}$$

satisfying

$$\frac{1}{2}|z_\mathcal{T}|^2 \leq V_2(t, z_\mathcal{T}) \leq \frac{\lambda_{max}(RR^\top)}{2}|z_\mathcal{T}|^2. \tag{33}$$

Its derivative yields

$$\begin{aligned} \dot{V}_2(t, z_\mathcal{T}) = & - c\ z_\mathcal{T}^\top RR^\top E_\mathcal{T}^\top E_\mathcal{T} RR^\top z_\mathcal{T} \\ & - c\ z_\mathcal{T}^\top RR^\top E_{L\mathcal{T}}^\top E_{L\mathcal{T}} \nabla W_L(t, z_\mathcal{T}). \end{aligned} \tag{34}$$

Denote $z_L \in \mathbb{R}^M$ as the vector with zeros in the position of the edges is $\mathcal{E}_F$, i.e., $[z_L]_k = z_k$ if $e_k \in \mathcal{E}_L$ and $[z_L]_k = 0$ otherwise. Then, from (12), (10), and the structure of E_L, we have that the right-hand side of (34) may be rewritten as

$$\begin{aligned} \dot{V}_2(t, z_\mathcal{T}) = & - c\ z_\mathcal{T}^\top RR^\top E_\mathcal{T}^\top E_\mathcal{T} RR^\top z_\mathcal{T} \\ & - c\ z_L^\top E_L^\top E_L \nabla W_L(t, z_L). \end{aligned} \tag{35}$$

Hence, from the definition of $W_k(t, z_k)$ and the forward invariance of $\mathcal{D}$, we have that

$$\begin{aligned} \dot{V}_2(t, z_\mathcal{T}) \leq & - c\ z_\mathcal{T}^\top RR^\top E_\mathcal{T}^\top E_\mathcal{T} RR^\top z_\mathcal{T} \\ \leq & - c'\ V_2(t, z_\mathcal{T}) < 0, \end{aligned} \tag{36}$$

where $c' := 2c\lambda_{min}(E_\mathcal{T}^\top E_\mathcal{T})$, and $c' > 0$ from [15].

Then, from (33), (36), and forward invariance of $\mathcal{D}$ it follows that the origin of (22) is asymptotically stable with domain of attraction $\mathcal{D}$. ■

Remark 1: A direct consequence of Theorem 1 is that the fulfillment of the bounds (3) depends on the topology of the leader-follower network. Indeed, the condition (23) implies that, for a fixed control gain c, given the matrices $E_\mathcal{T}$ and $E_{L\mathcal{T}}$, or equivalently, a graph topology, there exists a maximum rate $\bar{\varphi}_k/\underline{\rho}_k$, characterizing the performance functions, so that the entire system satisfies the transient constraints. •

C. Homogeneous case

We now show that, in the homogeneous configuration, that is, when all agents are directly controlled by the external input given in (18), stronger stability properties are established following the same analysis as in the main statement.

Let us consider that all the agents are labeled as leaders, that is, $\mathcal{V} = \mathcal{V}_L$. Therefore, we have that $E_L = E$. Moreover, defining the function

$$\tilde{W}(t, z) = \sum_{k\leq M} \frac{1}{2} z_k^2 + W_k(t, z_k), \quad e_k \in \mathcal{E}, \tag{37}$$

the closed-loop system becomes

$$\dot{z}_\mathcal{T} = -cE_\mathcal{T}^\top E_\mathcal{T} \nabla \tilde{W}(t, z_\mathcal{T}). \tag{38}$$

Then, we have the following result.

Theorem 2: Consider the multi-agent system (1) interconnected over a connected undirected graph with $\mathcal{V} = \mathcal{V}_L$ and in closed loop with the control law (18). Then, given performance functions $\rho_k(t)$, $k \leq M$, satisfying Assumption 1, for all initial conditions such that $z_k(t_0) \in \mathcal{D}_k$, $k \leq M$, where z_k and $\mathcal{D}_k$ are defined, respectively, in (2) and (14), the consensus manifold is asymptotically stable with domain of attraction $\mathcal{D} := \bigcap_{k \leq M} \mathcal{D}_k$ and the trajectories of the closed-loop system satisfy (3), $\forall t \geq t_0$. □

Proof: Let a candidate Lyapunov function be given by

$$V_1(t, z_\mathcal{T}) = \frac{\gamma}{2} z_\mathcal{T}^\top (E_\mathcal{T}^\top E_\mathcal{T})^{-1} z_\mathcal{T} + \tilde{W}(t, z_\mathcal{T}), \tag{39}$$

where, with an abuse of notation, we write $\tilde{W}$ as a function of $z_\mathcal{T}$ since $\tilde{W}(t, z) = \tilde{W}(t, R^\top z_\mathcal{T})$. Note that, akin to (27), from (37) and (17), V_1 in (39) satisfies the bounds

$$\alpha_1' |z_\mathcal{T}|^2 \leq V_1(t, z_\mathcal{T}) \leq \beta' |\nabla \tilde{W}(t, z_\mathcal{T})|^2, \quad \alpha_1', \beta' > 0. \tag{40}$$

Now, the derivative of (39) yields

$$\begin{aligned} \dot{V}_1(t, z_\mathcal{T}) =& -c\gamma\, z_\mathcal{T}^\top \nabla \tilde{W}(t, z_\mathcal{T}) + \sum_{k \leq M} z_k W_k(t, z_k) \phi_k(t) \\ & - c \nabla \tilde{W}(t, z_\mathcal{T})^\top E_\mathcal{T}^\top E_\mathcal{T} \nabla \tilde{W}(t, z_\mathcal{T}), \end{aligned} \tag{41}$$

where ϕ_k was defined in (30). Therefore, setting $\gamma \geq \bar{\kappa}/c$, where $\bar{\kappa} := \max\{\bar{\varphi}_k / \underline{\rho}_k\}$, we obtain

$$\begin{aligned} \dot{V}_1(t, z_\mathcal{T}) \leq & -(c\gamma - \bar{\kappa}) \sum_{k \leq M} z_k W_k(t, z_k) \\ & - c\lambda_{min}(E_\mathcal{T}^\top E_\mathcal{T}) |\nabla \tilde{W}(t, z_\mathcal{T})|^2 \\ \leq & -\bar{c}\, V_1(t, z_\mathcal{T}) < 0 \end{aligned} \tag{42}$$

with $\bar{c} := \frac{c\lambda_{min}(E_\mathcal{T}^\top E_\mathcal{T})}{2\alpha_1' \beta'} > 0$.

Forward invariance of $\mathcal{D}$ follows from (42) and from the same arguments as in the proof of Theorem 1.

Next, note that $V_1(t, z_\mathcal{T})$ is positive definite on $\mathcal{D}$ and it satisfies the bounds (40). This means that $V_1(t, z_\mathcal{T}) \to 0$ as $z_\mathcal{T} \to 0$. Therefore, from (40) and (42) it follows that all trajectories of (38) starting in $\mathcal{D}$ converge to the origin. Hence, the origin is attractive for all trajectories $z_\mathcal{T}(t)$ starting in $\mathcal{D}$. It follows from the attractivity of the origin and the forward invariance of $\mathcal{D}$ that the origin of the closed-loop system (38) is (non-uniformly) asymptotically stable with domain of attraction $\mathcal{D}$. ■

Corollary 1: Consider the closed-loop system (38) under an additive essentially bounded disturbance $d(t)$, i.e.,

$$\dot{z}_\mathcal{T} = -cE_\mathcal{T}^\top E_\mathcal{T} \nabla \tilde{W}(t, z_\mathcal{T}) + E_\mathcal{T}^\top d(t). \tag{43}$$

Then, under the same assumptions of Theorem 2, (43) is non-uniformly-in-time input-to-state stable. □

Proof: The time derivative of the Lyapunov function V_1 in (39) along (43) yields

$$\begin{aligned} \dot{V}_1(t, z_\mathcal{T}) =& -c\gamma\, z_\mathcal{T}^\top \nabla \tilde{W}(t, z_\mathcal{T}) + \sum_{k \leq M} z_k W_k(t, z_k) \phi_k(t) \\ & - c \nabla \tilde{W}(t, z_\mathcal{T})^\top E_\mathcal{T}^\top E_\mathcal{T} \nabla \tilde{W}(t, z_\mathcal{T}) \\ & + \nabla \tilde{W}(t, z_\mathcal{T})^\top E_\mathcal{T}^\top d(t) + c\gamma\, z_\mathcal{T}^\top d(t). \end{aligned} \tag{44}$$

Let $\tilde{c} := \left[\frac{(c - \delta_1)}{2 \max \underline{\rho}_k^2} - \frac{\gamma \delta_2}{2}\right]$ and $\delta := \left(\frac{1}{2\delta_1} + \frac{\gamma}{2\delta_2}\right)$, where δ_1, δ_2 are small positive constants so that $\tilde{c} > 0$. Then, setting $\gamma \geq \bar{\kappa}/c$, applying Young's inequality to the last two terms of the right-hand side of (44), and using (42) and (40), we have that

$$\dot{V}_1(t, z_\mathcal{T}) \leq -\tilde{c}\, V_1(t, z_\mathcal{T}) + \delta |d(t)|^2. \tag{45}$$

To assert the fulfillment of the performance bounds in the presence of additive disturbances, we show that in the proximity of the limits imposed by the performance function, the first term on the right-hand side of (45) dominates over the bounded disturbance. To that end, let $\bar{d} := \sup_{t \geq 0} |d(t)|$ and let $\varepsilon(t) \in (0, \rho_k(t))$ be a small constant to be determined. Without loss of generality, let $z_\mathcal{T}$ be such that for at least one edge e_k we have $|z_k(t)| \geq \rho_k(t) - \varepsilon(t)$. Then, $|z_\mathcal{T}(t)| \geq \rho_k(t) - \varepsilon(t)$ and since V_1 is continuous, non-decreasing, and $V_1(t, s) \to \infty$ as $s \to \rho_k(t)$, for all t, it follows that there exists $\varepsilon^*(t, \bar{d})$ such that if $\varepsilon(t) \leq \varepsilon^*(t, \bar{d})$, $\dot{V}_1(t, z_\mathcal{T}) < 0$. The latter holds along trajectories starting from any initial condition $z_\mathcal{T}(t_0) \in \mathcal{D}$ which implies that $z_\mathcal{T}(t)$ cannot approach the boundary of $\mathcal{D}$ so the performance bounds are fulfilled for all t.

Now, note that from (16) and (40), and for all $z_\mathcal{T}(t) \in \mathcal{D}$, it holds that

$$\alpha_1' |z_\mathcal{T}|^2 \leq V_1(t, z_\mathcal{T}) \leq \alpha_2(\mu(t) |z_\mathcal{T}|), \tag{46}$$

where $\alpha_1' > 0$, $\alpha_2(s) = \kappa s^2$, with $\kappa > 0$, and $\mu(\cdot)$ is a continuous positive function $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ given by

$$\mu(t) = \frac{1}{\min_k \{\underline{\rho}_k\}} \max_k \left\{ \frac{1}{\sqrt{1 - \varsigma_k(t)^2}} \right\}, \quad \varsigma_k(t) := \frac{z_k(t)}{\rho_k(t)}.$$

Therefore, from (45) and (46), invoking [18, Proposition 3.1], it follows that the closed-loop system (43) is non-uniformly-in-time input-to-state stable. ■

IV. Numerical simulation

We illustrate the theoretical results via a numerical example. The simulation involves the formation control of six agents evolving in two dimensions and modeled by (1) in closed-loop with (18), where the desired displacements $z_k^d := \left[E^\top \otimes I_2\right] (x_i^d - x_j^d)$, for each $k \leq 7$, are set to $(0.6, 0.3)$, $(-0.6, 0.9)$, $(-0.6, 0.3)$, $(0, 0.6)$, $(-0.6, 0.9)$, $(0, -0, 6)$, and $(-0.6, 0.3)$. The interconnection topology is given by the connected undirected graph in Fig. 1. The set of leaders is given by $\mathcal{V}_L = \{2, 4, 6\}$, represented in gray in Fig. 1. The objective is for the agents to reach the desired formation in a prescribed time, that is, the formation errors must satisfy the performance constraints in (3) with

$$\rho_k(t) = 4.6\ e^{-(0.5t)^2} + 0.2, \quad \forall k \leq 7,$$

where only the leaders apply the control law in (18). The initial conditions of the six agents in the simulation are presented in Table I. The controller gain is set to $c = 1.2$.

It is also assumed that the leader agents are subject to an essentially bounded disturbance modeled as a smoothed vanishing step, that is,

$$\begin{aligned} \theta_i(t) &= -\sigma_i(t)\, [0.2\ 0.3]^\top, \quad i \leq \mathcal{V}_L \\ \sigma_i(t) &= -\tanh(2(t - 5)) - 1. \end{aligned}$$

TABLE I: Initial conditions

Index	1	2	3	4	5	6
x [m]	1.9	-2	5.5	-5.5	5.5	-5.5
y [m]	0	0	2	2	-2	-2

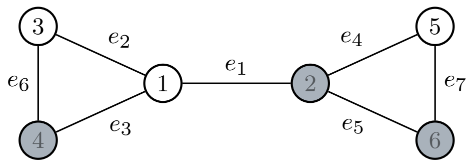

Fig. 1: Connected undirected graph representing the network topology with the leaders in gray.

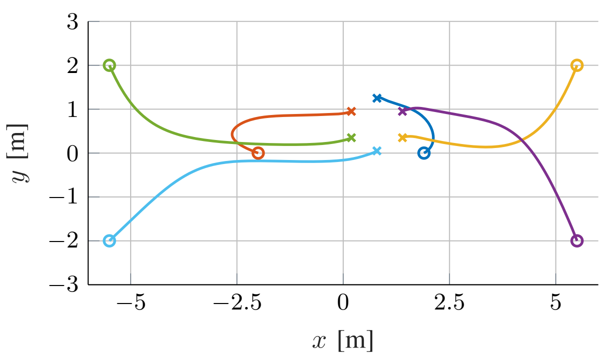

Fig. 2: Paths of the agents. The circles and the crosses represent, respectively, the initial and final positions.

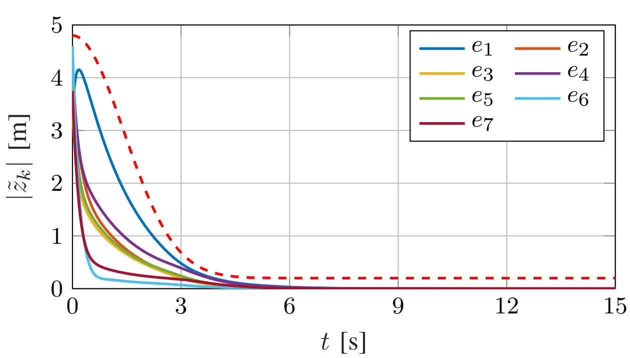

Fig. 3: Trajectories of the formation errors. The dashed line represents the performance bound.

It can be seen from Figs. 2 and 3 that the agents successfully reach the desired formation while satisfying the inter-agent performance bounds. Moreover, as seen in Fig. 3, the formation errors converge to the origin as soon as the disturbance vanishes. Although not proven in this letter, it may be conjectured, from the simulation results, that the input-to-state stability properties established in Corollary 1 may be extended to the leader-follower case under the assumptions of Theorem 1. The proof of this claim is under research.

V. Conclusions

Strong Lyapunov stability and robustness results for multi-agent systems under consensus-based controllers and with transient and steady-state constraints are considered in this letter. We propose a gradient-based agreement controller for leader-follower multi-agent systems that ensures the satisfaction of inter-agent transient constraints for all pairs of agents. We establish asymptotic stability of the agreement manifold in the leader-follower and, in the homogeneous case, nonuniform-in-time input-to-state stability. Although restricted to first-order systems, we argue that our preliminary results pave the way towards considering complex high-order nonlinear systems and more general cooperative tasks, which is the focus of our current research.

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