Navigation in Time-Varying Densities: An Operator Theoretic Approach

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Abstract— **This paper considers the problem of optimizing robot navigation with respect to a time-varying objective encoded into a navigation density function. We are interested in designing state feedback control laws that lead to an *almost everywhere stabilization* of the closed-loop system to an equilibrium point while navigating a region optimally and safely (that is, the transient leading to the final equilibrium point is optimal and satisfies safety constraints). Though this problem has been studied in literature within many different communities, it still remains a challenging non-convex control problem. In our approach, under certain assumptions on the time-varying navigation density, we use Koopman and Perron-Frobenius Operator theoretic tools to transform the problem into a convex one in infinite dimensional decision variables. In particular, the cost function and the safety constraints in the transformed formulation become linear in these functional variables. Finally, we present some numerical examples to illustrate our approach, as well as discuss the current limitations and future extensions of our framework to accommodate a wider range of robotics applications.**

I. Introduction

Robot navigation remains an important problem for a diverse community of researchers in the areas of autonomous driving, precision agriculture, service robots etc. Each of these different sub-areas have the common challenge of overcoming uncertainties in a dynamic environment, which might include other robotic agents and humans. Additionally, the robots may be tasked with optimizing utilities that are possibly time-varying under safety and mission constraints. Traditionally, one of the ways this problem has been tackled within nonlinear controls community, is through the use of navigation functions that encode the control objectives as well as safety constraints. These navigation functions can then be used to extract controllers via their gradients [1]. Although sub-optimal, this approach is efficient to compute, and scales gracefully with the dimensions of the system. However, it is well known that gradient-based approaches need additional care to overcome local-minima problems [2]. Navigation problems have also been studied extensively in an optimal control setting [3], which in addition to producing less conservative trajectories compared to navigation function based approaches, also do not suffer from singularity issues. However, optimal control approaches, particularly those based on dynamic programming principles, often suffer from computational intractability as the state dimensions increase. Direct optimization like collocation or single/multi-shooting methods are better suited for handling higher dimensions [4], but are not guaranteed to provide globally optimal solutions due to non-convexity of the problem. Model Predictive Control (MPC) based planning for robots ensures optimality over a receding time-horizon, with robustness to model uncertainties and dynamic environments [5][6].

This paper is motivated by the need for control design methods that can generate optimal trajectories under safety constraints in an efficient manner, and yet are also straightforward to implement, through a convex reformulation. We are particularly interested in this paper to obtain state feedback controllers in analytical form, because they render the close-loop robot dynamics amenable for analysis. For example, once these feedback controllers are synthesized in closed-form, they can be further used alongside Lyapunov functions for safety and performance verification, under the assumptions that the models of the system are well known and accurate. We develop tools for designing stabilizing feedback controllers that are also optimal with respect to known, but time-varying navigation density functions.

In our work, we adopt an operator theoretic paradigm to analyze and transform the time-varying optimal navigation problem into a convex program in infinite dimensions. Koopman Operators describe the linear evolution of a system through a lifted higher dimensional space, whereas Perron-Frobenius Operators are useful for understanding how densities evolve under a given dynamical flow. Please see [7][8] for recent surveys. Koopman Operator theory has risen in popularity not just for data-driven modelling [9][10], but also for control synthesis [11][12][13], and Lyapunov analysis [14][15]. More recently, the Perron-Frobenius Operator has been exploited to frame robot navigation problems into a convex optimization problem. For example, in [16], the authors consider probabilistic safety during robot navigation. In [17], an off-road navigation problem is solved by a data-driven approach using Perron-Frobenius based formulation. We extend these prior works by introducing time-varying density functions into the optimal navigation formulation, which we then convert into an amenable, linear infinite-dimensional optimization problem. Solving this reformulated problem yields feedback controllers that are almost everywhere stabilizing, while maximizing the navigation objective. Additionally, safety constraints can also be incorporated, and appear in our reformulated setup as linear inequalities in the decision variables (which belong to a function space).

The rest of our paper is organized as follows. Section

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II provides some essential background on Koopman and Perron-Frobenius operators and related results on stability. Next, in Section III, we formulate our optimal navigation problem, which is then analyzed and solved in Section IV with the help of operator theoretic tools described earlier in Section II. Some numerical examples illustrating our approach are presented in Section V, and finally, Section VI provides concluding remarks and our future directions of interest.

Notations: $\mathbb{R}$ and $\mathbb{N}$ are the set of real and natural numbers, respectively. We denote the set of essentially bounded functions defined over a set X as $\mathcal{L}_\infty(X)$. Similarly, the set of integrable functions $g : X \to X$ that satisfies $\int_X |g|_1 dx < \infty$ is denoted by $\mathcal{L}_1(X)$. The term ∇g means the gradient vector of function g, whereas the scalar quantity $\nabla \cdot g$ denotes the divergence of g. Given a linear operator $\mathbb{A}$, the notation $\mathbb{A}^k$ for any $k \in \mathbb{N}$ means the operator is applied k times. For example, $\mathbb{A}^2 g = \mathbb{A}(\mathbb{A}g)$. The support of a function or a density is denoted by $Supp(\cdot)$.

II. Preliminaries

Let us consider a nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \tag{1}$$

where state x is assumed to evolve in a compact set $X \subset \mathbb{R}^n$, and the vector field f is continuously differentiable on X. The flow map of this dynamical system, denoted by the function $\phi_t : X \to X$, is given as

$$\phi_t(x_0) = x_0 + \int_0^t f(x(s))ds, \quad x(0) = x_0.$$

The semigroup of Koopman Operators for system (1) defined over the (Banach) space of $\mathcal{L}_\infty(X)$ functions is given by the linear map $\mathcal{K}_t$ such that

$$\left[\mathcal{K}_t g\right](x) = g(\phi_t(x)), \ \forall g \in \mathcal{L}_\infty(X).$$

***Definition* 1.** *The dual of the Koopman Operator K_t, known as the Perron-Frobenius Operator associated with the dynamical system* (1)*, is defined as the linear map $\mathcal{P}_t : \mathcal{L}_1(X) \to \mathcal{L}_1(X)$ such that:*

$$\int_{\mathcal{D}} \left[\mathcal{P}_t h\right](x)dx = \int_{\phi_{-t}(\mathcal{D})} h(x)dx, \ \forall \mathcal{D} \subseteq X, \ \forall h \in \mathcal{L}_1(X). \tag{2}$$

In other words, $\left[\mathcal{P}_t h\right](x) = h(\phi_{-t}(x)) \left|\frac{\partial \phi_{-t}(x)}{\partial x}\right|$, where $|\cdot|$ is the determinant.

By taking the set $\mathcal{D}$ in equation (2) to be arbitrarily small, one can see that

$$h(x) \geq 0 \ \Rightarrow \ \left[\mathcal{P}_t h\right](x) \geq 0 \qquad \text{(Positivity property)}$$

and when $\mathcal{D}$ is the entire domain X, we note that

$$\int_X \left[\mathcal{P}_t h\right](x)dx = \int_X h(x)dx. \qquad \text{(Markov property)}$$

These two properties mean that the Perron-Frobenius Operator acting on a probability density function on X in particular, yields a function that is also a probability density. The duality between the two operators, which shall be utilized throughout this paper, is expressed as

$$\int_X [\mathcal{K}_t g](x)h(x)dx = \int_X g(x)[\mathcal{P}_t h](x)dx. \tag{3}$$

Contrary to the Koopman Operator, which describes the evolution of observables in a Banach space along the trajectories of a dynamical system, the Perron-Frobenius Operator can use used to describe the evolution of an ensemble of trajectories or densities defined over X. It is also important to note that given the linear operator

$$\mathbb{A}\delta \doteq -\nabla \cdot (\delta f), \tag{4}$$

the Perron-Frobenius operator describes the solution of the advection equation

$$\frac{\partial}{\partial t}\delta(x,t) = \mathbb{A}\delta(x,t), \quad \delta(x,0) = \delta_0(x) \tag{5}$$

linearly as $\mathcal{P}_t \delta_0 = e^{\mathbb{A}t}\delta_0(x)$.

Various stability notions for dynamical systems have been studied typically through the lens of Koopman Operators, like global asymptotic stability in [15], or contraction in [18] etc. Interestingly, stability results can also be described via Perron-Frobenius operators and Lyapunov measures [19]. Towards that end, we first provide the following definition:

***Definition* 2.** *(Almost everywhere stability [19]) Consider a measure μ_0 in the vector space $\mathcal{M}(X)$ of all real-valued measures defined over the Borel σ-algebra on X. The equilibrium point x_e of system* (1) *is said to be almost everywhere (a.e.) stable w.r.t. measure $\mu_0 \in \mathcal{M}(X)$ if*

$$\mu_0 \left\{ x \in X : \lim_{t \to \infty} \phi_t(x) \neq x_e \right\} = 0.$$

Lemma 1. *[20] The equilibrium point x_e of the system* (1) *with region of attraction A is a.e. stable w.r.t. measure μ_0 if there exists a non-negative function $\rho(x) \in C^1(X \backslash x_e) \cap \mathcal{L}_1(X \backslash A)$ such that*

$$\left[\mathbb{A}\rho\right](x) = -\rho_0(x), \tag{6}$$

where the non-negative function $\rho_0 \in \mathcal{L}_1(X)$ is the density corresponding to measure μ_0 (that is, $d\mu_0(x) = \rho_0(x)dx$).

Note that the the existence of density $\rho(x)$ satisfying equation (6) is also a necessary condition (in addition to being sufficient), if we consider a stronger notion of stability (*a.e. uniform stability*). Such a function $\rho(x)$ is also referred to as Lyapunov density. Please refer to [20] for a detailed presentation.

In the next two sections, we introduce our navigation problem, formulated via the linear operators described in this section, and how they can encode different robotic control objectives.

III. Optimal navigation problem formulation

We are ready to show how operator theoretic tools defined in the previous sections can be used to find optimal feedback controllers for robot navigation problems. Towards that end, we first introduce the control-affine dynamical system

$$\dot{x}(t) = f_0(x(t)) + \sum_{1\le i\le m} f_i(x(t))u_i(x(t)) \doteq f(x,u) \tag{7}$$

where $x \in X \subset \mathbb{R}^n$ is the state vector and control input vector $u = [u_1, \cdots, u_m]^\top$ belongs to the set $\mathcal{U}$ of all feedback controllers that are Lipschitz continuous on X and take values in some compact set $U \subset \mathbb{R}^m$. The functions f_i for $i = \{0, 1, \cdots, m\}$ are assumed to be continuously differentiable on the set X. We define the Koopman and Perron-Frobenius Operators in the same way for the controlled system (7) as the previous section, by simply considering the closed-loop dynamics. Thus for the remainder of the paper, the notation f refers to the right hand-side of the closed-loop system (7).

Consider the following optimal control problem, given a (time-invariant) density function $m : X \to \mathbb{R}_{\ge 0}$:

$$\max_{u\in\mathcal{U}} \int_0^\infty \int_{\mathcal{D}} m(\phi_t(x)) d\mu_0(x) dt, \tag{8}$$

where the measure μ_0 (corresponding to some absolutely continuous density δ_0 whose support is the set $Supp(\delta_0) = \mathcal{D} \subseteq X$) captures the distribution of the initial condition of the system. The flow map ϕ_t corresponds to the dynamics (7). Loosely speaking, equation (8) means that we want to control the evolution of the set of states with initial distribution δ_0, using feedback control u in way that maximizes the occupancy of the set $Supp(m)$, averaged over the density m. In addition to the objective (8), one may also have constraints on the navigation problem, such as

$$\int_0^\infty \int_{\mathcal{D}} 1_{X_a}(\phi_t(x)) d\mu_0(x) dt \le \gamma \tag{9}$$

where γ is some positive constant. The set $X_a \subset X$ may represent an unsafe set that needs to be avoided for example, with 1_{X_a} being its corresponding indicator function. For hard safety constraints, one can set γ to zero and show that the trajectories of the system (7) starting inside $Supp(\delta_0)$ do not enter X_a if and only if the left-hand side of equation (9) is zero. However, when γ is nonzero, one may interpret inequality (9) in a probabilistic sense, wherein the trajectories avoid the unsafe set with probability proportional to γ [16].

In this paper, we consider the optimal navigation problem under time-varying density functions, and present some results in the next section to allow a convex reformulation of the problem via the linear advection operator $\mathbb{A}$ acting on densities. For the sake of simplicity of presentation, we shall not consider constraints such as equation (9), although one may include time-varying constraints into our results in a straightforward manner.

IV. Main results

Let us now consider the optimal navigation problem with time-varying density function $m : X \times [0, \infty) \to \mathbb{R}_{\ge 0}$ as follows:

$$\begin{aligned} &\max_{u\in\mathcal{U}} \int_0^\infty \int_{\mathcal{D}} m(\phi_t(x), t) d\mu_0(x) dt, \\ &\textbf{subject to:} \\ &\frac{d}{dt}\phi_t(x) = f(\phi_t(x), u) \end{aligned} \tag{10}$$

where the set $\mathcal{U}$ is the set of all state-feedback controllers that stabilize the origin (a.e. uniformly). In order to make our problem more tractable for analysis, we shall impose certain restrictions on the structure of this time-varying density m. Thus, throughout our presentation, we shall assume that the following holds true:

Assumption 1. *The time-varying density $m(x,t)$ is assumed to be separable into time and space dependent terms as $m(x,t) = \sum_{i=1}^N w_i(t) b_i(x)$ with linearly independent basis b_i's. We also assume $\frac{d^k}{dt^k} w_i(t)$ are bounded for all $k \in \{0\} \cup \mathbb{N}$.*

Now, one may note that by using the duality relation between Koopman and Perron-Frobenius Operators given by equation (3) together with Assumption 1, we get

$$\begin{aligned} &\int_0^\infty \int_{\mathcal{D}} m(\phi_t(x), t) \delta_0(x) dx dt \\ = &\sum_{i=1}^N \int_0^\infty \int_{\mathcal{D}} w_i(t) b_i(\phi_t(x)) \delta_0(x) dx dt \\ = &\sum_{i=1}^N \int_0^\infty \int_{\mathcal{D}} w_i(t) \mathcal{K}_t b_i(x) \delta_0(x) dx dt \\ = &\sum_{i=1}^N \int_0^\infty \int_{\mathcal{D}} w_i(t) b_i(x) \mathcal{P}_t \delta_0(x) dx dt \\ = &\int_{\mathcal{D}} \sum_{i=1}^N b_i(x) \left(\int_0^\infty w_i(t) \mathcal{P}_t \delta_0(x) dt \right) dx. \end{aligned} \tag{11}$$

Next, let us define $\rho_{i,k}(x) \doteq \int_0^\infty \frac{d^k w_i}{dt^k} \mathcal{P}_t \delta_0(x) dt$. Note that for $k = 0$ in particular, $\rho_{i,0}(x) = \int_0^\infty w_i \mathcal{P}_t \delta_0(x) dt$ and thus the objective function is linear w.r.t. the vector $\left[\rho_{1,0}(x), \rho_{2,0}(x), \cdots, \rho_{N,0}(x)\right]$ from equation (11). The following proposition provides a helpful recurrence relation for computing these functions:

Proposition 1. *Let $u(x) \in \mathcal{U}$ be a given feedback controller. If there exist a function $\rho'(x)$ satisfying $\left[\mathbb{A}\rho'\right](x) = -\delta_0(x)$, then*

$$\rho_{i,k+1} = -\mathbb{A}\rho_{i,k} - \frac{d^k w_i(0)}{dt^k} \delta_0, \tag{12}$$

for all $k \ge 0$.

Proof. The operator $\mathbb{A}$ (defined in equation (4)) acting on the function $\rho_{i,k}$ gives

$$\begin{aligned} \mathbb{A}\rho_{i,k} &= -\nabla \cdot (\rho_{i,k} f) = \int_0^\infty \frac{d^k w_i}{dt^k} \mathbb{A}\mathcal{P}_t \delta_0(x) dt \\ &= \int_0^\infty \frac{d^k w_i}{dt^k} \frac{\partial}{\partial t} \mathcal{P}_t \delta_0(x) dt \end{aligned}$$

$$= \left[\frac{d^k w_i}{dt^k}\mathcal{P}_t\delta_0(x)\right]_0^\infty - \int_0^\infty \frac{d^{k+1}w_i}{dt^{k+1}}\mathcal{P}_t\delta_0(x)dx$$

Since the function ρ' satisfies equation (6) (that is, $\left[\mathbb{A}\rho'\right](x) = -\delta_0(x)$), it follows from Lemma 1 that the closed loop system (7) is a.e. stable, and $\lim_{t\to\infty}\mathcal{P}_t\delta_0(x) = 0$. Also, when $t = 0$, we know that $\mathcal{P}_0\delta_0 = \delta_0$. Thus, $\left[\frac{d^k w_i}{dt^k}\mathcal{P}_t\delta_0(x)\right]_0^\infty = -\frac{d^k w_i(0)}{dt^k}\delta_0(x)$, which gives us $\mathbb{A}\rho_{i,k} = -\frac{d^k w_i(0)}{dt^k}\delta_0 - \rho_{i,k+1}$. This completes the proof. □

In order to make our optimal feedback control problem well-posed, we shall only focus on a certain sub-class of time-varying densities $m(x,t)$, since allowing $m(x,t)$ to vary arbitrarily can make the problem increasingly challenging. We thus impose the following assumption on the time dependence of our density function $m(x,t)$:

Assumption 2. *We assume that the time-varying density $m(x,t)$ in equation* (10) *evolves according to a linear PDE of the form $\sum_{k=1}^{K+1}\sum_{j=1}^{M} a_{j,k}(x)\frac{\partial^{k-1}\partial^j}{\partial t^{k-1}\partial x^j}m(x,t) = 0$, for scalar functions $a_{j,k} \in C^1(X,\mathbb{R})$.*

This assumption allows us to solve for functions $\rho_{i,0}(x)$ using the recurrence relation (12). We can now present the following theorem, which reformulates our optimal navigation problem with time-varying densities.

Theorem 1. *Given a time-varying density $m(x,t)$ satisfying Assumptions 1 and 2, let $\alpha_{i,k}(x) = \sum_{j=1}^{M} a_{j,k}(x)\frac{d^j}{dx^j}b_i(x)$, and $\nu_{i,k} = \sum_{j=0}^{k-1}\frac{d^j w_i(0)}{dt^j}(-\mathbb{A})^{k-j-1}\delta_0$. The optimization problem* (10) *can be written equivalently as*

$$\begin{aligned}
&\max_{\rho,\rho',u}\int_D b(x)^\top\rho(x)dx\\
&\textbf{\textit{Subject to:}}\\
&\mathbb{C}(x)^\top\rho(x) = \upsilon(x),\\
&\mathbb{A}\rho'(x) = -\delta_0(x),
\end{aligned}\qquad(13)$$

where $\upsilon(x) = \sum_{i=1}^{N}\sum_{k=1}^{K+1}\alpha_{i,k}(x)\nu_{i,k}(x)$ is a scalar function, and $\mathbb{C}(x)$ is the vector of linear operators given by

$$\mathbb{C}(x) \doteq \left[\begin{bmatrix}\alpha_{11}(x) & \alpha_{12}(x) & \cdots & \alpha_{1K}(x)\\ \alpha_{21}(x) & \alpha_{22}(x) & \cdots & \alpha_{2K}(x)\\ \vdots & \vdots & \ddots & \vdots\\ \alpha_{N1}(x) & \alpha_{N2}(x) & \cdots & \alpha_{NK}(x)\end{bmatrix}\begin{bmatrix}1\\ -\mathbb{A}\\ \mathbb{A}^2\\ \vdots\\ (-\mathbb{A})^K\end{bmatrix}\right].$$

Proof. We first note that using Assumption 1, the objective function $\int_0^\infty\int_D m(\phi_t(x),t)d\mu_0(x)dt$ can be written as

$$\int_0^\infty\int_D m(\phi_t(x),t)d\mu_0(x)dt = \int_D b(x)^\top\rho_0(x)dx,$$

where $\rho_0(x)$ is the vector $\left[\rho_{1,0},\cdots,\rho_{N,0}\right]^\top$ (recall that $\rho_{i,0}(x) \doteq \int_0^\infty w_i(t)\mathcal{P}_t\delta_0(x)dt$). Additionally, the admissible set of controllers $\mathcal{U}$ for our problem (10) is taken to be a.e. uniform stabilizing, which implies (and is implied by) the existence of a function ρ' satisfying $\mathbb{A}\rho' = -\delta_0$ (see Lemma 1). Now by directly applying Proposition 1, we get

$$\begin{aligned}
\rho_{i,k+1} &= -\mathbb{A}\rho_{i,k} - \frac{d^k w_i(0)}{dt^k}\delta_0\\
\Rightarrow\ \rho_{i,k}(x) &= (-\mathbb{A})^k\rho_{i,0}(x) - \nu_{i,k}(x),\ \text{i.e.,}\\
\int_0^\infty\frac{d^k w_i}{dt^k}\mathcal{P}_t\delta_0(x)dt &= (-\mathbb{A})^k\rho_{i,0}(x) - \nu_{i,k}(x),
\end{aligned}\qquad(14)$$

where $\nu_{i,k}$ is defined as $\sum_{j=0}^{k-1}\frac{d^j w_i(0)}{dt^j}(-\mathbb{A})^{k-j-1}\delta_0$. We now use Assumption 2, and rewrite the PDE for $m(x,t)$ by rearranging its terms in the form:

$$\sum_{i=1}^{N}\sum_{k=1}^{K+1}\alpha_{i,k}(x)\frac{d^{k-1}}{dt^{k-1}}w_i(t) = 0,\qquad(15)$$

where $\alpha_{i,k}(x) = \sum_{j=1}^{M}a_{j,k}(x)\frac{d^j}{dx^j}b_i(x)$. Next, by right-multiplying the function $\alpha_{i,k}(x)$ on both sides of equation (14) and summing over indices i and k, we get:

$$0 \overset{(15)}{=} \sum_{i=1}^{N}\left[\sum_{k=1}^{K+1}\alpha_{i,k}(x)(-\mathbb{A})^{k-1}\right]\rho_{i,0}(x) - \sum_{i=1}^{N}\sum_{k=1}^{K+1}\alpha_{i,k}(x)\nu_{i,k}(x).$$

That is, ρ_0 is the unique solution to the equation

$$\mathbb{C}(x)^\top\rho(x) = \upsilon(x),\qquad(16)$$

where $\mathbb{C}(x)$ and $\upsilon(x)$ are as defined in the statement of the theorem. This concludes the equivalence of the two problems (10) and (13). □

Remark: At this point, we would like to comment that if the PDE describing the time-dependence of $m(x,t)$ is first order w.r.t the variable t (that is, $K = 1$), the problem (13) can further be expressed as a linear optimization problem. The decision variables u are combined with ρ and ρ', giving rise to new linear decision variables $\rho, u_i\rho', u_i\rho_j$, for $(i,j) \in \{1,\ldots,m\}\times\{1,\ldots,N\}$, in similar fashion as [16].

A. Trajectory dependent densities

In order to adapt the formulation (10)-(13) to broader practical applications (for example, coverage path planning), we consider the case when the density $m(x,t)$ varies depending on the state trajectory. Concretely, the right-hand side of equation (15) is now considered to be some function of $\phi_t(x)$ instead of zero:

$$\sum_{i=1}^{N}\sum_{k=1}^{K+1}\alpha_{i,k}(x)\frac{d^{k-1}}{dt^{k-1}}w_i(t) = g(\phi_t(x)) = \mathcal{K}_t g(x),\qquad(17)$$

This introduces an additional term on the left-hand side of equation (16):

$$\mathbb{C}(x)^\top\rho(x) = \upsilon(x) + \int_0^\infty\mathcal{K}_t g(x)\mathcal{P}_t\delta_0(x)dt.$$

The term on the right hand-side above involves integral of the Koopman and the Perron-Frobenius operators acting on functions over an infinite time horizon, and as such is difficult to compute. Thus, we assume the functions g to belong to the linear space spanned by the stable eigenfunctions associated with the Koopman operator $\mathcal{K}_t$. Note that since we restrict ourselves to a closed-loop dynamics that is a.e. uniformly stable, these stable eigenfunctions exist and are defined almost everywhere on $\mathcal{D}$. Now, for $g(x) = r^\top \theta(x) = \sum_i r_i \theta_i(x)$ where θ_i are stable eigenfunctions with associated (real) eigenvalues $\lambda_i < 0$, we get

$$\begin{aligned}
\sigma(x) &\doteq \int_0^\infty \mathcal{K}_t g(x) \mathcal{P}_t \delta_0(x) dt \\
&= \int_0^\infty \mathcal{K}_t \left(\sum_i r_i \theta_i(x) \right) \mathcal{P}_t \delta_0(x) dt \\
&= \int_0^\infty \left(\sum_i r_i \exp(\lambda_i t) \theta_i(x) \right) \mathcal{P}_t \delta_0(x) dt \\
&= \sum_i r_i \theta_i(x) \underbrace{\int_0^\infty \exp(\lambda_i t) \mathcal{P}_t \delta_0(x) dt}_{\doteq \sigma_{\lambda_i}(x)} \\
&= \sum_i r_i \theta_i(x) \sigma_{\lambda_i}(x) = r^\top \mathrm{diag}(\sigma_\lambda) \theta
\end{aligned}$$

Finally, using the fact that the function $\sigma_{\lambda_i}(x)$ solves the linear equation $(\mathbb{A} + \lambda_i I)\sigma_{\lambda_i} = -\delta_0$ (one can easily show this by following similar steps as the proof of Proposition 1), we have the following reformulation:

$$\begin{aligned}
&\max_{u \in \mathcal{U}} \int_0^\infty \int_{\mathcal{D}} m(\phi_t(x), t) \delta_0(x) dx dt, \\
&\textbf{Subject to:} \\
&\sum_{k=1}^{K+1} \sum_{j=1}^{M} a_{j,k} \left(\frac{\partial}{\partial x}\right)^j \left(\frac{\partial}{\partial t}\right)^{k-1} m(x,t) = g(\phi_t(x)), \\
&\frac{d}{dt}\phi_t(x) = f(\phi_t(x), u),
\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}
&\max_{\rho, \rho', u, \sigma_\lambda} \int_{\mathcal{D}} b(x)^\top \rho(x) dx \\
&\textbf{Subject to:} \\
&\mathbb{C}(x)^\top \rho(x) = v(x) + r^\top \mathrm{diag}(\sigma_\lambda)\theta, \\
&(\mathbb{A} + \mathrm{diag}(\lambda))\sigma_\lambda = -\delta_0 \\
&f^\top \nabla \theta = \lambda \theta, \\
&\mathbb{A}\rho' = -\delta_0(x) \\
&r^\top \theta = g(x)
\end{aligned}$$

We note that due to the nonlinear term $g(x)$ on the right hand-side of equation (17), the above reformulation is actually bi-linear in the decision variables and may not always be convex. However, for the case when the function $g(x)$ is linear, then r is a known constant, and θ is the identity map with λ, which makes the problem linear in the decision variables (and convex).

V. Numerical Examples

We now present some examples in this section to illustrate our methodology. As one may notice, a key challenge in implementation of our reformulation, as described by equation (10) is that the decision variables are infinite dimensional. We parameterize these decision functions using polynomials, thereby obtaining a finite dimensional problem instead, with scalar decision variables that represent the coefficients of the polynomials. One may choose other representations such as radial basis functions as in [17][16]. A particular advantage of using polynomials, is that one may satisfy the functional equality constraints exactly, but matching the monomial coefficients, as done in Example 1. However, this may not always be possible, say if the initial state density δ_0 or function $b(x)$ appearing in (13) are not polynomials. In such cases, as illustrated by Example 2, one may approximately satisfy the equality constraints by requiring the $\mathcal{L}_2$ norm (over $\mathcal{D}$) of the difference between the two sides of the constraints to be below some chosen threshold.

Example 1:
As an example, let us consider the following 2-d problem:

$$\text{Dynamics: } \dot{x} = u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2,$$

$$\text{Objective: } \max_{u \in \mathcal{U}} \int_0^\infty \int_{\mathcal{D}} \begin{bmatrix} \phi_t(x_1)\phi_t(x_2)\sin(\omega_1 t) + \\ \sin(\|\phi_t(x)\|^2)\cos(\omega_2 t) \end{bmatrix} \delta_0(x) dx dt,$$

where the density $\delta_0(x) = \left(1 - \frac{\|x\|^2}{2}\right)^4$ and $\mathcal{D} = [-1,1]^2$. This problem can then be written in the form of equation (13) with $b(x) = \begin{bmatrix} x_1 x_2 \\ 1 - \|x\|^2 \end{bmatrix}$, $\mathbb{C}(x) = \begin{bmatrix} \mathbb{A}^2 + \omega_1^2 & 0 \\ 0 & \mathbb{A}^2 + \omega_2^2 \end{bmatrix}$, and $v(x) = \begin{bmatrix} \omega_1 \\ \mathbb{A} \end{bmatrix} \delta_0$. When u is restricted to linear-feedback control, we obtain a solution $u^*(x) = Hx$ where $H = \begin{bmatrix} -93.25 & -278.88 \\ 74.69 & -146.77 \end{bmatrix}$. The matrix can be verified to be a Hurwitz matrix, which means that the closed-loop system is indeed a.e. stable. Note that by taking the feedback control to be linear, one can ensure that the functional equality conditions of equation (10) hold exactly, for polynomial candidate functions ρ and ρ' of degree greater than or equal to degree of δ_0.

Example 2:
Consider a 2-d single integrator system with the following objective:

$$\text{Dynamics: } \dot{x} = u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2,$$

$$\text{Objective: } \max_{u \in \mathcal{U}} \int_0^\infty \int_{\mathcal{D}} e^{-2\|\phi_t(x) - c\|^2 - \lambda t} \delta_0(x) dx dt,$$

where $\delta_0(x) = \left(1 - \frac{\|x - c_0\|^2}{8}\right)^4$ and $\mathcal{D} = [-1,1]^2$. The centers $c = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$, and $c_0 = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. Thus, following equation

(13), we have $b(x) = e^{-2\|x-c\|^2}$, $v(x) = -\delta_0$ and $\mathbb{C}(x) = \mathbb{A} - \lambda I$.The feedback controller $u(x)$ is parameterized as a degree four polynomial whereas the unknown densities $\rho(x), \rho'(x)$ are both parameterized as degree eight polynomials.

Figure 1 shows the polynomial vector field of the closed-loop system, and a trajectory of this system starting at $[0.5, 0.5]$, which is the center of the density δ_0. The trajectories move towards the point $[0.5, -0.5]$ initially. However, since that objective is exponentially decaying (with rate $\lambda = 0.01$), the robot isn't incentivized to move any closer as t becomes large, thus it then proceeds to converge to the origin.

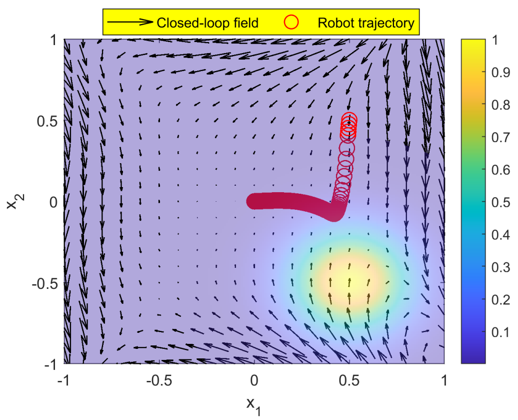

Fig. 1. Optimal trajectory of the robot starting at point $[0.5, 0.5]$ converge to the origin asymptotically, while maximizing the control objective. The density $m(x,t)$ at initial time $t = 0$ is shown via the colormap.

VI. Conclusion and Discussion

As a control problem, robot navigation is challenging, especially when faced with safety constraints and a dynamically changing control objective. In this paper, we consider the optimal navigation problem under time-varying navigation density functions (that in practice may encode, for example, the dynamic environment and/or time-dependent objectives), wherein the system is required to converge to an equilibrium point asymptotically from a given distribution of initial conditions, while optimizing the navigation objective. To this end, we propose an Operator theoretic approach to transform this problem into an amenable form that under certain conditions is linear in infinite dimensional decision variables. This allows us to compute stabilizing feedback control laws (in an *almost everywhere sense*, w.r.t to a given initial measure of initial conditions) in an efficient manner to achieve the time-varying optimal navigation task. An important next step for our work is to relax assumptions on the time-varying density (especially Assumption 1), which would allow for a much wider range of robotic control problems to be cast into our specific formulation. Additionally, we are interested in exploring further the computational tractability of our approach, namely solving an infinite-dimensional optimization problem such as (10) through a finite-dimensional approximations, and its associated limitations. Finally, we would like to extend our work to a data-driven setting when the densities and the system dynamics are not known a priori, leveraging research advances over the last few years in estimating Koopman and Perron Frobenius Operators from trajectory data.

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