Control Barrier Functions for Disjunctions of Signal Temporal Logic Tasks

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***Abstract*— In this work we consider the control problem of systems that are subject to disjunctions of Signal Temporal Logic (STL) tasks. Motivated by existing approaches encoding the STL tasks utilizing time-varying control barrier functions (CBFs), we propose a continuously differentiable function for encoding the STL constraints that is defined as the composition of a smooth approximator of the max operator and a set of functions ensuring the satisfaction of the corresponding STL tasks with a desired robustness, and derive conditions for the choice of the class $\mathcal{K}$ function (when the latter is considered to be linear) to ensure that the proposed function is a CBF. Then, a control law ensuring the satisfaction of the STL task is found as a solution to a computationally efficient QP.**

I. Introduction

Nowadays, autonomous systems are more and more involved in a variety of applications that require them to perform arbitrarily complex tasks constrained both in space and time. These tasks can be often expressed as boolean compositions of simpler spatio-temporal tasks such as "reach area A between 0 and 5 sec and maintain a distance d from area B between 0 and 10 sec".

Recently, Signal Temporal Logic (STL), a logic language introduced in [1] has been found to be an efficient tool for expressing complex, time constrained tasks such as the aforementioned example. Contrary to other logics STL is evaluated over continuous time signals and is equipped with a robustness metric expressing how well the STL task is satisfied. In literature various robustness metrics have been proposed to ensure the robustness of the STL tasks in space examples of which are [2], [3] and [4] while recently a novel temporal robustness metric was introduced in [5].

For control synthesis under STL tasks existing approaches may be categorized into two main groups: 1) works based on mixed integer linear programming (MILP) and 2) gradient based approaches. In the first category the STL tasks are encoded using integer variables and introduced as constraints to the problem while control laws are found as solutions to mixed integer (linear) programs as for example in [6]–[8]. Although these works consider all Boolean compositions of STL formulas they are computationally expensive. This is related to the choice of the optimization horizon which is often considered at least as large as the duration of the task.

On the other hand, gradient based approaches avoid integer encoding and thus are more computationally efficient. Examples of such works are [2], [9] and [10]–[13]. In the latter approaches which are closer to the work considered in this paper the satisfaction of the STL tasks is enforced using control barrier functions and/or control Lyapunov functions as in [10], [13] and feedback control laws are designed for continuous-time, nonlinear input-affine systems. Disjunctions of STL tasks have been recently considered in [11] and [12] utilizing control barrier functions and/or control Lyapunov functions. Yet, the problem of ensuring the existence of a control law over the horizon of the STL task [8] remains open. To tackle this problem, in our recent work [14] we propose updating the parameters determining the control barrier function online. Nevertheless, disjunctions of STL tasks are not considered there.

In this work, we consider formulas written as disjunctions of conjunctions of STL tasks. In order to avoid non-smooth analysis we introduce a differentiable function that under-approximates the max operator and propose a control law that is found as the solution to a computationally efficient QP. Under some regularity and topological assumptions on the predicate functions and the superlevel sets of the functions encoding the conjunctions of the STL tasks respectively, we derive a lower bound on the parameter determining the class $\mathcal{K}$ function, when the latter is chosen to be linear, and show that the proposed function is a CBF for a linear class $\mathcal{K}$ function satisfying the aforementioned bound.

II. Preliminaries and Problem Formulation

True and false are denoted by $\top, \bot$, respectively. Scalars and vectors are denoted by non-bold and bold letters respectively. The partial derivative of a function $\mathfrak{b}(\mathbf{x}, t)$ evaluated at $(\mathbf{x}', t')$ with respect to t and $\mathbf{x}$ is abbreviated by $\frac{\partial \mathfrak{b}(\mathbf{x}',t')}{\partial t} = \frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial t}\big|_{\substack{\mathbf{x}=\mathbf{x}' \\ t=t'}}$ and $\frac{\partial \mathfrak{b}(\mathbf{x}',t')}{\partial \mathbf{x}} = \frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}}\big|_{\substack{\mathbf{x}=\mathbf{x}' \\ t=t'}}$ respectively. The latter is considered to be a row vector. An extended class $\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is a locally Lipschitz continuous and increasing function with $\alpha(0) = 0$. The cardinality of a set $\mathcal{S}$ is denoted by $|\mathcal{S}|$ and its closure by $\bar{\mathcal{S}}$.

A. Signal Temporal Logic

Signal Temporal Logic (STL) determines whether a predicate μ is true or false. The predicate μ takes values from the set $\{\top, \bot\}$ based on the value of a predicate function $h : \mathbb{R}^n \to \mathbb{R}$ as follows: $\mu = \top$, if $h(\mathbf{x}) \geq 0$, or $\mu = \bot$, if $h(\mathbf{x}) < 0$. The basic STL formulas are given by:

$$\phi = \top \mid \mu \mid \neg\phi \mid \phi_1 \wedge \phi_2 \mid \phi_1 \, \mathcal{U}_{[a,b]} \, \phi_2,$$

where ϕ_1, ϕ_2 are STL formulas and $\mathcal{U}_{[a,b]}$ is the *until* operator defined over the interval $[a, b]$ with $0 \leq a \leq b < \infty$.

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The temporal operators *eventually* and *always* are defined as $\mathcal{F}_{[a,b]}\phi = \top \ \mathcal{U}_{[a,b]} \ \phi$ and $\mathcal{G}_{[a,b]}\phi = \neg\mathcal{F}_{[a,b]}\neg\phi$ respectively. Let $\mathbf{x} \models \phi$ denote the satisfaction of the formula ϕ by a signal $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. The formula ϕ is satisfiable if $\exists \ \mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ such that $\mathbf{x} \models \phi$. The semantics of STL are recursively defined in [1]. STL is equipped with a robustness metric determining how robustly an STL formula ϕ is satisfied at time t by a signal $\mathbf{x}$. These semantics are defined as in [15] but omitted here due to space limitations. Note that $\mathbf{x} \models \phi$ if $\rho^{\phi}(\mathbf{x}, 0) > 0$.

B. CBFs for Conjunctions of STL tasks

In this section we summarize the design of the control barrier function encoding an STL task $\phi'_j = \bigwedge_{i_j \in \mathcal{I}_j} \varphi_{i_j}$, where $\varphi_{i_j} = \mathcal{T}_{[a_{i_j}, b_{i_j}]}(h_{i_j}(\mathbf{x}) \geq 0)$ and $\mathcal{T} \in \{\mathcal{F}, \mathcal{G}\}$. In order to ensure satisfaction of the tasks $\varphi_{i_j}, i_j \in \mathcal{I}_j$, a temporal behavior, described by a piecewise linear function $\gamma_{i_j}(t)$, is designed in [16] that guarantees the satisfaction of the task φ_{i_j} with minimum robustness r, where r are tuning parameters. For every $i_j \in \mathcal{I}_j$ define the function:

$$\mathfrak{b}_{i_j}(\mathbf{x}, t) = -\gamma_{i_j}(t) - h_{i_j}(\mathbf{x}), \tag{1}$$

where $h_{i_j} : \mathbb{R}^n \to \mathbb{R}$ is the predicate function assumed to be continuously differentiable. The function $\gamma_{i_j}(t)$ is designed such that $\mathfrak{b}_{i_j}(\mathbf{x}(0), 0) > 0$ and $\mathfrak{b}_{i_j}(\mathbf{x}(t), t) < -r + h_{i_j}(\mathbf{x}(t))$, for every $t \geq t^*_{i_j}$, where $t^*_{i_j}$ is a tuning parameter. Hence, if $\mathfrak{b}_{i_j}(\mathbf{x}(t), t) \geq 0$, for every $t \geq t^*_{i_j}$, then $h_{i_j}(\mathbf{x}(t)) \geq r$. Given the functions (1) for every $i_j \in \mathcal{I}_j$, we can define the function $\mathfrak{b}_j(\mathbf{x}, t)$ encoding the STL task ϕ'_j as in [10]:

$$\mathfrak{b}_j(\mathbf{x}, t) = -\ln\Big(\sum_{i_j \in \mathcal{I}_j} o_{i_j}(t) \exp\left(-\mathfrak{b}_{i_j}(\mathbf{x}, t)\right)\Big), \tag{2}$$

where $o_{i_j}(t) \in \{0, 1\}$ is an integer variable which is equal to 1 for every $t \in T_{i_j}$ and 0 otherwise, where $T_{i_j} = [0, b_{i_j})$, if either $\varphi_{i_j} = \mathcal{F}_{[a_{i_j}, b_{i_j}]} \ (h_{i_j}(\mathbf{x}) \geq 0)$ or $\varphi_{i_j} = \mathcal{G}_{[0, b_{i_j}]} \ (h_{i_j}(\mathbf{x}) \geq 0)$, or $T_{i_j} = [0, a_{i_j}) \cup (a_{i_j}, b_{i_j})$ if $\varphi_{i_j} = \mathcal{G}_{[a_{i_j}, b_{i_j}]} \ (h_{i_j}(\mathbf{x}) \geq 0)$ with $a_{i_j} \neq 0$. Observe that due to the integer variables $o_{i_j}(t)$ the function $\mathfrak{b}_j(\mathbf{x}, t)$ is discontinuous with respect to t. The set of discontinuities of $\mathfrak{b}_j(\mathbf{x}, t)$ is defined as $\Sigma_j = \{b_{i_j} : \varphi_{i_j} = \mathcal{F}_{[a_{i_j}, b_{i_j}]} \ (h_{i_j}(\mathbf{x}) \geq 0) \text{ or } \varphi_{i_j} = \mathcal{G}_{[0, b_{i_j}]} \ (h_{i_j}(\mathbf{x}) \geq 0)\} \cup \{a_{i_j}, b_{i_j} : \varphi_{i_j} = \mathcal{G}_{[a_{i_j}, b_{i_j}]} \ (h_{i_j}(\mathbf{x}) \geq 0), a_{i_j} \neq 0\}$. The proposed CBF function $\mathfrak{b}_j(\mathbf{x}, t)$ is designed in such way that if it remains non-negative at all times, i.e., the sets $\mathcal{C}_j(t) = \{\mathbf{x} \in \mathbb{R}^n : \mathfrak{b}_j(\mathbf{x}, t) \geq 0\}$ are forward invariant, then $\mathbf{x} \models \phi'_j$. The last conclusion is a direct consequence of the fact that $\mathfrak{b}_j(\mathbf{x}, t)$ is an under approximation of $\min_{i_j \in \mathcal{A}_j(t)} \mathfrak{b}_{i_j}(\mathbf{x}, t)$, at fixed t, where $\mathcal{A}_j(t) = \{i_j \in \mathcal{I}_j : o_{i_j}(t) \neq 0\}$.

C. Problem Formulation

Consider the input-affine system:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)) + g(\mathbf{x}(t))\mathbf{u}(t), \tag{3}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$ is the state and input of the system at time t respectively, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz continuous functions. When it is clear from the context, we will omit the dependence on t from $\mathbf{x}(t), \mathbf{u}(t)$. In addition, we make the following assumption on $g(\mathbf{x})$:

Assumption 1. *The matrix $g(\mathbf{x})$ has full row rank for every $\mathbf{x} \in \mathbb{R}^n$.*

Assumption 1 is a standard assumption in CBF based approaches. Nevertheless, for systems with higher relative degree, one may consider higher order CBFs as for example in [17]. In this work, the system (3) is subject to an STL formula ϕ defined by the following STL fragment:

$$\psi = \top \mid \mu \mid \neg\mu \mid \psi_1 \wedge \psi_2 \tag{4a}$$

$$\varphi = \mathcal{G}_{[a,b]}\psi \mid \mathcal{F}_{[a,b]}\psi \mid \psi_1 \ \mathcal{U}_{[a,b]} \ \psi_2, \tag{4b}$$

$$\phi = \bigvee_{j \in \mathcal{J}} \bigwedge_{i_j \in \mathcal{I}_j} \varphi_{i_j}, \tag{4c}$$

where μ is a predicate, $\varphi_{i_j}, i_j \in \mathcal{I}_j$ are STL formulas of the form (4b), $[a, b] \subset \mathbb{R}_{\geq 0}$ and $\mathcal{J} \subset \mathbb{N}_{>0}$, where $|\mathcal{J}|$ determines the number of disjunctions of ϕ. Based on the above, we can state the problem considered in this work as follows:

Problem 1. *Consider the system dynamics* (3) *that is subject to an STL formula defined by* (4a)-(4c)*. Design a control law $\mathbf{u}(\mathbf{x}, t)$ such that $\rho^{\phi}(\mathbf{x}, 0) \geq r$, where $r > 0$ is a tuning parameter.*

III. Control Approach

In this section we will define the control barrier function (CBF) that encodes the spatio-temporal constraints induced by ϕ, defined by (4a)-(4c) and propose a feedback controller that ensures its satisfaction. We split the design of the CBF in two main steps. In the first step, for every $j \in \mathcal{J}$, we consider the CBF function $\mathfrak{b}_j : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined in (2), that encodes the satisfaction of ϕ'_j, defined as $\phi'_j = \bigwedge_{i_j \in \mathcal{I}_j} \varphi_{i_j}$. As discussed in Section II.B, the proposed functions $\mathfrak{b}_j(\mathbf{x}, t)$ ensure the satisfaction of ϕ'_j, when $\mathfrak{b}_j(\mathbf{x}, t) \geq 0$, for every $t \geq 0$. When the disjunction of a set of STL formulas is considered, and in order to avoid non-smooth analysis it is necessary to under-approximate the $\max$ operator with a differentiable function with respect to $\mathbf{x}$ [2]. Therefore, as a second step, given the STL formula ϕ defined by (4c), we define the function $\mathfrak{b} : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ as $\mathfrak{b}(\mathbf{x}, t) = \frac{\sum_{j \in \mathcal{J}} \mathfrak{b}_j(\mathbf{x}, t) \exp(\eta \mathfrak{b}_j(\mathbf{x}, t))}{\sum_{j \in \mathcal{J}} \exp(\eta \mathfrak{b}_j(\mathbf{x}, t))}$, where $\eta > 0$ is a parameter describing the quality of the approximation with a larger value of η resulting in a better approximation of the maximum value. Specifically, for a given point $(\mathbf{x}, t) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0}$, it holds that $\lim_{\eta \to +\infty} \mathfrak{b}(\mathbf{x}, t) = \max_{j \in \mathcal{J}} \mathfrak{b}_j(\mathbf{x}, t)$. Based on the above, and since $\mathfrak{b}(\mathbf{x}, t)$ is an under-approximation of the max operator, if we can ensure that $\mathfrak{b}(\mathbf{x}, t) \geq 0$, for every $t \geq 0$, then $\max_{j \in \mathcal{J}} \mathfrak{b}_j(\mathbf{x}, t) \geq 0$.

Although the function $\mathfrak{b}(\mathbf{x}, t)$ defined above provides a delicate way to encode the satisfaction of the disjunction of the formulas $\phi'_j, j \in \mathcal{J}$, it may enforce the non-negativity of the functions $\mathfrak{b}_j(\mathbf{x}, t)$ that were negative at past time instants, i.e., the system may be asked to satisfy a specification at time t that was violated at $t' < t$. To avoid such conservatism, for every $j \in \mathcal{J}$, we introduce a function $\mathfrak{o}_j : \mathbb{R}_{\geq 0} \to \{0, 1\}$ satisfying:

$$\mathfrak{o}_j(t) = 0 \Leftrightarrow \inf_{\tau \in [0,t]} \mathfrak{b}_j(\mathbf{x}(\tau), \tau) < 0, \tag{5}$$

where $\mathbf{x} : [0, t] \to \mathbb{R}^n$ is assumed to be absolutely continuous and $t \in [0, \tau_{\max})$, where $\tau_{\max} \leq +\infty$. Based on the deactivation functions $\mathfrak{o}_j(t), j \in \mathcal{J}$, we may define the CBF function considered in this work as:

$$\mathfrak{b}(\mathbf{x}, t) = \frac{\sum_{j \in \mathcal{J}} \mathfrak{o}_j(t) \mathfrak{b}_j(\mathbf{x}, t) \exp\left(\eta \mathfrak{b}_j(\mathbf{x}, t)\right)}{\sum_{j \in \mathcal{J}} \mathfrak{o}_j(t) \exp\left(\eta \mathfrak{b}_j(\mathbf{x}, t)\right)}. \tag{6}$$

In order to ensure that (6) is well defined at all times, we pose the following assumption:

Assumption 2. *There exists at least one $j \in \mathcal{J}$ such that $\mathfrak{o}_j(t) \neq 0$ for every $t \geq 0$.*

Assumption 2 implies that for a given function $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, there should always exist at least one $j \in \mathcal{J}$ such that $\mathbf{x}(t) \in \mathcal{C}_j(t)$ is true at all times, i.e., the STL formula ϕ_j' is satisfied (provided that $\mathcal{C}_j(t) \neq \emptyset$ for every $t \geq 0$). This is required to ensure that the denominator of $\mathfrak{b}(\mathbf{x}, t)$ is non-zero at all times (assuming that function $\mathfrak{b}_j(\mathbf{x}, t), j \in \mathcal{J}$ is bounded from below for any $\mathbf{x} \in \mathcal{C}_j(t)$ and $t \in \mathbb{R}_{\geq 0}$).

Let $\sigma_j = \inf\{t \in \mathbb{R}_{\geq 0} : \mathfrak{b}_j(\mathbf{x}(t), t) < 0\}$ denote the first time instant at which $\mathfrak{b}_j(\mathbf{x}(t), t)$ becomes negative for a given function $\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. If $\mathfrak{o}_j(t) \neq 0$ for every t, then $\sigma_j = +\infty$. Without loss of generality we assume that $\sigma_j > 0$ for every $j \in \mathcal{J}$, i.e., $\mathfrak{b}_j(\mathbf{x}(0), 0) \geq 0$ for all $j \in \mathcal{J}$. Based on the above, and given the deactivation policy defined in (5), $\mathfrak{b}(\mathbf{x}, t)$ is differentiable only in $\Omega_\sigma = \bigcup_{t \in \mathbb{R}_{\geq 0} \setminus \Sigma} \mathcal{C}(t) \times \{t\}$, where $\Sigma \subseteq \bigcup_{j \in \mathcal{J}} \Sigma_j \cup \{\sigma_j \in \mathbb{R}_{\geq 0} : \sigma_j \neq +\infty\}$ and $\mathcal{C}(t) = \{\mathbf{x} \in \mathbb{R}^n : \mathfrak{b}(\mathbf{x}, t) \geq 0\}$, which is assumed to satisfy $\mathcal{C}(t) \subset \mathcal{D}$, for every $t \geq 0$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is an open, bounded set. Let $\Omega = \bigcup_{t \geq 0} \mathcal{C}(t) \times \{t\}$.

Definition 1. *The function $\mathfrak{b} : \Omega \to \mathbb{R}$ is a control barrier function (CBF) for* (3)*, if there exists an extended class $\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that, for all $(\mathbf{x}, t) \in \Omega_\sigma$:*

$$\sup_{\mathbf{u} \in \mathbb{R}^m} \left[\frac{\partial \mathfrak{b}(\mathbf{x}, t)}{\partial \mathbf{x}} (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \frac{\partial \mathfrak{b}(\mathbf{x}, t)}{\partial t} \right] \geq -\alpha(\mathfrak{b}(\mathbf{x}, t)). \tag{7}$$

According to Definition 1, if $\mathfrak{b}(\mathbf{x}, t)$ is a CBF and $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} g(\mathbf{x}) \neq 0$, for every $(\mathbf{x}, t) \in \Omega_\sigma$, i.e., the Slater's constraint qualification is satisfied [18], then we may define the control law $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, for any $(\mathbf{x}, t) \in \Omega_\sigma$ as the solution to the following QP problem:

$$\min_{\mathbf{u} \in \mathbb{R}^m} \mathbf{u}^T \mathbf{u}, \tag{8}$$

subject to:

$$\frac{\partial \mathfrak{b}(\mathbf{x}, t)}{\partial \mathbf{x}} (f(\mathbf{x}) + g(\mathbf{x})\mathbf{u}) + \frac{\partial \mathfrak{b}(\mathbf{x}, t)}{\partial t} \geq -\alpha(\mathfrak{b}(\mathbf{x}, t)). \tag{8a}$$

If Slater's constraint qualification is not true for a point $(\mathbf{x}, t) \in \Omega_\sigma$, then the feasibility of (8a) is ensured under the following assumption:

Assumption 3. *For every point $(\mathbf{x}, t) \in \Omega_\sigma$ satisfying $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} g(\mathbf{x}) = 0$, it holds that $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial t} \geq -\alpha(\mathfrak{b}(\mathbf{x}, t))$.*

Assumption 3 is introduced to ensure that $\mathbf{x}(t) \in \mathcal{C}(t)$ holds even in cases of singularity of (8a). Intuitively, it states that the rate of change of $\mathfrak{b}(\mathbf{x}, t)$ for fixed $\mathbf{x}$ cannot be unbounded and independent of the value of $\mathfrak{b}(\mathbf{x}, t)$ and can often be ensured by appropriately choosing the class $\mathcal{K}$ function as will be shown in Section IV. Based on the above, we may introduce our first result on the satisfaction of the STL formula ϕ as follows:

Theorem 1. *Consider the system* (3)*, the STL formula ϕ defined by* (4a)-(4c)*, and the CBF function $\mathfrak{b} : \Omega \to \mathbb{R}$, defined in* (6) *that is continuously differentiable on Ω_σ. Let Assumptions 1- 3 hold. Let further $\mathbf{x}(0) \in \mathcal{C}(0)$. Then, for a given class $\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}_{\geq 0}$, and an open, bounded set $\mathcal{D} \subseteq \mathbb{R}^n$ satisfying $\mathcal{C}(t) \subset \mathcal{D}$ for every $t \geq 0$, the control law $\mathbf{u}(\mathbf{x}, t)$ found as the optimal solution of* (8) *ensures that $\rho^\phi(\mathbf{x}, 0) \geq r > 0$.*

Proof. When $(\mathbf{x}, t) \in \Omega_\sigma$ the proof follows similar arguments to the proof of [19, Cor. 1]. What remains to be shown is the non-negativity of $\mathfrak{b}(\mathbf{x}, t)$ for $(\mathbf{x}, t) \in \Omega_\sigma' = \bigcup_{t \in \Sigma} \mathcal{C}(t) \times \{t\}$. To show this, we consider 3 cases. First, consider the case of $(\mathbf{x}, \sigma) \in \Omega_\sigma'$ with $\sigma \in \{\sigma_{j'}, j' \in \mathcal{J} : \sigma_{j'} \neq +\infty\} \setminus \bigcup_{j' \in \mathcal{J}_1} \Sigma_{j'}$, where $\mathcal{J}_1 = \{j' \in \mathcal{J} : \mathfrak{o}_{j'}(\sigma) \neq 0\}$. By Assumption 2, $\mathcal{J}_1 \neq \emptyset$ holds. Hence, if $\lim_{t \to \sigma^-} \mathfrak{b}(\mathbf{x}(t), t) \geq 0$, then, for every $j' \in \mathcal{J}_1$, we have $\lim_{t \to \sigma^-} \mathfrak{b}_{j'}(\mathbf{x}(t), t) = \mathfrak{b}_{j'}(\mathbf{x}(\sigma), \sigma)$ implying $\mathfrak{b}(\mathbf{x}(\sigma), \sigma) \geq 0$. Second, assume that $(\mathbf{x}, \sigma) \in \Omega_\sigma'$ with $\sigma \in \Sigma_j \setminus \{\sigma_j, j \in \mathcal{J} : \sigma_j \neq +\infty\}$ for any $j \in \mathcal{J}$ satisfying $\sigma_j > \sigma$, non-negativity of $\lim_{t \to \sigma^-} \mathfrak{b}(\mathbf{x}(t), t)$ implies $\lim_{t \to \sigma^-} \mathfrak{b}_j(\mathbf{x}(t), t) \geq 0$, for every $j \in \mathcal{J}$ with $\mathfrak{o}_j(\sigma) \neq 0$. By construction of $\mathfrak{b}_j(\mathbf{x}(t), t)$, it holds that $\lim_{t \to \sigma^-} \mathfrak{b}_j(\mathbf{x}(t), t) \leq \mathfrak{b}_j(\mathbf{x}(\sigma), \sigma)$, hence $\mathfrak{b}(\mathbf{x}(\sigma), \sigma) \geq 0$ follows. Finally, assume that $(\mathbf{x}, \sigma) \in \Omega_\sigma'$ with $\sigma \in \{\sigma_{j'}, j' \in \mathcal{J} : \sigma_{j'} \neq +\infty\} \cap \bigcup_{j \in \mathcal{J}_1} \Sigma_j$, then combining the arguments used in the previous two cases we may conclude that $\mathfrak{b}(\mathbf{x}(\sigma), \sigma) \geq 0$. We have now shown that $\mathbf{x}(t) \in \mathcal{C}(t)$ for every $t \geq 0$. Let $\mathcal{J}' = \{j \in \mathcal{J} : \sigma_j = +\infty\}$ and observe that $\mathbf{x} \in \mathcal{C}(t)$ implies $\mathbf{x} \in \mathcal{C}_j(t), j \in \mathcal{J}'$ for every $t \geq 0$. Then, by construction of $\mathfrak{b}_j(\mathbf{x}, t)$ and [19, Cor. 1], we have $\rho^{\phi_j'}(\mathbf{x}, 0) \geq r > 0$. The result follows by considering that $\rho^\phi(\mathbf{x}, 0) = \max_{j \in \mathcal{J}} \rho^{\phi_j'}(\mathbf{x}, 0) \geq \max_{j \in \mathcal{J}'} \rho^{\phi_j'}(\mathbf{x}, 0)$. ∎

IV. Control Barrier Functions for Disjunctions

In Section III, we proposed a control law ensuring the satisfaction of ϕ which was based on the assumption that $\mathfrak{b}(\mathbf{x}, t)$ is a CBF function. In this section we derive sufficient conditions for the function $\mathfrak{b}(\mathbf{x}, t)$ to be a CBF function by ensuring the existence of the class $\mathcal{K}$ function for (7) to hold. As discussed in Section III, equation (7) might not hold, when Slater's constraint is violated which is equivalent to $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$ due to Assumption 1. Therefore, in this section we seek conditions under which Assumption 3 is satisfied by determining the nature of the critical points of $\mathfrak{b}(\mathbf{x}, t)$ and the value of $\mathfrak{b}(\mathbf{x}, t)$ at the critical points. In the following we make the following assumption on the functions $h_{i_j}(\mathbf{x})$ and $\mathfrak{b}_j(\mathbf{x},t), j \in \mathcal{J}$:

Assumption 4. *Let $h_{i_j}: \mathbb{R}^n \to \mathbb{R}$ be the predicate function corresponding to $\varphi_{i_j}, i_j \in \mathcal{I}_j, j \in \mathcal{J}$ and $\mathfrak{b}_j : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, defined as in (2). Then, we make the following assumptions:*

i) The predicate functions $h_{i_j}(\mathbf{x})$ are twice differentiable with continuous second derivatives and satisfy $h_{i_j}^{\max} = \sup_{\mathbf{x}\in\mathbb{R}^n} h_{i_j}(\mathbf{x}) < +\infty$.
ii) For fixed t the functions $\mathfrak{b}_j(\mathbf{x},t), j \in \mathcal{J}$ are quasiconcave, i.e., the superlevel sets of $\mathfrak{b}_j(\mathbf{x},t)$ are convex [20, Sec. 3.4.1]. In addition, $\mathcal{C}_j(t) \neq \emptyset$, for every $t \geq 0$.
iii) For any $j \in \mathcal{J}$, there exists $\delta > 0$ such that $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$ implies $\mathfrak{b}_j(\mathbf{x},t) \geq \delta$.

The first part of Assumption 4 ensures that all functions $\mathfrak{b}_j(\mathbf{x},t)$ and thus $\mathfrak{b}(\mathbf{x},t)$ are twice differentiable with continuous second derivatives with respect to $\mathbf{x}$ for fixed t. The second part of Assumption 4 ensures that all superlevel sets of $\mathfrak{b}_j(\mathbf{x},t)$ are non-empty and convex while (iii) implies that the critical points of $\mathfrak{b}_j(\mathbf{x},t)$ are in the interior of $\mathcal{C}_j(t)$ for every $t \in \mathbb{R}_{\geq 0}\backslash\Sigma_j$. The latter assumption is not restrictive and can often be ensured in practice by appropriately tuning the parameters determining the functions $\gamma_{i_j}(t)$.

Differentiating the barrier function $\mathfrak{b}(\mathbf{x},t)$ defined in (6) with respect to $\mathbf{x}$ we get:

$$\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} = \sum_{j\in\mathcal{J}} \mathfrak{o}_j(t)\kappa_j(\mathbf{x},t)\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}}, \tag{9}$$

where

$$\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} = \sum_{i_j\in\mathcal{A}_j(t)} \underbrace{\frac{\exp\left(-\mathfrak{b}_{i_j}(\mathbf{x},t)\right)}{\sum_{i_j\in\mathcal{I}_j} o_{i_j}(t)\exp\left(-\mathfrak{b}_{i_j}(\mathbf{x},t)\right)}}_{\lambda_{i_j}(\mathbf{x},t)} \frac{\partial h_{i_j}(\mathbf{x})}{\partial \mathbf{x}} \tag{10}$$

and where $\kappa_j(\mathbf{x},t) = \frac{\left(1+\eta(\mathfrak{b}_j(\mathbf{x},t)-\mathfrak{b}(\mathbf{x},t))\right)\exp\left(\eta\mathfrak{b}_j(\mathbf{x},t)\right)}{\sum_{j\in\mathcal{J}}\mathfrak{o}_j(t)\exp\left(\eta\mathfrak{b}_j(\mathbf{x},t)\right)}$ and $\mathcal{B}(t) = \{j \in \mathcal{J} : \mathfrak{o}_j(t) \neq 0\}$ for every $t \geq 0$. Note that $\lambda_{i_j}(\mathbf{x},t) \in (0,1]$ and $\sum_{i_j\in\mathcal{I}_j}\lambda_{i_j}(\mathbf{x},t) = 1$. By Assumption 2, $\mathcal{B}(t) \neq \emptyset$, for every $t \geq 0$. Initially, we assume that $\mathcal{B}(t)$ is known for every $t \geq 0$. Then, we begin the analysis by making the following observation:

Lemma 1. *Let $\mathfrak{b} : \Omega \to \mathbb{R}$ be a twice differentiable function with respect to $\mathbf{x}$ on Ω_σ, defined by (6). Then, $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$ is true if and only if $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}}, j \in \mathcal{B}(t)$ are linearly dependent.*

Proof. If $\mathcal{B}(t)$ is a singleton or there exists a $j \in \mathcal{B}(t)$ such that $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$, the result follows trivially. Next, assume that $\mathcal{B}(t)$ is not a singleton and $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}}, j \in \mathcal{B}(t)$ are linearly independent. Then, by the linear independence of the vectors and (9), we have that $1 + \eta(\mathfrak{b}_j(\mathbf{x},t) - \mathfrak{b}(\mathbf{x},t)) = 0$, $\forall j \in \mathcal{B}(t)$, or equivalently $\mathfrak{b}(\mathbf{x},t) = \mathfrak{b}_j(\mathbf{x},t) + \frac{1}{\eta}$, for every $j \in \mathcal{B}(t)$. This implies that $\mathfrak{b}_j(\mathbf{x},t) = \mathfrak{b}_{j'}(\mathbf{x},t)$, for any $j, j' \in \mathcal{B}(t)$. Substituting the latter to (6), we have $\mathfrak{b}(\mathbf{x},t) = \mathfrak{b}_j(\mathbf{x},t)$ which leads to contradiction. Hence, the result follows. ∎

Lemma 1 determines the relation of the gradients of the barriers corresponding to the conjunctions at the critical points of $\mathfrak{b}(\mathbf{x},t)$. This relation allows us to draw conclusions on the value of $\mathfrak{b}(\mathbf{x},t)$ at the critical points as follows:

Lemma 2. *Consider the function $\mathfrak{b}(\mathbf{x},t)$, defined in (6) and let Assumptions 2 and 4 hold. Assume further that $\mathcal{C}(t)$ has a non-empty interior for every $t \geq 0$. For every fixed $t \in \mathbb{R}_{\geq 0}\backslash\Sigma$, let $p_j(\mathbf{x},t) = 1 + \eta(\mathfrak{b}_j(\mathbf{x},t) - \mathfrak{b}(\mathbf{x},t)), j \in \mathcal{B}(t)$. In addition, for every $j \in \mathcal{B}(t)$ and fixed $t \in \mathbb{R}_{\geq 0}\backslash\Sigma$, assume that the following property holds:*

$$p_j(\mathbf{x},t)\lambda_{\max}\left(\frac{\partial^2 \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}^2}\right) + \eta(1+p_j(\mathbf{x},t))\left\|\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}}\right\|^2 < 0, \tag{11}$$

for every $\mathbf{x} \in \mathcal{M}(t) = \left\{\mathbf{z} \in \mathcal{C}(t) : \frac{\partial \mathfrak{b}(\mathbf{z},t)}{\partial \mathbf{z}} = \mathbf{0}, p_j(\mathbf{z},t) > 0, \forall j \in \mathcal{B}(t)\right\}$. Then, there exists a constant parameter $\bar{\mathfrak{b}}_t > 0$ such that $\mathfrak{b}(\mathbf{x},t) \geq \bar{\mathfrak{b}}_t$ is true for every $\mathbf{x} \in \mathcal{C}(t)$ satisfying $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$.

Proof. If there exists $j \in \mathcal{B}(t)$ such that $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$, then by Assumption 4, $\mathfrak{b}_j(\mathbf{x},t) \geq \delta$. Thus, $\mathfrak{b}(\mathbf{x},t) > 0$ holds. Specifically, we have that $\mathfrak{b}(\mathbf{x},t) \geq \delta_1$, where $\delta_1 = \delta$, if $\mathcal{B}(t)$ is a singleton, or $\delta_1 = \frac{\delta\exp(\eta\delta)}{\sum_{j\in\mathcal{J}}\mathfrak{o}_j(t)\exp(\eta M_j(t))}$ otherwise, where $M_j(t)$ is an upper bound of $\mathfrak{b}_j(\mathbf{x},t)$, for every $\mathbf{x} \in \mathcal{C}(t)$ and $j \in \mathcal{B}(t)$. Note that $M_j(t) < \infty$ is ensured for every $j \in \mathcal{B}(t)$ since $h_{i_j}^{\max} < \infty$ holds for every $i_j \in \mathcal{A}_j(t)$. In that case, the bound can be defined as $M_j(t) = -\ln\left(\sum_{i_j\in\mathcal{I}_j} o_{i_j}(t)\exp\left(\gamma_{i_j}(t) - h_{i_j}^{\max}\right)\right)$. For the rest of the analysis, we will assume that $\mathcal{B}(t)$ is not a singleton and $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} \neq \mathbf{0}$, for every $j \in \mathcal{B}(t)$. Then, due to the linear dependence of $\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}}, j \in \mathcal{B}(t)$ (shown in Lemma 1), there exists $j \in \mathcal{B}(t)$ such that $\mathfrak{b}(\mathbf{x},t) \neq \mathfrak{b}_j(\mathbf{x},t) + \frac{1}{\eta}$. If $\mathfrak{b}(\mathbf{x},t) > \mathfrak{b}_j(\mathbf{x},t) + \frac{1}{\eta}$ holds or there exists $j' \in \mathcal{B}(t)$ with $j' \neq j$ such that $\mathfrak{b}(\mathbf{x},t) \geq \mathfrak{b}_{j'}(\mathbf{x},t) + \frac{1}{\eta}$, then $\mathfrak{b}(\mathbf{x},t) \geq \frac{1}{\eta}$. Next, assume that $\mathfrak{b}(\mathbf{x},t) < \mathfrak{b}_j(\mathbf{x},t) + \frac{1}{\eta}$ for every $j \in \mathcal{B}(t)$. By (9), the latter assumption implies that $\kappa_j(\mathbf{x},t) > 0$, and thus $p_j(\mathbf{x},t) > 0$, for every $j \in \mathcal{B}(t)$. Differentiating (9) with respect to $\mathbf{x}$ and focusing on the points satisfying $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}} = \mathbf{0}$ we have:

$$\frac{\partial^2 \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}^2} = \sum_{j\in\mathcal{B}(t)} \lambda_j^+(\mathbf{x},t)\Bigg[\eta(1+p_j(\mathbf{x},t))\frac{\partial \mathfrak{b}_j(\mathbf{x},t)^T}{\partial \mathbf{x}} \times \tag{12}$$
$$\times \frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} + p_j(\mathbf{x},t)\frac{\partial^2 \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}^2}\Bigg],$$

where $\lambda_j^+(\mathbf{x},t) = \frac{\exp(\eta\mathfrak{b}_j(\mathbf{x},t))}{\sum_{j\in\mathcal{J}}\mathfrak{o}_j(t)\exp(\eta\mathfrak{b}_j(\mathbf{x},t))} > 0$. When $p_j(\mathbf{x},t) > 0, \forall j \in \mathcal{B}(t)$ the matrices $A_j, j \in \mathcal{B}(t)$ defined as $A_j = \eta(1+p_j(\mathbf{x},t))\frac{\partial \mathfrak{b}_j(\mathbf{x},t)^T}{\partial \mathbf{x}}\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}} + p_j(\mathbf{x},t)\frac{\partial^2 \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}^2}$ are symmetric as the sum of a positive semi-definite and a symmetric matrix. Invoking Weyl's inequalities [21, Th. 4.3.1], we have that:

$$\lambda_{\max}(A_j) \leq p_j(\mathbf{x},t)\lambda_{\max}\left(\frac{\partial^2 \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}^2}\right) + \eta(1+p_j(\mathbf{x},t))\left\|\frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial \mathbf{x}}\right\|^2.$$

Due to (11), it follows that $\lambda_{\max}(A_j) < 0$, for every $j \in \mathcal{B}(t)$. Therefore, $\frac{\partial^2 \mathfrak{b}(\mathbf{x},t)}{\partial \mathbf{x}^2} < \mathbf{0}_n$ holds and the critical points of $\mathfrak{b}(\mathbf{x}, t)$ are (local) maxima. Since $\mathcal{C}(t)$ has a non-empty interior for every t there exists a $\rho > 0$ such that $\mathfrak{b}(\mathbf{x}_{\max}, t) \geq \rho$. Setting $\bar{\mathfrak{b}}_t = \min\left\{\delta_1, \frac{1}{\eta}, \rho\right\}$, the result follows. ∎

Lemma 2 establishes a positive lower bound on $\mathfrak{b}(\mathbf{x}, t)$ at the critical points. This allows us to determine a class $\mathcal{K}$ function such that Assumption 3 is satisfied as depicted in the following lemma:

Lemma 3. *Let assumptions of Lemma 2 hold. Then, there exists an $\alpha_{\mathcal{B}} > 0$ such that Assumption 3 is satisfied considering the linear class $\mathcal{K}$ function $\zeta \mapsto \alpha_{\mathcal{B}}\zeta$.*

Proof. Let the set of the critical points of $\mathfrak{b}(\mathbf{x}, t)$ at fixed time t be denoted as $CR(t)$. If $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial t} > 0$ for $\mathbf{x} \in CR(t) \cap \mathcal{C}(t)$ and fixed t, then Assumption 3 is satisfied irrespective of the value of the barrier function $\mathfrak{b}(\mathbf{x}, t)$ at $\mathbf{x} \in CR(t) \cap \mathcal{C}(t)$. This holds for example in $CR(t) \cap \{\mathbf{x} \in \mathbb{R}^n : p_j(\mathbf{x}, t) < 0, \forall j \in \mathcal{B}(t)\}$, if the intersection is non-empty. The above is a direct consequence of the fact that $\mathfrak{b}(\mathbf{x}, t) \geq 0$ for $\mathbf{x} \in \mathcal{C}(t)$ which implies that $-\alpha(\mathfrak{b}(\mathbf{x}, t)) \leq 0$. Next, we focus on cases when there exists at least one $j \in \mathcal{B}(t)$ satisfying $p_j(\mathbf{x}, t) > 0$. Computing the derivative of $\mathfrak{b}(\mathbf{x}, t)$ with respect to t, we have the following:

$$\begin{aligned} \frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial t} &= \sum_{j \in \mathcal{J}} \mathfrak{o}_j(t) \kappa_j(\mathbf{x},t) \frac{\partial \mathfrak{b}_j(\mathbf{x},t)}{\partial t} \\ &\geq -\Delta(t) \sum_{j \in \mathcal{B}^+(t)} \kappa_j(\mathbf{x},t) \sum_{i_j \in \mathcal{I}_j} o_{i_j}(t) \lambda_{i_j}(\mathbf{x},t) \\ &= -\Delta(t) \sum_{j \in \mathcal{B}^+(t)} \kappa_j(\mathbf{x},t), \end{aligned} \tag{13}$$

where $\mathcal{B}^+(t) = \{j \in \mathcal{B}(t) : \kappa_j(\mathbf{x}, t) \geq 0\}$, $\Delta(t) = \max_{j \in \mathcal{B}^+(t)} \max_{i_j \in \mathcal{A}_j(t)} \left|\frac{d\gamma_{i_j}(t)}{dt}\right|$, $\mathcal{A}_j(t) \subseteq \mathcal{I}_j$ is the set of active tasks of the j-th conjunction at time t (defined in Section II.B) and $\gamma_{i_j} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the temporal function corresponding to the formula φ_{i_j} that was designed to ensure its satisfaction with a desired robustness $r > 0$. Additionally, for fixed t and $\mathbf{x} \in CR(t) \cap \mathcal{C}(t)$, due to Lemma 2 we have:

$$\begin{aligned} \sum_{j \in \mathcal{B}^+(t)} \kappa_j(\mathbf{x},t) \leq & \sum_{j \in \mathcal{B}^+(t)} p_j(\mathbf{x},t) \\ \leq & \sum_{j \in \mathcal{B}^+(t)} (1 + \eta(\min_{i_j \in \mathcal{A}_j(t)} \mathfrak{b}_{i_j}(\mathbf{x},t) - \bar{\mathfrak{b}}_t)) \\ \leq & \, K(t), \end{aligned} \tag{14}$$

where $K(t)$ is defined as $K(t) = \sum_{j \in \mathcal{B}^+(t)} (1 + \eta(\min_{i_j \in \mathcal{A}_j(t)}(-\gamma_{i_j}(t) + \max_{\mathbf{x} \in \overline{CR(t) \cap \mathcal{C}(t)}} h_{i_j}(\mathbf{x})) - \bar{\mathfrak{b}}_t))$. Combining (13)-(14) and taking the supremum over t, we have that $\frac{\partial \mathfrak{b}(\mathbf{x},t)}{\partial t} \geq -\Delta_{\mathcal{B}}$, where $\Delta_{\mathcal{B}} = \sup_{t \geq 0} \Delta(t) K(t)$. Then, the proof concludes by choosing $\alpha_{\mathcal{B}} \geq \frac{\Delta_{\mathcal{B}}}{\mathfrak{b}_{\mathcal{B}}}$, where $\mathfrak{b}_{\mathcal{B}} = \inf_{t \geq 0} \bar{\mathfrak{b}}_t$. ∎

So far, we have assumed that $\mathcal{B}(t), t \geq 0$ is given. Nevertheless, these sets are defined online according to the

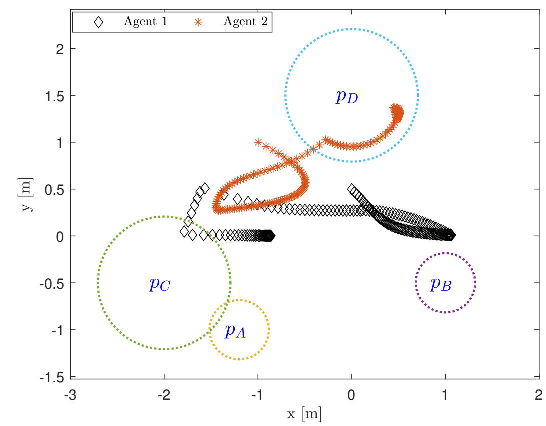

Fig. 1: Agents' trajectories.

deactivation policy defined in (5). Taking advantage of the aforementioned analysis, we can now show that $\mathfrak{b}(\mathbf{x}, t)$ is a CBF function according to Definition 1 accounting for every possible choice of $\mathcal{B}(t), t \geq 0$ as follows:

Theorem 2. *Consider the system* (3)*, the STL formula ϕ defined in* (4a)-(4c) *and the function $\mathfrak{b}(\mathbf{x}, t)$, defined in* (6)*. Let Assumption 1 hold. Let further $\{\mathcal{B}_\nu(t) : t \geq 0\}_{\nu=0}^{N_\nu}$ be a sequence of non-empty, non-increasing sets with respect to t with $\mathcal{B}_\nu(0) = \mathcal{J}$ and $\mathcal{B}_\nu(t) \in 2^{\mathcal{J}}, \forall t \geq 0$. Consider all possible subsequences $\{\mathcal{B}_{\nu_k}(t) : t \geq 0\}$ for which $\bigcap_{j \in \mathcal{B}_{\nu_k}(t)} \mathcal{C}_j(t) \neq \emptyset$ holds for every $t \geq 0$. If the conditions of Lemma 2 hold for every $\mathcal{B}_{\nu_k}(t), t \geq 0$, then $\mathfrak{b}(\mathbf{x}, t)$ is a CBF function.*

Proof. From Lemma 3 there exists a constant $\alpha_{\mathcal{B}_{\nu_k}} > 0$ that ensures the validity of Assumption 3 for every subsequence $\mathcal{B}_{\nu_k}$ for which $\bigcap_{j \in \mathcal{B}_{\nu_k}(t)} \mathcal{C}_j(t) \neq \emptyset$, is true for every $t \geq 0$. Then, the result follows by choosing $\alpha = \sup_{\nu_k} \alpha_{\mathcal{B}_{\nu_k}}$. ∎

Theorem 2 ensures that the STL formula ϕ can always be satisfied by (3) when the satisfaction of any admissible conjunction of formulas $\bigwedge_{j \in \mathcal{J}_2} \phi'_j$ is enforced provided that $\bigcap_{j \in \mathcal{J}_2} \mathcal{C}_j(t)$ is non-empty at all times.

V. Simulation Results

Consider two autonomous agents whose dynamics are coupled and given by the differential equation:

$$\dot{\mathbf{x}} = \begin{bmatrix} -x_1^3 - x_3 \\ -2x_2 \\ -5x_1 - 3x_3 \\ -x_4^2 - x_2 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \mathbf{u},$$

where $\mathbf{p}_1 = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, $\mathbf{p}_2 = \begin{bmatrix} x_3 & x_4 \end{bmatrix}^T$ is the position of agent 1 and 2 respectively. Note that $g(\mathbf{x})$ is a constant matrix satisfying $\det(g(\mathbf{x})) = 2$. Thus, Assumption 1 is satisfied. The system is subject to STL task $\phi = (\neg \phi'_2 \Rightarrow \phi'_1)$, where: $\phi'_1 = (\|\mathbf{p}_1 - \mathbf{p}_A\|^2 \leq 0.1) \, \mathcal{U}_{[1,5]} \, (\|\mathbf{p}_2 - \mathbf{p}_B\|^2 \leq 0.1)$ and $\phi'_2 = \mathcal{F}_{[2,4]}(\|\mathbf{p}_1 - \mathbf{p}_C\|^2 \leq 0.5) \wedge \mathcal{G}_{[3,6]}(\|\mathbf{p}_2 - \mathbf{p}_D\|^2 \leq 0.5)$, and where $\mathbf{p}_A = \begin{bmatrix} -1.2 & -1 \end{bmatrix}^T$, $\mathbf{p}_B = \begin{bmatrix} 1 & -0.5 \end{bmatrix}^T$,

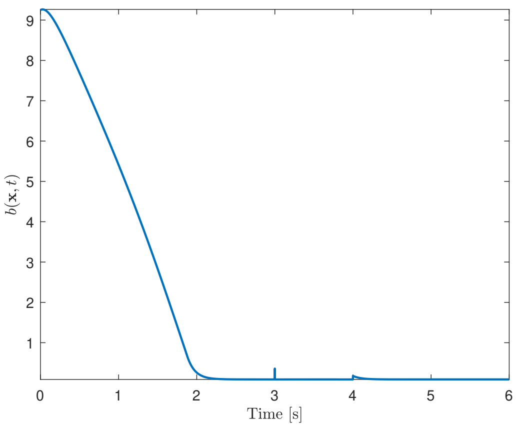

Fig. 2: Evolution of $\mathfrak{b}(\mathbf{x}(t),t)$ over time.

$\mathbf{p}_C = \begin{bmatrix}-2 & -0.5\end{bmatrix}^T$, $\mathbf{p}_D = \begin{bmatrix}0 & 1.5\end{bmatrix}^T$. Observe that the satisfaction of ϕ is equivalent to the satisfaction of $\phi_1' \vee \phi_2'$. Here, convexity of the superlevel sets of $\mathfrak{b}_j(\mathbf{x},t), j \in \{1,2\}$ is ensured since $\mathfrak{b}_j(\mathbf{x},t), j \in \{1,2\}$ are concave for fixed t due to the concavity of the predicate functions [20]. The initial condition of the system is chosen as $\mathbf{x}(0) = \begin{bmatrix}0 & 0.5 & -1 & 1\end{bmatrix}^T$ and $\eta = 30$. Here, the temporal functions are designed as in [16], where the satisfaction of the until formula is ensured by enforcing the satisfaction of $\mathcal{G}_{[1,5]}(\|\mathbf{p}_1 - \mathbf{p}_A\|^2 \leq 0.1) \wedge \mathcal{F}_{[5,5]}(\|\mathbf{p}_2 - \mathbf{p}_B\|^2 \leq 0.1)$. The satisfaction of ϕ is desired with a robustness $r = 0.09$. All computations were performed on an Intel Core i7-8665U with 16GB RAM using MATLAB. The QP problem was solved with `quadprog` at a frequency of 1kHz. In Figures 1 and 2 the agents' trajectories and the evolution of $\mathfrak{b}(\mathbf{x}(t),t)$ are shown. Observe that $\mathfrak{b}(\mathbf{x},t)$ remains non-negative at all times which by Theorem 1 implies the satisfaction of ϕ with a minimum robustness r (specifically, $\inf_{t\in[0,5]} \mathfrak{b}(\mathbf{x}(t),t) \geq 0.0883$). Initially, the two agents move towards satisfying both formulas $\phi_j', j = 1,2$. Nevertheless, ϕ_1' is violated for the first time at $t = 0.805$ sec and stops affecting the value of $\mathfrak{b}(\mathbf{x},t)$ thanks to the deactivation policy introduced in (5). After 0.805 sec the system moves towards satisfying ϕ_2'. Finally, note that the small spikes at time instants 3 and 4 sec are due to the discontinuity of $\mathfrak{b}_2(\mathbf{x},t)$ at the time instants at which the tasks defining ϕ_2' are deactivated.

VI. Conclusions

In this work we considered an STL fragment including disjunctions of temporal operators and proposed a control law found as a solution to a QP problem utilizing a time-varying control barrier function defined in such way to under approximate the max operator. In addition, under some regularity and topological assumptions on the barrier functions encoding the subtasks and superlevel sets respectively we have provided sufficient conditions for the proposed function to be a control barrier function.

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