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Periodic Behaviors for Discrete-time Second-order Multi-agent Systems with Input Saturation Constraints

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Abstract—**This paper considers the existence of periodic behaviors for discrete-time second-order multi-agent systems with input saturation constraints. We first consider the case where the agent dynamics is double integrator and establish conditions on the feedback gains of the linear consensus control law for achieving periodic behaviors. This in turn shows that the previous established sufficient condition for reaching global consensus has a necessary aspect since these two conditions are exclusive. We further consider all other second-order agent dynamics and show that these multi-agent systems under the linear consensus law exhibit periodic solutions provided the feedback gains satisfy certain conditions. Simulation results are used to validate the theoretical findings.**

Index Terms—**Input saturation, multi-agent systems, periodic behaviors.**

I. Introduction

In the multi-agent systems literature, the consensus problem, where the goal is to achieve asymptotic agreement on agents' states, has been extensively studied. Various distributed control laws have been proposed to achieve consensus through neighboring information exchange, e.g., [1]–[4]. Recently, global consensus and semi-global consensus for multi-agent systems with input saturation constraints have been considered, e.g., [5]–[10]. In particular, it has been shown that the multi-agent system with continuous-time double integrator agent dynamics in the presence of input saturation constraints, achieves global consensus under all locally linear consensus control laws [5]. On the other hand, we have previously shown that part of locally linear consensus control laws render global consensus for the discrete-time counterpart [6]. However whether global consensus is achieved when the other part of locally linear consensus laws are used is still unknown.

This motivates our study. In particular, we show that the multi-agent system under the linear consensus control law exhibits periodic behaviors if the feedback gains satisfy certain conditions. This in turn implies that global consensus is not achieved. We further investigate the existence of the periodic behavior for all other second-order agent dynamics including asymptotic stable, marginally stable, and unstable dynamics. We show that these multi-agent systems under the linear consensus feedback law exhibit periodic behaviors provided the feedback gains satisfy certain conditions.

The contribution of this paper is three-fold: 1) compared to [11] where the existence of the periodic phenomenon has been shown for discrete-time single integrator multi-agent systems, the considered systems are second-order. Such an extension is not only challenging since the dynamics may diverge without control laws, but also useful since many physical systems can be modeled as second-order systems; 2) compared to [12] where only double integrator agent dynamics has been considered, we consider all other second-order agent dynamics including asymptotically stable, marginally stable and unstable dynamics; and 3) this paper also extends the existence of periodic behavior for individual discrete-time system [13]–[15], to multi-agent systems. Although this paper only studies the second-order multi-agent systems, these systems are known as a key benchmark for dynamical behavior of nonlinear multi-agent systems. By fully understanding these systems, we make a key step in understanding the abilities of linear consensus control law for achieving periodic behaviors.

The remainder of the paper is organized as follows: In Section II, we present the motivation and formulate the considered problem. In Section III, we show that the multi-agent system with the double integrator dynamics under the linear consensus control law exhibit periodic solutions provided that the feedback gains satisfy certain conditions. In Section IV, we further show the existence of periodic behaviors for multi-agent systems with all other second-order agent dynamics. Simulation examples are offered in Section V. Finally, Section VI concludes the paper.

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II. Motivation and Problem Formulation

Consider a multi-agent system of N identical discrete-time second-order linear systems

$$y_i(k+1) = Ay_i(k) + B\sigma(u_i(k)), \quad i \in \{1, \dots, N\}, \tag{1}$$

where $y_i = [x_i; v_i] \in \mathbb{R}^2$, $u_i \in \mathbb{R}$, $\sigma(u_i)$ is the saturation function: $\sigma(u_i) = \text{sgn}(u_i)\min\{1, |u_i|\}$, and the pair (A, B) describes the agent dynamics.

The network among agents is described by an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, with the set of agents $\mathcal{V} = \{1, \ldots, N\}$, the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and the weighted adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$, where $a_{ij} > 0$ if and only if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. We also assume that $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$. The set of neighboring agents of agent i is defined as $\mathcal{N}_i = \{j \in \mathcal{V} | a_{ij} > 0\}$. The linear consensus feedback control law with gain parameters α and β is given by:

$$u_i(k) = \sum_{j \in \mathcal{N}_i} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_j(k) - y_i(k)). \tag{2}$$

For the case where the agent dynamics is double integrator, i.e.,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3}$$

we have shown in [6] that the multi-agent system (1) achieves global consensus under the linear consensus control law (2) if the feedback gain parameters satisfy the following condition

$$0 < \sqrt{3}\alpha < \beta < \frac{3}{2\lambda_N}, \tag{4}$$

where λ_N is the largest eigenvalue of the Laplacian matrix associated with the network.

It is natural to ask whether global consensus is still achieved if the condition (4) is not satisfied. In this paper, we will show that the if the condition (4) is not fulfilled, the multi-agent system may exhibit a non-converging and non-diverging behavior, i.e., the periodic solution. In particular, we explicitly construct the linear consensus control law with appropriated gain parameters under which the multi-agent system exhibits the periodic solution in the following sense.

Definition 1: A solution $y_i(k)$ of the multi-agent system (1) under the linear consensus control law (2) is a periodic solution with period $T > 0$, if for some initial states $y_i(0)$ for $i \in \{1, \ldots, N\}$, we have $y_i(k + T) = y_i(k)$ for all $i \in \{1, \ldots, N\}$, and for all $k = 0, 1, \ldots$.

III. Main Results

Our main result is given below.

Theorem 1: Consider the multi-agent system (1) with the pair (A, B) given by (3) under the linear consensus control law (2). Suppose that $\mathcal{G}$ is connected. If the feedback gain parameters α and β satisfy

$$0 < \alpha < \beta < \tfrac{3}{2}\alpha. \tag{5}$$

Then there exists initial states such that the corresponding solution of the multi-agent system is periodic with the period $T = 2m$, where

$$m \geq \tfrac{4(\alpha - \beta) + \frac{2}{\bar{a}}}{3\alpha - 2\beta}, \tag{6}$$

and

$$\bar{a} = \min_{\substack{(i,j) \in \mathcal{E} \\ i \in S_e, \, j \in S_o}} a_{ij}, \tag{7}$$

with S_e and S_o defined in the proof.

Proof: Since the graph is connected, without loss of generality, we assume that agent 1 is the root agent. We define the following sets based on whether the distance between agent $i \in \mathcal{V}$ and the root agent 1 is even or odd:

$$S_e = \{i | d(i, 1) = 0, 2, \ldots\}, \, S_o = \{i | d(i, 1) = 1, 3, \ldots\}, \tag{8}$$

where the distance between two nodes i and j, $d(i, j)$ is the number of edges of a path between i and j minimized over all possible paths.

We prove the theorem by explicitly constructing periodic solutions with an even period $T = 2m$. The periodic solution that we will construct is such that the input sequences (2) for all the agents are always in saturation, and for $i \in S_e$,

$$u_i(k) \geq 1, k = 0, \ldots, m{-}1, \; u_i(k) \leq -1, k = m, \ldots, 2m{-}1, \tag{9}$$

and for $i \in S_o$,

$$u_i(k) \leq -1, k = 0, \ldots, m{-}1, \; u_i(k) \geq 1, k = m, \ldots, 2m{-}1. \tag{10}$$

In what follows, we will show that (9) and (10) are satisfied for certain m and initial states $x_i(0)$ and $v_i(0)$ for $i \in \mathcal{V}$, and that the solution is periodic with period $T = 2m$ for these initial states. The proof has three steps.

Step 1: It is sufficient to show that $x_i(T) = x_i(0)$ and $v_i(T) = v_i(0)$ for all $i \in \mathcal{V}$. It follows from (1), (9) and (10) that $v_i(T) = v_i(0)$ for all $i \in \mathcal{V}$. It is also easy to obtain that $x_i(2m) = x_i(0) + 2mv_i(0) + m^2$ for $i \in S_e$ and $x_i(2m) = x_i(0) + 2mv_i(0) - m^2$ for $i \in S_o$. Thus, in order to have $x_i(T) = x_i(0)$ for all $i \in \mathcal{V}$, we must have that

$$\begin{cases} v_i(0) = -m/2, & i \in S_e, \\ v_i(0) = m/2, & i \in S_o. \end{cases} \tag{11}$$

Step 2: We show that the $2m$ inequalities in either (9) or (10) can be reduced to two inequalities by appropriately choosing initial states $x_i(0)$ for some $i \in \mathcal{V}$.

Step 2.1: For agent $j \in S_o$, we have

$$\begin{aligned} u_j(k) &= \sum_{i \in \mathcal{N}_j} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)) \\ &= \sum_{i \in \mathcal{N}_j \cap S_e} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)) \\ &\quad + \sum_{i \in \mathcal{N}_j \cap S_o} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)). \end{aligned}$$

Choosing $x_i(0) = x_j(0)$ for $i \in S_o$ if $(i, j) \in \mathcal{E}$, and using $v_i(0) = v_j(0)$ for all $i, j \in S_o$, gives

$$u_j(k) = \sum_{i \in \mathcal{N}_j \cap S_e} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)). \tag{12}$$

Similarly, for agent $i \in S_e$, choosing $x_j(0) = x_i(0)$ for $j \in S_e$ if $(i, j) \in \mathcal{E}$, yields

$$u_i(k) = \sum_{j \in \mathcal{N}_i \cap S_o} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)).$$

Step 2.2: Let us now focus on any edge $(i, j) \in \mathcal{E}$, such that $i \in S_e$ and $j \in S_o$. We first note that $0 < \alpha < \beta$ from (5) implies that $\beta > \alpha - \frac{1}{2}k\alpha$ for $k = 0, \ldots, m - 1$, which yields $-\frac{\alpha m}{2} + \beta > \frac{1}{2}\alpha(-m-k+2)$. Since $m-k-1 \geq 0$, multiplying the above inequality on both sides with $m - k - 1$ yields

$$-\tfrac{\alpha m}{2}(m-k-1) + \beta(m-k-1) \geq \alpha[\tfrac{k(k-1)}{2} - \tfrac{(m-1)(m-2)}{2}].$$

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This is equivalent to that

$$\begin{aligned} &a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(m-1)-y_j(m-1)) \\ &\geq a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(k)-y_j(k)) \end{aligned} \tag{13}$$

for $k=0,\dots,m-1$, since $v_i(0)=-\frac{m}{2}$ for $i\in S_e$, $v_j(0)=\frac{m}{2}$ for $j\in S_o$, and $a_{ij}\geq 0$.

Step 2.3: Since the inequality (13) holds for each $i\in\mathcal{N}_j\cap S_e$, where $j\in S_o$, adding them up together with (12) yields that

$$u_j(m-1)\geq u_j(k),\quad k=0,\dots,m-1,\quad j\in S_o.$$

Hence, for $j\in S_o$, $u_j(m-1)\leq -1$ implies that $u_j(k)\leq -1$ for all $k=0,\dots,m-1$.

A similar argument shows that

$$u_j(2m-1)\leq u_j(k),\quad k=m,\dots,2m-1,\quad j\in S_o.$$

Hence, for $j\in S_o$, $u_j(2m-1)\geq 1$ implies that $u_j(k)\geq 1$ for all $k=m,\dots,2m-1$.

Similarly, we can show that for $i\in S_e$, $u_i(m-1)\geq 1$ implies that $u_i(k)\geq 1$ for $k=0,\dots,m-1$, and that $u_i(2m-1)\leq -1$ implies that $u_i(k)\leq -1$ for $k=m,\dots,2m-1$.

To summarize, if there is an edge connecting agents within S_e or S_o, we set their initial states the same, i.e.,

$$x_i(0)=x_j(0)\ \text{ for } (i,j)\in\mathcal{E},\ \text{ if } i,j\in S_e,\ \text{ or } i,j\in S_o. \tag{14}$$

Then inequalities (9) or (10) are reduced to $u_j(m-1)\leq -1$ and $u_j(2m-1)\geq 1$ for $j\in S_o$, or $u_i(m-1)\geq 1$ and $u_i(2m-1)\leq -1$ for $i\in S_e$.

Step 3: It is clear that these two inequalities are satisfied provided that for each edge $(i,j)\in\mathcal{E}$, where $i\in S_e$ and $j\in S_o$ the following two conditions

$$\begin{aligned} &a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(m-1)-y_j(m-1)) \\ =&a_{ij}\Big\{\alpha\left[x_i(0)-x_j(0)-2m+2\right]+\beta(m-2)\Big\}\leq -1, \\ &a_{ij}\begin{bmatrix}\alpha & \beta\end{bmatrix}(y_i(2m-1)-y_j(2m-1)) \\ =&a_{ij}\Big\{\alpha\left[x_i(0)-x_j(0)+m-2\right]-\beta(m-2)\Big\}\geq 1, \end{aligned}$$

are satisfied. They are equivalent to

$$\begin{aligned} \tfrac{1}{a_{ij}}+(\beta-\alpha)(m-2)&\leq\alpha(x_i(0)-x_j(0)) \\ &\leq 2\alpha(m-1)-\beta(m-2)-\tfrac{1}{a_{ij}}. \end{aligned} \tag{15}$$

We see that suitable $x_i(0)$ and $x_j(0)$, where $i\in S_e$, $j\in S_o$, and $(i,j)\in\mathcal{E}$, exist if

$$\tfrac{1}{a_{ij}}+(\beta-\alpha)(m-2)\leq 2\alpha(m-1)-\beta(m-2)-\tfrac{1}{a_{ij}}. \tag{16}$$

For $m>2$, (16) is equivalent to $\beta\leq\frac{3m-4}{2m-4}\alpha-\frac{1}{a_{ij}(m-2)}$. If we take the value of m to be sufficiently large, we obtain that

$$\beta\leq\lim_{m\to+\infty}\left[\tfrac{3m-4}{2m-4}\alpha-\tfrac{1}{a_{ij}(m-2)}\right]=\tfrac{3}{2}\alpha.$$

Therefore for any α and β which satisfy (5), if the condition (6) holds, then (16) is satisfied.

From the above analysis, we see that the solution of the multi-agent system is periodic with period $T=2m$, where m satisfies (6), for initial states satisfying (11), (14), and (15). ■

Remark 1: For the multi-agent system (1) in the absence of input saturation constraints, where the agent dynamics is a double integrator and the undirected network is connected, it is shown in [16, Corollary 1] that the condition

$$0<\alpha<\beta<\frac{\alpha}{2}+\frac{2}{\lambda_N} \tag{17}$$

is a necessary and sufficient condition for achieving consensus. Note that condition (5) overlaps (17) provided that $\alpha<\frac{2}{\lambda_N}$. In view of this, the existence of the periodic solution shown in Theorem 1 is clearly due to the input saturation constraints.

Remark 2: Condition (5) implies that $0<\sqrt{3}\alpha<\beta<\frac{3}{2\lambda_N}$, which is a sufficient condition for achieving global consensus in the presence of input saturation constraints [6, Theorem 2], has a necessary aspect since these conditions are exclusive in the sense that $\frac{3\alpha}{2}<\sqrt{3}\alpha$.

Remark 3: In [13], [14], the periodic behaviors have been considered for individual discrete-time system. Theorem 1 extends this result to multi-agent systems for the double integrator case. In particular, the specified input is explicitly designed based on the saturation and the linear consensus control law.

IV. Periodic behaviors for other agent dynamics

In this section, we further investigate the existence of periodic behaviors for multi-agent systems with all controllable second-order agent dynamics. Without loss of generality, we assume that the pair (A,B) is in the following controllable canonical form:

$$A=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix},\ B=\begin{bmatrix}0\\1\end{bmatrix}, \tag{18}$$

since otherwise the system can be transferred into this form via a non-singular state transformation [17, Theorem 9.2].

Our main result for this case is given below.

Theorem 2: Consider the multi-agent system (1) with the pair (A,B) given by (18) under the linear consensus control law (2). Suppose that $\mathcal{G}$ is connected and that A has no eigenvalues at ± 1 and $\pm j$. If the feedback gain parameters α and β satisfy

$$(1-a_0+a_1)\alpha+(1-a_0-a_1)\beta\geq\tfrac{(1-a_0)^2+a_1^2}{2\bar{a}}, \tag{19a}$$

$$(1-a_0-a_1)\alpha-(1-a_0+a_1)\beta\geq\tfrac{(1-a_0)^2+a_1^2}{2\bar{a}}, \tag{19b}$$

where $\bar{a}$ is defined by (7). Then there exist initial states such that the corresponding solution of the multi-agent system is periodic with period $T=4$.

Proof: A periodic solution with period $T=4$ is such that the input sequences (2) for all the agents are always in saturation. Moreover, it holds that

$$\begin{cases} u_i(k)\geq 1, & k=0,1,\\ u_i(k)\leq -1, & k=2,3,\end{cases}\quad i\in S_e, \tag{20}$$

$$\begin{cases} u_i(k)\leq -1, & k=0,1,\\ u_i(k)\geq 1, & k=2,3,\end{cases}\quad i\in S_o. \tag{21}$$

In what follows, we will show that (20) and (21) are satisfied for certain initial states $y_i(0)$, $i\in\mathcal{V}$, and that the solution is periodic with period $T=4$. Again, the proof is carried out in three steps.

Step 1: It follows from (1) and (20) that for $i \in S_e$, $y_i(4) = A^4 y_i(0) + A^3 B + A^2 B - AB - B$. Thus, in order to have $y_i(0) = y_i(4)$ for $i \in S_e$, we need

$$\begin{aligned} y_i(0) &= (I - A^4)^{-1}(A^3 B + A^2 B - AB - B) \\ &= -(I + A^2)^{-1}(I + A)B, \end{aligned}$$

where we have used the assumption on the eigenvalues of A.

By plugging in the matrices A and B given in (18) into this equation, we obtain that

$$y_i(0) = -\frac{1}{(1-a_0)^2 + a_1^2} \begin{bmatrix} 1 - a_0 + a_1 \\ 1 - a_0 - a_1 \end{bmatrix} \text{ for } i \in S_e. \quad (22)$$

Similarly, in order to have $y_i(0) = y_i(4)$ for $i \in S_o$, we need

$$y_i(0) = \frac{1}{(1-a_0)^2 + a_1^2} \begin{bmatrix} 1 - a_0 + a_1 \\ 1 - a_0 - a_1 \end{bmatrix} \text{ for } i \in S_o. \quad (23)$$

Step 2: In this step, we show that the four inequalities in either (20) or (21) can be reduced to two inequalities. For agent $j \in S_o$, we have

$$\begin{aligned} u_j(k) &= \sum_{i \in \mathcal{N}_j} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)) \\ &= \sum_{i \in \mathcal{N}_j \cap S_e} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)) \\ &\quad + \sum_{i \in \mathcal{N}_j \cap S_o} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)). \end{aligned}$$

Taking into account that $x_i(0) = x_j(0)$ and $v_i(0) = v_j(0)$ for all $i, j \in S_o$, we obtain

$$u_j(k) = \sum_{i \in \mathcal{N}_j \cap S_e} a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(k) - y_j(k)). \quad (24)$$

From (24) and initial states given by (22) and (23), it is easy to verify that for $j \in S_o$, $u_j(k+2) \geq 1$ are equivalent to $u_j(k) \leq -1$ for $k = 0, 1$. Similarly, for $i \in S_e$, $u_i(k+2) \leq -1$ is equivalent to $u_i(k) \geq 1$ for $k = 0, 1$. Thus, the inequalities (20) and (21) are equivalent to the following inequalities: $u_i(0) \geq 1$ and $u_i(1) \geq 1$ for $i \in S_e$, and $u_j(0) \leq -1$ and $u_j(1) \leq -1$ for $j \in S_o$.

Step 3: These two inequalities are satisfied for each agent provided that for each edge $(i, j) \in \mathcal{E}$, where $i \in S_e$ and $j \in S_o$, the two conditions

$$\begin{aligned} &a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(0) - y_j(0)) \\ &= -\frac{2a_{ij}}{(1-a_0)^2 + a_1^2} [(1 - a_0 + a_1)\alpha + (1 - a_0 - a_1)\beta] \leq -1, \\ &a_{ij} \begin{bmatrix} \alpha & \beta \end{bmatrix} (y_i(1) - y_j(1)) \\ &= -\frac{2a_{ij}}{(1-a_0)^2 + a_1^2} [(1 - a_0 - a_1)\alpha - (1 - a_0 + a_1)\beta] \leq -1, \end{aligned}$$

are satisfied. It is easy to see that this is the case, if the feedback gain parameters α and β satisfy (19).

From the above analysis, it follows that the solution of the multi-agent system is periodic with period $T = 4$, for initial states satisfying (22) and (23). ■

Remark 4: Note that the periodic behavior presented in Theorem 2 has a period $T = 4$, which is independent of the network topology, while the feedback gain parameters for achieving this periodic behavior depend on the minimal edge weights of the underlying graph as given by (19). This is in contrast to the double integrator case in Theorem 1, where the feedback gain parameters for achieving periodic behaviors do not depend on the network topology, however, the periodic T depends on the minimal edge weights.

Remark 5: In [13, Corollary 21.10], it has been shown that for the time-invariant system $x(k+1) = Ax(k) + Bu(k)$, $x(0) = x_0$, if A^K has no unity eigenvalue, then for every K-periodic input signal $u(k)$ there exists an x_0 such that the corresponding solution is K-periodic. In view of this, Theorem 2 extends this result to multi-agent systems for the second-order dynamics. In particular, A^4 has no unity eigenvalue because of the assumption that A has no eigenvalues at ± 1 and $\pm j$. Moreover, a 4-periodic input is constructed based on the saturation and the linear consensus control law with the gain parameters satisfying (19).

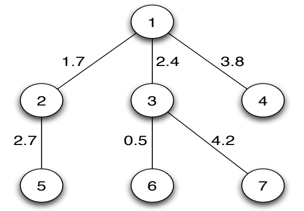

Fig. 1. Network with seven agents

V. Illustrative Examples

In this section, we present examples to illustrate the results. The network consists of $N = 7$ agents and the topology is given by the undirected weighted graph depicted in Fig. 1.

A. Double Integrator Case

For the double integrator case, we choose the feedback gain parameters $\alpha = 1$ and $\beta = 1.2$ so the sufficient condition for achieving a periodic behavior (5) is satisfied. It is easy to see that $\bar{a} = a_{36} = 0.5$, and therefore we choose $m = 11$ so that (6) is satisfied. From the proof of Theorem 1, we see that the multi-agent system exhibits a periodic solution with $T = 22$ if the initial states satisfy (11) and (15) with $m = 11$, i.e., $v_i(0) = -5.5$ for $i \in S_e = \{1, 5, 6, 7\}$, $v_i(0) = 5.5$ for $i \in S_o = \{2, 3, 4\}$, $2.3882 \leq x_1(0) - x_2(0) \leq 8.6118$, $2.2167 \leq x_1(0) - x_3(0) \leq 8.7833$, $2.0632 \leq x_1(0) - x_4(0) \leq 8.9368$, $2.1704 \leq x_5(0) - x_2(0) \leq 8.8296$, $3.8000 \leq x_6(0) - x_3(0) \leq 7.2000$, and $2.0381 \leq x_7(0) - x_3(0) \leq 8.9619$. We then choose $x_1(0) = 21$, $x_2(0) = 16$, $x_3(0) = 17$, $x_4(0) = 15$, $x_5(0) = 23$, $x_6(0) = 22$ and $x_7(0) = 24$ so that the above conditions are satisfied. With these initial states, the multi-agent system with input saturation constraints exhibits a periodic behavior of period 22 as shown in Fig. 2, where state trajectories for agents 1, 2, 5 and 7 are given.

B. Unstable Case

Next, we consider the case where the agent dynamics is given by (18). We begin by considering where $a_0 = 0$ and $a_1 = -2$, which results in an unstable eigenvalue at 2. For

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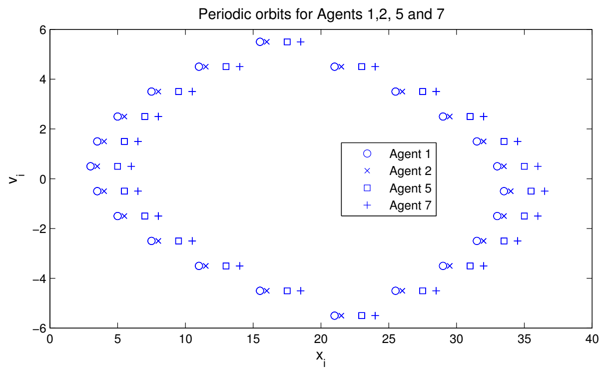

Fig. 2. Periodic solutions of period 22 for the double integrator case

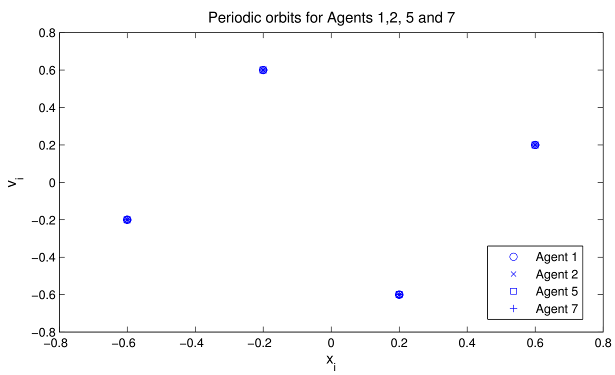

Fig. 3. Periodic behavior of period 4 for the unstable case

this case, (19) becomes $\frac{-\beta+5}{3} \leq \alpha \leq 3\beta - 5$. We then choose the feedback gain parameters $\alpha = 1$ and $\beta = 3$. According to the proof of Theorem 2, the multi-agent system exhibits a periodic solution of period $T = 4$ if the initial states satisfy (22) and (23), that is, $x_i(0) = 0.2$, $v_i(0) = -0.6$ for $i \in S_e = \{1, 5, 6, 7\}$ and $x_i(0) = -0.2$, $v_i(0) = 0.6$ for $i \in S_o = \{2, 3, 4\}$. Fig. 3 shows that the multi-agent system exhibits a periodic behavior with $T = 4$.

C. Marginally Stable Case

We then consider the case where $a_0 = 1$ and $a_1 = -1$, which results in eigenvalues $\frac{1}{2} \pm \frac{\sqrt{3}}{2}j$ on the unit circle. For this case, (19) becomes $-\beta + 1 \leq \alpha \leq \beta - 1$. We then choose the feedback gain parameters $\alpha = 0.5$ and $\beta = 2$. Fig. 4 shows that the multi-agent system exhibits a periodic behavior with $T = 4$ for initial states $x_i(0) = 1$, $v_i(0) = -1$ for $i \in S_e = \{1, 5, 6, 7\}$ and $x_i(0) = -1$, $v_i(0) = 1$ for $i \in S_o = \{2, 3, 4\}$, which satisfy (22) and (23).

VI. Conclusions

In this paper, we considered second-order discrete-time multi-agent systems with input saturation constraints. We showed that these multi-agent systems under linear consensus control laws exhibit periodic behaviors. We explicitly characterized conditions on feedback gain parameters and initial states for achieving periodic behaviors. An interesting future direction is to extend the results to high-order multi-agent systems.

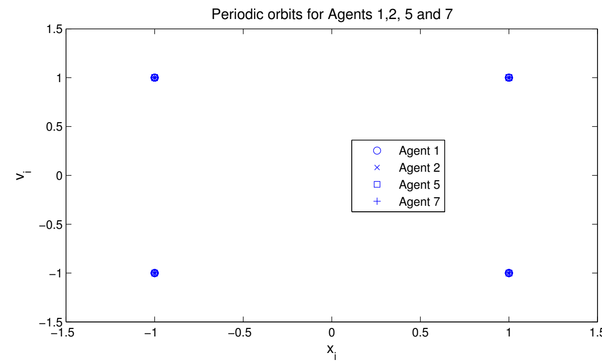

Fig. 4. Periodic behavior of period 4 for the marginally stable case

References

[1] J. Tsitsiklis, "Problems in decentralized decision making and computation," Ph.D. dissertation, MIT, Cambridge, MA, 1984.

[2] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Control*, vol. 48, no. 6, pp. 988–1001, Jun. 2003.

[3] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Autom. Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[4] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Trans. Autom. Control*, vol. 50, no. 5, pp. 655–661, May 2005.

[5] Z. Meng, Z. Zhao, and Z. Lin, "On global leader-following consensus of identical linear dynamic systems subject to actuator saturation," *Syst. & Contr. Letters*, vol. 62, no. 2, pp. 132–142, Feb. 2013.

[6] T. Yang, Z. Meng, D. V. Dimarogonas, and K. H. Johansson, "Global consensus for discrete-time multi-agent systems with input saturation constraints," *Automatica*, vol. 50, no. 2, pp. 499–506, Feb. 2014.

[7] T. Yang, A. A. Stoorvogel, H. F. Grip, and A. Saberi, "Semi-global regulation of output synchronization for heterogeneous networks of non-introspective, invertible agents subject to actuator saturation," *Int. J. Robust & Nonlinear Control*, vol. 24, no. 3, pp. 548–566, Feb. 2014.

[8] H. Su, Z. Chen, J. Lam, and Z. Lin, "Semi-global leader-following consensus of linear multi-agent systems with input saturation via low gain feedback," *IEEE Trans. Circuits Syst. I, Reg. Papers*, vol. 60, no. 7, pp. 1881–1889, Jul. 2013.

[9] Z. Zhao and Z. Lin, "Semi-global leader-following consensus of multiple linear systems with position and rate limited actuators," *Int. J. Robust & Nonlinear Control*, vol. 25, no. 13, pp. 2083–2100, Sep. 2015.

[10] Y. Su and J. Huang, "Cooperative semi-global robust output regulation for a class of nonlinear uncertain multi-agent systems," *Automatica*, vol. 50, no. 4, pp. 1053–1065, Apr. 2014.

[11] Y. Chen, J. Lü, F. Han, and X. Yu, "On the cluster consensus of discrete-time multi-agent systems," *Syst. & Contr. Letters*, vol. 60, no. 7, pp. 517–523, Jul. 2011.

[12] T. Yang, Z. Meng, D. V. Dimarogonas, and K. H. Johansson, "Periodic behaviors in multi-agent systems with input saturation constraints," in *Proc. 52nd IEEE Conf. Decision and Control*, Florence, Italy, Dec. 2013, pp. 4467–4472.

[13] W. Rugh, *Linear System Theory*, 2nd ed. Prentice Hall, 1996.

[14] T. Yang, A. A. Stoorvogel, and A. Saberi, "Dynamic behavior of the discrete-time double integrator with saturated locally stabilizing linear state feedback laws," *Int. J. Robust & Nonlinear Control*, vol. 23, no. 17, pp. 1899–1931, Nov. 2013.

[15] T. Yang, A. A. Stoorvogel, A. Saberi, and K. H. Johansson, "Further results on saturated globally stabilizing linear state feedback control laws for single-input neutrally stable planar systems," in *Proc. IEEE European Control Conf.*, Zurich, Switzerland, Jul. 2013, pp. 2728–2733.

[16] D. Xie and S. Wang, "Consensus of second-order discrete-time multi-agent systems with fixed topology," *J. Math. Analysis and Appl.*, vol. 387, no. 1, pp. 8–16, Mar. 2012.

[17] J. Lygeros and F. Ramponi, "Lecture notes on linear system theory," 2013, available at http://control.ee.ethz.ch/~ifalst/docs/Notes/LectureNotes.pdf.