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Brief paper

On robust synchronization of heterogeneous linear multi-agent systems with static couplings[☆]

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ABSTRACT

This paper addresses cooperative control problems in heterogeneous groups of linear dynamical agents that are coupled by diffusive links. We study networks with parameter uncertainties, resulting in heterogeneous agent dynamics, and we analyze the robustness of their output synchronization. The networks under consideration consist of non-identical double-integrators and harmonic oscillators. The geometric approach to linear control theory reveals structural requirements for non-trivial output synchronization in such networks. Furthermore, a clock synchronization problem and a circular motion coordination problem are discussed as applications corresponding to these two network types. The results are illustrated by numerical simulations.

1. Introduction

Consensus and synchronization problems in networks of dynamical agents are typically solved with diffusive couplings, i.e., distributed control laws based on the output differences of neighboring agents. Well-known examples are the classical consensus protocol (Olfati-Saber & Murray, 2004; Ren & Beard, 2005) and its extensions to double-integrators (Ren & Atkins, 2007), harmonic oscillators (Ren, 2008), and general linear agents (Scardovi & Sepulchre, 2009; Wieland, Kim, & Allgöwer, 2011). In this context, a major challenge is robust synchronization in heterogeneous linear networks, i.e., multi-agent systems consisting of non-identical linear agents (Grip, Yang, Saberi, & Stoorvogel, 2012; Lunze, 2012; Wieland & Allgöwer, 2009; Wieland, Sepulchre, & Allgöwer, 2011; Wu & Allgöwer, 2012). In Wieland and Allgöwer (2009) and Wieland, Sepulchre et al. (2011), a necessary condition for synchronization in heterogeneous linear networks is presented. The result is formulated as an internal model principle for synchronization and states that the agents have to embed a common internal model in order to be able to synchronize.

In this paper, we study cooperative control problems in heterogeneous linear networks, i.e., in diffusively coupled multi-agent systems with general high-order linear dynamics subject to parameter perturbations, which cause non-identical agent dynamics. In particular, we focus on output synchronization problems. The main goal is to develop a deeper understanding of the effects of heterogeneity in the agent dynamics on the dynamic behavior of the diffusively coupled multi-agent system and its implications for distributed control design. The contributions are the following.

We analyze the dynamic behavior of selected heterogeneous linear multi-agent systems. For each network, we discuss the implications of the internal model principle for synchronization, highlight the importance of the network topology, and assess the robustness of synchronization with respect to parameter uncertainties in the agent dynamics. Firstly, we consider a network of non-identical double-integrators, which achieves output synchronization if the output is position only, in Section 4. Afterwards, in Section 5, we study state synchronization in the same network. The structural requirements for synchronization are not met in this case, but it turns out that the synchronization error remains small,

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depending on the graph topology and the heterogeneity in the network. Secondly, in Section 6, we consider a network of harmonic oscillators with perturbed frequencies. We show that the internal model condition is not satisfied and that static diffusive couplings have a stabilizing effect in such networks. In particular, the network is rendered asymptotically stable if and only if there are oscillators with different frequencies in a certain region of the network. A preliminary version of these results has been presented in Seyboth, Dimarogonas, Johansson, and Allgöwer (2012). Moreover, we present two application examples: a clock synchronization problem and a motion coordination problem for mobile robots. The latter shows that heterogeneity may significantly impair the performance of cooperative control strategies designed for identical agents.

2. Preliminaries: notation and graph theory

For a vector $v \in \mathbb{R}^n$, $\mathrm{diag}(v)$ and $\mathrm{diag}(v_1, \dots, v_n)$ both denote the diagonal matrix with the entries v_i, $i = 1, \dots, n$, of v on the diagonal. The all ones and all zeros vectors are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively, and $I = \mathrm{diag}(\mathbf{1})$ is the identity matrix. The null space and image of a linear map defined by a matrix M are denoted by $\ker(M)$ and $\mathrm{im}(M)$, respectively. The norm $\|\cdot\|$ is understood as 2-norm for vectors and induced 2-norm for matrices. The spectrum of a square matrix M is denoted by $\sigma(M)$, which is to be understood as the set of roots of the characteristic polynomial of M, i.e., it respects the multiplicity of the eigenvalues. For symmetric matrices $M = M^\mathsf{T}$, $M > 0$ ($M \geq 0$) stands for positive (semi-)definiteness, while $M < 0$ ($M \leq 0$) stands for negative (semi-)definiteness. For a complex number $z \in \mathbb{C}$, $\mathbf{Re}(z)$ is the real part and $\mathbf{Im}(z)$ the imaginary part of z. The closed right-half complex plane is denoted by $\bar{\mathbb{C}}^+$. Let $\dot{x} = Ax, x \in \mathbb{R}^n$, be a linear dynamical system. A subspace $\mathcal{U} \subseteq \mathbb{R}^n$ is called invariant with respect to $\dot{x} = Ax$, or shortly A-invariant, if $x(0) \in \mathcal{U}$ implies $x(t) \in \mathcal{U}$ for all t. For convergence to a subspace $\mathcal{U}$, we write $x(t) \to \mathcal{U}$ as $t \to \infty$ as shorthand notation for $\forall \epsilon > 0 \exists \tau > 0 \forall t \geq \tau$: $\mathrm{dist}(x(t), \mathcal{U}) < \epsilon$, where $\mathrm{dist}(x(t), \mathcal{U}) = \inf_{\zeta \in \mathcal{U}} \|x(t) - \zeta\|$.

The network topology is modeled by a time-invariant directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A_\mathcal{G})$. Each vertex v_k in the set $\mathcal{V} = \{v_1, \dots, v_N\}$ corresponds to a dynamical subsystem (agent) k in the network. There is a directed edge from vertex v_j to v_k, i.e., $(v_j, v_k) \in \mathcal{E}$, if and only if v_k is influenced by (receives information from) v_j. A consecutive sequence of directed edges is called a directed path. The adjacency matrix $A_\mathcal{G} \in \mathbb{R}^{N \times N}$ describes the graph structure and edge weights, i.e., $a_{kj} > 0 \Leftrightarrow (v_j, v_k) \in \mathcal{E}$ and $a_{kj} = 0$ otherwise. A graph $\mathcal{G}$ is called undirected if $(v_j, v_k) \in \mathcal{E} \Leftrightarrow (v_k, v_j) \in \mathcal{E}$ and $a_{kj} = a_{jk}$. The Laplacian matrix $L \in \mathbb{R}^{N \times N}$ is defined as $L = \mathrm{diag}(A_\mathcal{G}\mathbf{1}) - A_\mathcal{G}$. By construction, L is a Metzler matrix and has zero row sums, i.e., $L\mathbf{1} = \mathbf{0}$. The vector of ones $\mathbf{1}$ is the eigenvector corresponding to the zero eigenvalue $\lambda_1(L) = 0$. All eigenvalues of L are contained in the closed right-half plane. The zero eigenvalue $\lambda_1(L) = 0$ is simple and all other eigenvalues have positive real parts $\mathbf{Re}(\lambda_k(L)) > 0$, $k \in \{2, \dots, N\}$, if and only if $\mathcal{G}$ is connected (Ren & Beard, 2005). An induced subgraph of $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a graph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ with $\tilde{\mathcal{V}} \subseteq \mathcal{V}$ and $\tilde{\mathcal{E}} = \{(v, w) \in \mathcal{E} : v, w \in \tilde{\mathcal{V}}\}$.

Definition 2.1 (*(Strongly) Connected Graph*)**.** A graph $\mathcal{G}$ is called connected if it contains a directed spanning tree, i.e., if there exists a vertex v_k such that there is a path from v_k to every other vertex $v_j \in \mathcal{V}$. A graph $\mathcal{G}$ is called strongly connected if there exists a directed path from any vertex to any other vertex in $\mathcal{V}$.

Definition 2.2 (*iSCC, Wieland, 2010*)**.** An independent strongly connected component (iSCC) of a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an induced subgraph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ which is maximal, subject to being strongly connected, and satisfies $(v, \tilde{v}) \notin \mathcal{E}$ for any $v \in \mathcal{V} \setminus \tilde{\mathcal{V}}$ and $\tilde{v} \in \tilde{\mathcal{V}}$.

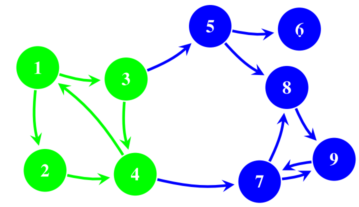


Fig. 1. A connected directed graph $\mathcal{G}$.

If $\mathcal{G}$ is connected, then $\mathcal{G}$ has exactly one iSCC (Wieland, 2010). Furthermore, in this case, $\mathrm{rank}(L) = N-1$ and the null space of L^T is spanned by a non-negative vector $p \in \mathbb{R}^N$, i.e., $p \geq \mathbf{0}$ element-wise. The k-th element p_k is positive, if and only if $v_k \in \mathcal{V}_{\mathrm{iSCC}}$ (Wieland, 2010). The vector p is the left eigenvector of L corresponding to eigenvalue zero, i.e., $p^\mathsf{T} L = \mathbf{0}^\mathsf{T}$. We normalize p such that $p^\mathsf{T}\mathbf{1} = 1$. If $\mathcal{G}$ is strongly connected, then $\mathcal{V}_{\mathrm{iSCC}} = \mathcal{V}$ and $p > \mathbf{0}$ element-wise. Fig. 1 shows an example of a directed graph which is connected but not strongly connected. Its iSCC consists of $\mathcal{V}_{\mathrm{iSCC}} = \{v_1, v_2, v_3, v_4\}$, and any vertex in $\mathcal{V}_{\mathrm{iSCC}}$ is the root of a spanning tree. For further details, see Godsil and Royle (2001), Wieland (2010) and Wieland, Kim et al. (2011).

3. Synchronization in heterogeneous linear networks

It has been shown in Wieland and Allgöwer (2009) and Wieland, Sepulchre et al. (2011) that the geometric approach to linear systems theory (Basile & Marro, 1992; Wonham, 1985) is useful for the analysis of synchronization problems in networks of linear systems. In this section, we review the main result of Wieland and Allgöwer (2009), i.e., the internal model principle for synchronization. We consider a heterogeneous group of N linear agents,

$$\begin{aligned} \dot{x}_k &= A_k x_k + B_k u_k \\ y_k &= C_k x_k, \end{aligned} \tag{1}$$

with state $x_k \in \mathbb{R}^{n_k}$, input $u_k \in \mathbb{R}^{q_k}$, and output $y_k \in \mathbb{R}^p$, for $k \in \mathcal{N}$, where $\mathcal{N}$ is the index set $\mathcal{N} = \{1, \dots, N\}$. The agents are interconnected by static diffusive couplings

$$u_k = K_k \sum_{j=1}^{N} a_{kj}(y_j - y_k), \tag{2}$$

where $K_k \in \mathbb{R}^{q_k \times p}$ is a coupling gain matrix and a_{kj} are the elements of the adjacency matrix $A_\mathcal{G}$ of the underlying communication graph $\mathcal{G}$. The network of N agents (1) with couplings (2) is said to reach output synchronization, if

$$y_j(t) - y_k(t) \to 0 \quad \text{as } t \to \infty$$

for all pairs $j, k \in \mathcal{N}$. Furthermore, non-trivial output synchronization is reached if, additionally, the closed-loop system has no asymptotically stable equilibrium set on which $y_k(t) = 0$ for all $k \in \mathcal{N}$. We impose the following standing assumption.

Assumption 3.1. (A_k, C_k) is detectable for all $k \in \mathcal{N}$.

The closed-loop system (1), (2) can be compactly written as $\dot{x} = (\hat{A} - \hat{B}\hat{K}(L \otimes I_p)\hat{C})x$, where $x = [x_1^\mathsf{T} \cdots x_N^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{\hat{n}}$ and $\hat{n} = \sum_{k=1}^{N} n_k$ is the state dimension of the overall network, and with the block diagonal matrices $\hat{A} = \mathrm{diag}(A_1, \dots, A_N)$, $\hat{B} = \mathrm{diag}(B_1, \dots, B_N)$, $\hat{C} = \mathrm{diag}(C_1, \dots, C_N)$, and $\hat{K} = \mathrm{diag}(K_1, \dots, K_N)$. Output synchronization is reached if all solutions $x(t)$ converge to the synchronous subspace $\mathcal{S} \subseteq \mathbb{R}^{\hat{n}}$, which is defined as the subspace on which the outputs $y_k = C_k x_k$ of all agents are identical, i.e., $\mathcal{S} = \{x \in \mathbb{R}^{\hat{n}} : C_1 x_1 = \cdots = C_n x_n\}$. The internal model principle for synchronization is a necessary condition for non-trivial output synchronization. This condition guarantees the existence of a non-trivial invariant subspace $\mathcal{S}^* \subseteq \mathcal{S}$ such that $x(t) \to \mathcal{S}^*$ as $t \to \infty$ for all initial conditions and the dynamics restricted to $\mathcal{S}^*$ are not asymptotically stable. In the original publication (Wieland & Allgöwer, 2009), the necessary condition is formulated for dynamic diffusive couplings instead of static diffusive couplings (2). Here, we do not distinguish between system states and controller states of an agent. Instead, we formulate the result for a general, diffusively coupled heterogeneous linear network consisting of (1), (2). In the present setup, the internal model principle for synchronization of Wieland and Allgöwer (2009) and Wieland, Sepulchre et al. (2011) can be stated as follows.

Theorem 3.2. *A necessary condition for the existence of K_k, $k \in \mathcal{N}$, which ensure non-trivial output synchronization of a heterogeneous linear network of N agents (1) with static diffusive couplings (2), is that there exist an integer $m > 0$ and matrices $\Pi_k \in \mathbb{R}^{n_k \times m}$ with full column rank, $S \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{p\times m}$, where $\sigma(S) \subset \bar{\mathbb{C}}^+$ and (S, R) is observable, such that*

$$A_k \Pi_k = \Pi_k S, \tag{3}$$

$$C_k \Pi_k = R, \tag{4}$$

for all $k \in \mathcal{N}$. Furthermore, in this case there exists a $w_0 \in \mathbb{R}^m$ such that $\lim_{t\to\infty} \|y_k(t) - Re^{St} w_0\| = 0$ for all $k \in \mathcal{N}$.

Remark 3.3. Eq. (3) is equivalent to A_k-invariance of $\mathrm{im}(\Pi_k)$. In particular, every autonomous agent k has an A_k-invariant subspace $\mathrm{im}(\Pi_k) \subseteq \mathbb{R}^{n_k}$, such that the dynamics restricted to this subspace are identical for all agents. Eq. (4) guarantees that the outputs match on these subspaces.

Remark 3.4. Since Π_k has full column rank, every eigenvalue of S is an eigenvalue of A_k, i.e., $\sigma(S) \subseteq \sigma(A_k)$ for all $k \in \mathcal{N}$. Consequently, the eigenvalues of S are a subset of the largest common subset $\bigcap_{k=1}^{N} \sigma(A_k)$ of all agent's spectra. If the agents in the network have no eigenvalues in common, then (non-trivial) synchronization is impossible.

In words, the internal model principle for synchronization in heterogeneous networks of linear systems states that the agents can only synchronize to a trajectory generated by a dynamical system $\dot{w} = Sw$ contained in the dynamics of each agent. This internal model (S, R) generates the synchronous output trajectories of the network.

Remark 3.5. Theorem 3.2 presents a necessary condition for output synchronization. It has been shown in Wieland, Sepulchre et al. (2011) that, under mild assumptions on the network connectivity and given the agents (1) are stabilizable, the internal model condition is also sufficient for the existence of dynamic diffusive couplings which solve the output synchronization problem. The existence of static gain matrices K_k in (2), however, is a structured static output feedback problem (Syrmos, Abdallah, Dorato, & Grigoriadis, 1997).

4. Double-integrators with partial output

In this section, we focus on output synchronization in a network of non-identical double-integrator agents. The agents are described by

$$\dot{x}_k = \begin{bmatrix} 0 & 1+\delta_k \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_k, \qquad y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k, \tag{5}$$

with parameter $\delta_k \in \mathbb{R}$, $k \in \mathcal{N}$. The couplings are given by

$$u_k = \begin{bmatrix} 1 \\ \alpha \end{bmatrix} \sum_{j=1}^{N} a_{kj}(y_j - y_k). \tag{6}$$

The objective is synchronization of the outputs $y_k(t)$ to a common ramp function. Such networks appear, for instance, in distributed clock synchronization as discussed in Remark 4.2. It is easy to check that the internal model principle for synchronization (Theorem 3.2) is satisfied with

$$\Pi_k = \begin{bmatrix} 1 & 0 \\ 0 & (1+\delta_k)^{-1} \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{7}$$

As stated next, the network (5), (6) indeed reaches output synchronization to a ramp function generated by (S, R).

Theorem 4.1. *Consider a network of N double-integrator agents (5) interconnected by static diffusive couplings (6). Suppose that the graph $\mathcal{G}$ is undirected and connected and let $\alpha > 0$. Then, for all parameters $\delta_k > -1$, $k \in \mathcal{N}$, the network reaches output synchronization and the common trajectory is a ramp function, i.e., for all $k, j \in \mathcal{N}$, it holds that $y_k(t) - y_j(t) \to 0$ and $\dot{y}_k(t) \to c$ as $t \to \infty$ for some $c \in \mathbb{R}$.*

Proof. Let $s_k, v_k \in \mathbb{R}$ be the components of $x_k = [s_k \;\; v_k]^\mathsf{T}$ and define the stack vectors $s = [s_1 \;\cdots\; s_N]^\mathsf{T}$ and $v = [v_1 \;\cdots\; v_N]^\mathsf{T}$. Then, (5), (6) can be compactly written as $\dot{s} = -Ls + Dv$ and $\dot{v} = -\alpha Ls$. We perform the change of variables $\tilde{s} = Hs$, where $H = I - \frac{1}{N}\mathbf{1}\mathbf{1}^\mathsf{T}$ defines an orthogonal projection on $\mathrm{im}(\mathbf{1})^\perp$, i.e., on the subspace orthogonal to $\mathrm{im}(\mathbf{1})$. It will be shown later that the projected state $\tilde{s}(t)$ is bounded for all $t \geq 0$, which allows us to apply LaSalle's invariance principle. Note that $L\mathbf{1} = \mathbf{0}$ and $\mathbf{1}^\mathsf{T} L = \mathbf{0}^\mathsf{T}$. Therefore, it holds that $LH = HL = L$ and hence $L\tilde{s} = LHs = Ls$. The change of variable yields

$$\begin{bmatrix} \dot{\tilde{s}} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -L & HD \\ -\alpha L & 0 \end{bmatrix} \begin{bmatrix} \tilde{s} \\ v \end{bmatrix}. \tag{8}$$

Note that the state $\tilde{s}$ is restricted to the subspace $\mathrm{im}(\mathbf{1})^\perp$. Since the system is linear, the solution $s(t)$, $v(t)$ exists for all times $t \geq 0$. Hence, we can analyze the behavior of the state component $\tilde{s}(t)$ in $\mathrm{im}(\mathbf{1})^\perp$ using the Lyapunov function $V = \alpha \tilde{s}^\mathsf{T} L \tilde{s} + v^\mathsf{T} D v$. V is positive definite since $D > 0$ and $\tilde{s}^\mathsf{T} L \tilde{s} > 0$ for all $\tilde{s} \in \mathrm{im}(\mathbf{1})^\perp$, $s \neq \mathbf{0}$. In particular, $V \geq 0$ for all $\tilde{s}$, v and $V = 0 \Leftrightarrow \tilde{s} = v = \mathbf{0}$. The Lie-Derivative $\dot{V}$ can be computed as $\dot{V} = -2\alpha \tilde{s}^\mathsf{T} LL\tilde{s}$. Hence, $\dot{V}$ is negative semi-definite and the set $\mathcal{S}$ on which $\dot{V}$ vanishes is given by $\mathcal{S} = \{\tilde{s}, v : \tilde{s} = \mathbf{0}\}$. Since V is positive definite and $\dot{V}$ is negative semi-definite, we can conclude that the trajectories $\tilde{s}(t)$, $v(t)$ are bounded for all $t \geq 0$. Hence, LaSalle's invariance principle (LaSalle, 1967) is applicable. It follows that $\tilde{s}(t)$, $v(t)$ converge to the largest invariant set $\mathcal{J}$ contained in $\mathcal{S}$. The trajectories contained in $\mathcal{S}$ have to satisfy $\tilde{s}(t) = \mathbf{0}$ and $\dot{\tilde{s}}(t) = \mathbf{0}$ for all $t \geq 0$. The dynamics (8) restricted to $\mathcal{S}$ are $\dot{\tilde{s}} = HDv$ and $\dot{v} = \mathbf{0}$, and therefore $HDv(t) = \mathbf{0}$, or equivalently, $Dv(t) \in \mathrm{im}(\mathbf{1})$. Thus, the largest invariant set $\mathcal{J} \subset \mathcal{S}$ is $\mathcal{J} = \{\tilde{s}, v : \tilde{s} = \mathbf{0}, Dv \in \mathrm{im}(\mathbf{1})\} \subset \mathcal{S}$. By LaSalle's invariance principle, the solutions of (8) converge to $\mathcal{J}$ asymptotically. Therefore, in original coordinates, we can conclude that, for some $c \in \mathbb{R}$, it holds that $\dot{v}(t) \to \mathbf{0}$, $Dv(t) \to c\mathbf{1}$, $\dot{s}(t) \to c\mathbf{1}$, and $s(t) \to \mathrm{im}(\mathbf{1})$ as $t \to \infty$. ■

The network (5), (6) reaches non-trivial output synchronization robustly with respect to the parameter perturbations. Exact synchronization is achieved despite the heterogeneous agent dynamics. Fig. 2 shows simulation results for an example network with 5 nodes.

Remark 4.2 (*Clock Synchronization*)**.** It has been shown in Carli, Chiuso, Schenato, and Zampieri (2011) and Carli and Lovisari (2012) that a distributed synchronization protocol for a network of non-identical clocks can be derived based on a discrete-time model similar to (5), (6). Each node k in the network has a register $\tau_k(t)$ which periodically increments its value by one with period Δ_k. The

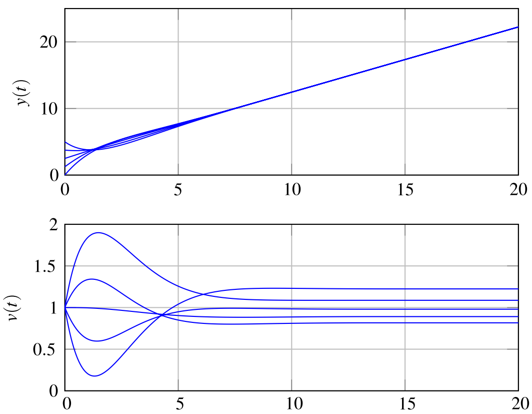


Fig. 2. This simulation of a network (5), (6) with 5 nodes with $\alpha = 0.5$, and $d_1 = 0.9$, $d_2 = 1.2$, $d_3 = 1.1$, $d_4 = 1.0$, $d_5 = 0.8$ and undirected cycle graph $\mathcal{G}$ shows that the outputs $y_k(t)$ synchronize.

period Δ_k is an unknown and perturbed value of some nominal Δ. Hence the time estimate of node k is given by $y_k(t) = \Delta\tau_k(t)$. In order to synchronize clocks, every node is equipped with a processor that communicates with neighboring nodes in the network and post-processes the register value $\tau_k(t)$. The evolution of the time estimate is described by the linear discrete-time system

$$x_k((h+1)T) = \begin{bmatrix} 1 & d_kT \\ 0 & 1 \end{bmatrix} x_k(hT) + u_k(hT)$$

and $y_k(hT) = [1 \ \ 0]x_k(hT)$, where $d_k = \Delta/\Delta_k$ and T is the communication period. The synchronization protocol proposed in Carli et al. (2011) is given by $u_k(hT) = [1 \ \ \alpha]^\mathsf{T} \sum_{j=1}^N a_{kj}(y_j(hT) - y_k(hT))$. For undirected connected graphs $\mathcal{G}$, necessary and sufficient conditions on α and the spectrum of the Laplacian L for synchronization of the nominal network are given in Carli et al. (2011). In the proof, the authors assume that $D = \mathrm{diag}(d_1, \ldots, d_N) = I$, i.e., the clocks are identical. Then, by continuity arguments, it can be concluded that the network also synchronizes for sufficiently small perturbations $D \neq I$. Explicit bounds on the tolerable parameter variation have recently been presented in Carli and Lovisari (2012). Suppose that the register $\tau_k(t)$ is a continuous ramp function $\tau_k(t) = d_kt$. Then, the limit $T \to 0$ corresponds to continuous communication between neighboring clocks and the dynamics of the network are described by (5), where $\delta_k = d_k - 1$, and the couplings are given by (6). A similar model has been studied in Lovisari and Jönsson (2011). In this context, Theorem 4.1 shows that, in case of continuous communication, the control law (6) achieves synchronization for arbitrary $d_k > 0$. In discrete-time, with $T > 0$, synchronization is not reached for all $\alpha > 0$, $d_k > 0$. However, Theorem 4.1 suggests that this discrete-time heterogeneous network can always be synchronized via sufficiently fast sampling, i.e., sufficiently small T. In fact, the results in Carli and Lovisari (2012) confirm this observation.

5. Double-integrators with full output

We focus again on synchronization in a network of non-identical double-integrator agents. In contrast to the previous section, we consider the full output $y_k = x_k$, i.e.,

$$\dot{x}_k = \begin{bmatrix} 0 & 1+\delta_k \\ 0 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u_k, \qquad y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k, \tag{9}$$

where $\delta_k \in \mathbb{R}$, $k \in \mathcal{N}$, and the state couplings

$$u_k = \sum_{j=1}^N a_{kj}(x_j - x_k). \tag{10}$$

A candidate matrix S of Theorem 3.2 for the matrix A_k has to fulfill $\sigma(S) \subseteq \sigma(A_k) = \{0, 0\}$ for all $k \in \mathcal{N}$. Thus, there are three candidates

$$S_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad S_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad S_3 = 0.$$

Since $C_k = I$, condition (4) yields $\Pi_k = \Pi_j$ for all $k, j \in \mathcal{N}$. Therefore, a necessary condition for non-trivial state synchronization in the double-integrator network is that there exists a matrix Π with full column rank such that $A_k\Pi = \Pi S_l$ for some $l \in \{1, 2, 3\}$ and all $k \in \mathcal{N}$. In general, $\delta_k \neq -1$ and thus there is no solution for S_1. There is also no solution for S_2. The necessary condition in Theorem 3.2 is not fulfilled for any $S \in \mathbb{R}^{2\times 2}$. Note that there exist A_k-invariant subspaces on which the dynamics of all agents are identical, given by $\mathrm{im}(\Pi_k)$ with Π_k as in (7). However, (3) and (4) cannot be satisfied at the same time for $C_k = I$ and S_1 or S_2. In other words, in the present network, the dynamics of the agents are compatible but the outputs do not match, which is a structural difference to the network of non-identical harmonic oscillators discussed in Section 6.

Note that for S_3, the necessary condition is fulfilled for $\Pi = [1 \ \ 0]^\mathsf{T}$. This is not surprising since the internal model S_3 is contained in A_k as the lower right entry. Hence exact synchronization to a trajectory generated by a single-integrator may be possible. However, exact synchronization to a trajectory generated by a double-integrator model is impossible. The following theorem characterizes the dynamic behavior of the network (9), (10).

Theorem 5.1. *Consider a network of N double-integrator agents (9) interconnected by static diffusive couplings (10). Suppose that the directed graph $\mathcal{G}$ is connected. Furthermore, suppose that there exists a pair k, j of agents such that $\delta_k \neq \delta_j$. Let $x_k = [s_k \ v_k]^\mathsf{T}$, $p^\mathsf{T}L = \mathbf{0}^\mathsf{T}$, and $p^\mathsf{T}\mathbf{1} = 1$. Then, $v(t) \to \mathbf{1}p^\mathsf{T}v_0$ as $t \to \infty$ and the states $s(t)$ do not synchronize but asymptotically grow with constant and identical speed. In particular, $(s(t) - s_\perp) \to \mathrm{im}(\mathbf{1})$ and $\dot{s}(t) \to \mathbf{1}(p^\mathsf{T}v_0 + c)$ as $t \to \infty$, where $c \in \mathbb{R}$ and the asymptotic disagreement $s_\perp \in \mathbb{R}^N$ with $\mathbf{1}^\mathsf{T}s_\perp = 0$ are given by*

$$\begin{bmatrix} s_\perp \\ c \end{bmatrix} = \begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^\mathsf{T} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \delta p^\mathsf{T} v_0 \\ 0 \end{bmatrix}. \tag{11}$$

Proof. With stack vectors s, v, and matrix $\Delta = \mathrm{diag}(\delta)$, the dynamics (9), (10) can be written as

$$\dot{s} = -Ls + (I + \Delta)v, \tag{12}$$
$$\dot{v} = -Lv. \tag{13}$$

The network (13) converges to consensus, i.e., for $t \to \infty$, $v(t) \to \mathbf{1}p^\mathsf{T}v_0$, where $v(0) = v_0$, cf., Wieland (2010). Suppose that $s, v \in \mathrm{im}(\mathbf{1})$. Then, $\dot{s} = (I + \Delta)v \notin \mathrm{im}(\mathbf{1})$ since $\delta \notin \mathrm{im}(\mathbf{1})$ by assumption. Thus, $\mathrm{im}(\mathbf{1})$ is not invariant for (12) and the states $s(t)$ do not synchronize. Let $\xi = \dot{s}$. Then, with (12) and (13), $\dot{\xi} = -L\xi - (I + \Delta)Lv$. It holds that $Lv(t) \to L\mathbf{1}p^\mathsf{T}v_0 = \mathbf{0}$ as $t \to \infty$, and $\xi(t)$ converges exponentially to a solution of the unforced system $\dot{\xi} = -L\xi$. Hence, for $t \to \infty$, $\dot{s}(t) = \xi(t) \to \mathrm{im}(\mathbf{1})$. Asymptotically, the states $s(t)$ grow with constant and identical velocity. With (12) and $v(t) \to \mathbf{1}p^\mathsf{T}v_0$, it follows that

$$-Ls(t) + \delta p^\mathsf{T}v_0 \to \mathrm{im}(\mathbf{1}) \tag{14}$$

as $t \to \infty$. The state s can be decomposed into a sum of two components, one component in the subspace $\mathrm{im}(\mathbf{1})$ and the other, denoted by $s_\perp$, in the orthogonal complement $\mathrm{im}(\mathbf{1})^\perp$. We are interested in the component $s_\perp$ since it determines the distance of s from $\mathrm{im}(\mathbf{1})$. Since $L\mathbf{1} = \mathbf{0}$, it holds that $Ls = Ls_\perp$ and therefore with

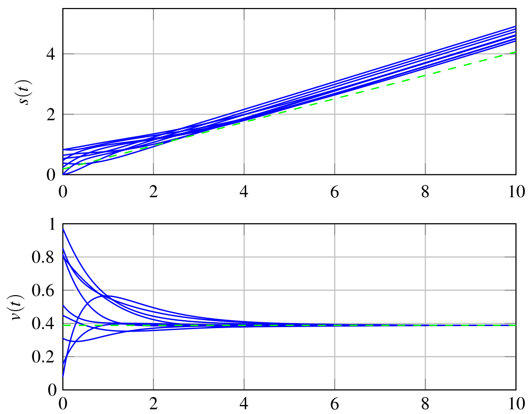


Fig. 3. Simulation result for a network of 9 agents (9) with (10), $\mathcal{G}$ as in Fig. 1, random non-identical parameters $\delta_k \in [-0.5, 0.5]$ with $\mathbf{1}^\mathsf{T}\delta = 0$, and random initial conditions. The dashed lines (– – –) indicate the asymptotic solution in the nominal case ($\delta = \mathbf{0}$). The second states $v(t)$ of all agents reach consensus (bottom). The first states $s(t)$ grow with constant and identical speed but with constant offsets $s_\perp$ according to (11) (top).

(14), $-Ls_\perp + \delta p^\mathsf{T} v_0 \in \operatorname{im}(\mathbf{1})$. This can be rewritten as $Ls_\perp + c\mathbf{1} = \delta p^\mathsf{T} v_0$ for some $c \in \mathbb{R}$, or equivalently,

$$\begin{bmatrix} L & \mathbf{1} \\ \mathbf{1}^\mathsf{T} & 0 \end{bmatrix} \begin{bmatrix} s_\perp \\ c \end{bmatrix} = \begin{bmatrix} \delta p^\mathsf{T} v_0 \\ 0 \end{bmatrix}. \tag{15}$$

It holds that $\operatorname{im}(L)^\perp = \ker(L^\mathsf{T}) = \operatorname{im}(p)$, where $p^\mathsf{T} L = \mathbf{0}^\mathsf{T}$, $p^\mathsf{T}\mathbf{1} = 1$. Since $p^\mathsf{T}\mathbf{1} \neq 0$, it follows that $\operatorname{im}([L\ \mathbf{1}]) = \mathbb{R}^N$, i.e., the rank of the matrix $[L\ \mathbf{1}] \in \mathbb{R}^{N\times(N+1)}$ is N. It holds that $[L\ \mathbf{1}][\mathbf{1}^\mathsf{T}\ 0]^\mathsf{T} = \mathbf{0}$ and $[\mathbf{1}^\mathsf{T}\ 0][\mathbf{1}^\mathsf{T}\ 0]^\mathsf{T} \neq 0$. Therefore, the coefficient matrix in (15) has full rank $N + 1$ and (15) has the unique solution (11). With (12), we can finally conclude that $\dot{s}(t) \to \mathbf{1}(p^\mathsf{T} v_0 + c)$ as $t \to \infty$, i.e., the constant c is the deviation of the agents' velocity from the nominal case, in which $\dot{s}(t) \to \mathbf{1}p^\mathsf{T} v_0$ as $t \to \infty$. ∎

Theorem 5.1 shows that networks of double-integrators (9) with static diffusive couplings (10) have a certain robustness with respect to heterogeneity in the dynamics, in the sense that they synchronize approximately for small perturbations δ_k, $k \in \mathcal{N}$. The quantity $\|s_\perp\|$ can be seen as an asymptotic synchronization error since $\lim_{t\to\infty} \operatorname{dist}(s(t), \operatorname{im}(\mathbf{1})) = \|s_\perp\|$. According to (15), $s_\perp$ scales inversely with L. This shows that the underlying graph plays an important role: stronger couplings decrease the error proportionally. The velocities of the agents synchronize for arbitrary parameters δ_k. Both the final velocity and the asymptotic offsets between the agents can be computed explicitly according to (11), depending on the graph topology, parameters δ, and the initial states. A numerical example is shown in Fig. 3. In the context of coupled Kuramoto models, such a behavior (motion with common frequency and constant phase offsets) is also called phase locking, cf., Dörfler and Bullo (2012). The analysis above demonstrates that, in contrast to the previous section, a heterogeneous network may fail to reach exact synchronization but can still reach synchronization approximately, even if the internal model principle for synchronization is not fulfilled.

6. Harmonic oscillators

In this section, networks of non-identical harmonic oscillators are analyzed. In these networks, exact non-trivial synchronization is impossible. The structural difference to the previous network is that there exists no solution to Eqs. (3), even if (4) is disregarded. As an application example, it is shown that a certain multi-agent control algorithm is not robust with respect to parameter uncertainty. The dynamical agents are described by

$$\dot{x}_k = \begin{bmatrix} 0 & 1 \\ -(\omega+\delta_k)^2 & 0 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k, \qquad y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k \tag{16}$$

and coupled through

$$u_k = \sum_{j=1}^{N} a_{kj}(y_j - y_k), \tag{17}$$

for $k \in \mathcal{N}$. The frequencies of the individual oscillators are perturbed by the parameters $\delta_k \in \mathbb{R}$ and deviate from the nominal frequency $\omega \in \mathbb{R}$. It is shown in Ren (2008) that the oscillator network reaches state synchronization when $\delta_k = 0$ for all $k \in \mathcal{N}$ and $\mathcal{G}$ is connected. Suppose instead there exist two agents $k, j \in \mathcal{N}$ such that $\delta_k \neq \delta_j$, i.e., not all oscillators have identical frequencies. Then the intersection of the agents' spectra $\bigcap_{k=1}^{N} \sigma(A_k)$ is empty and exact non-trivial synchronization is impossible since the necessary condition in Theorem 3.2 is not fulfilled, cf., Remark 3.4. In geometric terms, there exist no A_k-invariant subspaces on which the dynamics of all agents are identical. Eq. (3) cannot be satisfied, even if (4) is disregarded. The following result characterizes the dynamic behavior of the network.

Theorem 6.1. *Consider a network of N harmonic oscillators (16) interconnected by static diffusive couplings (17). Suppose that the directed graph $\mathcal{G}$ is connected. Then, the network is asymptotically stable if and only if there exist $k, j \in \mathcal{N}$ within the iSCC of $\mathcal{G}$ such that $\delta_k \neq \delta_j$. Otherwise, the oscillators within the iSCC reach non-trivial output synchronization.*

Proof. Define $D = \operatorname{diag}((\omega+\delta_1)^2, \ldots, (\omega+\delta_N)^2)$, $x_k = [s_k\ v_k]^\mathsf{T} \in \mathbb{R}^2$, $s = [s_1\ \cdots\ s_N]^\mathsf{T}$, and $v = [v_1\ \cdots\ v_N]^\mathsf{T}$. Then,

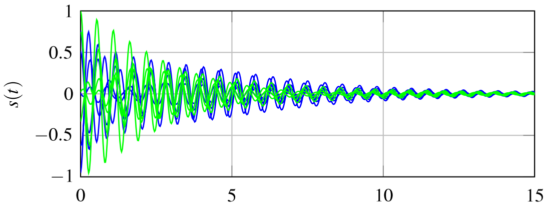
$$\begin{bmatrix} \dot{s} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -D & -L \end{bmatrix} \begin{bmatrix} s \\ v \end{bmatrix}.$$

At first, we assume that the graph $\mathcal{G}$ is strongly connected. Afterwards, we will relax this assumption and prove stability for general connected graphs. It is shown in Ren (2008) that the network reaches (non-trivial) state synchronization if the frequencies of all oscillators are identical. Hence, it remains to show that the network is rendered asymptotically stable by frequency perturbations within the iSCC of $\mathcal{G}$. We consider the Lyapunov function $V = s^\mathsf{T} PDs + v^\mathsf{T} Pv$, where $P = \operatorname{diag}(p)$ and p is the left eigenvector of L corresponding to zero. Since $\mathcal{G}$ is strongly connected, $\mathcal{V}_{\mathrm{iSCC}} = \mathcal{V}$ and $P > 0$. The Lie-derivative of V is $\dot{V} = -v^\mathsf{T}(PL + L^\mathsf{T} P)v$, which is negative semi-definite (Zhang, Lewis, & Qu, 2012). The set on which $\dot{V} = 0$ is given by $\mathcal{J} = \{s, v : v \in \operatorname{im}(\mathbf{1})\}$. Since $L\mathbf{1} = \mathbf{0}$, the dynamics on $\mathcal{J}$ are given by $\dot{s} = v$ and $\dot{v} = -Ds$. Every solution in $\mathcal{J}$ has to satisfy $\dot{v}(t) = -Ds(t) \in \operatorname{im}(\mathbf{1})$ and $D\dot{s}(t) = Dv(t) \in \operatorname{im}(\mathbf{1})$. This can only be true if $s = v = \mathbf{0}$ since $D \neq I$. By LaSalle's invariance principle, it follows that the origin $s = v = \mathbf{0}$ is asymptotically stable.

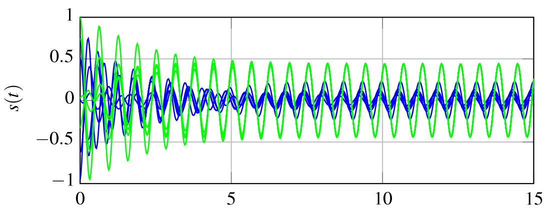
Suppose now that $\mathcal{G}$ is connected but not necessarily strongly connected. Then, there exists a vertex permutation such that the Laplacian matrix reduces to the Frobenius normal form

$$L = \begin{bmatrix} L_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ L_{m1} & \cdots & L_{mm} \end{bmatrix},$$

where L_{ii}, $i = 1, \ldots, m$, are irreducible square matrices, each L_{ii}, $i = 2, \ldots, m$, has at least one row with positive row sum, and L_{11} is the Laplacian associated to the unique iSCC of $\mathcal{G}$ (Brualdi & Ryser, 1991). We have seen that $x_k(t) \to 0$ as $t \to \infty$ for all $k : v_k \in \mathcal{V}_{\mathrm{iSCC}}$. It remains to show that this implies $x_j(t) \to 0$ as $t \to \infty$ for $j \in \mathcal{V} \setminus \mathcal{V}_{\mathrm{iSCC}}$. We partition the vectors s, v according

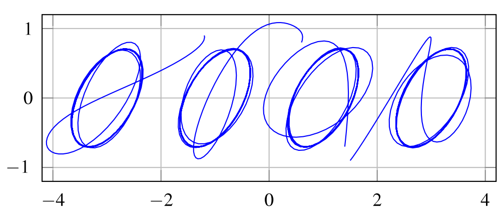


(a) Non-identical oscillators in the iSCC stabilize the network.

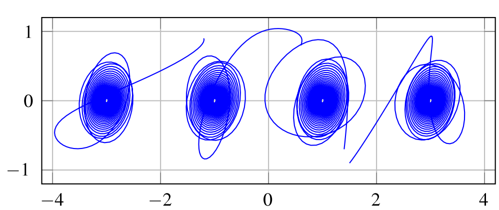


(b) Identical oscillators in the iSCC synchronize and excite the perturbed oscillators outside the iSCC.

Fig. 4. Simulation results for a network of 9 oscillators with $\mathcal{G}$ as in Fig. 1, nominal frequency $\omega = 10$, and random offsets $\delta_k \in [0, 2]$ (a) for all $k = 1, \ldots, 9$ and (b) identical frequencies within the iSCC. The plots show the first states of the oscillators within (———) and outside (———) the iSCC of $\mathcal{G}$.



(a) Control law (19), identical $m_k = 1$.



(b) Control law (19), non-identical $m_k \in [0.75\ 1.25]$.

Fig. 5. Motion coordination of $N = 4$ mobile robots. The simulation with $\alpha = 1.8$, masses $m_1 = 0.75$, $m_2 = 0.92$, $m_3 = 1.08$, $m_4 = 1.25$, and directed cycle graph $\mathcal{G}$ demonstrates that the agents indeed asymptotically converge to fixed points, see (b). For comparison, the nominal case with $m_k = 1$, $k \in \mathcal{N}$ is shown in (a).

to the size of the blocks on the diagonal of L, i.e., $s = [s_1^\top, \ldots, s_m^\top]^\top$, $v = [v_1^\top, \ldots, v_m^\top]^\top$, and $D = \mathrm{diag}(D_{11}, \ldots, D_{mm})$. This yields $\dot{s}_i = v_i$ and $\dot{v}_i = -D_{ii}s_i - \sum_{l=1}^{i-1} L_{il}v_l - L_{ii}v_i$, for $i = 2, \ldots, m$. Each L_{ii} has at least one row with positive row sum. Therefore it is possible to decompose $L_{ii} = \tilde{L}_{ii} + M_{ii}$, such that $\tilde{L}_{ii}$ is the Laplacian matrix corresponding to a strongly connected graph $\tilde{\mathcal{G}}_{ii}$ and M_{ii} is a non-negative diagonal matrix with at least one positive element (Brualdi & Ryser, 1991). Now we prove asymptotic stability block-wise by induction. For block $i = 1$, exponential stability follows since L_{11} corresponds to a strongly connected graph. For any block $i > 1$, it can be shown that $s_i(t), v_i(t) \to 0$ as $t \to \infty$ if $s_l(t), v_l(t) \to 0$ as $t \to \infty$ for $l = 1, \ldots, i-1$ by the following argument. If $s_l(t), v_l(t) \to 0$ as $t \to \infty$ for $l = 1, \ldots, i-1$, then the dynamics of s_i, v_i are asymptotically described by $\dot{s}_i = v_i$, $\dot{v}_i = -D_{ii}s_i - (\tilde{L}_{ii} + M_{ii})v_i$. Consider the Lyapunov function $V_i = s_i^\top P_{ii}D_{ii}s_i + v_i^\top P_{ii}v_i$, where $P_{ii} = \mathrm{diag}(\tilde{p}_i)$ and $\tilde{p}_i$ is the left eigenvector of $\tilde{L}_{ii}$ corresponding to zero. Since $\tilde{\mathcal{G}}_{ii}$ is strongly connected, $P_{ii} > 0$ and hence V_i is positive definite. Furthermore, we obtain $\dot{V}_i = -v_i^\top(P_{ii}\tilde{L}_{ii} + \tilde{L}_{ii}^\top P_{ii})v_i - 2v_i^\top P_{ii}M_{ii}v_i$. It holds that $(P_{ii}\tilde{L}_{ii} + \tilde{L}_{ii}^\top P_{ii}) \geq 0$ and $\ker(P_{ii}\tilde{L}_{ii} + \tilde{L}_{ii}^\top P_{ii}) = \mathrm{im}(\mathbf{1})$. Since $P_{ii}M_{ii}$ is a non-negative diagonal matrix with at least one positive element, $\mathbf{1}^\top P_{ii}M_{ii}\mathbf{1} > 0$ and therefore $\dot{V}_i < 0$. This proves that $s_i(t), v_i(t) \to 0$ as $t \to \infty$. By induction, we conclude that $s(t), v(t) \to 0$ as $t \to \infty$. ■

Theorem 6.1 shows that the network topology again plays a crucial role for the dynamic behavior of the oscillator network. The network is asymptotically stable if and only if there is a pair of oscillators inside the iSCC of the underlying graph, which do not have identical frequencies. If the graph is strongly connected, then all nodes belong to the iSCC and the network is stabilized whenever there exist any two oscillators with non-identical frequencies. Furthermore, Theorem 6.1 shows that (non-trivial) synchronization of harmonic oscillators via static diffusive couplings is not at all robust with respect to parametric uncertainties causing variations of the frequencies. It suffices to change the frequency of one single oscillator in the iSCC by an arbitrarily small δ_k in order to render the entire network asymptotically stable. Fig. 4 shows two numerical examples.

Remark 6.2 (*Motion Coordination*). In Ren (2008), a motion coordination problem for a group of mobile robots is presented as an application of distributed oscillator synchronization. The mobile robots are modeled as point-mass agents in the plane with force inputs and are equipped with distributed controllers, which coordinate their motion such that all agents move synchronously on identical elliptic paths. Suppose now that the agents have non-identical and unknown masses $m_k > 0$, i.e., each agent is modeled as

$$\dot{s}_k = v_k, \qquad m_k\dot{v}_k = u_k, \tag{18}$$

where $s_k, v_k \in \mathbb{R}^2$ are the position and velocity of agent k in the plane, $k \in \mathcal{N}$. The control law proposed in Ren (2008) is

$$u_k = -\alpha(s_k - c_{s,k}) - \sum_{j=1}^{N} a_{kj}(v_k - v_j), \tag{19}$$

where $c_{s,k} \in \mathbb{R}^2$ is a constant offset which defines the relative position of agent k in the formation and $\alpha > 0$. Note that in Ren (2008), the network has a dedicated leader node which is the only root of a spanning tree of the graph $\mathcal{G}$. Here we consider general connected directed graphs, and leader–follower topologies are included as a special case. The constant offsets $c_{s,k}$, $k \in \mathcal{N}$, can be set to zero in the stability analysis since they represent a constant shift of the motion in the plane. The dynamics of agent k with its controller are $\dot{s}_k = v_k$, $\dot{v}_k = -\frac{\alpha}{m_k}s_k - \sum_{j=1}^{N}\frac{a_{kj}}{m_k}(v_k - v_j)$. Let L be the Laplacian associated to the graph $\hat{\mathcal{G}}$ with weights $\hat{a}_{kj} = a_{kj}/m_k$, and $D = \mathrm{diag}(\alpha_1, \ldots, \alpha_N)$, $\alpha_k = \alpha/m_k$. Then, the dynamics of the network (18), (19) match (16), (17). Consequently, the network of N point-mass agents (18), (19) is asymptotically stable if and only if there exists a pair $k, j \in \mathcal{N}$ in the iSCC of $\mathcal{G}$ such that $m_k \neq m_j$. This fact shows that the motion coordination algorithm proposed in Ren (2008) is not robust with respect to non-identical and possibly uncertain masses of the robots. In this example, the stabilizing effect due to the parameter uncertainties in the network is an issue. Numerical simulations are shown in Fig. 5.

7. Conclusions

Output synchronization problems are significantly more complex in heterogeneous multi-agent systems than in homogeneous multi-agent systems. In the present paper, various important heterogeneous linear networks have been studied. These networks illustrated the structural requirements for exact non-trivial output synchronization, known as the internal model principle for synchronization. The first network consisted of double-integrator agents and achieved non-trivial output synchronization. As was shown in the second network, the same network with full output $y_k = x_k$ cannot synchronize to a double-integrator trajectory since it does not fulfill the necessary condition in Theorem 3.2 due to (4). However, the synchronization error turned out to be small for large coupling gains. The third network consisted of harmonic oscillators and does not fulfill the necessary condition in Theorem 3.2 due to (3). In this case, non-trivial output synchronization is impossible, irrespective of the output matrices of the agents. Heterogeneous networks of non-identical harmonic oscillators were shown to be rendered asymptotically stable, if and only if there are non-identical oscillators inside the iSCC of the underlying connected and directed graph. This shows that the presence of heterogeneity in the network crucially affects the dynamic behavior of the network if the heterogeneity is located in a particular region of the graph, i.e., the iSCC.

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