FEL3330 Networked and Multi-agent Control Systems
Lecture 8: Formation Control 2

September 27, 2016

Today's lecture

- Distance based formations
- Rigidity
- Persistence
- Leader-follower networks

Formation Control

- Convergence to desired relative states with respect to neighbors
- Can be distinguished between position based and distance based
- Other relative states can be considered such as relative orientation

Distance-based formations

- Goal: Convergence to desired relative *distances* in $\mathbb{R}^2$.
- Consider undirected and static graph $G_f = (V, E_f, w)$.
- A scalar weight $w(i,j) = d_{ij} > 0$ is associated to each edge $(i,j) \in E$ representing the desired relative distance of agents i, j.
- The formation configuration is called *feasible* if the set $D = \left\{ q \in \mathbb{R}^{pN} \mid ||q_i - q_j|| = d_{ij},\ \forall (i,j) \in E_f \right\}$ of feasible formation configurations is nonempty. $p = 2, 3$ typically.

Distance-based formations

- Scale invariance: $D' = aD, a > 0$.
- Translational invariance (not rotational): let $q \in D$, ie, $||q_i - q_j|| = d_{ij} \forall (i,j) \in E_f$. Then any configuration $x = [x_1^T, \ldots, x_N^T]^T$ with $x_i = q_i + \tau$ for an arbitrary $\tau \in \mathbb{R}^p$ satisfies the formation specification.
- What about both translational and rotational invariance?

Rigidity

- Let $q = [q_1^T, \ldots, q_N^T]^T$ a set of feasible points.
- Framework $G(q) = (G_f, q)$. Trajectory of framework is any $x(t)$ starting from $x_i(0) = q_i, i = 1, \ldots, N$.
- Edge consistent if $||x_i - x_j||$ is constant $\forall (i,j) \in E_f$.
- Rigid trajectory: if $||x_i - x_j||$ is constant $\forall i, j \in V, i \neq j$.
- A framework is rigid iff all edge consistent trajectories are rigid trajectories.
- Maintaining formation graph edge distances maintains ALL distances!

Infinitesimal rigidity

- Edge consistency: $||x_i(t) - x_j(t)|| = d_{ij}$ and then

$$(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0, \forall (i,j) \in E_f$$

- Assigning constant+instantenuous $\dot{x}_i = u_i$ satisfying the above at $t = 0$: infinitesimal motion u of the framework.
- For $t = 0$, $x_i(0) = q_i, i = 1, \ldots, N$ and then $(x_i - x_j)^T(\dot{x}_i - \dot{x}_j) = 0, \forall (i,j) \in E_f$ becomes

$$R(G(q))u = 0$$

where R is the rigidity matrix.

Infinitesimal rigidity

- $G(q)$ infinitesimally rigid if $R(G(q))u = 0$ for all infinitesimal motions u.
- Rank condition for planar graphs: $G(q) = (G_f, q)$ with $N \leq 2$ and $p = 2$ infinitesimally rigid if and only if

$$\mathrm{rank} R(G(q)) = 2N - 3$$

- Result: infinitesimal rigidity implies rigidity but not necessarily the other way round.

Graph rigidity

- $G(q) = (G_f, q)$ particular realization of G_f.
- G_f is (generically) rigid if it has an infinitesimally rigid realization.
- For G_f rigid and $G(q) = (G_f, q)$ infinitesimally rigid, then q is called generic configuration and $G(q)$ generic realization of G_f.
- If G_f is (generically) rigid, then the set of all generic configurations of G_f is a dense open subset of $\mathbb{R}^{pN}$.

Minimal rigidity

- G_f is minimally rigid if it is rigid but does not remain rigid after the removal of a single edge.
- Planar graphs: G_f with $N \leq 2$ and $p = 2$ is minimally rigid if and only if
    1. $|E_f| = 2N - 3$ and
    2. Any induced subgraph with $N' \leq N$ vertices has no more than $2N' - 3$ edges.

Persistence

- To distinguish directed case from undirected, the term graph persistence instead of graph rigidity is used.
- And now we look at directed graphs.
- $i \to j$: i responsible for maintaining the correct distance from j, and j may be unconscious of i.

Persistence

Theorem

A rigid graph is minimally persistent if and only if either

1. *there are three vertices that have one outgoing edge, and the remaining vertices have two outgoing edges, or*
2. *there is one vertex that has no outgoing edge, one vertex that has one outgoing edge, and the remaining vertices have two outgoing edges.*

Leader-follower networks

Leaders can be important in various properties

- Network Controllability
- Containment control (not covered)
- Leader-follower ratio can be also a factor in various problems (see HW2)

Network Controllability

- Goal: how to control a multi-agent system to any configuration with some agents as free inputs.
- Simplest case: $\dot{x}_i = -\frac{1}{|N_i|}\sum_{j\in\mathcal{N}_i}(x_i - x_j), i = 1, \ldots, N-1$
- One leader (free input): $\dot{x}_N = u_N$
- Denote $y_i = x_i, i = 1, \ldots, N-1, z = x_N$.
- Then $\dot{y} = -Fy - rz$ where F is the matrix obtained from $\Delta^{-1}L$ after deleting the last row and column, and r is the vector of the first $N-1$ elements of the deleted column.

Network Controllability

- Controllability of the followers through the motion of the leader z is examined through the controllability matrix of the system:

$$C = \begin{pmatrix} -r & Fr & -F^2 r & \dots & (-1)^n F^{n-1} r \end{pmatrix}$$

- Main result: The system is controllable iff (i) F has distinct eigenvalues and (ii) the eigenvectors of F are not orthogonal to r.

Remarks

- It can be shown that the complete graph is uncontrollable, whereas a tree graph is controllable (connectivity is not always good).
- Extensions to multiple leaders in later works.

Next Lecture

Sensing Constraints 2: Swarming-sensor networks

- Swarm aggregation
- Swarm dispersion
- Collision avoidance