FEL3330 Networked and Multi-agent Control Systems
Lecture 6:
Sensing constraints 1

September 22, 2016

Today's lecture

- Connectivity
- Connectivity maintenance for static interaction graphs
- Connectivity maintenance for dynamic interaction graphs
- Robust connectivity maintenance under bounded controls

Connectivity

- Motiavation: Mobile robots with limited sensing range (e.g, omnidirectional sensors)
- Δ-proximity graph: $\{v_i, v_j\} \in \mathcal{E} \iff |x_i - x_j| \leq \Delta$
  Notational convention $|\cdot| := \|\cdot\|_2$
- Δ-proximity graph is a dynamic interaction graph
- Static interaction graph (SIG): Communication links are assumed fixed between the agents

Cooperative robots

- Single integrator dynamics: $\dot{x}_i = u_i$
- SIG $(\mathcal{V}, \mathcal{E})$: $\{i, j\} \in \mathcal{E} \iff j \in \mathcal{N}_i$
- Decentralized relative position contol laws:

$$u_i = \sum_{j \in \mathcal{N}_i} f(x_i - x_j)$$

- Antisysmetric f: $f(x) = -f(-x) \Longrightarrow f(x_i - x_j) = -f(x_j - x_i)$
- Consensus special case:

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j), x_i \in \mathbb{R}^n$$

Cooperative robots

- Apply component operator to reduce to n concsensus problems in $\mathbb{R}^N$

$$c_l(x) := (x_{1,l}, \ldots, x_{N,l}), l = 1, \ldots, n, x_i = (x_{i,1}, \ldots, x_{i,n})$$

- We then get

$$\dot{x}_{i,l} = -\sum_{j \in \mathcal{N}_i} (x_{i,l} - x_{j,l}), l = 1, \ldots, n, i = 1, \ldots, N$$

- and thus

$$c_l(\dot{x}) = -L(\mathcal{G}) c_l(x), l = 1, \ldots, n$$

- If we apply this control law to the dynamic Δ-proximity graph with $\{i, j\} \in \mathcal{E}(t) \iff |x_i(t) - x_j(t)| \leq \Delta$ we might loose connectivity.

Weighted graph based feedback

- Feedback components $f_i : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$f_i(x_i - x_j) := -w(x_i - x_j)(x_i - x_j)$$

- with nonlinear weight function $w : \mathbb{R}^n \to \mathbb{R}_{>0}$ positive and symmetric
- We obtain the decentarlized contol law

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} w(x_i - x_j)(x_i - x_j), i = 1, \ldots, N$$

- and componentwise

$$c_l(\dot{x}) = -D(\mathcal{G})W(x)D(G)^T c_l(x), l = 1, \ldots, n$$

- State dependent weighted Laplacian

$$L_w(x) = D(\mathcal{G})W(x)D(G)^T$$

Weighted graph based feedback

- State dependent weighted Laplacian

$$L_w(x) = D(\mathcal{G})W(x)D(G)^T$$

- Properties of $L_w(x)$ (for each x)
  - symmteric
  - positive semidefinite
  - assuming that $\mathcal{G}$ is connected the only zero eigenvalue corrsponds to $\mathrm{span}(\mathbf{1})$
- Critical edge distance δ with initial tolerance $\epsilon < \delta$
- ϵ shrinking of a δ constrained realization of the SIG $\mathcal{G}$

$$\mathcal{D}_\delta^\epsilon = \{x \in \mathbb{R}^{Nn} : |\ell_{ij}| \leq \delta - \epsilon \text{ for all } \{i,j\} \in \mathcal{E}\}, \ell_{ij}(x) = x_i - x_j$$

Weighted graph based feedback

- Edge tension function

$$V_{ij}(x) = \begin{cases} \frac{|\ell_{ij}(x)|^2}{\delta - |\ell_{ij}(x)|}, & \text{if } \{i,j\} \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

- with partial derivatives

$$\frac{\partial V_{ij}(x)}{\partial x_i} = \begin{cases} \frac{2\delta - |\ell_{ij}(x)|}{(\delta - |\ell_{ij}(x)|)^2}(x_i - x_j)^T, & \text{if } \{i,j\} \in \mathcal{E}, \\ 0, & \text{otherwise.} \end{cases}$$

- Total energy of $\mathcal{G}$

$$V(x) = \frac{1}{2} \sum_{i \in \mathcal{N}} \sum_{j \in \mathcal{N}} V_{ij}(x)$$

Weighted graph based feedback

LEMMA

Given an initial position $x_0 \in D_\delta^\epsilon$, for a given $\epsilon \in (0, \delta)$, if the SIG $\mathcal{G}$ is conncted then the set $\Omega(\delta, x_0) = \{x : V(x) \leq V(x_0)\}$ is invarinat under the control law

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} \frac{2\delta - |\ell_{ij}(x)|}{(\delta - |\ell_{ij}(x)|)^2}(x_i - x_j) \tag{1}$$

Weighted graph based feedback

THEOREM

Consider the connected SIG $\mathcal{G}$ with initial condition $x_0 \in D_\delta^\epsilon$ and a given $\epsilon \in (0, \delta)$. Then the multiagent system under the conrol law (1) converges asymptoticaly to the static centroid $\bar{x}$.

Dynamic graphs

- Dynamic Δ-proximity graph with

$$\{i, j\} \in \mathcal{E}(t) \iff |x_i(t) - x_j(t)| \leq \Delta$$

- Add new edge $\{i, j\}$ when crossing the switching threshold

$$|l_{ij}| \leq \Delta - \epsilon$$

- Switching protocol

$$\sigma(i, j)[t] = \begin{cases} 0, & \text{if } \sigma(i, j)(s) = 0, \forall s \in [0, t) \text{ and } |l_{ij}| > \Delta - \epsilon \\ 1, & \text{otherwise.} \end{cases}$$

Dynamic graphs

THEOREM

Consider the an initail position $x_0 \in D^{\epsilon}_{\Delta}(\mathcal{G}_0)$ where $\epsilon \in (0, \Delta)$ is the switching threshold and $\mathcal{G}_0$ is the initial Δ-disk DIG. Assume that the graph $\mathcal{G}_\sigma$ induced by the indicator function is initially connected. with initial condition $x_0 \in D^{\epsilon}_{\delta}$ and a given ϵ. Then the control law

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i^\sigma} \frac{2\Delta - |\ell_{ij}(x)|}{(\Delta - |\ell_{ij}(x)|)^2}(x_i - x_j)$$

with the swithcing protcol $\sigma(i, j)$ as previously defined assymptotically converges to $\mathrm{span}(\mathbf{1})$.

Systems Description

- Consider the single integrator multi-agent system

$$\dot{x}_i = u_i, x_i \in \mathbb{R}^n, i = \{1, \ldots, N\} := \mathcal{N}$$

- Network graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ undirected & connected
- $\mathcal{V} := \mathcal{N} \qquad j \in \mathcal{N}_i \iff \{i, j\} \in \mathcal{E}$
- Design decentralized control laws

$$u_i = f_i(x_i, x_{j_1}, \ldots, x_{j_{|\mathcal{N}_i|}}) + v_i, i \in \mathcal{N}$$

with free input terms v_i.

Motivation

- Design bounded feedback laws which guarantee
  - connectivity maintenance of the multi-robot network
  - invariance of systems solutions inside a bounded domain
  - robustly wrt free input terms
- Exploit bounds on dynamics and acceptable bound on free input terms to extract a discretized model of the continuous time system[1]
- Exploit invariance to extract a finite transition system

[1] D. B. and D. V. Dimarogonas, Decentralized Abstractions for Feedback Interconnected Multi-Agent Systems, CDC 2015

Problem Description

ASSUMPTIONS

- Multi agent network with static interaction graph.
- Network initially connected.

GOAL

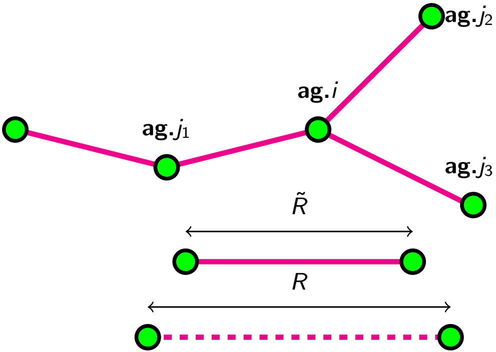
- Specify apriori bounds on the initial distances between interconnected agents
- Design bounded (for bounded inter-agent distances) control laws
- Guarantee connectivity maintenance robustly wrt. free inputs

EXTENSION

- Guarantee invariance of solutions inside a spherical domain.

Connectivity Assumptions

- Agents i, j connected iff $\{i, j\} \in \mathcal{E}$ and $|x_i - x_j| \leq R$
- Initial Connectivity Hypothesis: $\forall \{i, j\} \in \mathcal{E}$: $|x_i(0) - x_j(0)| \leq \tilde{R} < R$



Potential field based Controllers[2]

- $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ continuous; increasing.
- Define the potential function

$$P(\rho) = \int_0^{\rho} r(s)sds, \rho \in \mathbb{R}_{\geq 0}$$

- Gradient of the potential function

$$\nabla_{x_i} P(|x_i - x_j|) = r(|x_i - x_j|)(x_i - x_j)$$

- Select the control law

$$\begin{aligned} u_i &= \sum_{j \in \mathcal{N}_i} \nabla_{x_j} P(|x_i - x_j|) + v_i \\ &= -\sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i \end{aligned}$$

[2] M. Ji and M. Egerstedt, Distributed Coordination Control of Multi-Agent Systems While Preserving Connectedness, 2007

Dynamics in Compact Form

- Overall dynamics

$$c_l(\dot{x}) = -L_w(x)c_l(x) + c_l(v), l = 1, \dots, n$$
$$c_l(x) := (x_{1,l}, \dots, x_{N,l}), x_i = (x_{i,1}, \dots, x_{i,n})$$

- Weighted Laplacian

$$L_w(x) := D(\mathcal{G})W(x)D(\mathcal{G})^T$$

- $D(\mathcal{G})$: incidence matrix
- Edge weights

$$W(x) := \mathrm{diag}\{w_1(x), \dots, w_M(x)\} := \mathrm{diag}\{r(|x_i - x_j|), \{i,j\} \in \mathcal{E}\}$$

Energy Function

- For each $\{i, j\} \in \mathcal{E}$ define

$$V_{ij}(x) = P(|x_i - x_j|), x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nn}$$

- Define the energy function

$$V := \frac{1}{2} \sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} V_{ij}$$

- Partial derivatives

$$\frac{\partial}{\partial x_i} V(x) = \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j)^T$$

- Componentwise derivative of energy function

$$c_l \left(\frac{\partial}{\partial x} V(x) \right) = c_l(x)^T L_w(x), l = 1, \ldots, n$$

Derivative along System Trajectories

- Derivative of energy function

$$\dot{V} = -\sum_{l=1}^{n} c_l\left(\frac{\partial}{\partial x}V(x)\right)c_l(\dot{x})$$

$$\leq -\underbrace{\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x)}_{\textit{term 1}} + \underbrace{\left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right|}_{\textit{term 2}}$$

- Lower bound for term 1

$$\sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) \geq [\lambda_2(\mathcal{G})r(0)]^2|x^{\perp}|^2$$

- Upper bound for term 2

$$\left|\sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v)\right| \leq \sqrt{N}|D(\mathcal{G})^T||\Delta x|r(|\Delta x|_\infty)|v|_\infty$$

$$\Delta x := \text{stack vector of } x_i - x_j, \{i,j\} \in \mathcal{E}$$

$$|\Delta x|_\infty := \max\{|x_i - x_j| : \{i,j\} \in \mathcal{E}\}$$

Derivative along System Trajectories

- Requirement on $\dot{V}$

$$|\Delta x|_\infty \geq \tilde{R} \Rightarrow \dot{V} \leq 0 \qquad (\#)$$

- A sufficient condition for $(\#)$ is that

$$|v|_\infty \leq \frac{1}{K} r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R}; \qquad K := \frac{2\sqrt{N(N-1)}|D(\mathcal{G})^T|}{\lambda_2(\mathcal{G})^2}$$

- Network remains connected for all times if

$$\underbrace{MP(\tilde{R})}_{\substack{\textit{worst case}\\ \textit{initial energy}}} \leq \underbrace{P(R)}_{\substack{\textit{minimum energy}\\ \textit{required to}\\ \textit{loose connectivity}}} \quad \& \quad (\#)$$

Connectivity Result

PROPOSITION 1

Consider the control law

$$u_i = -\sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i$$

and a constant $\delta > 0$. Assume that $r(\cdot)$, δ and the maximum initial distance $\tilde{R}$ satisfy the restrictions

$$\delta \leq \frac{1}{K} r(0)^2 \frac{s}{r(s)}, \forall s \geq \tilde{R}; \qquad K := \frac{2\sqrt{N(N-1)}|D(\mathcal{G})^T|}{\lambda_2(\mathcal{G})^2}$$

and

$$MP(\tilde{R}) \leq P(R)$$

Then the network remains connected for all $t \geq 0$ provided that the inputs terms v_i satisfy the bound

$$|v_i(t)| \leq \delta, \forall t \geq 0$$

Illustrative Controller Selection

- Recall that

$$u_i = -\sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i$$

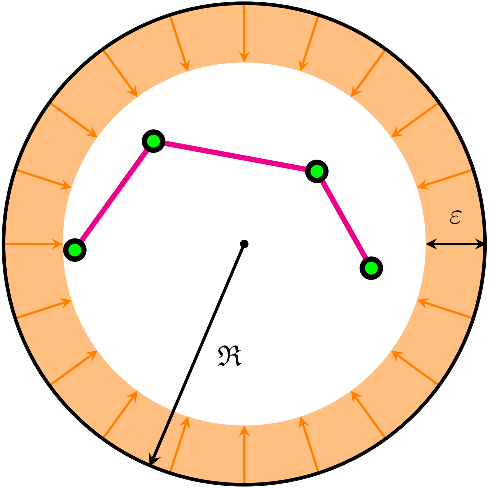
- Selection of a linear and a nonlinear control law providing the same bound on $|v|_\infty$
- Linear case

$$r(s) := 1, s \geq 0 \quad \& \quad \tilde{R} \leq \frac{1}{\sqrt{M}} R$$

- Nonlinear case

$$r(s) := \begin{cases} 1, & s \in [0, \tilde{R}] \\ \frac{s}{\tilde{R}}, & s \in (\tilde{R}, R] \\ \frac{R}{\tilde{R}}, & s \in (R, \infty) \end{cases} \quad \& \quad \tilde{R} \leq \left(\frac{2}{3M - 1} \right)^{\frac{1}{3}} R$$

Invariance for a Spherical Domain



Repulsive Dynamics

- Define the vector field $g : B(\mathfrak{R}) \to \mathbb{R}^n$ as

$$g(x) := \begin{cases} -c\delta^{\frac{\varepsilon+|x|-\mathfrak{R}}{\varepsilon}} \frac{x}{|x|}, & \text{if } x \in N_\varepsilon \\ 0, & \text{if } x \in D_\varepsilon \end{cases}$$

- Regions

$$N_\varepsilon := \{x \in \mathbb{R}^n : \mathfrak{R} - \varepsilon \leq |x| < \mathfrak{R}\} \quad D_\varepsilon := B(\mathfrak{R}) \setminus N_\varepsilon$$

- Control law

$$u_i = g(x_i) - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i$$

Derivative along System Trajectories

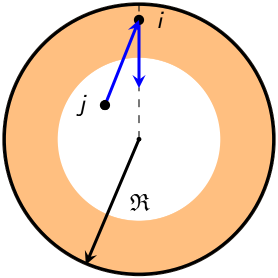
- Derivative of (same as before) energy function

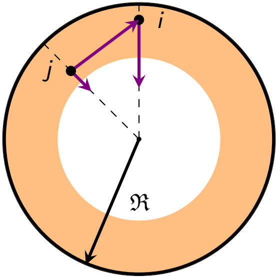
$$\dot{V} \leq \underbrace{\sum_{i=1}^{N} \frac{\partial}{\partial x_i} V(x) g(x_i)}_{\text{extra term}} - \sum_{l=1}^{n} c_l(x)^T L_w(x)^2 c_l(x) + \left| \sum_{l=1}^{n} c_l(x)^T L_w(x) c_l(v) \right|$$

- Focus on the extra term

$$\begin{aligned} &\sum_{i=1}^{N} \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|) \langle (x_i - x_j), g(x_i) \rangle \\ &= \sum_{\{i \in \mathcal{N} : x_i \in N_\varepsilon\}} \sum_{j \in \mathcal{N}_i^{D_\varepsilon}} r(|x_i - x_j|) \langle (x_i - x_j), g(x_i) \rangle \\ &+ \sum_{\{i,j\} \in \mathcal{E}^{N_\varepsilon}} r(|x_i - x_j|) [\langle (x_i - x_j), g(x_i) \rangle + \langle (x_j - x_i), g(x_j) \rangle] \end{aligned}$$

Sign of the Extra Terms





Invariance Result

THEOREM

Consider the control law

$$u_i = g(x_i) - \sum_{j \in \mathcal{N}_i} r(|x_i - x_j|)(x_i - x_j) + v_i$$

where

$$g(x) := \begin{cases} -c\delta \frac{\varepsilon + |x| - \mathfrak{R}}{\varepsilon} \frac{x}{|x|}, & \text{if } x \in N_\varepsilon \\ 0, & \text{if } x \in D_\varepsilon \end{cases}$$

with $c > 1$ and assume that $r(\cdot)$, the maximum initial distance $\tilde{R}$ and the bound δ on the inputs v_i satisfy the conditions in Proposition 1.

Then the solution of the closed loop system remains in D for all $t \geq 0$ and the network remains connected for all $t \geq 0$.