FEL3330 Networked and Multi-agent Control Systems
Lecture 4: Agreement Protocols 2

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Today's lecture

- Lyapunov approaches to agreement
- Directed and switching topologies
- Using the Edge Laplacian

Lyapunov approaches to agreement

- Reveal further structural issues of the network with respect to convergence
- Can handle time-variation, more general dynamics, robustness

Agreement over undirected graphs

- N agents with $\dot{x}_i = u_i, i \in V = \{1, \dots, N\}$
- Agreement algorithm: $u_i = -\sum_{j \in N_i} (x_i - x_j)$
- $\dot{x} = -Lx$, where L is the Laplacian matrix of the graph $G = (V, E)$
- Agreement set: subspace spanned by $\mathbf{1}$
- Recall that if G is undirected connected then

$$x \to \mathcal{A} = \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\},$$

and the agreement point is equal to the initial average of the agents.

Proof 3

- Use $V = \frac{1}{2}x^T x$ as a candidate Lyapunov function.
- Then from $\dot{V} = -x^T L x$ and LaSalle's invariance principle convergence to $\mathcal{A}$ follows.

Proof 4: using the Edge Laplacian

- Convergence proof using the $D^T D$ matrix
- Denote the vector x_e as the vector of edge differences given the prescribed orientation for D.
- We have $Lx = Dx_e$ and $x_e = D^T x$.
- The proof is based on the fact that $\dot{x} = -Lx$ implies $\dot{x}_e = -D^T D x_e$, and using $V = \frac{1}{2} x_e^T x_e$ as a Lyapunov function candidate.

Lyapunov analysis for balanced graphs

- Define $\delta_i = x_i - a$ for each $i \in V$.
- Decomposition of x: $x(t) = a\mathbf{1} + \delta(t)$ as in the undirected case.
- Disagreement dynamics: $\dot{\delta} = -L\delta$
- Mirror graph $(G, \hat{E}, \hat{w}) : \hat{E} = E \bigcup \tilde{E}$ where $\tilde{E}$ same edges with reversed order and $\hat{w}_{ij} = \hat{w}_{ji} = \frac{w_{ij}+w_{ji}}{2}$

Lyapunov analysis for balanced graphs

- $L_s = \frac{L+L^T}{2}$ is a valid weighted Laplacian for $\hat{G}$, ie, $L_s = L(\hat{G})$ iff G is balanced.
- Then we have $x^T L x = x^T L_s x$ for all x.
- Using $V = \frac{1}{2}\delta^T \delta$ as a Lyapunov function candidate it can be shown that
$$||\delta(t)|| \leq ||\delta(0)||e^{-\lambda_2(\hat{G})t}$$
- Natural extension to switching between all possible weakly connected and balanced graphs.

General directed graphs

- We have seen that an iff condition for consensus is that G contains a rooted out-branching.
- Lyapunov arguments rely on properties of stochastic matrices: non-negative matrices with row sums equal to one.
- For all directed graphs G and $\delta > 0$, $e^{-\delta L}$ is stochastic and $(e^{-\delta L})_{ij} > 0$ if and only if $i = j$ or there is a directed path from j to i in G.
- Corollary: G has a rooted out-branching if and only if, for any $\delta > 0$, $e^{-\delta L}$ has at least one column with all positive elements.

Lyapunov function for general directed graphs

- Examine sampled-data version of closed loop system: $z(k+1) = e^{-\delta L} z(k)$, where $z(k) = x(k\delta), k \in \mathbb{N}$.
- Pick $V(z) = \max_i z_i - \min_i z_i$.
- Since (for the case of rooted out-branching) there exists one column with strictly positive elements, V is strictly decreasing as long as all states are not equal.
- V is thus a strong discrete-time Lyapunov function and vanishes at the agreement subspace.

Extension to switching graphs

- Loss of connectivity
- If there exists a $T = m\delta < \infty$, $m \in \mathbb{N}$ such that on every interval of the form $\{\tau, \tau + T\}, \tau = l\delta$, $\forall l \in \mathbb{N}$, the union

$$\bigcup_{t=\tau}^{t=\tau+T} G(t)$$

contains a rooted out-branching then convergence to agreement is achieved.

Edge agreement

- Recall $\dot{x}_e = -D^T D x_e = -L_e x_e$: edge dynamics.
- Desired equilibrium is the origin $x_e = 0$.
- Recall also the property associating tree structure with positive definiteness of $D^T D$. What is the role of cycles?

Role of cycles in edge agreement

- Any connected graph G contains a spanning tree.
- Consider $D = \begin{bmatrix} D_T & D_C \end{bmatrix}$ where D_T corresponds to edges of the spanning tree and D_C to the rest (edges completing its cycles).
- Then $L = DD^T = D_T D_T^T + D_C D_C^T$ and

$$D^T D = \begin{bmatrix} D_T^T \\ D_C^T \end{bmatrix} \begin{bmatrix} D_T & D_C \end{bmatrix} = \begin{bmatrix} D_T^T D_T & D_T^T D_C \\ D_C^T D_T & D_C^T D_C \end{bmatrix}$$

- The edges are also decomposed as $x_e = [x_T^T \quad x_C^T]^T$.

Role of cycles in edge agreement

- There exists a matrix R such that $D^T D = R^T D_T^T D_T R$. Can be shown that $R = [I \quad T]$ where $T = (D_T^T D_T)^{-1} D_T^T D_C$.
- Then $\dot{x}_e = -D^T D x_e = -L_e x_e$ is equivalent to $R^T \dot{x}_T = -R^T D_T^T D_T R R^T x_T$.
- Edge agreement is described by

$$\dot{x}_T = -D_T^T D_T R R^T x_T$$

and

$$x_c = T^T x_T$$

Next Lecture

Formation Control 1

- Position based formations
- Formation infeasibility
- Flocking behavior