FEL3330 Networked and Multi-agent Control Systems
Lecture 3: Agreement Protocols 1

September 7, 2016

Today's lecture

- Definition of agreement
- Agreement for undirected static graphs
- Graph theoretic issues for directed graphs
- Agreement for directed static graphs

Consensus/Agreement/Rendezvous

- Consider the multiple sensor network measuring temperature in the first lecture
- Goal: reach the best estimate of temperature given the available limited information
- It seems that the average of all measurements is the optimal choice
- Problem: design an algorithm that converges to the average for all sensors given *relative* information exchange

Agreement over static graphs

- N agents with $\dot{x}_i = u_i, i \in V = \{1, \ldots, N\}$
- Neighboring set: $N_i = \{j \in V | (i, j) \in E\}$
- Available information for i: $(x_i - x_j), j \in N_i$
- Agreement algorithm: $u_i = -\sum_{j \in N_i} (x_i - x_j)$
- $\dot{x} = -Lx$, where L is the Laplacian matrix of the graph $G = (V, E)$
- Agreement set: subspace spanned by $\mathbf{1}$

$$\mathcal{A} = \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\}$$

Agreement over undirected graphs

Theorem: If G is undirected connected then

$$x \to \mathcal{A} = \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\},$$

and the agreement point is equal to the initial average of the agents. Moreover, the worst-case convergence rate is equal to $\lambda_2(G)$.

Proof 1

- Using the spectral factorization of L: $L = U \Lambda U^T$ where $U = [u_1, \ldots, u_N]^T$ consists of orthonormal set of eigenvectors of L and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$
- Solution of $\dot{x} = -Lx$ is $x(t) = e^{-Lt} x_0$
- Using $L = U \Lambda U^T$ we get $x(t) = \sum_{i=1}^{N} e^{-\lambda_i t} (u_i^T x_0) u_i$
- Proof follows by using $\lambda_1 = 0$ and $\lambda_i > 0, i = 2, \ldots, N$ for connected undirected graphs.

Convergence point

- $x(t) \to \frac{\mathbf{1}^T x_0}{N}\mathbf{1}$
- Let $a(t) = \frac{\mathbf{1}^T x(t)}{N} = \frac{1}{N}\sum_i x_i(t)$ denote the average of the agents' states.
- Then:

$$\dot{a} = \frac{1}{N}\sum_i \dot{x}_i = -\frac{1}{N}\sum_i \sum_{j \in N_i} (x_i(t) - x_j(t)) = 0$$

- Then $a(t) = a(0) = \dfrac{1}{N}\sum_i x_i(0) \equiv a$

Proof 2

- Define $\delta_i = x_i - a$ for each $i \in V$.
- Decomposition of x: $x(t) = a\mathbf{1} + \delta(t)$
- Disagreement dynamics: $\dot{\delta} = -L\delta$
- Using $V = \frac{1}{2}\delta^T\delta$ as a Lyapunov function candidate it can be shown that

$$||\delta(t)|| \leq ||\delta(0)||e^{-\lambda_2 t}$$

Minimal requirements

- Spanning tree: subgraph of G with the same number of vertices that is also a tree
- G containing a spanning tree is a necessary and sufficient condition for average agreement

Switching graphs

- But what happens when the communication graph loses/adds edges over time?
- Time-varying sets: $N_i(t) = \{j \in V | (i, j) \in E(t)\}$
- $u_i = - \sum_{j \in N_i(t)} (x_i - x_j)$
- $s(t) : \mathbb{R}_{\geq 0} \to I_{G_c^N}$ switching signal
- G_c^N: all connected graphs with N vertices

Switching graphs

- We have $\dot{x}(t) = -L(G_{s(t)})x(t)$ and $a(t) = a(0) = \frac{1}{N}\sum_i x_i(0) = a$
- Then $\dot{\delta} = -L(G_{s(t)})\delta$ and $V = \frac{1}{2}\delta^T\delta$ can be used as a common Lyapunov function for the switched system
- It can be shown that

$$||\delta(t)|| \leq ||\delta(0)||e^{-\min_{k \in I_{G_c^N}} \{\lambda_2(G_k)\}t}$$

Directed graphs

- A directed graph is a *rooted out-branching* (or directed spanning tree) if (1) it does not contain a directed cycle and (2) there exists a vertex v_r (the root) such that there exists a directed path from v_r to every other vertex.
- A directed graph with N vertices contains a rooted out-branching if and only if $\mathrm{rank} L(G) = N - 1$. Then the null space of $L(G)$ is spanned by $\mathbf{1}$.
- Proof uses weighted directed graph version of matrix-tree theorem. Other proofs also exist.

Spectrum of directed graphs

- Let $d_{max}^{in} = \max_i d_{in}(v_i)$. Then the eigenvalues of L lie in the region $\{z \in \mathbb{C} : \|z - d_{max}^{in}\| \leq d_{max}^{in}\}$.
- Direct consequence of Gersgorin's disk theorem.
- Therefore all eigenvalues of L have non-negative real-parts.

Agreement for directed graphs

- Theorem: For a weighted directed graph G, $x \to \mathcal{A}$ from all initial conditions if and only if G contains a rooted out-branching.
- Proof uses Jordan decomposition of L along with the fact that the (algebraic and geometric) multiplicity of zero eigenvalue is one.
- It can be shown that $x(t) \to (p_1 q_1^T)x_0$ where p_1, q_1 are right and left eigenvectors of L corresponding to the zero eigenvalue with $p_1^T q_1 = 1$. Choosing $p_1 = \mathbf{1}$, $x(t) \to (q_1^T x_0)\mathbf{1}$ where $q_1^T \mathbf{1}$=1.
- $q^T x(t)$ remains invariant, where q is the left eigenvector of L wrt 0 eigenvalue.

What about average consensus

- Has been shown to hold for *balanced* graphs:

$$\sum_i w(i,j) = \sum_i w(j,i)$$

  for all $i \in V$ which are weakly connected.
- Balanced graphs: in-degree and out-degree are equal.
- $\mathbf{1}^T L = L\mathbf{1} = 0$
- Then $(p_1 q_1^T)x_0$ coincides with initial average.
- Proof relies on the fact that weakly connected and balanced implies strong connectivity and thus existence of a rooted out-branching.
- Connection to disagreement dynamics analysis (next lecture).

Next Lecture

Agreement Protocols 2

- Lyapunov based approaches
- More on switching graphs
- Agreement using the edge Laplacian