FEL3330 Networked and Multi-Agent Control Systems

Lecture 3: Elements of consensus/agreement algorithms

- Definition of consensus/rendezvous
- Consensus for undirected fixed and switching graphs
- Graph theoretic issues for directed graphs
- Consensus for directed fixed graphs

Consensus/Agreement/Rendezvous

- Consider the multiple sensor network measuring temperature in the first lecture
- Goal: reach the best estimate of temperature given the available limited information
- It seems that the average of all measurements is the optimal choice
- Problem: design an algorithm that converges to the average for all sensors given *relative* information exchange

Agreement algorithms for static graphs

- N agents with $\dot{x}_i = u_i, i \in V = \{1, \ldots, N\}$
- Neighboring set: $N_i = \{j \in V | (i,j) \in E\}$
- Available information for i: $(x_i - x_j), j \in N_i$
- Agreement algorithm: $u_i = -\sum_{j \in N_i} (x_i - x_j)$
- $\dot{x} = -Lx$, where L is the Laplacian matrix of the graph $G = (V, E)$

Convergence and Performance

Theorem: If G is undirected connected then

$$x \to \mathcal{A} \triangleq \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\},$$

and the agreement point is equal to the initial average of the agents. Moreover, the worst-case convergence rate is equal to $\lambda_2(G)$.

Proofs

- Proof 1: Convergence proof using the $B^T B$ matrix
- Denote $\bar{x}$ the vector of edge differences.
- We have $Lx = B\bar{x}$ and $\bar{x} = B^T x$
- The proof is based on the fact that $\dot{x} = -Lx$ implies $\dot{\bar{x}} = -B^T B\bar{x}$, and using $V = \frac{1}{2}\bar{x}^T\bar{x}$ as a Lyapunov function candidate.

Proofs

- Proof 2: Performance analysis using the disagreement vector

- Let $a(t) = \frac{1}{N} \sum_i x_i(t)$ denote the average of the agents' states. Then:

$$\dot{a} = \frac{1}{N} \sum_i \dot{x}_i = -\frac{1}{N} \sum_i \sum_{j \in N_i} (x_i(t) - x_j(t)) = 0$$

- Then $a(t) = a(0) = \dfrac{1}{N} \sum_i x_i(0) \equiv a$

Disagreement dynamics

- Define $\delta_i = x_i - a$ for each $i \in V$.
- Decomposition of x: $x(t) = a\mathbf{1} + \delta(t)$
- Disagreement dynamics: $\dot{\delta} = -L\delta$
- Using $V = \frac{1}{2}\delta^T\delta$ as a Lyapunov function candidate it can be shown that

$$||\delta(t)|| \leq ||\delta(0)||e^{-\lambda_2 t}$$

Switching graphs

- But what happens when the communication graph loses/adds edges over time?
- Time-varying sets: $N_i(t) = \{j \in V | (i,j) \in E(t)\}$
- $u_i = - \sum_{j \in N_i(t)} (x_i - x_j)$
- $s(t) : \mathbb{R}_{\geq 0} \to I_{G_c^N}$ switching signal
- G_c^N: all connected graphs with N vertices

Switching graphs

- We have $\dot{x}(t) = -L(G_{s(t)})x(t)$ and $a(t) = a(0) = \frac{1}{N}\sum_i x_i(0) = a$

- Then $\dot{\delta} = -L(G_{s(t)})\delta$ and $V = \frac{1}{2}\delta^T\delta$ can be used as a common Lyapunov function for the switched system

- It can be shown that

$$||\delta(t)|| \leq ||\delta(0)||e^{-\min\limits_{k \in I_{G_c^N}} \{\lambda_2(G_k)\}t}$$

(Fixed) Directed graphs

- What happens when the edges are directed?
- Neighboring set: $N_i = \{j \in V | (i,j) \in E\}$
- Adjacency matrix (simple graph)

$$A = A(G) = [a_{ij}], a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E, \\ 0 & \text{if otherwise.} \end{cases}$$

- In general $a_{ij} \neq a_{ji}$ in this case!
- Degree matrix

$$\Delta = \Delta(G) = \mathrm{diag}(d_1, \ldots, d_N), d_i = \sum_j a_{ij} = |N_i|$$

(Fixed) Directed graphs

- Strongly connected digraph: any two nodes can be connected through a path that follows the direction of edges of the graph.
- Directed tree: a digraph where every vertex is a tail to exactly one edge, except of the "root" vertex.
- A spanning tree of a graph is a directed tree formed by graph edges that connect all the vertices of the graph.

(Fixed) Directed graphs

- First result: If G is strongly connected then $\lambda_1 = 0$ is simple with corresponding eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Second result: $\lambda_1 = 0$ is simple iff G contains a spanning tree.

Convergence to agreement set

- First result: If G is strongly connected then

$$x \to \mathcal{A} \triangleq \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\},$$

globally asymptotically.

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$$x \to \mathcal{A} \triangleq \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\}$$

globally asymptotically iff G contains a spanning tree.

What about average consensus?

- Has been shown to hold for *balanced* graphs:

$$\sum_i a_{ij} = \sum_i a_{ji}$$

for all $i \in V$.

- Theorem: Assume G is strongly connected. Then

$$x \to \mathcal{A} \triangleq \{x \in \mathbb{R}^N | x_i = x_j, \forall i, j\},$$

and the agreement point is equal to the initial average of the agents iff G is balanced.

Next Lecture

Communication constraints

- Connectivity
- Quantization
- Time-delays
- Event-triggered control