FEL3330 Networked and Multi-agent Control Systems
Lecture 2: Graphs and Matrices

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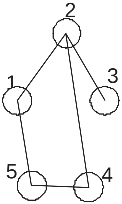
Graph theoretic approach

- Limitations in communication/sensing do now allow each agent to communicate with everyone else
- Modelling of limitations through graphs
- Graph based abstractions: do not include exact information of what is shared or communication protocol
- Give high level description of how agents (vertices) interact through edges (pairs of vertices)

Graph theoretic approach

- Finite, undirected and simple graphs (abbr. "graphs")



$G = (V, E)$

- Agents are the vertices $V = V(G) = \{1, \ldots, N\}$
- Alternative notation: $V = V(G) = \{v_1, \ldots, v_N\}$
- Edges $E = E(G) \subset V \times V$ are pairs of agents that can communicate (are adjacent)
- Notation: $(i, j) \in E \Leftrightarrow i \sim j$
- Undirected: $(i, j) \in E \Leftrightarrow (j, i) \in E$

Paths and cycles

- Neighboring set: $N_i(= N(i)) = \{j \in V | (i, j) \in E\}$
- Path of length m in G: sequence of distinct vertices $i_0, i_1, \ldots, i_m$ s.t. $(i_k, i_{k+1}) \in E, \forall k = 0, 1, \ldots, m-1$.
- A path is a cycle when $i_0 = i_m$ and all other vertices are distinct.
- Forest: a graph with no cycles

Connectedness

- G is connected when there is a path between any pair of its vertices.
- Otherwise it is called disconnected.
- Connected components: elements of minimal partitioning of a graph s.t. each element is connected.
- Connected graphs have one connected component. Disconnected have more than one.
- Connected forest is called a tree.
- Interesting cases: path graphs, complete graphs, cycle graphs, star graphs.

Weighted graphs and path length

- Weighted graphs: $w : E \to \mathbb{R}$ associates a weight to each edge. Notation: $G = (V, E, w)$.
- Length of a path: sum of all weights of edges through the path.
- Can use shortest path algorithms for each pair of agents.

Directed graphs

- Assigns orientation to edge set E. Is also called digraph.
- $(v_i, v_j) \in E$ is now an ordered pair with v_i being the head and v_j the tail.
- Previous notions can be extended this case.
- Strong connectedness: there exists a directed path between any pair of vertices.
- Weak connectedness: it is connected when viewed as disoriented graph, ie, without assigning orientations to edges.
- Show examples of digraphs that are weakly but not strongly connected.

The adjacency matrix and the degree matrix

- We want to associate matrices with (undirected) graphs.
- Neighboring set: $N_i = \{j \in V | (i, j) \in E\}$
- $d_i (= d(i))$ denotes the number of adjancent vertices to i, ie, cardinality $|N_i|$ of the set N_i.
- Adjacency matrix (undirected, simple graph)

$$A = A(G) = [a_{ij}], a_{ij} = \begin{cases} 1 & \text{if } (i, j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

- Degree matrix

$$\Delta = \Delta(G) = \mathrm{diag}(d_1, \ldots, d_N), d_i = \sum_j a_{ij} = |N_i|$$

The incidence matrix

- Orientation of G: assignment of direction to each edge.
- Incidence matrix of oriented graph with $M = |E|$ edges, labeled as $E = \{e_1, \ldots, e_M\}$:

$$D(= D(G)) = [d_{ij}], d_{ij} = \begin{cases} 1 & \text{if } i \text{ is the head of } e_j, \\ -1 & \text{if } i \text{ is the tail of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

- Incidence matrix of digraph defined based on the given orientation.

The Laplacian matrix and its eigenvalues

- $L = L(G) = \Delta(G) - A(G)$.
- Alternative definition: $L = DD^T$, independent of orientation.
- Symmetric and positive semi-definite matrix.
- Eigenvalues $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_N(G)$
- For a connected G, $L(G)$ has a simple zero eigenvalue with the corresponding eigenvector $\mathbf{1} = [1, \ldots, 1]^T$. Equivalent condition.
- Thus $\lambda_2(G) > 0$ for a connected graph.

Laplacians for directed and/or weighted graphs

- Weighted graphs: $W = \mathrm{diag}(w(e_1), \ldots, w(e_M))$.
- Weighted graph Laplacian: $L_w(G) = D(G)WD(G)^T$. Equivalent to starting from weighted versions of adjacency and degree matrix.
- Directed weighted graphs. Need to cope with asymmetric features.
- Weighted in-degree of vertex i: $d_{in}(v_i) = \sum_{\{j|(j,i)\in E\}} w(j,i)$. How much agent i is influenced by its neighbors.
- Adjanceny matrix:

$$A = A(G) = [a_{ij}], a_{ij} = \begin{cases} w(j,i) & \text{if } (j,i) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Laplacians for directed and/or weighted graphs, ctd.

- Degree matrix:

$$\Delta = \Delta(G) = \mathrm{diag}(d_{in}(v_1), \ldots, d_{in}(v_N))$$

- $L = L(G) = \Delta(G) - A(G)$.
- Again, matrix with zero row sums.
- In all cases $\mathbf{1} \in \mathcal{N}(L)$, where $\mathbf{1}$ is a vector of ones and $\mathcal{N}(L)$ is the null space of L.

Edge Laplacian

- For an undirected graph, $L_e = D^T D$ is called edge Laplacian.
- If G is a tree, then $D^T D$ is positive definite.
- Thus if G is a tree, then $\lambda_{\min}(D^T D) > 0$.
- Proven using the cycle space of G, which (can be shown to be) equivalent to the null space of D. See lecture notes for more details.

Eigenvalue bounds 1

Two important relations resulting from the symmetry of L and the variational characterization of the eigenvalues of symmetric matrices are as follows:

$$\lambda_2(G) = \min_{x \perp \mathbf{1}, ||x||=1} x^T L x$$

and

$$\lambda_N(G) = \max_{||x||=1} x^T L x$$

Eigenvalue bounds 2: Cheeger's inequality

- Let $S \subset V$ and $S^C = V \setminus S$.
- $\varepsilon(S) = \text{card}\{(i,j) \in E | (i \in S, j \in S^C) \vee (i \in S^C, j \in S)\}$
- $\varepsilon(S)$ is # edges needed to be cut to separate S from S^C
- $\phi(S) = \frac{\varepsilon(S)}{\min\{|S|,|S^C|\}}$: represents the "cut-ratio" for the case that the smallest set of agents that is cut is lost from the network
- Isoperimetric number of G: $\phi(G) = \min_{S \in 2^V}\{\phi(S)\}$: what is the worst case of number of losing vertices vs. how many edges need to be cut. Measure of network robustness.
- Cheeger's inequality: $\phi(G) \geq \lambda_2(G) \geq \frac{\phi(G)^2}{2\max_{i \in V}\{d_i\}}$
- $\phi(G)$ and thus $\lambda_2(G)$ are a metric of connectivity of the graph.

Next Lecture

Agreement Protocolls 1

- State agreement definition
- Agreement for static and undirected graphs
- Directed graphs