# Decentralized Feedback Stabilization and Collision Avoidance of Multiple Agents

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**Abstract**

The navigation function methodology, established in previous work for centralized multiple robot navigation, is extended for decentralized navigation. In contrast to the centralized case, each agent plans its actions without knowing the destinations of the other agents. Asymptotic stability is guaranteed by the existence of a global Lyapunov function for the whole system, which is actually the sum of the separate navigation functions. The collision avoidance and global convergence properties are verified through simulations.

## 1 Introduction

Multi-agent Navigation is a field that has recently gained increasing attention both in the robotics and the control communities, due to the need for autonomous control of more than one mobile robotic agents in the same workspace. While most efforts in the past had focused on centralized planning, specific real-world applications have lead researchers throughout the globe to turn their attention to decentralized concepts. The basic motivation for this work comes from two application domains: (i) decentralized conflict resolution in air traffic management ([12]) and (ii) the field of micro robotics ([26],[14]), where a team of autonomous micro robots must cooperate to achieve manipulation precision in the sub micron level.

Decentralized navigation approaches are more appealing to centralized ones, due to their reduced computational complexity and increased robustness with respect to agent failures. The main focus of work in this domain has been cooperative and formation control of multiple agents, where so much effort has been devoted to the design of systems with variable degree of autonomy ([7],[9],[28], [36]). There have been many different approaches to the decentralized motion planning problem. Open loop approaches use game theoretic and optimal control theory to solve the problem taking the constraints of vehicle motion into account; see for example [2],[5], [13],[30], [34], [35]. On the other hand, closed loop approaches use tools from classical Lyapunov theory and graph theory to design control laws and achieve the convergence of the distributed system to a desired configuration both in the concept of cooperative ([6], [17], [18], [24]) and formation control ([1], [3], [8], [19], [29], [33]). A few approaches use computer science based tools to treat the problem;see for example [11], [22], [23]. However, the latter fail to guarantee convergence of the multi-agent system.

Closed loop strategies are apparently preferable to open loop ones, mainly because they provide robustness with respect to modelling uncertainties and agent failures and guaranteed convergence to the desired configurations. However, a common point of most work in this area is devoted to the case of point agents. Although this allows for variable degree of decentralization, it is far from realistic in real world applications. For example, in conflict resolution in Air Traffic Management, two aircraft are not allowed to approach each other closer than a specific "alert" distance. The construction of closed loop methods for distributed non-point multi-agent systems is both evident and appealing.

A closed loop approach for single robot navigation was proposed by Koditschek and Rimon [15] in their seminal work. This navigation functions' framework had all the sought qualities but could only handle single, point-sized, robot navigation. In [20] this method was successfully extended to take into account the volume of each robot while a decentralized version of this work has been presented by the authors in [4].

In this paper we make the following assumptions:

- Each agent has global knowledge of the position of the others at each time instant.
- Each agent has knowledge only of its own desired destination but not of the others.
- We consider spherical agents.
- The workspace is bounded and spherical.
- The dynamics of each agent are holonomic.

Our assumption regarding the spherical shape of the agents does not constrain the generality of this work since it has been proven that navigation properties are invariant under diffeomorphisms ([15]). Arbitrarily shaped agents diffeomorphic to spheres can be taken into account. Methods for constructing analytic diffeomorphisms are discussed in [32] for point agents and in [31] for rigid body agents.

The second assumption makes the problem decentralized. Clearly, in the centralized case a central authority has knowledge of everyones goals and positions at each time instant and it coordinates the whole team so that the desired specifications (destination convergence and collision avoidance) are fulfilled. In the current situation no such authority exists and we have to deal with the limited knowledge of each agent. This is of course the first step towards a variable degree of decentralization. This paper presents the first to the authors knowledge extension of centralized multi-agent control using navigation functions, to a decentralized scheme.

An extension of this work to nonholonomic agents has been provided in [21].

The rest of the paper is organized as follows: The rest of the paper is organized as follows: section II presents the system definition and problem statement. Section III outlines the concept of navigation functions and describes their extension to the decentralized case to obtain the feedback control law. In section IV simulation results are presented for a number of non-trivial multi agent navigational tasks. Section V summarizes the conclusions and indicates our current research. The proofs of the technical Propositions of section III are provided in the Appendix.

## 2 System and Problem Definition

Consider a system of $N$ agents operating in the same workspace $W \subset \mathcal{R}^2$. Each agent $i$ occupies a disc: $R = \{q \in \mathcal{R}^2 : \| q - q_i \| \leq r_i\}$ in the workspace where $q_i \in \mathcal{R}^2$ is the center of the disc and $r_i$ is the radius of the agent. The configuration space is spanned by $q = [q_1, \ldots, q_N]^T$. The motion of each agent are described by the single integrator:

$$\dot{q}_i = u_i, i \in \mathcal{N} = [1, \ldots, N] \tag{1}$$

The desired destinations of the agents are denoted by the index $d$: $q_d = [q_{d1}, \ldots, q_{dN}]^T$ The following figure shows a three-agent conflict situation:

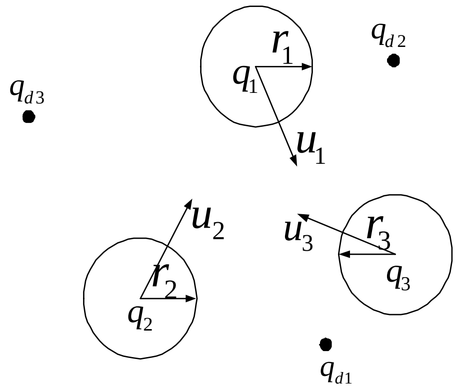


Figure 1: A conflict scenario with three agents.

The multi agent navigation problem treated in this paper can be stated as follows: " *Derive a set of control laws (one for each agent) that drives the team of agents from any initial configuration to a desired goal configuration avoiding, at the same time, collisions. Each agent has global knowledge of the team configuration but is unaware of the other agents desired destinations* ".

## 3 Decentralized Navigation Functions(DNF's)

### 3.1 Preliminaries

In this section we review the navigation function method introduced in the seminal paper of Koditscheck and Rimon [15] for single point robot navigation.

Navigation functions (NF's) are real valued maps realized through cost functions $\varphi(q)$, whose negated gradient field is attractive towards the goal configuration and repulsive wrt obstacles. It has been shown by Koditscheck and Rimon that strict global navigation (i.e. the system $\dot{q} = u$ under a control law of the form $u = -\nabla\varphi$ admits a globally attracting equilibrium state) is not possible, and a smooth vector field on any sphere world with a unique attractor, must have at least as many saddles as obstacles [15]. Our assumption about spherical agents and obstacles does not constrain the generality of this work since it has been proven that navigation properties are invariant under diffeomorphisms.

A navigation function can be defined as follows:

**Definition 1 [15]**:*Let $F \subset \mathcal{R}^{2N}$ be a compact connected analytic manifold with boundary. A map $\varphi : F \rightarrow [0,1]$ is a navigation function if:(1) it is analytic on $F$, (2) it has only one minimum at $q_d \in int(F)$, (3) its Hessian at all critical points (zero gradient vector field) is full rank, and (4)* $\lim_{q \to \vartheta F} \varphi(q) = 1$.

Strictly speaking, the continuity requirements for the navigation functions are to be $C^2$. The property 1 of Definition 1 follows the intuition provided by the authors of [15], that is preferable to use closed form mathematical expressions to encode actuator commands instead of "patching together" closed form expressions on different portions of space, so as to avoid branching and looping in the control algorithm. Analytic navigation functions, through their gradient provide a direct way to calculate the actuator commands, and once constructed they provide a provably correct control algorithm for every environment that can be diffeomorphically transformed to a sphere world. In this paper, we further relax this requirement by using a non-analytic, merely $C^1$ navigation function. The discontinuity however, takes place outside of the region where critical points of the potential function occur, so it does not affect the navigation properties of the proposed function.

A function $\varphi$ that has a unique minimum on $F$ is called *polar*. By using a polar function on a compact connected manifold with boundary, all initial conditions will either be brought to a saddle point or to the unique minimum of the function.

A scalar valued function $\varphi$ whose Hessian at all critical points is full rank is called *Morse*. The corresponding critical points are called *non-degenerate*. The requirement in Definition 1 that a navigation function must be a Morse function, establishes that the initial conditions that bring the system to saddle points are sets of measure zero [27]. In view of this property, all initial conditions away from sets of measure zero are brought to the unique minimum.

The last property of Definition 1 guarantees that the resulting vector field is transverse to the boundary of $F$. This establishes that the system will be safely brought to $qd$, avoiding collisions.

## 3.2 DNF's vs MRNF's

In [20], the navigation functions method has been extended to the case of multiple mobile robots with the use of Multi-Robot navigation functions (MRNF's).

In the form of a centralized setup [20], where a central authority has knowledge of the current positions and desired destinations of all agents, the sought control law is of the form: $u = -K\nabla\varphi(q)$ where $K$ is a gain. In the decentralized case addressed in this work, each agent has knowledge of only the current positions of the others, and not of their desired destinations. Hence each agent has a different navigation law.

Following the procedure of [15],[20], we consider the following class of decentralized navigation functions(*DNF's*):

$$\varphi_i \triangleq \sigma_d \circ \sigma \circ \hat{\varphi}_i = \left(\frac{\gamma_i}{\gamma_i + G_i}\right)^{1/k} \tag{2}$$

which is a composition of $\sigma_d \triangleq x^{1/k}$, $\sigma \triangleq \frac{x}{1+x}$ and the cost function $\hat{\varphi}_i \triangleq \frac{\gamma_i}{G_i}$,where $\gamma_i^{-1}(0)$ denotes the desirable set(i.e. the goal configuration) and $G_i^{-1}(0)$ the set that we want to avoid(i.e. collisions with other agents).A suitable choice is:

$$\gamma_i = (\gamma_{di} + f_i)^k \tag{3}$$

where $\gamma_{di} = \| q_i - q_{di} \|^2$, is the squared metric of the current agent's configuration $q_i$ from its destination $q_{di}$. The definition of the function $f_i$ will be given later. Function $G_i$ has as arguments the coordinates of all agents, i.e. $G_i = G_i(q)$, in order to express all possible collisions of agent $i$ with the others. The proposed navigation function for agent i, wrt that proposed in [20] is

$$\varphi_i(q) = \frac{\gamma_{di} + f_i}{\left((\gamma_{di} + f_i)^k + G_i\right)^{1/k}} \tag{4}$$

By using the notation $\tilde{q}_i \triangleq [q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_N]^T$, the decentralized NF can be rewritten as

$$\varphi_i = \varphi_i(q_i, \tilde{q}_i) = \varphi_i(q_i, t)$$

that is, the potential function in hand contains a *time-varying* element which corresponds to the movement in time of all the other agents apart from $i$. This element is neglected in the case of a single agent moving in an environment of static obstacles ([15]), but in this case the term $\frac{\partial \varphi_i}{\partial t}$ is nonzero.

## 3.3 Control Strategy

The proposed feedback control strategy for agent $i$ is defined as

$$u_i = -K_i \frac{\partial \varphi_i}{\partial q_i} \tag{5}$$

where $K_i > 0$ a positive gain.

## 3.4 Construction of the $G$ function

In the proposed decentralized control law, each agent has a different $G_i$ which represents its relative position with all the other agents. In contrast to the centralized case, in which a central authority has global knowledge of the positions and desired destinations of the whole team and plans a global $G$ function accordingly, in the decentralized case, each member $i$ of the team has its own $G_i$ function, which encodes the different proximity relations with the rest. The main difference of the DNF's and the MRNF's in [20] from the NF's introduced in [15] lies in the structure of the function $G$. While there were attempts to prove convergence and collision avoidance to the straightforward extension of [15] to the multiple moving agents case, only collision avoidance properties were established. Furthermore simulation results motivated us to consider a different approach to [20] for the decentralized setup.

We review now the construction of the "collision" function $G_i$ for each agent $i$. The "Proximity Function"between agents $i$ and $j$ is given by

$$\beta_{ij} = \|q_i - q_j\|^2 - (r_i + r_j)^2 \tag{6}$$

Consider now the situation in figure 2. There are 5 agents and we proceed to define the function $G_R$ for agent $R$.

**Definition 2**: *A* relation *with respect to agent $R$ is every possible collision*

*scheme that can occur in a multiple agents scene with respect R.*
**Definition 3**: *A* binary relation *with respect to agent R is a relation between agent R and another.*
**Definition 4**: *The* relation level *in the number of binary relations in a relation.*

We denote by $(R_j)_l$ the $j$th relation of level-$l$ with respect to agent $R$. With this terminology in hand, the collision scheme of figure (2a) is a level-1 relation (one binary relation) and that of figure (2b) is a level-3 relation (three binary relations), always with respect to the specific agent $R$. We use the notation

$$(R_j)_l = \{\{R, A\}, \{R, B\}, \{R, C\}, \ldots\}$$

to denote the set of binary relations in a relation with respect to agent $R$, where $\{A, B, C, ...\}$ the set of agents that participate in the specific relation. For example, in figure 1b:

$$(R_1)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_3\}\}$$

where we have set arbitrarily $j = 1$.

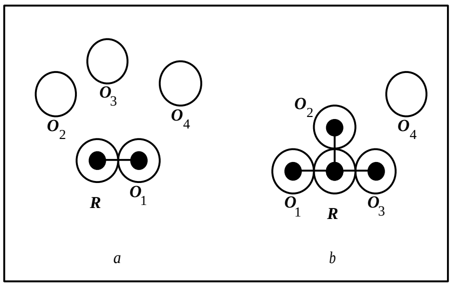


Figure 2: Part $a$ represents a level-1 relation and part $b$ a level-3 relation wrt agent $R$.

The complementary set $(R_j^C)_l$ of relation $j$ is the set that contains all the relations of the same level apart from the specific relation $j$. For example in figure 1b:

$$\left(R_1^C\right)_3 = \{(R_2)_3, (R_3)_3, (R_4)_3\}$$

where

$$(R_2)_3 = \{\{R, O_1\}, \{R, O_2\}, \{R, O_4\}\}$$
$$(R_3)_3 = \{\{R, O_1\}, \{R, O_3\}, \{R, O_4\}\}$$

$$(R_4)_3 = \{\{R, O_2\}, \{R, O_3\}, \{R, O_4\}\}$$

A "Relation Proximity Function" (RPF) provides a measure of the distance between agent $i$ and the other agents involved in the relation. Each relation has its own RPF. Let $R_k$ denote the $k^{th}$ relation of level $l$. The RPF of this relation is given by:

$$(b_{R_k})_l = \sum_{j\in(R_k)_l} \beta_{\{R,j\}} \tag{7}$$

where the notation $j \in (R_k)_l$ is used to denote the agents that participate in the specific relation of agent $R$. In the proofs, we also use the simplified notation $b_r = \sum_{j\in P_r} \beta_{ij}$ for simplicity, where $r$ denotes a relation and $P_r$ denotes the set of agents participating in the specific relation wrt agent $i$.

For example, in the relation of figure (2b) we have

$$(b_{R_1})_3 = \sum_{m\in(R_1)_3} \beta_{\{R,m\}} = \beta_{\{R,O_1\}} + \beta_{\{R,O_2\}} + \beta_{\{R,O_3\}}$$

A "Relation Verification Function" (RVF) is defined by:

$$(g_{R_k})_l = (b_{R_k})_l + \frac{\lambda(b_{R_k})_l}{(b_{R_k})_l + (B_{R_k^C})_l^{1/h}} \tag{8}$$

where $\lambda, h$ are positive scalars and

$$(B_{R_k^C})_l = \prod_{m\in(R_k^C)_l} (b_m)_l$$

where as previously defined, $(R_k^C)_l$ is the complementary set of relations of level-$l$, i.e. all the other relations with respect to agent $i$ that have the same number of binary relations with the relation $R_k$. Continuing with the previous example we could compute, for instance,

$$\left(B_{R_1^C}\right)_3 = (b_{R_2})_3 \cdot (b_{R_3})_3 \cdot (b_{R_4})_3$$

which refers to level-3 relations of agent R.

For simplicity we also use the notation $(B_{R_k^C})_l \equiv \tilde{b}_i = \prod_{m\in(R_k^C)_l} b_m$. The RVF can be written as $g_i = b_i + \frac{\lambda b_i}{b_i+\tilde{b}_i^{1/h}}$ It is obvious that for the highest level $l = n-1$ only one relation is possible so that $(R_k^C)_{n-1} = \emptyset$ and $(g_{R_k})_l = (b_{R_k})_l$ for $l = n-1$. The basic property that we demand from RVF is that it assumes the value of zero if a relation holds, while no other relations of the same or other levels hold. In other words it should indicate which of all possible relations holds. We have he following limits of RVF (using the simplified notation): (a) $\lim_{b_i \to 0} \lim_{\tilde{b}_i \to 0} g_i \left(b_i, \tilde{b}_i\right) = \lambda$ (b) $\lim_{\substack{b_i \to 0 \\ \tilde{b}_i \neq 0}} g_i \left(b_i, \tilde{b}_i\right) = 0$. These limits guarantee that RVF will behave in the way we want it to, as an indicator of a specific collision.

The function $G_i$ is now defined as

$$G_i = \prod_{l=1}^{n_L^i} \prod_{j=1}^{n_{R_l}^i} (g_{R_j})_l \tag{9}$$

where $n_L^i$ the number of levels and $n_{R_l}^i$ the number of relations in level-$l$ with respect to agent $i$.

The definition of the G function in the multiple moving agents situation is slightly different than the one introduced by the authors in ([15]). The collision scheme in that approach involved a single moving point agent in an environment with static obstacles. A collision with more than one obstacle was therefore impossible and the obstacle function was simply the product of the distances of the agent from each obstacle. In our case however, this is inappropriate, as can be seen in figure 2. The control law of agent A should

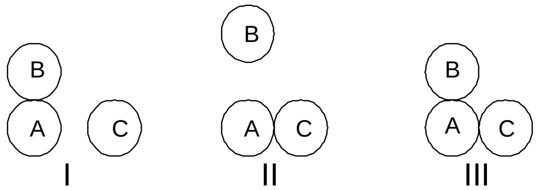


Figure 3: I,II are level-1 relations with respect to A, while III is level-2. The RVFs of the level-1 relations are nonzero in situation III.

distinguish when agent A is in conflict with B, C, or B and C simultaneously. Mathematically, the first two situations are level-1 relations and the third a level-2 relation with respect to A. Whenever the latter occurs, the RVF of the level-2 relation tends to zero while the RVFs of the two separate level-1 relations (A,B and A,C) are nonzero. The key property of an RVF is that it tends to zero only when the corresponding relation holds. Hence it serves as an analytic switch that is activated (tends to zero) only when the relation it represents is realized.

### 3.5 An example

As an example, we will present steps to construct the function $G$ with respect to a specific agent in a team of 4 agents indexed 1 through 4. We construct the function $G_1$ wrt agent 1. We begin by defining the Relation Proximity Functions (eq.(7)) in every level (Table 1):

| Relation | Level 1 | Level 2 | Level 3 |
|---|---|---|---|
| 1 | $(b_1)_1 = \beta_{12}$ | $(b_1)_2 = \beta_{12} + \beta_{13}$ | $(b_1)_3 = \beta_{12}+$ $+\beta_{13} + \beta_{14}$ |
| 2 | $(b_2)_1 = \beta_{13}$ | $(b_2)_2 = \beta_{12} + \beta_{14}$ | - |
| 3 | $(b_3)_1 = \beta_{14}$ | $(b_3)_2 = \beta_{13} + \beta_{14}$ | - |

**Table 1**

It is now easy to calculate the Relation Verification Functions for each relation based on equation (8). For example, for the second relation of level 2, the complement (term $(B_{R_k^C})_l$ in eq.(8)) is given by $(B_{2^C})_2 = (b_1)_2 \cdot (b_3)_2$ and substituting in (8), we have

$$(g_2)_2 = (b_2)_2 + \frac{\lambda\,(b_2)_2}{(b_2)_2 + ((b_1)_2 \cdot (b_3)_2)^{1/h}}$$

The function $G_1$ is then calculated as the product of the Relation Verification Functions of all relations.

### 3.6 The $f$ function

The key difference of the decentralized method with respect to the centralized case is that the control law of each agent ignores the destinations of the others. By using $\varphi_i = \frac{\gamma_{di}}{\left((\gamma_{di})^k + G_i\right)^{1/k}}$ as a navigation function for agent $i$, there is no potential for $i$ to cooperate in a possible collision scheme when its initial condition coincides with its final destination. In order to overcome this limitation,we add a function $f_i$ to $\gamma_i$ so that the cost function $\varphi_i$ attains positive values in proximity situations even when $i$ has already reached its destination. A preliminary definition for this function was given in [4], [37]. Here, we modify the previous definitions to ensure that the destination point is a non-degenerate local minimum of $\varphi_i$ with minimum requirements on assumptions. We define the function $f_i$ by:

$$f_i(G_i) = \begin{cases} a_0 + \sum_{j=1}^{3} a_j G_i^j, \ G_i \leq X \\ 0, \ G_i > X \end{cases} \tag{10}$$

where $X, Y = f_i(0) > 0$ are positive parameters the role of which will be made clear in the following. The parameters $a_j$ are evaluated so that $f_i$ is maximized when $G_i \to 0$ and minimized when $G_i = X$. We also require that $f_i$ is continuously differentiable at $X$. Therefore we have:

$$a_0 = Y, a_1 = 0, a_2 = \frac{-3Y}{X^2}, a_3 = \frac{2Y}{X^3}$$

We require that $Y \leq \frac{\Theta_1}{k}$ where $\Theta_1$ is an arbitrarily large positive gain. This will help in obtaining a lower bound of $k$ analytically in the stability analysis that follows. The parameter $X$ serves as a sensing parameter that activates the $f_i$ function whenever possible collisions are bound to occur. The only requirement we have for $X$ is that it must be small enough to guarantee that $f_i$ vanishes whenever the system has reached its equilibrium, i.e. when everyone has reached its destination. In mathematical terms:

$$X < G_i \left(q_{d1}, \ldots, q_{dN}\right) \forall i \tag{11}$$

That's the minimum requirement we have regarding knowledge of the destinations of the team.

The resulting navigation function is no longer analytic but merely $C^1$ at $G_i = X$. However, by choosing $X$ large enough, the resulting function is analytic in a neighborhood of the boundary of the free space so that the characterization of its critical points can be made by the evaluation of its Hessian. Hence, the parameter $X$ must be chosen small enough in order to satisfy (11) but large enough to include the region described above. Clearly, this is a tradeoff the control design has to pay in order to achieve decentralization. Intuitively, the destinations should be far enough from one another.

## 3.7 Proof of Correctness

Let $\varepsilon > 0$ . Define $B_{j,l}^i(\varepsilon) \equiv \{q : 0 < (g_{R_j}^i)_l < \varepsilon\}$. Following [15],[20] we discriminate the following topologies for the function $\varphi_i$:

1. The destination point: $q_{di}$

2. The free space boundary: $\partial F(q) = G_i^{-1}(\delta), \delta \to 0$

3. The set near collisions: $F_0(\varepsilon) = \bigcup_{l=1}^{n_L^i} \bigcup_{j=1}^{n_{R,l}^i} B_{j,l}^i(\varepsilon) - \{q_{di}\}$

4. The set away from collisions: $F_1(\varepsilon) = F - (\{q_{di}\} \cup \vartheta F \cup F_0(\varepsilon))$

The following theorem allows us to derive results for the function $\varphi_i$ by examining the simpler function $\hat{\varphi}_i(q) = \frac{\gamma_i}{G_i}$:
**Theorem 1 [15]**:*Let $I_1, I_2$ be intervals, $\hat{\varphi} : F \to I_1$ and $\sigma : I_1 \to I_2$ be analytic. Define the composition $\varphi : F \to I_2$ to be $\varphi = \sigma \circ \hat{\varphi}$. If $\sigma$ is monotonically increasing on $I_1$, then the set of critical points of $\varphi$ and $\hat{\varphi}$ coincide and the (Morse) index of each critical point is identical.*

A key point in the discrimination between centralized and decentralized navigation functions is that the latter contain a time-varying part which depends on the movement of the other agents. Using the same procedure as in [20],[15] we first prove that the construction of each $\varphi_i$ guarantees collision avoidance:
**Proposition 1**: *For each fixed t, the function $\varphi_i(q_i, \cdot)$ is a navigation function if the parameters $h, k$ assume values bigger than a finite lower bound..*
**Proof Sketch**: The set of critical points of $\varphi_i$ is defined as $C_{\varphi_i} = \{q : \partial\varphi_i/\partial q_i = 0\}$ . A critical point is non-degenerate if $\partial^2\varphi_i/\partial^2 q_i$ has full rank at that point.The statement of the proposition is guaranteed by the following Lemmas:
**Lemma 2**: *If the workspace is valid, the destination point $q_{di}$ is a non-degenerate local minimum of $\varphi_i$.*
**Lemma 3**: *All critical points of $\varphi_i$ are in the interior of the free space.*
**Lemma 4**: *For every $\varepsilon > 0$, there exists a positive integer $N(\varepsilon)$ such that if $k > N(\varepsilon)$ then there are no critical points of $\hat{\varphi}_i$ in $F_1(\varepsilon)$.*
**Lemma 5**: *There exists an $\varepsilon_0 > 0$ such that $\hat{\varphi}_i$ has no local minimum in $F_0(\varepsilon)$, as long as $\varepsilon < \varepsilon_0$.*
**Lemma 6**: *There exist $\varepsilon_1 > 0$ and $h_1 > 0$, such that the critical points of $\hat{\varphi}_i$ are non-degenerate as long as $\varepsilon < \varepsilon_1$ and $h > h_1$.*

The complete proofs of the Lemmas can be found in the Appendix. Lemmas 2-5 guarantee the polarity of the proposed DNF, whilst Lemma 6 guarantees the non-degeneracy of the critical points. By choosing $k, h$ that satisfy the above Lemmas, the statement of Proposition 1 is proved.

This however does not guarantee global convergence of the system state to the destination configuration. This is achieved by using a Lyapunov function for the *whole* system which is *time invariant* that is a function that depends on the positions of all the agents. The candidate Lyapunov function that we use in this paper is simply the sum of the DNF's of all agents. Specifically we prove the following:

**Proposition 2**: *The time-derivative of $\varphi = \sum_{i=1}^{N} \varphi_i$ is negative definite across the trajectories of the system up to a set of initial conditions of measure zero if the parameters $h, k$ assume values bigger than a finite lower bound.*

# 4 Simulations

To demonstrate the navigation properties of our decentralized approach, we present two simulations of four holonomic agents that have to navigate from an initial to a final configuration, avoiding collision with each other. Each agent has no knowledge of the desired destinations of the other agents. In the following figure A-$i$,T-$i$ denote the initial condition and desired destination of agent $i$ respectively. The chosen configurations constitute non-trivial setups since the straight-line paths connecting initial and final positions of each agent are obstructed by other agents. Screenshots I-VI in each simulation show the evolution in time of the four member team.

In the second simulation the initial positions of agents 2,3,4 coincide with their desired destinations. We see how the $f$ function forces these 3 robots to cooperate in order to let robot 1 reach its target.

# 5 Conclusions

In this paper, a methodology for multiple mobile agent navigation is presented. The methodology extends the centralized agent navigation established in [20] to a decentralized approach to the problem under input constraints. As in [20], the agent potentials are formed by appropriately constructed agent proximity potentials, which capture all the possible multi agent proximity situations. The great advantages of the method are (i) its relatively low complexity with respect to the number of agents, compared to centralized approaches to the problem and (ii) its application to non-point agents. The effectiveness of the methodology is verified through non-trivial computer simulations.

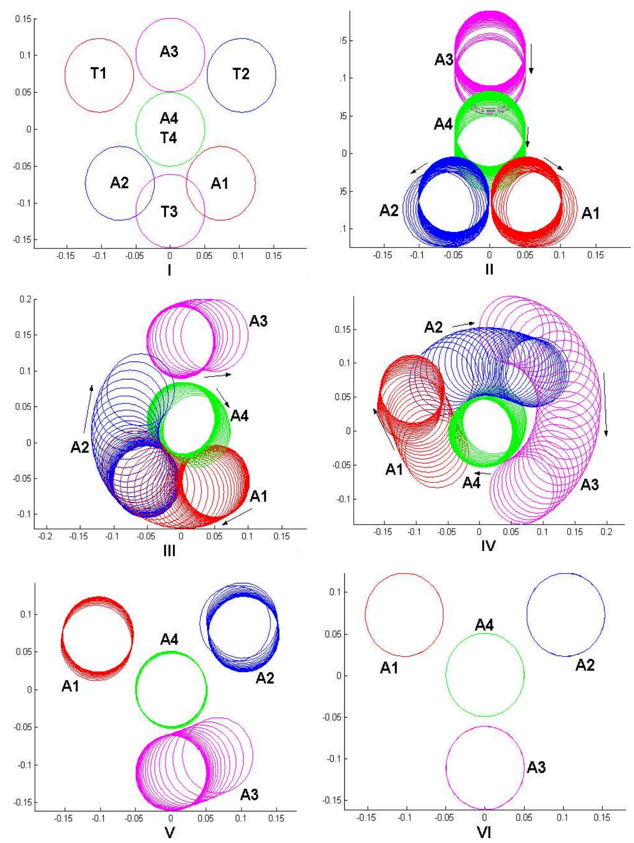


Figure 4: Simulation A

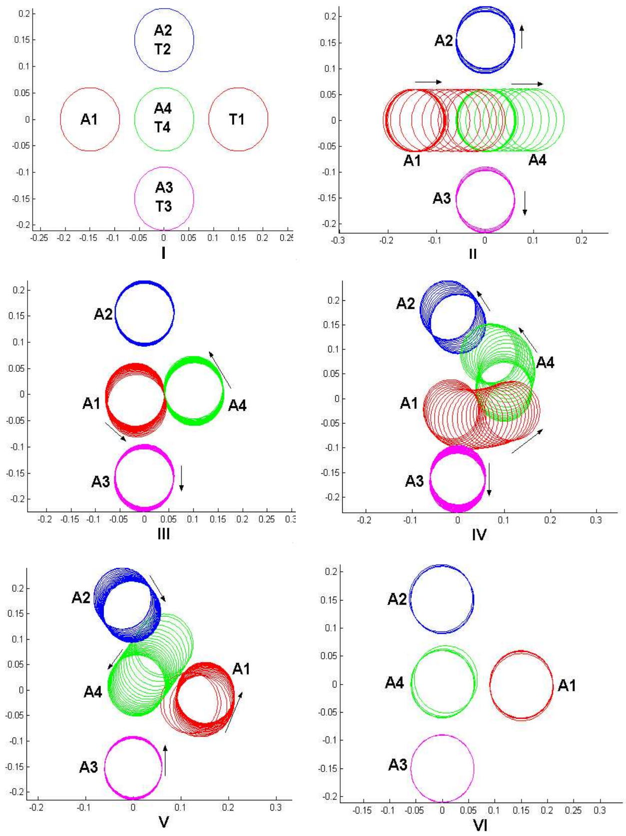


Figure 5: Simulation B

Current research directions are towards applying the methodology to the cases where each agent has limited knowledge of the positions of the others and where there is some form of uncertainty in the agent movement.

# 6 Proofs

We first provide the proofs of propositions 2-5 and then the proof of proposition 1.

Before proceeding with our proof, we introduce some simplifications concerning terminology. To simplify notation we denote by $q$ instead of $q_i$ the current agent configuration, by $q_d$ instead of $q_{di}$ its goal configuration, by $G$ instead of its "$G$" function and by $q_j$ the configurations of the other agents. In the proof sketches of Lemmas 2-6 we use the notation $\frac{\partial}{\partial q_i}(\cdot) \triangleq \nabla(\cdot)$ and $\frac{\partial^2}{\partial q_i^2}(\cdot) \triangleq \nabla^2(\cdot)$

**Proof of Lemma 2**: At steady state, the function $f$ vanishes due to the constraint $X < G_i(q_{d1}, \ldots, q_{dN})\ \forall i$. Taking the gradient of the definition of $\varphi$ we have:

$$\nabla\varphi(q_d) = \frac{1}{\left((\gamma_d)^k + G\right)^{2/k}}\left(\left((\gamma_d)^k + G\right)^{1/k}\nabla(\gamma_d) - (\gamma_d)\nabla\left((\gamma_d)^k + G\right)^{1/k}\right) = 0$$

since both $\gamma_d$ and $\nabla(\gamma_d)$ vanish by definition at $q_d$. The Hessian at $q_d$ is

$$\begin{aligned}\nabla^2\varphi(q_d) &= \tfrac{1}{\left((\gamma_d)^k + G\right)^{2/k}}\left(\left((\gamma_d)^k + G\right)^{1/k}\nabla^2(\gamma_d) - (\gamma_d)\nabla^2\left((\gamma_d)^k + G\right)^{1/k}\right) = \\ &= G^{-1/k}\cdot\nabla^2(\gamma_d) = 2G^{-1/k}I\end{aligned}$$

which is non-degenerate.$\diamondsuit$

**Proof of Lemma 3**: Let $q_0$ be a point in $\vartheta F$ and suppose that $(g_{R_a})_b(q_0) = 0$ for some relation $a$ of level $b$. If the workspace is valid:$\left(g_{R_j}\right)_l(q_0) > 0$ for any level-l and $j \neq a$ since only one RVF can hold at a time. Using the terminology previously defined, and setting $g_i \equiv (g_{R_a})_b(q_0) = 0$ , it follows that $\bar{g}_i > 0$. Taking the gradient of $\varphi$ at $q_0$ , we obtain:

$$\nabla\varphi\left(q_0\right) = \frac{1}{\left((\gamma_d+f)^k+G\right)^{2/k}}\left(\left((\gamma_d+f)^k+G\right)^{1/k}\nabla\left(\gamma_d+f\right)-\left(\gamma_d+f\right)\nabla\left(\left(\gamma_d+f\right)^k+G\right)^{1/k}\right)=$$
$$=\frac{1}{\left((\gamma_d+f)^k+G\right)^{2/k}}\left(\left((\gamma_d+f)^k+G\right)^{1/k}\nabla\left(\gamma_d+f\right)-\right.$$
$$\left.-\frac{1}{k}\left(\gamma_d+f\right)\left((\gamma_d+f)^k+G\right)^{\frac{1}{k}-1}\left(\nabla\left(\gamma_d+f\right)^k+\nabla G\right)\right)=$$
$$=\frac{1}{\left((\gamma_d+f)^k+G\right)^{2/k}}\left(\left((\gamma_d+f)^k+G\right)^{1/k}\nabla\left(\gamma_d+f\right)-\right.$$
$$\left.-\left(\gamma_d+f\right)^k\left((\gamma_d+f)^k+G\right)^{\frac{1}{k}-1}\nabla\left(\gamma_d+f\right)-\frac{1}{k}\left(\gamma_d+f\right)\left((\gamma_d+f)^k+G\right)^{\frac{1}{k}-1}\nabla G\right)$$

Since $G = g_i \cdot \bar{g}_i = 0$, we have

$$\nabla\varphi\left(q_0\right)=\frac{1}{(\gamma_d+f)^2}\left(\left(\gamma_d+f\right)\nabla\left(\gamma_d+f\right)-\left(\gamma_d+f\right)\nabla\left(\gamma_d+f\right)-\frac{1}{k}\left(\gamma_d+f\right)^{2-k}\nabla G\right)=$$
$$=-\frac{1}{k}\left(\gamma_d+f\right)^{-k}\nabla G=$$
$$=-\frac{1}{k}\left(\gamma_d+f\right)^{-k}\left(g_i\nabla\bar{g}_i+\bar{g}_i\nabla g_i\right)=$$
$$=-\frac{1}{k}\left(\gamma_d+f\right)^{-k}\bar{g}_i\nabla g_i\neq 0$$

$\diamondsuit$

**Proof of Lemma 4**:At a critical point $q \in C_{\hat{\varphi}} \bigcap F_1\left(\varepsilon\right)$ we have:

$$\hat{\varphi}=\frac{\gamma}{G}\Rightarrow\nabla\hat{\varphi}=\frac{1}{G^2}\left(G\nabla\gamma-\gamma\nabla G\right)\overset{\nabla\hat{\varphi}=0}{\Rightarrow}G\nabla\gamma=\gamma\nabla G\Rightarrow$$
$$\Rightarrow G\nabla\left(\gamma_d+f\right)^k=\left(\gamma_d+f\right)^k\nabla G\Rightarrow kG\nabla\left(\gamma_d+f\right)=\left(\gamma_d+f\right)\nabla G$$

Taking the magnitude of both sides yields:

$$kG\left\|\nabla\left(\gamma_d+f\right)\right\|=\left(\gamma_d+f\right)\left\|\nabla G\right\|$$

A sufficient condition for the above equality not to hold is given by:

$$\frac{\left(\gamma_d+f\right)\left\|\nabla G\right\|}{G\left\|\nabla\left(\gamma_d+f\right)\right\|}<k,\mathrm{forall}q\in F_1(\varepsilon)$$

An upper bound for the left side is given by:

$$\frac{(\gamma_d+f)\|\nabla G\|}{G\|\nabla(\gamma_d+f)\|}<\frac{(\gamma_d+f)}{\|\nabla(\gamma_d+f)\|}\cdot\sum_{l=1}^{n_L}\sum_{j=1}^{n_{R,l}}\frac{\overline{G}_{j,l}}{G}\left\|\nabla\left(g_{R_j}\right)_l\right\|<$$
$$<\frac{1}{\varepsilon}\cdot\frac{\left(\max\limits_W\{\gamma_d\}+\max\limits_W\{f\}\right)\cdot\sum\limits_{l=1}^{n_L}\sum\limits_{j=1}^{n_{R,l}}\max\limits_W\left\|\nabla\left(g_{R_j}\right)_l\right\|}{\min\limits_W\|\nabla(\gamma_d+f)\|}=$$
$$=\frac{1}{\varepsilon}\cdot\frac{\left(\max\limits_W\{\gamma_d\}+Y\right)\cdot\sum\limits_{l=1}^{n_L}\sum\limits_{j=1}^{n_{R,l}}\max\limits_W\left\|\nabla\left(g_{R_j}\right)_l\right\|}{\min\limits_W\|\nabla(\gamma_d+f)\|}$$

since: $\left(g_{R_j}\right)_l \geq \varepsilon.\diamondsuit$
**Proof of Lemma 5**: : If $q \in F_0(\varepsilon) \cap C_{\hat{\varphi}}$ , where $C_{\hat{\varphi}}$ is the set of critical points, then $q \in B_i^L(\varepsilon)$ for at least one set $\{L, i\}, i \in \{1...n_{R,L}\}, L \in \{1...n_L\}$, with $n_L$ the number of levels and $n_{R,L}$ the number of relations in level $L$. We will use a unit vector as a test direction to demonstrate that $(\nabla^2\hat{\varphi})(q)$ has at least one negative eigenvalue. At a critical point,

$$(\nabla\hat{\varphi})(q) = \frac{1}{G^2}\left(k \cdot G \cdot (\gamma_d + f)^{k-1} \cdot \nabla(\gamma_d + f) - (\gamma_d + f)^k \cdot \nabla G\right) = 0$$

Hence,

$$k \cdot G \cdot \nabla(\gamma_d + f) = (\gamma_d + f) \cdot \nabla G \tag{12}$$

The Hessian at a critical point is:

$$\left(\nabla^2\hat{\varphi}\right)(q) = \frac{1}{G^2}\left(G \cdot \nabla^2(\gamma_d + f)^k - (\gamma_d + f)^k \cdot \nabla^2 G\right)$$

and expanding

$$\begin{aligned}(\nabla^2\hat{\varphi})(q) = \tfrac{(\gamma_d+f)^{k-2}}{G^2}\Big\{kG \cdot \Big[(\gamma_d + f) \cdot \nabla^2(\gamma_d + f) + (k-1)\nabla(\gamma_d + f)\nabla(\gamma_d + f)^T\Big] - \\ -(\gamma_d + f)^2\nabla^2 G\Big\}\end{aligned} \tag{13}$$

Taking the outer product of both sides of equation (12), we get:

$$(kG)^2\nabla(\gamma_d + f)\nabla(\gamma_d + f)^T = (\gamma_d + f)^2\nabla G\nabla G^T \tag{14}$$

Substituting equation (14) in equation (13), we get:

$$\left(\nabla^2\hat{\varphi}\right)(q) = \frac{(\gamma_d + f)^{k-1}}{G^2}\left\{kG \cdot \nabla^2(\gamma_d + f) + \left(1 - \frac{1}{k}\right)\frac{(\gamma_d + f)}{G}\nabla G\nabla G^T - (\gamma_d + f)\nabla^2 G\right\}$$

We choose the test vector (unit magnitude) to be:$\hat{u} = \frac{\nabla b_i(q_c)^\perp}{\left\|\nabla b_i(q_c)^\perp\right\|}$. By its definition $\hat{u}$ is orthogonal to $\nabla b_i$ at a critical point $q_c$, and so the following properties hold:$\hat{u}^T \cdot \nabla b_i = 0$ and $\nabla b_i^T \cdot \hat{u} = 0$. With $\nabla^2(\gamma_d + f) = 2 \cdot \mathrm{I} + \nabla^2 f$, we form the quadratic form:

$$\begin{aligned}\tfrac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T(\nabla^2\hat{\varphi})(q)\hat{u} = 2kG + kG \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \left(1 - \tfrac{1}{k}\right)\tfrac{(\gamma_d+f)}{G}\hat{u}^T\nabla G\nabla G^T\hat{u} - \\ -(\gamma_d + f)\hat{u}^T\nabla^2 G\hat{u}\end{aligned} \tag{15}$$

where:

$$\nabla^2 f = \sum_{j=2}^{3} j a_j \left[(j-1)\, G^{j-2} \nabla G \nabla G^T + G^{j-1} \nabla^2 G\right]$$

Expanding the term $\hat{u}^T \cdot \nabla G \cdot \nabla G^T \cdot \hat{u}$, we get:

$$\begin{aligned}
&\hat{u}^T \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = \hat{u}^T \cdot (g_i \cdot \nabla \bar{g}_i + \bar{g}_i \cdot \nabla g_i) \cdot \left(g_i \cdot \nabla \bar{g}_i^T + \bar{g}_i \cdot \nabla g_i^T\right) \cdot \hat{u} = \\
&= g_i^2 \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} + g_i \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla g_i^T \cdot \hat{u} + \\
&+ \bar{g}_i \cdot g_i \cdot \hat{u}^T \cdot \nabla g_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} + \bar{g}_i^2 \cdot \hat{u}^T \cdot \nabla g_i \cdot \nabla g_i^T \cdot \hat{u} = \\
&= g_i^2 \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} + 2 \cdot g_i \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla g_i^T \cdot \hat{u} + \\
&+ \bar{g}_i^2 \cdot \hat{u}^T \cdot \nabla g_i \cdot \nabla g_i^T \cdot \hat{u}
\end{aligned} \tag{16}$$

We also have:

$$\hat{u}^T \cdot \nabla g_i = -\frac{\lambda \cdot b_i}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \cdot \hat{u}^T \cdot \nabla \tilde{b}_i^{1/h} \text{and} \nabla g_i^T \cdot \hat{u} = -\frac{\lambda \cdot b_i}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u}$$

and equation (16) simplifies to:

$$\hat{u}^T \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = g_i^2 \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} - \hat{u}^T \cdot \nabla \bar{g}_i \cdot \frac{2 \cdot \lambda \cdot G \cdot b_i}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u} +$$

$$+ \bar{g}_i^2 \cdot \frac{\lambda^2 \cdot b_i^2}{\left(b_i + \tilde{b}_i^{1/h}\right)^4} \cdot \hat{u}^T \cdot \nabla \tilde{b}_i^{1/h} \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u}$$

Using $g_i = c_i \cdot b_i$, , where: $c_i = 1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}}$, , we get:

$$\hat{u}^T \cdot \nabla G \cdot \nabla G^T \cdot \hat{u} = g_i^2 \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u} - \frac{2 \cdot \lambda \cdot G \cdot g_i}{c_i \left(b_i + \tilde{b}_i^{1/h}\right)^2} \cdot \hat{u}^T \cdot \nabla \bar{g}_i \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u} +$$

$$+ \bar{g}_i^2 \cdot \frac{\lambda^2 \cdot g_i^2}{c_i^2 \left(b_i + \tilde{b}_i^{1/h}\right)^4} \cdot \hat{u}^T \cdot \nabla \tilde{b}_i^{1/h} \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u}$$

Hence:

$$\left(1 - \frac{1}{k}\right) \frac{\hat{u}^T \nabla G \nabla G^T \hat{u}}{G} = g_i \cdot \eta_i \tag{17}$$

where:

$$\eta_i = \left(1 - \frac{1}{k}\right) \cdot \left[ \begin{array}{l} \frac{\hat{u}^T \cdot \nabla \bar{g}_i \cdot \nabla \bar{g}_i^T \cdot \hat{u}}{\bar{g}_i} - 2 \cdot \lambda \cdot \frac{\hat{u}^T \cdot \nabla \bar{g}_i \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u}}{c_i \left(b_i + \tilde{b}_i^{1/h}\right)^2} \\ + \lambda^2 \cdot \bar{g}_i \cdot \frac{\hat{u}^T \cdot \nabla \tilde{b}_i^{1/h} \cdot \left(\nabla \tilde{b}_i^{1/h}\right)^T \cdot \hat{u}}{c_i^2 \left(b_i + \tilde{b}_i^{1/h}\right)^4} \end{array} \right]$$

Expanding the term $\hat{u}^T \cdot \nabla^2 G \cdot \hat{u}$ from equation (13), we get:

$$\begin{aligned} &\hat{u}^T \cdot \nabla^2 G \cdot \hat{u} = \hat{u}^T \cdot \nabla^2 (g_i \cdot \bar{g}_i) \cdot \hat{u} = \\ &= \hat{u}^T \cdot (g_i \cdot \nabla^2 \bar{g}_i + 2 \cdot \nabla g_i \cdot \nabla \bar{g}_i + \bar{g}_i \cdot \nabla^2 g_i) \cdot \hat{u} = \\ &= g_i \cdot \hat{u}^T \cdot \nabla^2 \bar{g}_i \cdot \hat{u} + 2 \cdot \hat{u}^T \cdot \nabla g_i \cdot \nabla \bar{g}_i \cdot \hat{u} + \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 g_i \cdot \hat{u} \end{aligned} \tag{18}$$

We have:

$$\hat{u}^T \cdot \nabla^2 g_i \cdot \hat{u} = \left(1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}}\right) \cdot \hat{u}^T \cdot \nabla^2 b_i \cdot \hat{u} + b_i \cdot \hat{u}^T \cdot A_i \cdot \hat{u} \tag{19}$$

where:

$$A_i = \lambda \cdot \left[ 2 \frac{\left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right) \left(\nabla b_i + \nabla \tilde{b}_i^{1/h}\right)^T}{\left(b_i + \tilde{b}_i^{1/h}\right)^3} - \frac{\left(\nabla^2 b_i + \nabla^2 \tilde{b}_i^{1/h}\right)}{\left(b_i + \tilde{b}_i^{1/h}\right)^2} \right]$$

Using $\nabla^2 b_i = 2 \cdot l \cdot \mathrm{I}_2$, the term $\hat{u}^T \cdot \nabla^2 b_i \cdot \hat{u}$ becomes:

$$\hat{u}^T \cdot \nabla^2 b_i \cdot \hat{u} = 2 \cdot l$$

We define:$\upsilon_i = 2 \cdot l \geq 2$. Substituting $\upsilon_i, c_i = 1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}}$ in equation (19), we get:

$$\hat{u}^T \cdot \nabla^2 g_i \cdot \hat{u} = c_i \cdot \upsilon_i + b_i \cdot \hat{u}^T \cdot A_i \cdot \hat{u}$$

and equation (18) becomes:

$$\hat{u}^T \cdot \nabla^2 G \cdot \hat{u} = g_i \cdot \hat{u}^T \cdot \nabla^2 \bar{g}_i \cdot \hat{u} - 2 \cdot \frac{b_i \cdot \lambda}{\left(b_i + b_i^{1/h}\right)^2} \cdot \hat{u}^T \cdot \nabla b_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u} + \bar{g}_i \cdot \left(c_i \cdot \upsilon_i + b_i \cdot \hat{u}^T \cdot A_i \cdot \hat{u}\right)$$

Using $g_i = c_i \cdot b_i$, we get

$$\begin{aligned} &\hat{u}^T \cdot \nabla^2 G \cdot \hat{u} = g_i \cdot \hat{u}^T \cdot \nabla^2 \bar{g}_i \cdot \hat{u} - 2 \cdot g_i \cdot \frac{\lambda}{c_i \left(b_i + b_i^{1/h}\right)^2} \cdot \hat{u}^T \cdot \nabla b_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u} + \\ &+ \bar{g}_i \cdot c_i \cdot \upsilon_i + \tfrac{\bar{g}_i \cdot g_i}{c_i} \cdot \hat{u}^T \cdot A_i \cdot \hat{u} = \\ &= g_i \cdot \xi_i + \upsilon_i \cdot \bar{g}_i \cdot c_i \end{aligned} \tag{20}$$

where:

$$\xi_i = \hat{u}^T \cdot \nabla^2 \bar{g}_i \cdot \hat{u} + \frac{\bar{g}_i}{c_i} \cdot \hat{u}^T \cdot A_i \cdot \hat{u} - 2 \frac{\lambda}{c_i \left(b_i + \tilde{b}_i^{1/h}\right)^2} \cdot \hat{u}^T \cdot \nabla \tilde{b}_i^{1/h} \cdot \nabla \bar{g}_i \cdot \hat{u}$$

Substituting equations (19) and (20) in equation (15), we get:

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\hat{u}^T\left(\nabla^2\hat{\varphi}\right)(q)\,\hat{u} = 2kG + kG\cdot\hat{u}^T\cdot\nabla^2 f\cdot\hat{u} + g_i\cdot\eta_i - (\gamma_d+f)\left(g_i\cdot\xi_i + \upsilon_i\cdot\bar{g}_i\cdot c_i\right) =$$
$$= \left(2kG + \upsilon_i\cdot\bar{g}_i\cdot(\gamma_d+f)\cdot c_i\right) + g_i\cdot\left[k\cdot\bar{g}_i\cdot\hat{u}^T\cdot\nabla^2 f\cdot\hat{u} + (\gamma_d+f)\cdot\eta_i - (\gamma_d+f)\cdot\xi_i\right] \tag{21}$$

For the analysis to come, we compute the inner products:

$$\nabla\gamma_d^T\cdot\nabla\gamma_d = \|\nabla\gamma_d\|^2 = 4\gamma_d$$

$$\nabla G^T\cdot\nabla G = \left(g_i\nabla\bar{g}_i^T + \bar{g}_i\nabla g_i^T\right)\cdot\left(g_i\nabla\bar{g}_i + \bar{g}_i\nabla g_i\right) = g_i^2\|\nabla\bar{g}_i\|^2 + 2g_i\bar{g}_i\nabla g_i^T\nabla\bar{g}_i + \bar{g}_i^2\|\nabla g_i\|^2$$

$$\nabla G^T\cdot\nabla\gamma_d = g_i\nabla\bar{g}_i^T\nabla\gamma_d + \bar{g}_i\nabla g_i^T\nabla\gamma_d$$

$$\begin{aligned}
\nabla f^T\cdot\nabla G &= \left(\sum_{j=1}^{3} j a_j G^{j-1}\right)\nabla G^T\cdot\nabla G = \\
&= \left(\sum_{j=1}^{3} j a_j g_i^{j-1}\bar{g}_i^{j-1}\right)\cdot\left(g_i^2\|\nabla\bar{g}_i\|^2 + 2g_i\bar{g}_i\nabla g_i^T\nabla\bar{g}_i + \bar{g}_i^2\|\nabla g_i\|^2\right) = \\
&= \|\nabla\bar{g}_i\|^2\left(\sum_{j=1}^{3} j a_j g_i^{j}\bar{g}_i^{j-1}\right)\cdot g_i + 2\nabla g_i^T\nabla\bar{g}_i\left(\sum_{j=1}^{3} j a_j g_i^{j-1}\bar{g}_i^{j}\right)\cdot g_i + \\
&+ \|\nabla g_i\|^2\left(\sum_{j=1}^{3} j a_j g_i^{j-1}\bar{g}_i^{j+1}\right) = \\
&= \|\nabla\bar{g}_i\|^2\left(\sum_{j=1}^{3} j a_j g_i^{j}\bar{g}_i^{j-1}\right)\cdot g_i + 2\nabla g_i^T\nabla\bar{g}_i\left(\sum_{j=1}^{3} j a_j g_i^{j-1}\bar{g}_i^{j}\right)\cdot g_i + \\
&+ \|\nabla g_i\|^2\left(a_1\bar{g}_i^3 + g_i\cdot\sum_{j=2}^{3} j a_j g_i^{j-2}\bar{g}_i^{j+1}\right) = \\
&= \left[\|\nabla\bar{g}_i\|^2\left(\sum_{j=1}^{3} j a_j g_i^{j}\bar{g}_i^{j-1}\right) + 2\nabla g_i^T\nabla\bar{g}_i\left(\sum_{j=1}^{3} j a_j g_i^{j-1}\bar{g}_i^{j}\right) + \right. \\
&\left. + \|\nabla g_i\|^2\left(\sum_{j=2}^{3} j a_j g_i^{j-2}\bar{g}_i^{j+1}\right)\right]\cdot g_i + a_1\bar{g}_i^3\|\nabla g_i\|^2
\end{aligned}$$

Hence:

$$\nabla f^T\cdot\nabla G = z_0\left(g_i,\bar{g}_i,\nabla g_i,\nabla\bar{g}_i\right)\cdot g_i + z_1\left(\bar{g}_i,\nabla g_i\right)$$

where:

$$z_0\left(g_i,\bar{g}_i,\nabla g_i,\nabla\bar{g}_i\right) = \|\nabla\bar{g}_i\|^2\left(\sum_{j=1}^{3} j a_j g_i^{j}\bar{g}_i^{j-1}\right) + 2\nabla g_i^T\nabla\bar{g}_i\left(\sum_{j=1}^{3} j a_j g_i^{j-1}\bar{g}_i^{j}\right) +$$
$$+ \|\nabla g_i\|^2\left(\sum_{j=2}^{3} j a_j g_i^{j-2}\bar{g}_i^{j+1}\right)$$

and:

$$z_1\left(\bar{g}_i, \nabla g_i\right) = a_1 \bar{g}_i^3 \left\|\nabla g_i\right\|^2$$

Taking the inner product of (12) with $\nabla\left(\gamma_d + f\right)$, we get:

$$\begin{aligned}
&kG\nabla\left(\gamma_d + f\right)^T \cdot \nabla\left(\gamma_d + f\right) = \left(\gamma_d + f\right)\nabla G^T \cdot \nabla\left(\gamma_d + f\right) \Rightarrow \\
&kG\left(\nabla\gamma_d^T + \nabla f^T\right) \cdot \left(\nabla\gamma_d + \nabla f\right) = \left(\gamma_d + f\right)\nabla G^T \cdot \left(\nabla\gamma_d + \nabla f\right) \Rightarrow \\
&kG\nabla\gamma_d^T \cdot \nabla\gamma_d + kG\nabla\gamma_d^T \cdot \nabla f + kG\nabla f^T \cdot \nabla\gamma_d + kG\nabla f^T \cdot \nabla f = \\
&= \left(\gamma_d + f\right)\nabla G^T \cdot \nabla\gamma_d + \left(\gamma_d + f\right)\nabla G^T \cdot \nabla f \Rightarrow
\end{aligned}$$

$$\begin{aligned}
&4kG\gamma_d = \left(\gamma_d + f\right)\nabla G^T \cdot \nabla\gamma_d + \left(\gamma_d + f\right)\nabla G^T \cdot \nabla f - 2kG\nabla\gamma_d^T \cdot \nabla f - kG\nabla f^T \cdot \nabla f = \\
&= \left(\gamma_d + f\right)\left(g_i\nabla\bar{g}_i^T\nabla\gamma_d + \bar{g}_i\nabla g_i^T\nabla\gamma_d\right) + \left(\gamma_d + f\right)\left(z_0 \cdot g_i + z_1\right) - \\
&-k\bar{g}_i\left(2\nabla\gamma_d^T \cdot \nabla f - \nabla f^T \cdot \nabla f\right) \cdot g_i = \\
&= \gamma_d\nabla\bar{g}_i^T\nabla\gamma_d \cdot g_i + \gamma_d\bar{g}_i\nabla g_i^T\nabla\gamma_d + f\nabla\bar{g}_i^T\nabla\gamma_d \cdot g_i + \\
&+\left(a_0 + g_i \cdot \sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right)\bar{g}_i\nabla g_i^T\nabla\gamma_d + \gamma_d z_0 \cdot g_i + \gamma_d z_1 + f z_0 \cdot g_i + \\
&+\left(a_0 + g_i \cdot \sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) z_1 - k\bar{g}_i\left(2\nabla\gamma_d^T \cdot \nabla f - \nabla f^T \cdot \nabla f\right) \cdot g_i = \\
&= \gamma_d\nabla\bar{g}_i^T\nabla\gamma_d \cdot g_i + \gamma_d\bar{g}_i\nabla g_i^T\nabla\gamma_d + f\nabla\bar{g}_i^T\nabla\gamma_d \cdot g_i + a_0\bar{g}_i\nabla g_i^T\nabla\gamma_d + \\
&+\bar{g}_i\nabla g_i^T\nabla\gamma_d\left(\sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) \cdot g_i + \gamma_d z_0 \cdot g_i + \gamma_d z_1 + f z_0 \cdot g_i + \\
&+a_0 z_1 + z_1\left(\sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) \cdot g_i - k\bar{g}_i\left(2\nabla\gamma_d^T \cdot \nabla f - \nabla f^T \cdot \nabla f\right) \cdot g_i = \\
&= \left[\gamma_d\nabla\bar{g}_i^T\nabla\gamma_d + f\nabla\bar{g}_i^T\nabla\gamma_d + \bar{g}_i\nabla g_i^T\nabla\gamma_d\left(\sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) + \gamma_d z_0 + f z_0 + \right. \\
&\left. +z_1\left(\sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) - k\bar{g}_i\left(2\nabla\gamma_d^T \cdot \nabla f - \nabla f^T \cdot \nabla f\right)\right] \cdot g_i + \\
&+\left(\gamma_d\bar{g}_i\nabla g_i^T\nabla\gamma_d + a_0\bar{g}_i\nabla g_i^T\nabla\gamma_d + \gamma_d z_1 + a_0 z_1\right)
\end{aligned}$$

Hence:

$$4kG\gamma_d = z_2\left(g_i, \bar{g}_{i,}\nabla g_i, \nabla\bar{g}\right) \cdot g_i + z_3\left(\bar{g}_i, \nabla g_i\right) \tag{22}$$

where:

$$\begin{aligned}
&z_2\left(g_i, \bar{g}_i, \nabla g_i, \nabla\bar{g}_i\right) = \gamma_d\nabla\bar{g}_i^T\nabla\gamma_d + f\nabla\bar{g}_i^T\nabla\gamma_d + \bar{g}_i\nabla g_i^T\nabla\gamma_d\left(\sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) + \gamma_d z_0 + f z_0 + \\
&+z_1\left(\sum_{j=1}^{3} a_j g_i^{j-1}\bar{g}_i^j\right) - k\bar{g}_i\left(2\nabla\gamma_d^T \cdot \nabla f - \nabla f^T \cdot \nabla f\right)
\end{aligned}$$

and

$$z_3\left(\bar{g}_i, \nabla g_i\right) = \gamma_d \bar{g}_i \nabla g_i^T \nabla \gamma_d + a_0 \bar{g}_i \nabla g_i^T \nabla \gamma_d + \gamma_d z_1 + a_0 z_1 = \\ = \left(\gamma_d + a_0\right) \bar{g}_i \nabla g_i^T \nabla \gamma_d + z_1 \left(\gamma_d + a_0\right)$$

Substituting equation (22) in equation (21), we get:

$$\begin{aligned}
&\tfrac{G^2}{(\gamma_d+f)^{k-1}} \hat{u}^T \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = \left(\tfrac{z_2}{2\gamma_d} \cdot g_i + \tfrac{z_3}{2\gamma_d} - \upsilon_i \cdot \bar{g}_i \cdot (\gamma_d + f) \cdot c_i\right) + g_i \cdot \Big[k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \\
&+ (\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i\Big] = \\
&= \left(\tfrac{z_2}{2\gamma_d} \cdot g_i + \tfrac{z_3}{2\gamma_d} - \upsilon_i \cdot \bar{g}_i \cdot \left(\gamma_d + a_0 + \sum_{j=1}^{3} a_j g_i^j \bar{g}_i^j\right) \cdot c_i\right) + g_i \cdot \Big[k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \\
&+ (\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i\Big] = \\
&= \left(\tfrac{z_2}{2\gamma_d} \cdot g_i + \tfrac{z_3}{2\gamma_d} - \upsilon_i \cdot \bar{g}_i \cdot (\gamma_d + a_0) \cdot c_i - \upsilon_i \cdot g_i \cdot c_i \cdot \sum_{j=1}^{3} a_j g_i^{j-1} \bar{g}_i^{j+1}\right) + \\
&+ g_i \cdot \Big[k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + (\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i\Big] = \\
&= \left(\tfrac{z_3}{2\gamma_d} - \upsilon_i \cdot \bar{g}_i \cdot (\gamma_d + a_0) \cdot c_i\right) + g_i \cdot \Big[k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \\
&+ (\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i + \tfrac{z_2}{2\gamma_d} - \upsilon_i \bar{g}_i c_i \left(\sum_{j=1}^{3} a_j g_i^{j-1} \bar{g}_i^j\right)\Big]
\end{aligned} \tag{23}$$

From the previous analysis we have found:

$$z_3\left(\bar{g}_i, \nabla g_i\right) = \left(\gamma_d + a_0\right) \bar{g}_i \nabla g_i^T \nabla \gamma_d + z_1 \left(\gamma_d + a_0\right)$$

and

$$z_1\left(\bar{g}_i, \nabla g_i\right) = a_1 \bar{g}_i^3 \left\|\nabla g_i\right\|^2 = 0$$

So:

$$z_3\left(\bar{g}_i, \nabla g_i\right) = \left(\gamma_d + a_0\right) \bar{g}_i \nabla g_i^T \nabla \gamma_d$$

and equation (23) becomes:

$$\begin{aligned}
&\tfrac{G^2}{(\gamma_d+f)^{k-1}} \hat{u}^T \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = \bar{g}_i \cdot c_i \cdot \left(\tfrac{1}{2\gamma_d}\left(\gamma_d + a_0\right) \nabla b_i^T \nabla \gamma_d - \upsilon_i \cdot (\gamma_d + a_0)\right) \\
&+ g_i \cdot \Big[k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \\
&(\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i + \tfrac{z_2}{2\gamma_d} - \upsilon_i \bar{g}_i c_i \left(\sum_{j=1}^{3} a_j g_i^{j-1} \bar{g}_i^j\right) - \sigma_i\Big] = \\
&= \bar{g}_i \cdot c_i \cdot \left(1 + \tfrac{a_0}{\gamma_d}\right) \cdot \left(\tfrac{1}{2} \nabla b_i^T \nabla \gamma_d - \upsilon_i \cdot \gamma_d\right) + g_i \cdot \Big[k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \\
&(\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i + \tfrac{z_2}{2\gamma_d} - \upsilon_i \bar{g}_i c_i \left(\sum_{j=1}^{3} a_j g_i^{j-1} \bar{g}_i^j\right) - \sigma_i\Big]
\end{aligned} \tag{24}$$

where:

$$\sigma_i = \frac{\lambda \bar{g}_i}{2c_i \left(b + \tilde{b}^{1/h}\right)^2} \left(\nabla b + \nabla \tilde{b}^{1/h}\right)^T \cdot \nabla \gamma_d$$

Setting:

$$\tilde{\mu}_i = \left(1 + \frac{a_0}{\gamma_d}\right) \cdot \mu_i$$

where:

$$\mu_i = \frac{1}{2} \nabla b_i^T \nabla \gamma_d - \upsilon_i \cdot \gamma_d$$

equation (24) becomes:

$$\begin{aligned} &\tfrac{G^2}{(\gamma_d+f)^{k-1}} \hat{u}^T \left(\nabla^2 \hat{\varphi}\right)(q)\, \hat{u} = \bar{g}_i \cdot c_i \cdot \tilde{\mu}_i + g_i \cdot \Big[ k \cdot \bar{g}_i \cdot \hat{u}^T \cdot \nabla^2 f \cdot \hat{u} + \\ &+ (\gamma_d + f) \cdot \eta_i - (\gamma_d + f) \cdot \xi_i + \tfrac{z_2}{2\gamma_d} - \upsilon_i \bar{g}_i c_i \left( \sum_{j=1}^{3} a_j g_i^{j-1} \bar{g}_i^j \right) - \sigma_i \Big] \end{aligned} \tag{25}$$

The second term is proportional to $g_i$ and can be made arbitrarily small by a suitable choice of $\varepsilon$ but can still be positive, so the first term should be strictly negative.

From the result of Lemma 7, we have:

$$\max_{q \in F_0} \{\mu_i\} = \frac{2}{l} \cdot \left( \begin{array}{l} \frac{1}{l} \cdot \sqrt{\left\|\sum q_j\right\|^2 - l \cdot \sum \|q_j\|^2 + l \cdot \left(\sum (r + r_j)^2 + \varepsilon\right)} \\ - \left\|l \cdot q_d - \sum q_j\right\| \end{array} \right) \cdot \left\| l \cdot q_d - \sum q_j \right\|$$

For $\varepsilon$ small enough, $\max\limits_{q \in F_0} \{\mu_i\}$ is negative. Moreover, the term $\left(1 + \frac{a_0}{\gamma_d}\right)$ is always greater than one, since we have assumed that $a_0 > 0$, and $\gamma_d > 0$ for $q \in F_0(\varepsilon)$. Thus for $\varepsilon$ small enough, $\tilde{\mu}_i$ is also negative. So, for $\tilde{\mu}_i$, according to Lemma 1, it is sufficient to make sure that:

$$\frac{1}{l} \cdot \sqrt{\left\|\sum q_j\right\|^2 - l \cdot \sum \|q_j\|^2 + l \cdot \left(\sum (r + r_j)^2 + \varepsilon\right)} < \left\| l \cdot q_d - \sum q_j \right\| \Rightarrow$$

$$\varepsilon < l \cdot \left\| l \cdot q_d - \sum q_j \right\|^2 + \sum \|q_j\|^2 - \frac{1}{l} \cdot \left\|\sum q_j\right\|^2 - \sum (r + r_j)^2 \equiv \varepsilon_0$$

An other constraint arises from the fact that $\varepsilon > 0.$ . So for a valid workspace it will be:

$$l \cdot \left\| l \cdot q_d - \sum q_j \right\|^2 + \sum \|q_j\|^2 - \frac{1}{l} \cdot \left\|\sum q_j\right\|^2 > \sum (r + r_j)^2$$

$\diamondsuit$

**Lemma 7**:

$$\max_{q\in F_0}\{\mu_i\} = \frac{2}{l}\cdot\left(\begin{array}{l}\frac{1}{l}\cdot\sqrt{\left\|\sum q_j\right\|^2 - l\cdot\sum\left\|q_j\right\|^2 + l\cdot\left(\sum(r+r_j)^2+\varepsilon\right)} \\ -\left\|l\cdot q_d - \sum q_j\right\|\end{array}\right)\cdot\left\|l\cdot q_d - \sum q_j\right\|$$

**Proof**: We have

$$\begin{array}{l}\mu_i = \frac{1}{2}\cdot\nabla b_i^T\cdot\nabla\gamma_d - \upsilon_i\cdot\gamma_d = \frac{1}{2}\cdot\left(2\cdot\sum(q-q_j)^T\right)\cdot 2\cdot(q-q_d) - 2\cdot l\cdot(q-q_d)^T\cdot(q-q_d) = \\ = 2\cdot\left[\left(\sum(q-q_j)^T\right)\cdot(q-q_d) - l\cdot(q-q_d)^T\cdot(q-q_d)\right] = \\ = 2\cdot\left[\sum(q-q_j)^T - l\cdot(q-q_d)^T\right]\cdot(q-q_d) = \\ = 2\cdot\left[\left(l\cdot q - \sum q_j\right)^T - (l\cdot q - l\cdot q_d)^T\right]\cdot(q-q_d) = \\ = 2\cdot\left(l\cdot q_d - \sum q_j\right)^T\cdot(q-q_d)\end{array} \tag{26}$$

The Gradient and the Hessian of $\mu_i$ are:

$$\nabla\mu_i = 2\cdot\left(l\cdot q_d - \sum q_j\right), \nabla^2\mu_i = 0_2$$

This proves that $\nabla\mu_i$ is constant and, according to the Kuhn Tucker conditions, $\mu_i$ attains its maximum and minimum values on the boundary of any compact set. We are interested in finding the maximum value that $\mu_i$ may attain under the constraint that $b_i \le \varepsilon$ . We form the constraint function:

$$h_i(q) = \sum\left\|q-q_j\right\|^2 - \sum(r+r_j)^2 - \varepsilon \le 0$$

It can be seen that $h_i$ is convex $(\nabla^2 h_i(q) = 2\cdot l\cdot \mathrm{I}_2 > 0)$ , and so the set $U = \{q : h_i(q) \le 0\}$ is a compact set. So $\mu_i$ attains its maximum $q^*$ on the boundary of $U$ i.e. $h_i(q^*) = 0$. According to Kuhn Tucker conditions, there exists a $\lambda \ge 0$ such that:

$$\lambda\cdot\nabla\mu_i(q^*) - \nabla h_i(q^*) = 0 \tag{27}$$

$$\lambda\cdot h_i(q^*) = 0 \tag{28}$$

From equation (27), solving for $q^*$, we have:

$$\begin{array}{l}\lambda\cdot\nabla\mu_i(q^*) = \nabla h_i(q^*) \Rightarrow \\ 2\cdot\lambda\cdot\left(l\cdot q_d - \sum q_j\right) = 2\cdot\sum(q^*-q_j) \Rightarrow \\ \lambda\cdot\left(l\cdot q_d - \sum q_j\right) = l\cdot q^* - \sum q_j \Rightarrow \\ q^* = \frac{1}{l}\cdot(1-\lambda)\cdot\sum q_j + l\cdot q_d\end{array}$$

Substituting in equation (26) we get:

$$\mu_i\left(q^*\right) = 2 \cdot \left(l \cdot q_d - \sum q_j\right)^T \cdot \left(\frac{1}{l} \cdot (1-\lambda) \cdot \sum q_j + \lambda \cdot q_d - q_d\right) = \\ = \frac{2\cdot(\lambda-1)}{l} \cdot \left\|l \cdot q_d - \sum q_j\right\|^2$$

From (28) we have

$$\sum \left\|q^* - q_j\right\|^2 - \sum \left(r + r_j\right)^2 - \varepsilon = 0 \Rightarrow \\ l \cdot \left\|q^*\right\|^2 - 2 \cdot \left(q^*\right)^T \cdot \sum q_j + \sum \left\|q_j\right\|^2 - \sum \left(r + r_j\right)^2 - \varepsilon = 0 \Rightarrow \\ l \cdot \left\|\frac{1}{l} \cdot (1-\lambda) \cdot \sum q_j + \lambda \cdot q_d\right\|^2 - 2 \cdot \left(\frac{1}{l} \cdot (1-\lambda) \cdot \sum q_j + \lambda \cdot q_d\right)^T \cdot \sum q_j + \\ + \sum \left\|q_j\right\|^2 - \sum \left(r + r_j\right)^2 - \varepsilon = 0 \Rightarrow \\ l \cdot \left(\frac{1}{l^2} \cdot (1-\lambda)^2 \cdot \left\|\sum q_j\right\|^2 + 2 \cdot \frac{\lambda}{l} \cdot (1-\lambda) \cdot q_d^T \cdot \sum q_j + \lambda^2 \cdot \left\|q_d\right\|^2\right) - \\ -\frac{2}{l} \cdot (1-\lambda) \cdot \left\|\sum q_j\right\|^2 - 2 \cdot \lambda \cdot q_d^T \cdot \sum q_j + \sum \left\|q_j\right\|^2 - \sum \left(r + r_j\right)^2 - \varepsilon = 0 \Rightarrow \\ \frac{1}{l} \cdot (1-\lambda)^2 \cdot \left\|\sum q_j\right\|^2 + 2 \cdot \lambda \cdot (1-\lambda) \cdot q_d^T \cdot \sum q_j + l \cdot \lambda^2 \cdot \left\|q_d\right\|^2 - \\ -\frac{2}{l} \cdot (1-\lambda) \cdot \left\|\sum q_j\right\|^2 - 2 \cdot \lambda \cdot q_d^T \cdot \sum q_j + \sum \left\|q_j\right\|^2 - \sum \left(r + r_j\right)^2 - \varepsilon = 0 \Rightarrow$$

$$\lambda^2 \cdot \left(\frac{1}{l} \cdot \left\|\sum q_j\right\|^2 - 2 \cdot \sum q_j + l \cdot \left\|q_d\right\|^2\right) - \frac{1}{l} \cdot \left\|\sum q_j\right\|^2 + \sum \left\|q_j\right\|^2 - \\ - \sum \left(r + r_j\right)^2 - \varepsilon = 0 \Rightarrow \\ \lambda^2 \cdot l \cdot \left\|l \cdot q_d - \sum q_j\right\|^2 - \frac{1}{l} \cdot \left\|\sum q_j\right\|^2 + \sum \left\|q_j\right\|^2 - \sum \left(r + r_j\right)^2 - \varepsilon = 0$$

and solving for $\lambda$, we get:

$$\lambda_{1,2} = \pm \frac{\sqrt{\frac{1}{l} \cdot \left\|\sum q_j\right\|^2 - \sum \left\|q_j\right\|^2 + \sum \left(r + r_j\right)^2 + \varepsilon}}{\sqrt{l} \cdot \left\|l \cdot q_d - \sum q_j\right\|} \Rightarrow$$

$$\lambda_{1,2} = \pm \frac{1}{l} \cdot \frac{\sqrt{\left\|\sum q_j\right\|^2 - l \cdot \sum \left\|q_j\right\|^2 + l \cdot \left(\sum \left(r + r_j\right)^2 + \varepsilon\right)}}{\left\|l \cdot q_d - \sum q_j\right\|}$$

Choosing for the maximum value the "+" option, and substituting in $\mu_i\left(q^*\right)$, we have:

$$\max_{q \in F_0} \left\{\mu_i\right\} = \frac{2}{l} \cdot \left(\frac{1}{l} \cdot \frac{\sqrt{\left\|\sum q_j\right\|^2 - l \cdot \sum \left\|q_j\right\|^2 + l \cdot \left(\sum \left(r + r_j\right)^2 + \varepsilon\right)}}{\left\|l \cdot q_d - \sum q_j\right\|} - 1\right) \cdot \left\|l \cdot q_d - \sum q_j\right\|^2$$

$$\max_{q \in F_0} \left\{\mu_i\right\} = \frac{2}{l} \cdot \left(\begin{array}{l} \frac{1}{l} \cdot \sqrt{\left\|\sum q_j\right\|^2 - l \cdot \sum \left\|q_j\right\|^2 + l \cdot \left(\sum \left(r + r_j\right)^2 + \varepsilon\right)} \\ - \left\|l \cdot q_d - \sum q_j\right\| \end{array}\right) \cdot \left\|l \cdot q_d - \sum q_j\right\|$$

**Proof of Lemma 6**: From equation (13) we have

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\left(\nabla^2\hat{\varphi}\right)=kG\nabla^2(\gamma_d+f)+\left(1-\frac{1}{k}\right)\frac{\gamma_d+f}{G}\nabla G\nabla G^T-(\gamma_d+f)\nabla^2 G$$

where

$$\nabla f=\left(\underbrace{\sum_{j=1}^{3} j a_j G_i^{j-1}}_{\sigma(G)}\right)\nabla G$$

At a critical point

$$\begin{aligned}
&kG\nabla(\gamma_d+f)=(\gamma_d+f)\nabla G\Rightarrow\\
&kG\nabla\gamma_d=(\gamma_d+f)\nabla G-kG\nabla f\Rightarrow\\
&kG\nabla\gamma_d=(\gamma_d+f-kG\sigma(G))\nabla G\Rightarrow\\
&G\nabla\gamma_d=\left\{\underbrace{\frac{\gamma_d+f}{k}-G\sigma(G)}_{-\sigma_i}\right\}\nabla G
\end{aligned}$$

Taking the magnitude from both sides we have

$$2kG=\frac{k\left|\sigma_i\right|^2}{2G\gamma_d}\left\|\nabla G\right\|^2$$

We also have

$$\nabla^2 f=\sigma\nabla^2 G+\sigma^*\nabla G\nabla G^T,\sigma^*=\sum_{j=2}^{3} j(j-1)a_j G^{j-2}$$

so that

$$\frac{G^2}{(\gamma_d+f)^{k-1}}\left(\nabla^2\hat{\varphi}\right)=2kG+\left\{kG\sigma^*+\left(1-\frac{1}{k}\right)\frac{\gamma_d+f}{G}\right\}\nabla G\nabla G^T+k\sigma_i\nabla^2 G$$

We have

$$kG\sigma^*+\left(1-\frac{1}{k}\right)\frac{\gamma_d+f}{G}=\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{G}+\sum_{j=2}^{3}\left\{kj(j-1)+\left(1-\frac{1}{k}\right)\right\}a_j G^{j-1}$$

Define

$$\xi = \left(1 - \frac{1}{k}\right) \frac{\gamma_d + Y}{kG} + \sum_{j=2}^{3} \left\{ kj(j-1) + \left(1 - \frac{1}{k}\right) \right\} \frac{a_j}{k} G^{j-1}$$

Choosing $\tilde{u} = \widehat{\nabla b_i}$ as a test direction we get

$$\frac{G^2}{k\left(\gamma_d + f\right)^{k-1}} \tilde{u}^T \left(\nabla^2 \hat{\varphi}\right) \tilde{u} = \underbrace{\frac{|\sigma_i|^2}{2G\gamma_d} \|\nabla G\|^2}_{L} + \underbrace{\xi \tilde{u}^T \nabla G \nabla G^T \tilde{u}}_{M} + \underbrace{\sigma_i \tilde{u}^T \nabla^2 G \tilde{u}}_{N}$$

with

$$L = \frac{|\sigma_i|^2}{2G\gamma_d} \left( g_i^2 \|\nabla \bar{g}_i\|^2 + \bar{g}_i^2 \|\nabla g_i\|^2 + 2G \nabla \bar{g}_i \nabla g_i \right), L_a = \frac{|\sigma_i|^2}{2G\gamma_d} 2G \nabla \bar{g}_i \nabla g_i$$

After some malnipulation

$$\tilde{u}^T \nabla G \nabla G^T \tilde{u} = g_i^2 \left(\tilde{u}^T \nabla \bar{g}_i\right)^2 + \bar{g}_i^2 \left(\tilde{u}^T \nabla g_i\right)^2 + 2G \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right)$$

$$\tilde{u}^T \nabla^2 G \tilde{u} = \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u + 2 \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right)$$

$$2\xi G + 2\sigma_i = 2\xi^*, \xi^* = -\frac{\gamma_d + Y}{k^2} + \sum_{j=2}^{3} a_j G^j \left\{ \frac{j}{k} + j(j-1) - \frac{1}{k^2} \right\}$$

$$\bar{M} = \xi \cdot 2G \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right)$$

$$\bar{M} + N = \underbrace{2\xi^* \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right)}_{\bar{M}_1} + \underbrace{\sigma_i \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u}_{\bar{N}}$$

$$\begin{aligned} & L_a + \bar{M}_1 = \tfrac{|\sigma_i|^2}{2G\gamma_d} 2G \nabla \bar{g}_i \nabla g_i + 2\xi^* \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right) = \\ & = \tfrac{|\sigma_i|^2}{\gamma_d} \left( \nabla \bar{g}_i \nabla g_i + \left(2 - 2 + \tfrac{2\xi^* \gamma_d}{|\sigma_i|^2}\right) \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right) \right) \geq \\ & \geq -\tfrac{|\sigma_i|^2}{\gamma_d} \|\nabla \bar{g}_i\| \left\| \nabla g_i - 2 \left(\tilde{u}^T \nabla g_i\right) \tilde{u} \right\| + \tfrac{|\sigma_i|^2}{\gamma_d} \left(2 + \tfrac{2\xi^* \gamma_d}{|\sigma_i|^2}\right) \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right) \end{aligned}$$

so that

$$\begin{aligned} & L + M + N \geq \tfrac{|\sigma_i|^2}{2G\gamma_d} \left\{ g_i^2 \|\nabla \bar{g}_i\|^2 + \bar{g}_i^2 \|\nabla g_i\|^2 - 2G \|\nabla \bar{g}_i\| \left\| \nabla g_i - 2 \left(\tilde{u}^T \nabla g_i\right) \tilde{u} \right\| \right\} + \\ & \tfrac{|\sigma_i|^2}{\gamma_d} \left(2 + \tfrac{2\xi^* \gamma_d}{|\sigma_i|^2}\right) \left(\tilde{u}^T \nabla g_i\right) \left(\nabla \bar{g}_i \tilde{u}\right) + \xi \cdot \left\{ g_i^2 \left(\tilde{u}^T \nabla \bar{g}_i\right)^2 + \bar{g}_i^2 \left(\tilde{u}^T \nabla g_i\right)^2 \right\} + \\ & \sigma_i \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u \end{aligned}$$

Following the recipe in [20] we get

$$h > \frac{1}{2} \frac{\max\left(\sum\limits_{\mu \in R_i^C} \|\nabla b_\mu\|\right)}{\min\left(\sqrt{\sum\limits_{j \in P_i} (r_i + r_j)^2}\right)} \Rightarrow \tilde{u}^T \nabla g_i \geq \|\nabla b_i\|$$

Define

$$c_i = 1 + \frac{\lambda}{b_i + \tilde{b}_i^{1/h}}, s_i = \frac{2\lambda}{\left(b_i + \tilde{b}_i^{1/h}\right)^2}$$

After some manipulation we have

$$\tilde{u}^T \nabla^2 g_i u < c_i \tilde{u}^T \nabla^2 b_i u + 8\lambda \|\nabla b_i\|^2 - b_i s_i - \frac{b_i s_i}{2} \left(\tilde{u}^T \nabla^2 b_i u - \tilde{u}^T \nabla^2 \tilde{b}_i^{1/h} u\right)$$

$$2 \leq \tilde{u}^T \nabla^2 b_i u \leq 2(N-1)$$

$$\sigma_i < 0 \Rightarrow \sigma_i \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u \geq \sigma_i g_i \tilde{u}^T \nabla^2 \bar{g}_i u + \\ + \sigma_i \bar{g}_i \left(2c_i(N-1) + 8\lambda \|\nabla b_i\|^2 - b_i s_i - \tfrac{b_i s_i}{2} \tilde{u}^T \nabla^2 \tilde{b}_i^{1/h} u\right)$$

We have

$$\xi \cdot \left\{g_i^2 \left(\tilde{u}^T \nabla \bar{g}_i\right)^2 + \bar{g}_i^2 \left(\tilde{u}^T \nabla g_i\right)^2\right\} = \left\{\xi g_i^2 + (\xi G)^2 - (\xi G)^2\right\} \left(\tilde{u}^T \nabla \bar{g}_i\right)^2 + \\ + \left\{\xi \bar{g}_i^2 + \left(\tfrac{|\sigma_i|^2}{\gamma_d} + \sigma_i\right)^2 - \left(\tfrac{|\sigma_i|^2}{\gamma_d} + \sigma_i\right)^2\right\} \left(\tilde{u}^T \nabla g_i\right)^2$$

so that

$$L + M + N \geq \left\{\xi g_i^2 - (\xi G)^2\right\} \left(\tilde{u}^T \nabla \bar{g}_i\right)^2 + \left\{\xi \bar{g}_i^2 - \left(\tfrac{|\sigma_i|^2}{\gamma_d} + \sigma_i\right)^2\right\} \left(\tilde{u}^T \nabla g_i\right)^2 \\ + \sigma_i \tilde{u}^T \left(g_i \nabla^2 \bar{g}_i + \bar{g}_i \nabla^2 g_i\right) u$$

After some manipulation we have

$$\xi_{\min} = \left(1 - \frac{1}{k}\right) \frac{\gamma_d + Y}{k \varepsilon \bar{g}_i} + \sum_{j=2}^{3} \left\{kj(j-1) + \left(1 - \frac{1}{k}\right)\right\} \frac{a_j}{k} (\varepsilon \bar{g}_i)^{j-1}$$

and we write

$$\left(1 - \frac{1}{k}\right) \frac{\gamma_d + Y}{k \varepsilon \bar{g}_i} = M \frac{1}{M} \left(1 - \frac{1}{k}\right) \frac{\gamma_d + Y}{k \varepsilon \bar{g}_i}$$

For

$$\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\varepsilon\bar{g}_i} > \left(2k+1-\frac{1}{k}\right)\frac{3Y}{X^2}\varepsilon\bar{g}_i \Leftrightarrow \varepsilon < \sqrt{\frac{\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\bar{g}_i}}{\left(2k+1-\frac{1}{k}\right)\frac{3Y}{X^2}\bar{g}_i}}$$

we have

$$\xi\bar{g}_i^2\left(\tilde{u}^T\nabla g_i\right)^2 \geq \frac{M-1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\varepsilon}\bar{g}_i\left\|\nabla b_i\right\|^2$$

so that

$$\frac{3}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\varepsilon}\bar{g}_i\left\|\nabla b_i\right\|^2+\sigma_i\bar{g}_i\left(2c_i(N-1)+8\lambda\left\|\nabla b_i\right\|^2-\frac{b_is_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}u\right)>0$$

for:

$$\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\varepsilon}\left\|\nabla b_i\right\|^2 > 2c_i\left|\sigma_i\right|(N-1) \Leftrightarrow \varepsilon < \frac{\min\left(\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k}\left\|\nabla b_i\right\|^2\right)}{\max\left(2c_i\left|\sigma_i\right|(N-1)\right)}$$

$$\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\varepsilon}\left\|\nabla b_i\right\|^2 > 8\lambda\left\|\nabla b_i\right\|^2\left|\sigma_i\right| \Leftrightarrow \varepsilon < \frac{\min\left(\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k}\right)}{\max\left(8\lambda\left|\sigma_i\right|\right)}$$

$$\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k\varepsilon}\left\|\nabla b_i\right\|^2 > \left|\sigma_i\right|\frac{\varepsilon s_i}{2}\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}u \Leftrightarrow \varepsilon < \sqrt{\frac{\min\left(\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k}\left\|\nabla b_i\right\|^2\right)}{\max\left(\frac{s_i}{2}\left|\tilde{u}^T\nabla^2\tilde{b}_i^{1/h}u\right|\left|\sigma_i\right|\right)}}$$

and we get

$$\begin{aligned}L+M+N \geq \frac{|\sigma_i|^2}{\gamma_d}\left(2+\frac{2\xi^*\gamma_d}{|\sigma_i|^2}\right)\left(\tilde{u}^T\nabla g_i\right)\left(\nabla\bar{g}_i\tilde{u}\right)+\xi g_i^2\left(\tilde{u}^T\nabla\bar{g}_i\right)^2+\\ \tfrac{M-4}{M}\left(1-\tfrac{1}{k}\right)\tfrac{\gamma_d+Y}{k\varepsilon}\bar{g}_i\left\|\nabla b_i\right\|^2+\sigma_i\tilde{u}^Tg_i\nabla^2\bar{g}_iu\end{aligned}$$

$$\begin{aligned}L+M+N \geq -\frac{|\sigma_i|^2}{\gamma_d}\left(2+\frac{2\xi^*\gamma_d}{|\sigma_i|^2}\right)\left\|\nabla b_i\right\|\left|\nabla\bar{g}_i\tilde{u}\right|+\\ \tfrac{M-4}{M}\left(1-\tfrac{1}{k}\right)\tfrac{\gamma_d+Y}{k\varepsilon}\bar{g}_i\left\|\nabla b_i\right\|^2-\varepsilon\left|\sigma_i\right|\left|\tilde{u}^T\nabla^2\bar{g}_iu\right|>0\end{aligned}$$

for:

$$\varepsilon < \frac{\min\left(\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k}\left\|\nabla b_i\right\|\right)}{\max\left(2\left(\frac{|\sigma_i|^2}{\gamma_d}+\xi G+\sigma_i\right)\left|\nabla\bar{g}_i\tilde{u}\right|\right)}$$

$$\varepsilon < \sqrt{\frac{\min\left(\frac{1}{M}\left(1-\frac{1}{k}\right)\frac{\gamma_d+Y}{k}\left\|\nabla b_i\right\|\right)}{\max\left(\left|\sigma_i\right|\left|\tilde{u}^T\nabla^2\bar{g}_iu\right|\right)}}$$

**Proof of Proposition 2**: In the proof of Proposition 2, the terms $\nabla(\cdot)$, $\nabla^2(\cdot)$ have their usual meaning and refer to the whole state space and not a single agent, namely $\nabla(\cdot) \triangleq \left[\frac{\partial}{\partial q_1}(\cdot), \ldots, \frac{\partial}{\partial q_N}(\cdot)\right]^T$ and $\nabla^2(\cdot) \triangleq \left[\frac{\partial^2}{\partial q_{ij}}(\cdot)\right]$.

We immediately note that the following proof is existential rather than computational. We show that a finite $k$ that renders the system almost everywhere asymptotically stable *exists*, but we do not provide an analytical expression for this lower bound. However, practical values of $k$ have been provided in the simulation section.
Let us recall that the Proximity function between agents $i$ and $j$ is given by:

$$\beta_{ij}(q) = \|q_i - q_j\|^2 - (r_i + r_j)^2 = q^T D_{ij} q - (r_i + r_j)^2$$

where the $2N \times 2N$ matrix $D_{ij}$ is defined in [20]:

$$D_{ij} = \begin{bmatrix} & & O_{2(i-1)\times 2N} & & \\ O_{2\times 2(i-1)} & I_{2\times 2} & O_{2\times 2(j-i-1)} & -I_{2\times 2} & O_{2\times 2(N-j)} \\ & & O_{2(j-i-1)\times 2N} & & \\ O_{2\times 2(i-1)} & -I_{2\times 2} & O_{2\times 2(j-i-1)} & I_{2\times 2} & O_{2\times 2(N-j)} \\ & & O_{2(N-j)\times 2N} & & \end{bmatrix}$$

We can also write $b_r^i = q^T P_r^i q - \sum\limits_{j\in P_r} (r_i + r_j)^2$ ,where $P_r^i = \sum\limits_{j\in P_r} D_{ij}$, and $P_r$ denotes the set of binary relations in relation $r$. It can easily be seen that $\nabla b_r^i = 2P_r^i q, \nabla^2 b_r^i = 2P_r^i$. We also use the following notation for the $r$-th relation wrt agent $i$:

$$g_r^i = b_r^i + \frac{\lambda b_r^i}{b_r^i + \left(\tilde{b}_r^i\right)^{1/h}}, \tilde{b}_r^i = \prod_{\substack{s\in S_r \\ s\neq r}} b_s^i,$$
$$\nabla \tilde{b}_r^i = \sum_{\substack{s\in S_r \\ s\neq r}} \underbrace{\prod_{\substack{t\in S_r \\ t\neq s,r}} b_i^t}_{\tilde{b}_{s,r}^i} \cdot 2P_s^i q$$

where $S_r$ denotes the set of relations in the same level with relation $r$. An easy calculation shows that

$$\nabla g_r^i = \ldots = 2\left[d_r^i P_r^i - w_r^i \tilde{P}_r^i\right] q \triangleq Q_r^i q, \tilde{P}_r^i \triangleq \sum_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i P_s^i$$

where $d_r^i = 1 + (1 - \frac{b_r^i}{b_r^i + (\widetilde{b_r^i})^{1/h}})\frac{\lambda}{b_r^i + (\widetilde{b_r^i})^{1/h}}$, $w_r^i = \frac{\lambda b_r^i (\widetilde{b_r^i})^{\frac{1}{h}-1}}{h(b_r^i + (\widetilde{b_r^i})^{1/h})^2}$. The gradient of the $G_i$ function is given by:

$$G_i = \prod_{r=1}^{N_i} g_r^i \Rightarrow \nabla G_i = \sum_{r=1}^{N_i} \underbrace{\prod_{\substack{l=1\\ l\neq r}}^{N_i} g_l^i}_{\tilde{g}_r^i} \nabla g_r^i = \sum_{r=1}^{N_i} \tilde{g}_r^i Q_r^i q \triangleq Q_i q$$

We define $\nabla G \triangleq \begin{bmatrix} \nabla G_1 \\ \vdots \\ \nabla G_N \end{bmatrix} = \begin{bmatrix} Q_1 \\ \vdots \\ Q_N \end{bmatrix} q \triangleq Qq$

Remembering that $u_i = -K_i \frac{\partial \varphi_i}{\partial q_i}$ and that $\varphi_i = \frac{\gamma_{di} + f_i}{\left((\gamma_{di}+f_i)^k + G_i\right)^{1/k}}$, $f_i = \sum_{j=0}^{3} a_i G_i^j$ the closed loop dynamics of the system are given by:

$$\dot{q} = \begin{bmatrix} -K_1 A_1^{-(1+1/k)} \left\{ G_1 \frac{\partial \gamma_{d1}}{\partial q_1} + \sigma_1 \frac{\partial G_1}{\partial q_1} \right\} \\ \vdots \\ -K_N A_N^{-(1+1/k)} \left\{ G_N \frac{\partial \gamma_{dN}}{\partial q_N} + \sigma_N \frac{\partial G_N}{\partial q_N} \right\} \end{bmatrix} = \ldots$$
$$= -A_K G \left(\partial \gamma_d\right) - A_K \Sigma Q q$$

where $(\partial \gamma_d) = \left[\frac{\partial \gamma_{d1}}{\partial q_1} \ldots \frac{\partial \gamma_{dN}}{\partial q_N}\right]^T$, $\sigma_i = G_i \sigma(G_i) - \frac{\gamma_{di} + f_i}{k}$, $\sigma(G_i) = \sum_{j=1}^{3} j a_j G_i^{j-1}$, $A_i = (\gamma_{di} + f_i)^k + G_i$ and the matrices

$$G \triangleq \underbrace{diag\left(G_1, G_1, \ldots, G_N, G_N\right)}_{2N \times 2N}$$

$$A_K \triangleq \underbrace{diag\begin{pmatrix} K_1 A_1^{-(1+1/k)}, K_1 A_1^{-(1+1/k)}, \ldots, \\ K_N A_N^{-(1+1/k)}, K_N A_N^{-(1+1/k)} \end{pmatrix}}_{2N \times 2N}$$

$$\Sigma \triangleq \underbrace{\left[ \underbrace{\Sigma_1}_{2N \times 2N}, \ldots, \underbrace{\Sigma_N}_{2N \times 2N} \right]}_{2N \times 2N^2},$$

$$\Sigma_i = diag\left(0, 0, \ldots, \underbrace{\sigma_i, \sigma_i}_{2i-1, 2i}, \ldots, 0, 0\right)$$

By using $\varphi = \sum\limits_i \varphi_i$ as a candidate Lyapunov function we have

$$\begin{array}{l} \varphi = \sum\limits_i \varphi_i \Rightarrow \dot{\varphi} = \left\{ \sum\limits_i \left( \nabla \varphi_i \right)^T \right\} \dot{q}, \\ \nabla \varphi_i = A_i^{-(1+1/k)} \left\{ G_i \nabla \gamma_{di} + \sigma_i \nabla G_i \right\} \end{array}$$

and after some trivial calculation

$$\sum_i \left( \nabla \varphi_i \right)^T = \ldots = \left( \partial \gamma_d \right)^T A_G + q^T Q^T A_\Sigma$$

where

$$A_G = \underbrace{diag \left( \begin{array}{l} G_1 A_1^{-(1+1/k)}, G_1 A_1^{-(1+1/k)}, \ldots, \\ G_N A_N^{-(1+1/k)}, G_N A_N^{-(1+1/k)} \end{array} \right)}_{2N \times 2N}$$

$$A_\Sigma = \underbrace{\left[ \begin{array}{l} \underbrace{A_{\Sigma_1}}_{2N \times 2N} \\ \vdots \\ \underbrace{A_{\Sigma_N}}_{2N \times 2N} \end{array} \right]}_{2N^2 \times 2N}, A_{\Sigma_i} = \underbrace{diag \left( \begin{array}{l} A_i^{-(1+1/k)} \sigma_i, \ldots, \\ A_i^{-(1+1/k)} \sigma_i \end{array} \right)}_{2N \times 2N}$$

So we have

$$\begin{array}{l} \dot{\varphi} = \left\{ \sum\limits_i \left( \nabla \varphi_i \right)^T \right\} \dot{q} = \ldots = \\ = - \left[ \begin{array}{cc} \left( \partial \gamma_d \right)^T & q^T \end{array} \right] \underbrace{\left[ \begin{array}{cc} M_1 & M_2 \\ M_3 & M_4 \end{array} \right]}_{M} \left[ \begin{array}{l} \partial \gamma_d \\ q \end{array} \right] \end{array}$$

where $M_1 = A_G A_K G$, $M_2 = A_G A_K \Sigma Q$, $M_3 = Q^T A_\Sigma A_K G$, $M_4 = Q^T A_\Sigma A_K \Sigma Q$.

We proceed by evaluating an expression for the elements of the matrix $Q$. Each $Q_i$ can be written as

$$Q_i = \left[ \begin{array}{ccc} Q_{11}^i & \cdots & Q_{1N}^i \\ \vdots & \ddots & \vdots \\ Q_{N1}^i & \ldots & Q_{NN}^i \end{array} \right], Q_{jk}^i \ : 2 \times 2$$

where the submatrices can be expressed as:

$$Q^i_{ii} = \sum_{r=1}^{N_i} \tilde{g}^r_i \left( d^i_r - w^i_r \sum_{\substack{s\in S_r\\ s\neq r}} \tilde{b}^i_{s,r} \right) l_r I_2$$

$$j \neq i : Q^i_{jj} = \sum_{\substack{r=1\\ j\in P_r}}^{N_i} \tilde{g}^r_i \left( d^i_r - w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2$$

$$- \sum_{\substack{r=1\\ j\notin P_r}}^{N_i} \tilde{g}^r_i \left( w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2$$

$$j \neq i : Q^i_{ij} = Q^i_{ji} = - \sum_{\substack{r=1\\ j\in P_r}}^{N_i} \tilde{g}^r_i \left( d^i_r - w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2$$

$$+ \sum_{\substack{r=1\\ j\notin P_r}}^{N_i} \tilde{g}^r_i \left( w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2$$

$$Q^i_{ii} = \sum_{r=1}^{N_i} \tilde{g}^r_i \left( d^i_r - w^i_r \sum_{\substack{s\in S_r\\ s\neq r}} \tilde{b}^i_{s,r} \right) l_r I_2$$

$$j \neq i : Q^i_{jj} = \sum_{\substack{r=1\\ j\in P_r}}^{N_i} \tilde{g}^r_i \left( d^i_r - w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2 - \sum_{\substack{r=1\\ j\notin P_r}}^{N_i} \tilde{g}^r_i \left( w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2$$

$$j \neq i : Q^i_{ij} = Q^i_{ji} = - \sum_{\substack{r=1\\ j\in P_r}}^{N_i} \tilde{g}^r_i \left( d^i_r - w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2 + \sum_{\substack{r=1\\ j\notin P_r}}^{N_i} \tilde{g}^r_i \left( w^i_r \sum_{\substack{s\in S_r\\ s\neq r\\ j\in P_s}} \tilde{b}^i_{s,r} \right) I_2$$

$$j \neq k \neq i \neq j : Q^i_{jk} = Q^i_{kj} = O_2$$

In the above notation, $l_r$ denotes the relation level of relation $r$ and $N_i$ the number of relations of agent $i$. An immediate conclusion of these equations is that the matrix $Q_i$ is symmetric, i.e. $Q_i = Q_i^T$. We also have

$Q^i_{ij} = Q^i_{ji} = -Q^i_{jj}$. In the following, we will use the notation $Q_{jk}$ both for the matrix as well as its diagonal elements which are equal. We examine the positive definiteness of the matrix $M$ by use of the following theorems:

**Theorem 2.1 (Gersgorin)** [10]: *Given a matrix $A \in \Re^{n\times n}$ then all its eigenvalues lie in the union of n discs:*

$$\bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \right\} \triangleq \bigcup_{i=1}^{n} R_i(A) \triangleq R(A)$$

*Each of these discs is called a Gersgorin disc of $A$.*

**Corollary 2.2** [10]: *Given a matrix $A \in \Re^{n\times n}$ and $n$ positive real numbers $p_1, \ldots, p_n$ then all its eigenvalues of $A$ lie in the union of n discs:*

$$\bigcup_{i=1}^{n} \left\{ z : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{\substack{j=1 \\ j\neq i}}^{n} p_j \, |a_{ij}| \right\}$$

A key point of Corollary 2.2 is that if we bound the first $n/2$ Gersgorin discs of a matrix $A$ sufficiently away from zero, then an appropriate choice of the numbers $p_1, \ldots, p_n$ renders the remaining $n/2$ discs sufficiently close to the corresponding diagonal elements. Hence, by ensuring the positive definiteness of the eigenvalues of the matrix $M$ corresponding to the first $n/2$ rows, then we can render the remaining ones sufficiently close to the corresponding diagonal elements. This fact will be made clearer in the analysis that follows.

Some useful bounds are obtained by the following lemma:

**Lemma 2.3**: *The following bounds hold for the terms* $Q_{ii}^i, Q_{ii}^j, \sigma_i$

$$\sigma_i(\varepsilon) \in \begin{cases} \left[ -Y\left(\frac{1}{k}+\frac{8}{9}\right) - \frac{\gamma_{di}}{k}, \underbrace{-\frac{Y}{k} - \frac{\gamma_{di}}{k}}_{\sigma_i(0)} \right], 0 \le \varepsilon \le \varepsilon^* \\ \left[ -Y\left(\frac{1}{k}+\frac{8}{9}\right) - \frac{\gamma_{di}}{k}, \underbrace{-\frac{\gamma_{di}}{k}}_{\sigma_i(X)} \right], X \ge \varepsilon \ge \varepsilon^* \end{cases}$$

$$0 < Q_{ii}^i < \left| Q_{ii}^i \right|_{\max} < \infty$$

*and*

$$0 < Q_{ii}^j < \left| Q_{ii}^j \right|_{\max} < \infty$$

**Proof**: : For $0 \le \varepsilon \le X$ we have

$$\sigma_i(\varepsilon) = \varepsilon\sigma(\varepsilon) - \frac{\gamma_{di} + f_i(\varepsilon)}{k} = \varepsilon\sigma(\varepsilon) - \frac{f_i(\varepsilon)}{k} - \frac{\gamma_{di}}{k}$$

$$0 < Q_{ii}^i < \left| Q_{ii}^i \right|_{\max} < \infty$$

and

$$0 < Q_{ii}^j < \left| Q_{ii}^j \right|_{\max} < \infty$$

and

$$\begin{aligned} &\varepsilon\sigma(\varepsilon) - \tfrac{f_i(\varepsilon)}{k} = \sum_{j=0}^{3}\left(j - \tfrac{1}{k}\right) a_j \varepsilon^j = \\ &= -\tfrac{a_0}{k} + \left(2 - \tfrac{1}{k}\right) a_2\varepsilon^2 + \left(3 - \tfrac{1}{k}\right) a_3 \varepsilon^3 = \\ &= \tfrac{Y}{kX^2} \underbrace{\left( -X^2 - 3(2k-1)\varepsilon^2 + 2(3k-1)\frac{\varepsilon^3}{X} \right)}_{\zeta_1(\varepsilon)} \end{aligned}$$

We have

$$\begin{aligned} &\zeta_1' = -6(2k-1)\varepsilon + 6(3k-1)\tfrac{\varepsilon^2}{X} = \\ &= 6\varepsilon\left\{ -(2k-1) + (3k-1)\tfrac{\varepsilon}{X} \right\} \end{aligned}$$

The critical values are $\zeta_1' = 0 \Leftrightarrow \varepsilon = 0$ or $\varepsilon = \frac{2k-1}{3k-1}X = \varepsilon^*$ and $\zeta_1'' = -6(2k-1) + 12(3k-1)\frac{\varepsilon}{X}$ Hence $\zeta_1''(0) = -6(2k-1) < 0$(local maximum),

$\zeta_1''(\varepsilon^*) = 6(2k-1) > 0$(local minimum). We have

$$\begin{array}{l} \varepsilon^*\sigma(\varepsilon^*) - \frac{f_i(\varepsilon^*)}{k} = \\ \frac{Y}{kX^2}\left(\begin{array}{l} -X^2 - 3(2k-1)\left(\frac{2k-1}{3k-1}\right)^2 X^2 + \\ 2(3k-1)\left(\frac{2k-1}{3k-1}\right)^3 X^2 \end{array}\right) = \\ = Y\left(-\frac{1}{k} - \frac{(2k-1)^3}{k(3k-1)^2}\right) \overset{k>>0}{\geq} -Y\left(\frac{1}{k}+\frac{8}{9}\right) \end{array}$$

It can easily be seen that $\sigma_i(\varepsilon)$ is strictly increasing for $\varepsilon > \varepsilon^*$ hence

$$\sigma_i(\varepsilon) \in \left\{ \begin{array}{l} \left[ -Y\left(\frac{1}{k}+\frac{8}{9}\right) - \frac{\gamma_{di}}{k}, \underbrace{-\frac{Y}{k} - \frac{\gamma_{di}}{k}}_{\sigma_i(0)} \right], 0 \leq \varepsilon \leq \varepsilon^* \\ \left[ -Y\left(\frac{1}{k}+\frac{8}{9}\right) - \frac{\gamma_{di}}{k}, \underbrace{-\frac{\gamma_{di}}{k}}_{\sigma_i(X)} \right], X \geq \varepsilon \geq \varepsilon^* \end{array} \right.$$

This establishes the negative definiteness of $\sigma_i$. We also have established that

$$|\sigma_i(\varepsilon)| \leq Y\left|\frac{1}{k}+\frac{8}{9}\right| + \left|\frac{\gamma_{di}}{k}\right|$$

We now turn our attention to the terms $Q_{ii}^i, Q_{ii}^j$. For all $i$ there is always at least one $r$ such that $\tilde{g}_i^r > 0$. We examine the term $d_r^i - w_r^i \sum\limits_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i$. For $\tilde{b}_r^i \to 0, b_r^i \neq 0$, we have $w_r^i \to 0, d_r^i \to 1$ so that $d_r^i - w_r^i \sum\limits_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i \to 1$. The same happens when $\tilde{b}_r^i, b_r^i \to 0$. For $\tilde{b}_r^i \neq 0, b_r^i \to 0$, $w_r^i \to 0, d_r^i > 1$, so that $d_r^i - w_r^i \sum\limits_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i > 1$. For $\tilde{b}_r^i, b_r^i \neq 0$, we have $d_r^i - w_r^i \sum\limits_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i > 1$ because in this case the term $\frac{w_r^i \sum\limits_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i}{\frac{\lambda\left(\tilde{b}_r^i\right)^{1/h}}{\left(b_r^i + \left(\tilde{b}_r^i\right)^{1/h}\right)^2}} < 1$ for sufficiently large $h$ which is always finite in a bounded workspace. Since $\tilde{g}_i^r > 0$ for at least one $r$, and recalling that $Q_{ii}^i = \sum\limits_{r=1}^{N_i} \tilde{g}_i^r \left( d_r^i - w_r^i \sum\limits_{\substack{s\in S_r \\ s\neq r}} \tilde{b}_{s,r}^i \right) l_r I_2$ we can see that we always have $Q^i_{ii} > 0$. The same procedure applies to terms of the form $Q^j_{ii}$. A finite upper bound for these terms always exists in a bounded workspace. This establishes the positive definiteness and boundness of the terms $Q^i_{ii}, Q^j_{ii}$. $\diamondsuit$

Let us examine the Gersgorin discs of the first half rows of the matrix $M$. We denote this procedure as $M_1 - M_2$, as the main diagonal elements of $M_1$ are "compared" with the corresponding raw elements of $M_2$. Note that the submatrices $M_1, M_2$ are both diagonal, therefore the only nonzero elements of raw $i$ of the $4N \times 4N$ matrix $M$ are the elements $M_{ii}, M_{i,2N+i}$ where of course $1 \leq i \leq 2N$ as we calculate the Gersgorin discs of the first half rows of the matrix $M$. We have:

$$
\begin{array}{l}
|z - M_{ii}| \leq \frac{1}{p_i} \sum\limits_{j \neq i} p_j \, |M_{ij}|, 1 \leq i \leq 2N \Rightarrow \\
\left| z - A_i^{-2(1+1/k)} K_i G_i^2 \right| \leq \frac{p_{2N+i}}{p_i} \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i Q^i_{ii} \right| \Rightarrow \\
\Rightarrow z \geq A_i^{-2(1+1/k)} K_i G_i^2 - \frac{p_{2N+i}}{p_i} \left| A_i^{-2(1+1/k)} \sigma_i K_i G_i Q^i_{ii} \right|
\end{array}
$$

We examine the following three cases:

- $G_i < \varepsilon$ At a critical point in this region, the corresponding eigenvalue tends to zero, so that the derivative of the Lyapunov function could achieve zero values. However, the third property of the definition indicates that $\varphi_i$ is a Morse function, hence its critical points are isolated[15]. Thus the set of initial conditions that lead to saddle points are sets of measure zero[27].

- $G_i > X$ The corresponding eigenvalue is guaranteed to be positive as long as:

$$
\begin{array}{l}
z > 0 \Leftarrow A_i^{-2(1+1/k)} K_i \left( G_i - \frac{p_{2N+i}}{p_i} \left| \sigma_i Q^i_{ii} \right| \right) > 0 \Leftarrow \\
G_i \geq X > \frac{p_{2N+i}}{p_i} \left| \sigma_i Q^i_{ii} \right| = \frac{\gamma_{di}}{k} \frac{p_{2N+i}}{p_i} \left| Q^i_{ii} \right| \Leftarrow \\
\Leftarrow k > \frac{(\gamma_{di})_{\max}}{X} \frac{p_{2N+i}}{p_i} \left| Q^i_{ii} \right|_{\max}
\end{array}
$$

- $0 < \varepsilon \leq G_i \leq X$

$$
\begin{array}{l}
z > 0 \Leftarrow \varepsilon > \left\{ Y \left| \frac{1}{k} + \frac{8}{9} \right| + \left| \frac{\gamma_{di}}{k} \right| \right\} \frac{p_{2N+i}}{p_i} \left| Q^i_{ii} \right|_{\max} \Leftarrow \\
\Leftarrow \varepsilon > 2 \max \left\{ \begin{array}{c} 2 \max \left\{ \frac{Y}{k}, \frac{8Y}{9} \right\}, \\ \left| \frac{(\gamma_{di})_{\max}}{k} \right| \end{array} \right\} \frac{p_{2N+i}}{p_i} \left| Q^i_{ii} \right|_{\max} \\
\overset{Y \leq \frac{\Theta_1}{k}}{\Leftarrow} k > 2 \max \left\{ \begin{array}{c} 2\sqrt{\frac{\Theta_1}{\varepsilon}}, \frac{16\Theta_1}{9\varepsilon}, \\ \frac{(\gamma_{di})_{\max}}{\varepsilon} \end{array} \right\} \frac{p_{2N+i}}{p_i} \left| Q^i_{ii} \right|_{\max}
\end{array}
$$

A key point is that there is no restriction on how to select the terms $\frac{p_{2N+i}}{p_i}$. This will help us in deriving bounds that guarantee the positive definiteness of the matrix $M$.

Let us examine the Gersgorin discs of the second half rows of the matrix $M$. Likewise, we denote this procedure as $M_3 - M_4$. The discs of Corollary 5.3 are evaluated:

$$\begin{array}{l} |z - M_{ii}| \le \sum\limits_{j \ne i} \frac{p_j}{p_i} |M_{ij}|, 2N+1 \le i \le 4N, 1 \le j \le 4N \Rightarrow \\ \Rightarrow |z - (M_4)_{ii}| \le R_i(M_3) + R_i(M_4) \end{array}$$

where

$$(M_4)_{ii} = \sum_j K_i A_i^{-(1+1/k)} A_j^{-(1+1/k)} \sigma_j \sigma_i Q_{ii}^i Q_{ii}^j$$

and

$$\begin{array}{l} R_i(M_3) = \sum\limits_{j=1}^{2N} \frac{p_j}{p_i} \left| (M_3)_{ij} \right| = \\ = \sum\limits_{j=1}^{2N} \frac{p_j}{p_i} \left| \sum\limits_l A_l^{-(1+1/k)} \sigma_l A_j^{-(1+1/k)} K_j G_j Q_{ij}^l \right| \\ R_i(M_4) = \sum\limits_{\substack{j=2N+1 \\ j \ne i}}^{4N} \frac{p_j}{p_i} \left| (M_4)_{ij} \right| = \\ = \sum\limits_{j \ne i} \frac{p_j}{p_i} \left| \sum\limits_l (A_l A_j)^{-(1+1/k)} \sigma_l \sigma_j K_j Q_{ij}^l Q_{jj}^j \right| \end{array}$$

A sufficient condition for the positive definiteness of the corresponding eigenvalue for raw $i$ is then:

$$\begin{array}{l} (M_4)_{ii} > R_i(M_3) + R_i(M_4) \Leftarrow \\ \Leftarrow (M_4)_{ii} > \max \{2R_i(M_3), 2R_i(M_4)\} \end{array}$$

We first show that we always have $R_i(M_3) \ge R_i(M_4)$. By taking into account the relations $Q_{jk}^i = Q_{kj}^i = 0, Q_{ij}^i = -Q_{jj}^i, j \ne i \ne k \ne j$ and expanding it is easy to see that

$$\begin{array}{l} R_i(M_3) = -\frac{1}{p_i} \sum\limits_{j=1}^{2N} p_j \left\{ \begin{array}{l} A_j^{-2(1+1/k)} \sigma_j K_j G_j Q_{ii}^j + \\ (A_j A_i)^{-(1+1/k)} \sigma_i K_j G_j Q_{jj}^i \end{array} \right\} = \\ = -\sum\limits_{\substack{j=1 \\ j \ne i}}^{2N} \frac{p_j}{p} \left\{ \begin{array}{l} \underbrace{A_j^{-2(1+1/k)} \sigma_j K_j G_j Q_{ii}^j}_{(I)} + \\ \underbrace{(A_j A_i)^{-(1+1/k)} \sigma_i K_j G_j Q_{jj}^i}_{(II)} \end{array} \right\} \\ -2\frac{p_i}{p} A_i^{-2(1+1/k)} \sigma_i K_i G_i Q_{ii}^i \end{array}$$

where without loss of generality we choose $p_i = p, 2N+1 \leq i \leq 4N$.We also have

$$R_i(M_4) = \sum_{j \neq i} \left\{ \begin{array}{l} \underbrace{A_j^{-2(1+1/k)} \sigma_j^2 K_j Q_{ii}^j Q_{jj}^j}_{(I)} + \\ \underbrace{\left(A_i A_j\right)^{-(1+1/k)} \sigma_i \sigma_j K_j Q_{jj}^i Q_{jj}^j}_{(II)} \end{array} \right\}$$

By comparing the terms $(I)$ and $(II)$ in the last two equations we have:

$$\begin{array}{l} (I) : -\frac{p_j}{p} A_j^{-2(1+1/k)} \sigma_j K_j G_j Q_{ii}^j \geq A_j^{-2(1+1/k)} \sigma_j^2 K_j Q_{ii}^j Q_{jj}^j \\ \Leftarrow -\frac{p_j}{p} \sigma_j G_j \geq \sigma_j^2 Q_{jj}^j \Leftrightarrow \sigma_j \left( \sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \right) \leq 0 \\ \overset{\sigma_j \leq 0}{\Leftrightarrow} \sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \geq 0 \\ (II) : -\frac{p_j}{p} \left(A_j A_i\right)^{-(1+1/k)} \sigma_i K_j G_j Q_{jj}^i \geq \\ \geq \left(A_i A_j\right)^{-(1+1/k)} \sigma_i \sigma_j K_j Q_{jj}^i Q_{jj}^j \\ \Leftarrow -\frac{p_j}{p} \sigma_i G_j \geq \sigma_i \sigma_j Q_{jj}^j \Leftrightarrow \sigma_i \left( \sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \right) \leq 0 \\ \overset{\sigma_i \leq 0}{\Leftrightarrow} \sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \geq 0 \end{array}$$

Thus, the condition $\sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \geq 0$ guarantees that $R_i(M_3) \geq R_i(M_4) \forall i$. Hence it suffices to show that $(M_4)_{ii} > 2R_i(M_3)$. The fact that $\sigma_j Q_{jj}^j + \frac{p_j}{p} G_j \geq 0$ is a direct conclusion of the results of procedure $M_1 - M_2$. For example, by the last bound on $k$ we have:

$$\begin{array}{l} k > 2 \max \left\{ 2\sqrt{\frac{\Theta_1}{\varepsilon}}, \frac{16\Theta_1}{9\varepsilon}, \frac{\left(\gamma_{dj}\right)_{\max}}{\varepsilon} \right\} \frac{p}{p_j} \left| Q_{jj}^j \right|_{\max} \\ \underset{G_j \geq \varepsilon}{\overset{Y \leq \frac{\Theta_1}{k}}{\Rightarrow}} G_j > 2 \max \left\{ 2 \max \left\{ \frac{Y}{k}, \frac{8Y}{9} \right\}, \left| \frac{\left(\gamma_{dj}\right)_{\max}}{k} \right| \right\} \frac{p}{p_j} \left| Q_{jj}^j \right|_{\max} \\ \Rightarrow G_j > \left\{ Y \left| \frac{1}{k} + \frac{8}{9} \right| + \left| \frac{\gamma_{dj}}{k} \right| \right\} \frac{p}{p_j} \left| Q_{jj}^j \right|_{\max} \\ \Rightarrow \frac{p_j}{p} G_j > \left| \sigma_j \right|_{\max} \left| Q_{jj}^j \right|_{\max} \Rightarrow \sigma_j Q_{jj}^j + \frac{p_j}{p} G_j > 0 \end{array}$$

The fact that $(M_4)_{ii} > 0$ is guaranteed by Lemma 5.4. This lemma also guarantees that there is always a finite upper bound on the terms

$$\left| (M_3)_{ij} \right| = \left| \sum_l A_l^{-(1+1/k)} \sigma_l A_j^{-(1+1/k)} K_j G_j Q_{ij}^l \right|$$

We have

$$
\begin{array}{l}
(M_4)_{ii} > 2R_i(M_3) = 2\sum\limits_{j=1}^{2N} \frac{p_j}{p}\left|(M_3)_{ij}\right| \Leftarrow \\
p > \frac{4N}{(M_4)_{ii}} \max\limits_j \left\{p_j\left|(M_3)_{ij}\right|\right\}, \\
2N+1 \le i \le 4N, 1 \le j \le 2N
\end{array}
$$

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